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Balanced generic circuits without long paths

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Abstract

We call a graph $G = (V, E)$ a (k, ℓ)-*circuit* if $|E| = k|V| - \ell + 1$ and every $X \subset V$ with $2 \leq |X| \leq |V| - 1$ satisfies $i(X) \leq k|X| - \ell$. A $(2, 3)$-circuit is also called a *generic circuit*. We say that a graph is *balanced* if the difference between its maximum and minimum degree is at most 1. J. Graver, B. Servatius and H. Servatius asked in [7] whether a balanced generic circuit always admits a decomposition into two disjoint Hamiltonian paths. We show that this does not hold, moreover there are balanced $(k, k + 1)$-circuits for all $k \geq 2$ which do not contain any Hamiltonian path nor a path longer than $|V|^\lambda$ for $\lambda > \frac{\log 8}{\log 9} \simeq 0,9464$.

1 Introduction

All graphs considered are undirected and simple (i.e., may not contain loops or multiple edges). Let $G = (V, E)$ be a graph, and k, ℓ nonnegative integers. Let $i(X)$ denote the number of edges induced by a subset X of V. G is called (k, ℓ)-*sparse* if $i(X) \leq k|X| - \ell$ for every subset $X \subseteq V$ with $|X| \geq 2$. It can easily be checked that the (k, ℓ)-sparse subgraphs of a given graph form the independent sets of a matroid, which is called a (k, ℓ)-*count-matroid* (see [18]). The circuits of these matroids that are the minimal not (k, ℓ)-sparse graphs are the graphs with exactly $k|V| - \ell + 1$ edges whose every proper subgraph is (k, ℓ)-sparse. Let K_n denote the complete graph on n vertices. If G is a circuit in the (k, ℓ)-count-matroid of $K_{|V|}$ we call it a (k, ℓ)-*circuit*. The $(2, 3)$-count-matroid is isomorphic to the 2-dimensional generic rigidity matroid (see [12]) and we call a $(2, 3)$-circuit also a *generic circuit*.

A well-known result of Nash-Williams [13] says that a graph G is decomposable into k spanning forests if and only if $i(X) \leq k(|X| - 1)$ for all nonempty subsets $X \subseteq V$. From this theorem it follows that every generic circuit is decomposable into two spanning trees, moreover we obtain the following proposition:

Proposition 1.1. A graph G is a generic circuit if and only if it can be decomposed into two spanning trees such that no pair of proper subtrees, except single vertices, spans the same vertex set.

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Having this result one may ask whether a tree decomposition with some special properties exist. Graver et al. posed the following problem:

Open question 1.2 ([7], Exercise 4.69)**.** Does every generic circuit with vertices of degree 3 and 4 only, have a two tree decomposition into two paths?

Note that the smallest $(k, k+1)$-circuit is K_{2k} since a graph with $k|V| - k$ edges cannot be simple if it has less than $2k$ vertices. A balanced $(k, k+1)$-circuit has vertices with degree $2k-1$ and $2k$ only and the number of vertices with degree $2k-1$ is exactly $2k$.

In this note we show balanced $(k, k+1)$-circuits for all $k \geq 2$ which do not contain a Hamiltonian path. Thus a balanced generic circuit does not always have a decomposition demanded in Open question 1.2. Moreover, we have a stronger result on the length of the longest paths in balanced $(k, k+1)$-circuits. For a graph G let $h(G)$ and $h^*(G)$ denote the length of a maximum cycle and the number of vertices in a maximum path of G, respectively. Following [9] we define the *shortness exponent* $\sigma(\mathcal{G})$ for a family $\mathcal{G}$ of graphs as follows:

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log h(G)}{\log |V(G)|}.$$

Concerning paths instead of cycles we also define the parameter $\sigma^*(\mathcal{G})$ in a similar way but with $h^*(G)$ in place of $h(G)$. Let $\mathcal{C}^{bal.}_{k,k+1}$ denote the family of balanced $(k, k+1)$-circuits. What we prove is as follows:

Theorem 1.3. $\sigma^*(\mathcal{C}^{bal.}_{k,k+1}) \leq \frac{\log 8}{\log 9}$ for all $k \geq 2$.

2 Preliminaries

Let $G = (V, E)$ be a graph. For $X, Y \subseteq V$ let $E(X, Y)$ denote the set of edges with one endpoint in X and another in Y. For a subset $X \subset V$ we call $E(X, V - X)$ the *edge-cut* corresponding to X. We also say that X corresponds to the edge-cut. If $2 \leq |X| \leq |V| - 2$ then the edge-cut is *nontrivial*. We say that a graph G is *essentially k-edge-connected*, if all nontrivial edge-cuts of G contain at least k edges.

Our construction will be based on the observation that balanced $(k, k+1)$-circuits are just the $2k$-regular essentially $(2k+2)$-edge-connected graphs minus a vertex in the following sense:

Lemma 2.1.

(i) Let $G = (V, E)$ be a balanced $(k, k+1)$-circuit. If we add a new vertex s to G and connect it to the vertices of G with degree $2k-1$ then the obtained graph $G' = (V', E')$ is $2k$-regular and essentially $(2k+2)$-edge-connected.

(ii) Let $G' = (V', E')$ be a $2k$-regular essentially $(2k+2)$-edge-connected graph. If we omit an arbitrary vertex s of G' then the obtained graph $G = (V, E)$ is a balanced $(k, k+1)$-circuit.

Proof:

(i) It is clear that G' is $2k$-regular. Suppose that it is not essentially $(2k+2)$-edge-connected. Then there is a subset $X \subseteq V'$, $2 \leq |X| \leq |V'|-2$ with $d_{G'}(X) \leq 2k$. (As G' is an Eulerian graph $d_{G'}(X) = 2k+1$ cannot hold.) We may assume that $s \notin X$. Then $2i_G(X) = 2i_{G'}(X) = (\sum_{v \in X} d_{G'}(v)) - d_{G'}(X) \geq 2k|X| - 2k$, a contradiction.

