Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-06. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Generalization of Chen's and Manalastas' conjecture

Dávid Herskovics

June 2012

Generalization of Chen's and Manalastas' conjecture

Dávid Herskovics *

Abstract

C.C. Chen and P. Manalastas, Jr. conjectured that all strong digraph with stability number α is spanned by the disjoint union of some k_i-handles where $\sum k_i \leq \alpha$. In this paper we introduce a stronger version of this conjecture and prove it to some special cases.

1 Introduction

C.C. Chen and P. Manalastas, Jr. published a paper in 1983 ([1]) in which they proved that every strong digraph with stability number at most 2 can be spanned with at most two circuits which are "nicely positioned": their intersection is either empty or a path. In the same paper they made a conjecture generalizing this theorem: every strong digraph can be spanned by at most $\alpha(D)$ circuits which are "nicely positioned". For the more precise way to put it we will use the wording of S. Bessy and S. Thomassé from [3].

Conjecture 1.1. *"Every strong digraph with stability α is spanned by the disjoint union of some k_i-handles, where $k_i > 0$ for all i and the sum of the ki being at most α."*

Here a k-handle is the union of the first k ears of an ear-decomposition.

The significance of this conjecture is shown by the fact that if true it would give a common generalization to important of the following results:

Theorem 1.2 (Bessy-Thomassé / conjectured by Gallai)**.** *[2] Every strong digraph with stability number α can be spanned by α circuit.*

Theorem 1.3 (Bessy-Thomassé)**.** *[3] Every strong digraph has a spanning strong subgraph with at most $n + 2\alpha - 2$ arcs.*

Theorem 1.4 (Thomassé)**.** *[4] Every strong digraph with stability number $\alpha > 1$ has a spanning arborescence with at most $\alpha - 1$ leaves.*

*Department of Operations Research, Eötvös Loránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary.

This is already a generalization of the classic Gallai-Milgram theorem:

Theorem 1.5 (Gallai-Milgram). *[5] Every digraph with stability number α is spanned by at most α disjoint directed paths.*

And the already mentioned Chen-Manalastas theorem:

Theorem 1.6 (Chen-Manalastas). *[1] Every strong digraph with stability number 2 can be spanned by one circuit or by two circuits whose intersection is a directed path or empty*

In this paper we will introduce a stronger version of the conjecture and prove its validity for some special cases.

Conjecture 1.7. *Let $D = (V, A)$ be a strongly connected digraph with stability number $\alpha = \alpha(D)$. Then there is a partition of V into $r \geq 1$ classes such that:*

1. *Each class V_i of the partition spans a strongly connected digraph D_i and $|V_i| \geq 2$,*
2. *$\alpha(D_1) + \ldots + \alpha(D_r) = \alpha(D)$,*
3. *Each D_i has a spanning k_i-handle with $k_i \leq \alpha(D_i)$.*

We will call a partition of V which satisfies the first two properties a *decomposition*, and a decomposition satisfying the third property a *valid decomposition*.

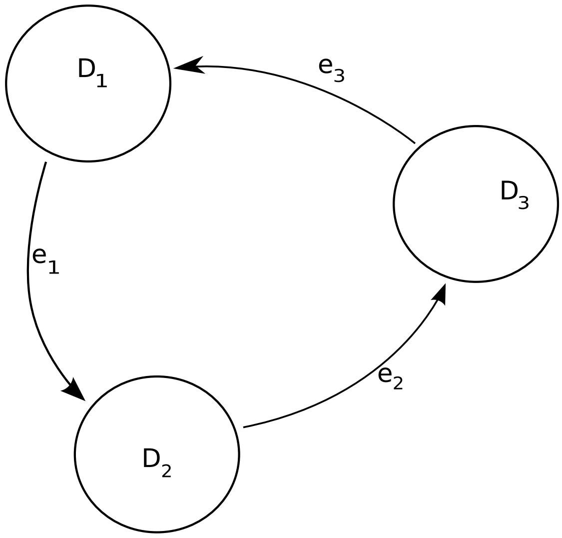
2 Results

Definition We will call an arc of a strong digraph *critical* if $D - e$ is not strong.

Theorem 2.1. *If D is a digraph in which every arc is critical then D has a valid decomposition*

For the proof we will use the consequences of two known results: the first is Knuth's theorem from his Wheels Within Wheels paper [6] and the already mentioned proof of Gallai's conjecture given by Bessy and Thomassé.

Theorem 2.2 (Knuth). *[6] For every strong digraph D with at least 2 vertices and for every circuit C in D there is a partition of the arcs and vertices of D into r (possibly 1) smaller strong digraphs $D_1, \ldots, D_n$ and arcs $e_1 = (x_1y_2), \ldots, e_r = (x_ry_1)$ where $e_1, \ldots, e_r$ belongs to C and $x_i, y_i \in D_i$ (see the figure below).*



Now we have the necessary tools to prove