(ii) Let $X \subseteq V$ be a subset with $2 \leq |X| \leq |V|-1$. Then the same calculation as above admits that $i_G(X) = i_{G'}(X) \leq k|X| - k - 1$. □

We call the graph G' obtained from a balanced $(k, k+1)$-circuit G as described in Lemma 2.1 (i) the *underlying regular graph* of G.

In the case of balanced generic circuits we obtain 4-regular essentially 6-edge-connected graphs. Now we can easily prove that not all balanced generic circuits can be decomposed into two Hamiltonian paths. It is clear that if a balanced generic circuit G admits such a decomposition, then the four end-vertices of the two Hamiltonian paths must be disjoint and can only be the four vertices with degree 3. Thus in the underlying 4-regular graph G' they extend to a decomposition into two Hamiltonian cycles. Therefore, it is sufficient to show a 4-regular essentially 6-edge-connected graph which does not have a decomposition into two Hamiltonian cycles.

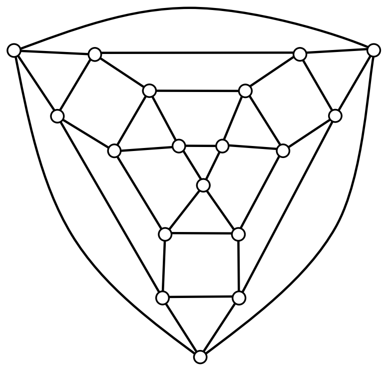


Figure 1: The graph given by Grünbaum and Malkevitch.

Grünbaum and Malkevitch [8] gave an example for a 4-regular 4-connected planar graph without a Hamiltonian decomposition (see Figure 1). One can easily check that it is essentially 6-edge-connected therefore this is a good example for our problem as well.[1] (For an elegant proof for this graph not to be decomposable, using the fact that it is the medial graph of the Herschel graph, see [3].)

3 3-regular 3-connected graphs without long paths

In our further constructions we will use 3-regular 3-connected graphs as a starting point. In this section we review some of their properties. Let us denote the set of these graphs with $\mathcal{B}$.

[1] Shortly before finishing this paper we learned that this result was observed independently in [11].

Examining their shortness parameters Bondy and Simonovits [5] constructed graphs certifying that $\sigma(\mathcal{B}) \leq \frac{\log 8}{\log 9} \simeq 0.9464$ and Jackson [10] gave the lower bound $\sigma(\mathcal{B}) \geq \log_2(1+\sqrt{5})-1 \simeq 0.6942$. The lower bound was recently improved in [2] to $\lambda \simeq 0.753$, where λ is the real root of $4^{1/x} - 3^{1/x} = 2$. Now considering the case of longest paths we have the same lower bound for $\sigma^*(\mathcal{B})$ as $h^*(G) \geq h(G)$ for any graph G. On the other hand Bondy and Locke [4] showed that $h^*(G) \leq \frac{3}{2}h(G) - 2$ for all 3-regular 3-connected graphs G, which implies $\sigma^*(\mathcal{B}) = \sigma(\mathcal{B})$. We can also observe that the construction of Bondy and Simonovits actually works for paths as well. The fact that a graph constructed such way does not contain a Hamiltonian path either was also used by Singleton [16]. In what follows we will briefly sketch the construction.

First we need some additional notation. Let $G = (V, E)$ be a graph, $X \subset V$. We will denote by $G[X]$ the subgraph of G induced by X. The edge set of $G[X]$ will be denoted by $I_G(X)$. Let $s \notin V$. We say that the (multi)graph $G' = (V', E')$ is obtained from G by *contracting* X into s if $V' = V - X + s$ and E' is obtained from E by deleting the edges $I_G(X)$ and replacing all edges $uv : u \in X$ and $v \in V - X$ with an edge sv. Note that this definition admits G' to have multiple edges but in our case the obtained graphs will be simple. We will also denote with G/X the graph obtained from G by contracting X. A *graph-fragment* is a 'graph' which can also include *semi-edges*, which are 'edges' with only one end-vertex (see Figure 2). The graph-fragment corresponding to X in G is the vertex set X with the edge set $I_G(X)$ and the semi-edge set corresponding to $E(X, V - X)$. Let $G' = (V', E')$ be a graph, $s \in V'$ and let A be a graph-fragment with vertex set X, $X \cap V' = \emptyset$. We say that the graph $G = (V, E)$ is obtained from G' by *inflating* the graph-fragment A into s if $V = (V' - s) \cup X$, the graph-fragment corresponding to X in G is A and if we contract X into s in G then we obtain G'.

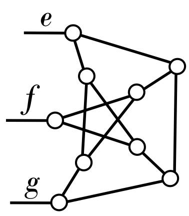


Figure 2: The graph-fragment A_P with three semi-edges e, f, g.

Considering 3-regular 3-connected graphs we can establish some special properties of these operations. First observe that if G is 3-regular then it is 3-connected if and only if it is 3-edge-connected.

Proposition 3.1. Let $G = (V, E)$ be a 3-regular 3-connected graph and let $C = E(X, V - X)$ be a nontrivial edge-cut of size 3. Then G/X and $G/(V - X)$ are also 3-regular 3-connected (simple) graphs.

Proof: They are simple because C contains of 3 independent edges by the 3-(vertex)-connectivity of G. The 3-connectivity follows from the fact that any edge-cut in G/X

or $G/(V-X)$ is also an edge-cut in G. □

Proposition 3.2. Let $G' = (V' + s, E')$ be a 3-regular 3-connected graph. Let $G = (V, E)$ be obtained from G' by inflating a graph-fragment with vertex-set X into s. If G/V' is 3-regular and 3-connected then so is G.

Proof: First observe that $G[X]$ and $G[V']$ are 2-connected because they are both obtained from a 3-connected graph by a deletion of a vertex. Now let $u, v \in G$ be different vertices. We need to show that $G - \{u, v\}$ is connected. If $\{u, v\} \subseteq V'$ then $(G - \{u, v\})/X$ is connected by the 3-connectivity of G' and inflating a connected fragment into s preserves connectivity. The case $\{u, v\} \subseteq X$ is similar. If $|\{u, v\} \cap V'| = 1$, say $u \in V'$ and $v \in X$ then $G - \{u, v\}$ consists of the two connected subgraphs $G[V'] - u$ and $G[X] - v$ and at least one edge connecting them. □

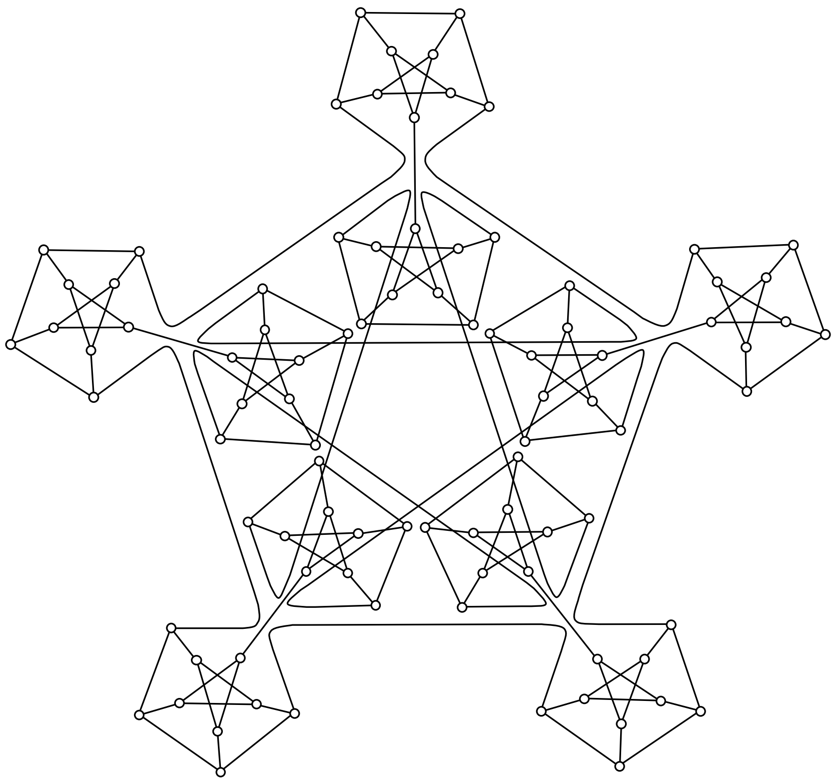


Figure 3: The graph S_1 constructed by Bondy and Simonovits.

Let $G_P = (V_P, E_P)$ be the Petersen graph and let A_P be the graph-fragment corresponding to $V_P - v$ in G_P with an arbitrary vertex $v \in V_P$ (see Figure 2). Now we can describe the sequence of graphs $S_0, S_1, \ldots, S_i, \ldots$ constructed by Bondy and Simonovits. Let $S_0 = G_P$ and if S_i is already constructed let S_{i+1} be obtained from S_i by inflating a copy of A_P into all its vertices (see Figure 3). By Proposition 3.2 all S_i are 3-connected. Observe that if a copy of A_P is included in a graph $G = (V, E)$

and P is a path in G with none of its ends in A_P then P avoids at least one vertex of A_P. Otherwise, if we contract the vertices $V - V(A_P)$ in G we would obtain a Hamiltonian cycle in the contracted graph which is isomorphic to the Petersen graph. Because of this $h^*(S_{i+1}) \leq 8h^*(S_i) + 2$. Since $|V(S_i)| = 10 \cdot 9^i$ finally we get $\lim_{i\to\infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}$.

Now we consider other properties of 3-regular 3-connected graphs. According to Petersen's theorem [15] all these graphs have a perfect matching. Moreover, it is an easy consequence of Tutte's theorem [17] that all edges of the graph are included in some perfect matching. We call a perfect matching M *admissible* if $|M \cap C| \leq 1$ for all edge-cuts C of size 3.

Lemma 3.3. Let $G = (V, E)$ be a 3-regular 3-connected graph and let $e \in E$ be an arbitrary edge of G. Then G has an admissible perfect matching which includes e.

Proof: If the size of every nontrivial edge-cut of G is greater than 3 then every perfect matching is admissible. Now suppose that there is a nontrivial edge-cut $C = E(X, V - X)$ of size 3. By Proposition 3.1 $G_1 = G/X$ and $G_2 = G/(V - X)$ are 3-regular and 3-connected too so by induction they both have an admissible perfect matching including an arbitrary designated edge.

Case 1: $e \in C$. Consider an admissible perfect matching M_i in G_i including e for $i = 1, 2$. We claim that the union M of M_1 and M_2 (with one copy of e) is an admissible perfect matching in G. To prove this let D be an edge-cut with the corresponding vertex-set Z. If Z or $V - Z$ is a subset of X or $V - X$ then D is also an edge-cut in G_1 or G_2 so it does not violate admissibility. Now suppose that D cuts both X and $V - X$. As we already observed in the proof of Proposition 3.2, $G[X]$ and $G[V - X]$ are 2-connected subgraphs. So $|D| \geq e(X \cap Z, X - Z) + e((V - X) \cap Z, (V - X) - Z) \geq 2 + 2 = 4$.

Case 2: $e \notin C$. We may assume by symmetry that $e \in I(V - X)$. Now consider an admissible perfect matching M_1 in G_1 which includes e. Let f be the edge in M_1 incident with the vertex in G_1 corresponding to X. Now let M_2 be an admissible perfect matching in G_2 which includes f. Just as in case 1 the perfect matching combining M_1 and M_2 is admissible in G. □

4 Our construction

Now we present our construction of balanced $(k, k+1)$-circuits which do not contain long paths. It is clear that $h^*(G) \leq h^*(G')$ for a $(k, k+1)$-circuit G and its underlying $2k$-regular graph G' so to prove Theorem 1.3 it suffices to show the same proposition for $2k$-regular essentially $(2k + 2)$-edge-connected graphs by Lemma 2.1.

For the construction we use a special operation similar to inflating that we call *blackberrizing*. This operation 'substitutes' all vertices of a t-regular graph with a special auxiliary graph that we call a *berry*. A berry is a simple graph that consists of two types of vertices that we call *inner vertices* and *outer vertices*, respectively (see Figure 4). We will use a berry with t outer vertices for blackberrizing a t-regular

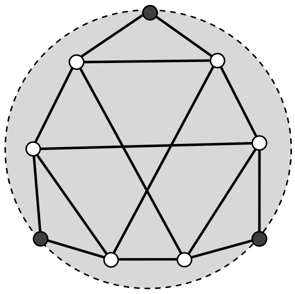


Figure 4: A copy of a berry which can be used in the case $k = 2$. The black vertices are the outer vertices, the white ones are the inner vertices.

graph $G = (V, E)$. We take $|V|$ disjoint copies of the berry and correspond them to the vertices of G. If two vertices are adjacent in G we choose one outer vertex of both corresponding berries and identify them. For the t neighbors of a vertex we use the t different outer vertices (in any order). Finally the inner vertices of a berry correspond to a certain vertex of the basic graph and an outer vertex correspond to an edge of the basic graph (see Figure 5).

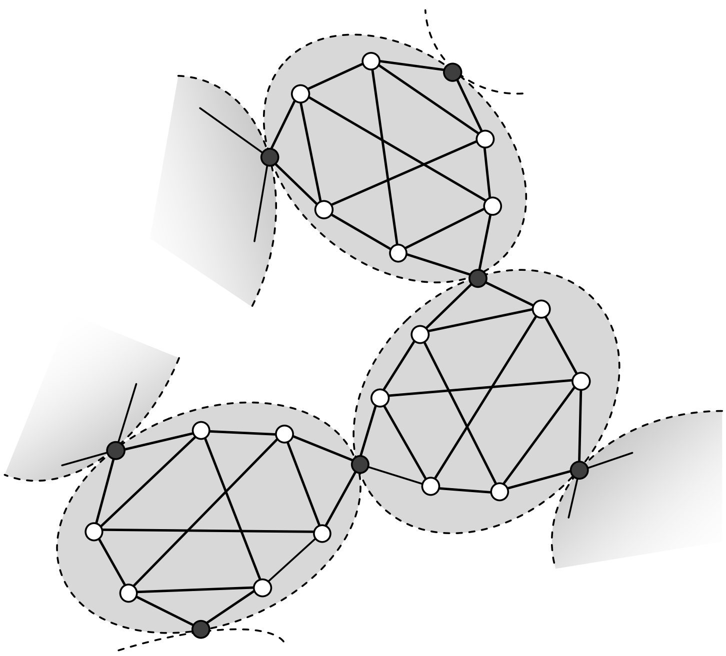


Figure 5: Adjacencies of berries in the blackberry graph.

To get $2k$-regular graphs by blackberrizing we need that the inner vertices of the berry are of degree $2k$ and the outer vertices are of degree k. In our construction we will use the 3-regular graphs described in Section 3 as basic graphs.

Since for odd k there are no graphs with three vertices of degree k while the other vertices are of degree $2k$ we need another operation too that we call *parallel blackberrizing*. This operation substitutes the vertices of a t-regular graph that has a perfect matching. The substitution will be practically as before, but we use the copies of two type of berries that have a designated outer vertex and we substitute one to one endpoint of each matching edges and one to the other endpoints. The designated outer vertex of a berry that is substituted to a node v will be identified with the designated outer vertex of the berry that is substituted to the other endpoint of the matching edge incident to v. Though this operation does not require that there are the same number of vertices in the two berries, from now we will assume that the two berries used in parallel blackberrizing have the same number of vertices. A graph that arises by (parallel) blackberrizing a regular graph is called a *blackberry graph*.

Therefore, when k is odd we will parallel blackberrize our graphs described in Section 3 that have an admissible perfect matching. We will use two berries which have inner vertices of degree $2k$, two non-designated outer vertices of degree k and a designated outer vertex of degree $k-1$ and $k+1$, respectively. Thus in both cases we get a $2k$-regular graph H.

Examining the properties of this operation we first show that if the basic graph does not contain a path 'too long', then neither does its blackberry graph. The proof is similar to the one in [14].

Proposition 4.1. Let S be a 3-regular graph and let H be its blackberry graph. Let b be the number of vertices in the berries used in the blackberrization. Then

(i) $h^*(H) \leq (b-1) \cdot h^*(S) + 1$,

(ii) $h^*(H) \leq |V(H)| - (|V(S)| - h^*(S))(b-3)$.

Proof: Take an arbitrary path P in H and let us consider the sequence of its edges $e_1, e_2, e_3, \ldots, e_m$. To this sequence we associate a sequence of vertices of S in the following way: we replace every edge by the vertex of S which the berry including the edge corresponds to. If the resulting sequence contains identical neighboring elements then we delete some until only one remains consecutively. Let $v_1, v_2, \ldots, v_r$ denote the remaining sequence.

It is easy to check that the consecutive members of this sequence are neighbors in S and thus it describes a walk in S. We claim that if $\{i,j\} \cap \{1,r\} = \emptyset$ with $i < j$ then $v_i \neq v_j$. Suppose for contradiction that $v_i = v_j$, let B be the berry corresponding to this vertex of S and let e_p and e_q be the edges of H corresponding to v_{i-1} and v_{j+1}, respectively. Then in the original path in H, between e_p and e_q we enter and leave B two times, which contradicts that B has only 3 outer vertices.

So if a vertex occurs more than once in the sequence $v_1, \ldots, v_r$ then it is v_1 or v_r. In this case delete v_1 or v_r or both until every vertex occurs exactly once. Now we obtained a path in S. For any edge e in H the vertex of S corresponding to the berry including e is present on this finally obtained path. We conclude that P contains edges from at most $h^*(S)$ berries. Since a path contains at most $b-1$ edges from a berry we have (i). On the other hand, if a path do not use any edge from a berry then it avoids all its inner vertices, which implies (ii). □

Now we show that if we choose appropriate berries the blackberry graph is essentially $(2k+2)$-edge-connected. To obtain this, we prove more general results for (parallel) blackberrizing.

Lemma 4.2. Let $t \geq 3$ and $\alpha \geq 3$ be two integers and let G be a 3-connected t-regular simple graph and let β be a positive integer with $\beta \leq \frac{\alpha}{3}$. Let B be a berry with t outer vertices. Let $\mathcal{O}$ and $\mathcal{I}$ denote the set of the outer and the inner vertices of B respectively. Assume that B satisfies the following properties:

(i) $d_B(v) = \alpha - \beta,\ \forall v \in \mathcal{O}$,

(ii) $d_B(v) = 2\alpha - 2\beta,\ \forall v \in \mathcal{I}$,

(iii) if $X \subseteq \mathcal{I}, |X| \geq 2$ then $d_B(X) \geq 2\alpha$.

(iv) if $X \subseteq \mathcal{I} \cup \mathcal{O}, 2 \leq |X| \leq |\mathcal{I} \cup \mathcal{O}| - 2$ then $d_B(X) \geq \alpha + \beta$.

Then the graph H obtained from G by blackberrizing with B is $(2\alpha-2\beta)$-regular and essentially 2α-edge-connected.

Lemma 4.3. Let $t \geq 3$ and $\alpha \geq 4$ be two integers. Let G be a 3-connected t-regular simple graph with an admissible perfect matching M and let β be a positive integer with $\beta \leq \frac{\alpha-1}{3}$. Let B_1 and B_2 be two berries with t outer vertices. Let $\mathcal{O}_i$ and $\mathcal{I}_i$ denote the set of outer and inner vertices of B_i respectively for $i = 1, 2$. Assume that B_i satisfies the following properties for $i = 1, 2$:

(i)' for the designated outer vertex $u_i \in \mathcal{O}_i :\ d_{B_i}(u_i) = \alpha - \beta + (-1)^i$ and $d_{B_i}(v) = \alpha - \beta,\ \forall v \in \mathcal{O}_i - \{u_i\}$,

(ii)' $d_{B_i}(v) = 2\alpha - 2\beta,\ \forall v \in \mathcal{I}_i$,

(iii)' if $X \subseteq \mathcal{I}_i, |X| \geq 2$ then $d_{B_i}(X) \geq 2\alpha$.

(iv)' if $X \subseteq \mathcal{I}_i \cup \mathcal{O}_i, 2 \leq |X| \leq |\mathcal{I}_i \cup \mathcal{O}_i| - 2$ then $d_{B_i}(X) \geq \alpha + \beta + 1$.

Then the graph H obtained from G by parallel blackberrizing with B_1 and B_2 is $(2\alpha - 2\beta)$-regular and essentially 2α-edge-connected.

To use the above lemmas the following propositions will be useful.

Proposition 4.4. Let B be a berry with **3** outer vertices. Let $\mathcal{O}$ and $\mathcal{I}$ denote the set of the outer and the inner vertices of B respectively. If B satisfies properties (i)-(iii) from Lemma 4.2, then it also satisfies property (iv).

Proposition 4.5. Let B_1 and B_2 be two berries with **3** outer vertices. Let $\mathcal{O}_i$ and $\mathcal{I}_i$ denote the set of outer and inner vertices of B_i respectively for $i = 1, 2$. Assume that B_i satisfies the properties (i)'-(ii)' from Lemma 4.3 and it satisfies the following property for $i = 1, 2$:

(iii)* if $X \subseteq \mathcal{I}_i, |X| \geq 2$ then $d_{B_i}(X) \geq 2\alpha + 1 + (-1)^i$.

Then it also satisfies property (iv)' for $i = 1, 2$.

We prove Proposition 4.4 and Proposition 4.5 parallel. If we need some special argument for the proof of Proposition 4.5 we will put it between brackets.

Proof: [Let $B := B_i$, $\mathcal{O} := \mathcal{O}_i$, $\mathcal{I} := \mathcal{I}_i$.] By taking a complement we can assume that $|X \cap \mathcal{O}| \leq 1$. If $|X \cap \mathcal{O}| = 0$, then (iv)['] follows by (iii)[*]. Otherwise there are two cases. If $|X \cap \mathcal{I}| \geq 2$, then $d_B(X \cap \mathcal{I}) \geq 2\alpha[+1 + (-1)^i]$ by (iii)[*]. Since $d_B(X) \geq d_B(X \cap \mathcal{I}) - d_B(X \cap \mathcal{O})$ (iv)['] follows by (i)[']. If $|X \cap \mathcal{I}| = 1$, then $|X| = 2$, $d_B(X \cap \mathcal{I}) = 2\alpha - 2\beta$ by (ii)['] and $d_B(X \cap \mathcal{O}) \geq \alpha - \beta + (-1)^i$. Since the berries are simple graphs $d_B(X) \geq d_B(X \cap \mathcal{I}) + d_B(X \cap \mathcal{O}) - 2 \geq d_B(X \cap \mathcal{I}) \geq 2\alpha - 2\beta \geq \alpha + \beta[+1]$ by (i)[']-(ii)['], $\beta \leq \frac{\alpha[-1]}{3}$ and $\alpha \geq 3$. □

Now we will prove Lemma 4.2 and Lemma 4.3. If we need some special argument for the proof of Lemma 4.3 we will put it between brackets again. In the proof we will also use properties (iv) and (iv)'.

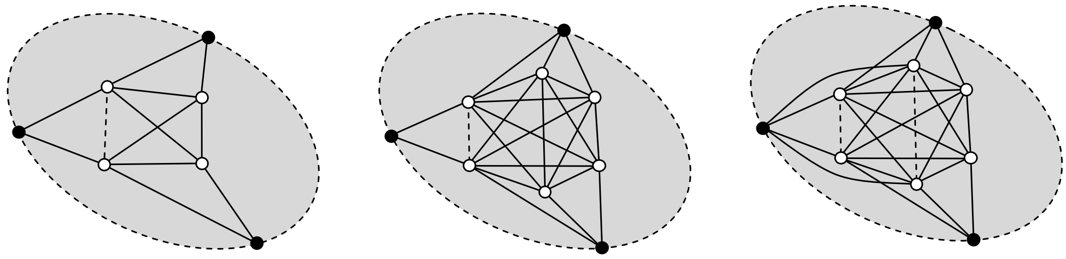
Proof: For $X \subset V(H)$ we call the corresponding edge-cut *proper* if X and $V - X$ each induce a berry. By 3-connectivity of G in a proper edge-cut there must be at least 3 berries cut. An edge-cut that cuts at least 3 berries includes at least $3(\alpha - \beta) \geq 2\alpha$ edges by (i)-(iv) and $\beta \leq \frac{\alpha}{3}$. [In (i)' there could happen that a berry-cut cuts only $\alpha - \beta - 1$ edges, but then it will belong to a matching-edge-cut in G. Since M is admissible there could be just one such a berry-cut if we cut only 3 berries. Hence if we cut 3 berries, then the edge cut includes at least $3(\alpha - \beta) - 1 \geq 2\alpha$ edges by (i)'-(iv)' and $\beta \leq \frac{\alpha-1}{3}$. If we cut at least 4 berries, then the edge-cut includes at least $4(\alpha - \beta - 1) \geq 2\alpha$ edges by (i)'-(iv)', $\beta \leq \frac{\alpha-1}{3}$ and $\alpha \geq 4$.] Therefore we only need to consider those cases when the edge-cut is non-proper and it cuts at most two berries.

If a nontrivial edge-cut cuts only one berry B', then for one of its corresponding sets $X \subseteq \mathcal{I}'$ where $\mathcal{I}'$ denotes the set of inner vertices of B'. Therefore since $|X| \geq 2$, $d(X) \geq 2\alpha$ by (iii)['].

Now consider the case when a nontrivial edge-cut cuts two berries B' and B''. Let $\mathcal{I}'$, $\mathcal{I}''$, $\mathcal{O}'$ and $\mathcal{O}''$ be the set of the inner and the outer vertices of B' and B'' respectively and let X be the set corresponding to the cut that is included by $V(B') \cup V(B'')$. Note that X contain an outer vertex if and only if it is an outer vertex of B' and B'' each because otherwise the edge-cut would cut another berry. Hence if $|X \cap V(B')| = |X \cap V(B'')| = 1$, then $|X \cap \mathcal{I}'| = |X \cap \mathcal{I}''| = 1$ because otherwise $|X| = 1$, a contradiction. Therefore in this subcase $d(X) = 2(2\alpha - 2\beta) \geq 2\alpha$ by (ii)['] and $\beta \leq \frac{\alpha}{3}$. Now by changing the indices we can assume that $|X \cap V(B')| \geq 2$. Then by (iv)['] $d_{B'}(X \cap V(B')) \geq \alpha + \beta[+1]$ and by (i)[']-(iv)['] $d_{B''}(X \cap V(B'')) \geq \alpha - \beta[-1]$ hence $d(X) \geq 2\alpha$. □

By *pinching* k edges of a graph we mean the following operation: take k independent edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ (that is they have no end-vertices in common), delete them, and add a new vertex s and the $2k$ edges $su_1, sv_1, su_2, sv_2, \ldots, su_k, sv_k$ to the graph.

Now we define the berries that we use to blackberrize the graphs described in Section 3. For any $k \geq 2$ we take the complete graph K_{2k} and we take a partition $\{V_2, V_3\}$ of its vertices with $|V_2| = |V_3| = k$. We add two new vertices v_2 and v_3 and we connect v_i to all vertices of V_i for $i = 1, 2$. For even k we pinch $\frac{k}{2}$ independent



(a) Berry B^2. (b) Berry B^3_1. (c) Berry B^3_2.

Figure 6: Outer vertices in black, inner vertices in white. The dashed edges are the pinched edges that are not included in the berry.

edges of K_{2k} with a new vertex v_1. We say that the graph we got is the berry B^k with the outer vertices v_1, v_2, v_3. (See Figure 6a.) For odd k we pinch $\frac{k+(-1)^i}{2}$ independent edges of K_{2k} with a new vertex v_1 so we get B^k_i with the designated outer vertex v_1 and non-designated outer vertices v_2, v_3 for $i = 1, 2$, respectively. (See Figure 6b and 6c.) (We note that in the case $k = 2$ it is easy to find smaller berries, e.g. a triangle which is a berry without any inner vertices works and using this berry the blackberry graph is in fact the line graph of the original one. One can see that if $k > 2$ the berry holding properties (i)['] and (ii)['] from Lemma 4.2 [Lemma 4.3] must have at least $2k$ inner vertices hence there are no simpler 'good' berries than the ones described here.)

We show that these berries hold the conditions of Lemma 4.2 [Lemma 4.3].

Proposition 4.6. Let $t = 3$, $\beta = 1$, let $k \geq 2$ be an even integer and let $\alpha = k + 1$. Then the berry $B := B^k$ holds the conditions of Lemma 4.2.

Proposition 4.7. Let $t = 3$, $\beta = 1$, let $k \geq 3$ be an odd integer and let $\alpha = k + 1$. Then the berries $B_1 := B^k_1$ and $B_2 := B^k_2$ hold the conditions of Lemma 4.3.

We give again a parallel proof for the two propositions.

Proof: By Proposition 4.4 [Proposition 4.5] it is enough to show that properties (i)['], (ii)['] and (iii)[*] hold. (i)['] and (ii)['] follows by definition hence we will show only (iii)[*]. Let $X \subseteq \mathcal{I}_{[i]}$ with $|X| \geq 2$. We need to show that $d_{B^k_{[i]}}(X) \geq 2\alpha[+1 + (-1)^i]$. It is easy to show that K_{2k} is essentially $(4k-4)$-edge-connected hence $d_{B^k_{[i]}}(X) \geq d_{K_{2k}}(X) + |X| \geq 4k - 4 + 2 = 4\alpha - 6$. For $\alpha = 3$: $4\alpha - 6 = 2\alpha$; for $\alpha \geq 4$: $4\alpha - 6 \geq 2\alpha + 2 \geq 2\alpha[+1 + (-1)^i$. □

Now we are ready to prove Theorem 1.3. Let us denote by G^k_i the graph obtained from the basic graph S_i by [parallel] blackberrizing it with the berry [berries] B^k [B^k_1 and B^k_2]. [For even k, $|V(B^k_1)| = |V(B^k_2)|$ and $|\mathcal{I}(B^k_1)| = |\mathcal{I}(B^k_2)|$ so we may simply denote these values with $|V(B^k)|$ and $|\mathcal{I}(B^k)|$]. By Lemma 4.2 [Lemma 4.3] G^k_i is $2k$-regular and essentially (2k+2)-edge-connected for all i. By Proposition 4.1 (i), $h^*(G^k_i) \leq |V(B^k)| \cdot h^*(S_i)$. Hence

$$\liminf_{i\to\infty} \frac{\log h^*(G_i^k)}{\log |V(G_i^k)|} \leq \lim_{i\to\infty} \frac{\log(|V(B^k)| \cdot h^*(S_i)}{\log(|\mathcal{I}(B^k)| \cdot |V(S_i)|)} = \lim_{i\to\infty} \frac{\log |V(B^k)| + \log h^*(S_i)}{\log |\mathcal{I}(B^k)| + \log |V(S_i)|} =$$

$$= \lim_{i\to\infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}.$$

We note that according to [10] this upper bound cannot be essentially improved by our method.

5 Small $(k, k+1)$-circuits without long paths

We call the *deficit* of a graph $G = (V, E)$ the value $\mathrm{df}(G) = |V(G)| - h^*(G)$. Our graphs G_i^k are rather big graphs: even S_1 has 90 vertices already before the blackberrization. They proved to be useful for showing Theorem 1.3 but do not provide small examples for $(k, k+1)$-circuits without Hamiltonian paths or with a fixed small deficit. Here we give a method to obtain some small graphs with this property.

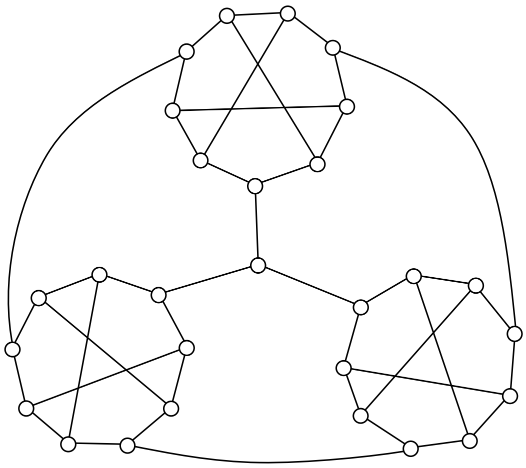


Figure 7: Singleton-graph.

First we note that similarly to the proof used for the Bondy-Simonovits-graphs in Section 3 we also get that inflating A_P into t vertices of an arbitrary 3-regular 3-connected graph G, $\mathrm{df}(G_P^t) \geq \mathrm{df}(G) + t - 2 + 8 \cdot \max\{t - h^*(G), 0\}$ for the obtained graph G_P^t. If we choose K_4 as initial graph and $t = 3$ we get a 3-regular 3-connected graph G with only 28 vertices which does not include a Hamiltonian path (see Figure 7 or [16, Figure 5.4]). We call this graph the Singleton-graph. Using this as basic graph in the blackberrization we obtain relatively small $(k, k+1)$-circuits without a Hamiltonian path. Counting precisely their deficit is $|\mathcal{I}| - 1$ by Proposition 4.1 (ii).

In the case $k = 2$ the smallest possible berry including inner vertices is the one with two adjacent inner vertices which are both adjacent to all 3 outer vertices (see Figure

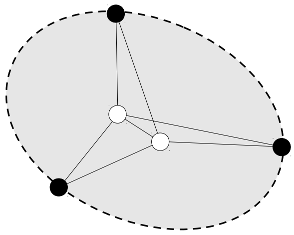


Figure 8: A copy of a berry with two adjacent inner vertices which are both adjacent to all 3 outer vertices. This berry can be used in case $k = 2$.

8). With this berry we obtain a blackberry graph with 98 vertices and deficit 2 from the Singleton-graph (see Figure 9). The resulting generic circuit with 97 vertices is the smallest generic circuit without Hamiltonian paths that the authors know.

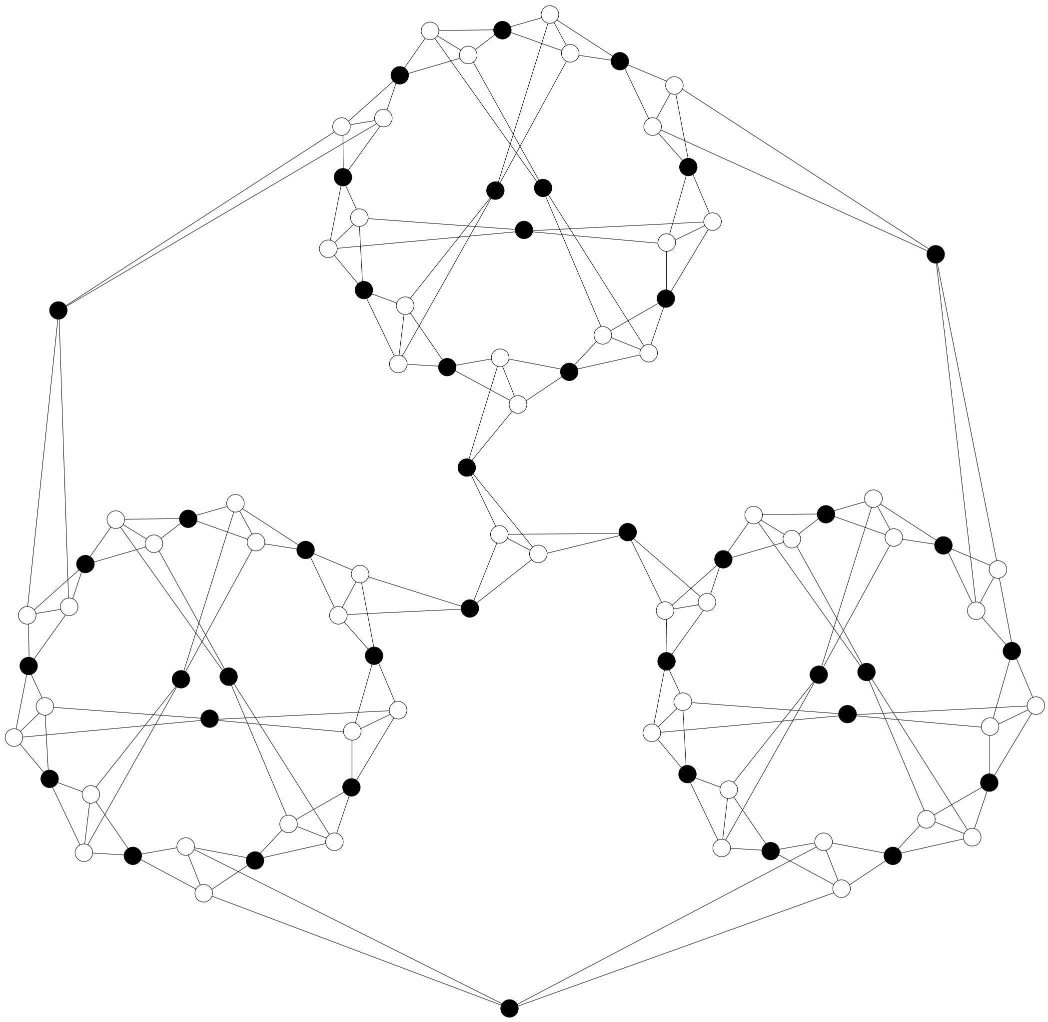


Figure 9: The blackberry graph of the Singleton-graph.

Let c be a positive integer. We give a method for deriving a not much bigger circuit with deficit $\mathrm{df}(G) + c$ from a circuit G constructed with berries including inner vertices. First observe that the pinching operation preserves $2k$-regularity and also essential $(2k+2)$-edge-connectivity.

Lemma 5.1. Let $k \geq 2$. If we pinch k edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ of a $2k$-regular, essentially $(2k+2)$-edge-connected graph $G = (V, E)$, then the obtained graph $G' = (V', E')$ is also $2k$-regular and essentially $(2k+2)$-edge-connected.

Proof: Let $E'(X, V' - X)$ be a nontrivial edge-cut in G'. We may suppose that $s \in V' - X$. If $|V' - X| \geq 3$ then $E(X, V - X)$ is a nontrivial edge-cut in G, and thus contains at least $2k+2$ edges. Suppose that the cut contains t edges $u_{i_1}v_{i_1}, \ldots, u_{i_t}v_{i_t}$ out of the pinched ones. We may assume that $u_{i_1}, \ldots, u_{i_t} \in X$. Then $su_{i_1}, \ldots, su_{i_t}$ and the $2k+2-t$ remaining original edges are $2k+2$ edges of the cut in G'. To complete the proof observe that in a $2k$-regular graph for a subset of vertices X with $|X| = 2$ the inequality $d(X) \geq 4k - 2 \geq 2k + 2$ always holds ($k \geq 2$). □

Now let G be a blackberry graph. If we successively perform c arbitrary pinching in every berry, the proof of Proposition 4.1 remains valid and we obtain that the deficit increased with c. The fact that the obtained graph is $2k$-regular and essentially $(2k+2)$-edge-connected follows directly from Lemma 5.1 without using Lemmas 4.2 or 4.3.

We only showed here that pinching preserves essential $(2k+2)$-edge-connectivity. In fact a much stronger theorem is true in the case $k = 2$. It is easy to check that 'replacing' a triangle with the berry in Figure 8 also preserves 4-regularity and essential 6-edge-connectivity. This, together with [2, Lemma 3.2] provides the following constructive characterization:

Theorem 5.2. A graph $G = (V, E)$ is 4-regular and essentially 6-edge-connected if and only if it can be obtained from K_5 by the following operations (see also Figure 10):

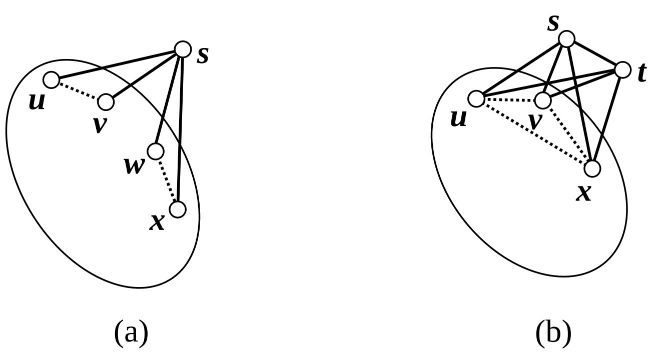


Figure 10: The two operations used in Theorem 5.2. The dashed edges are deleted from, the solid ones are added to the graph.

(a) pinch two independent edges,

(b) take a triangle uvw (which means that uv, uw, vw are all edges), delete its edges and add two new vertices s, t and edges $st, su, sv, sw, tu, tv, tw$ to the graph.

This theorem is also an immediate consequence of the constructive characterization of generic circuits given in [1] and of Lemma 2.1.

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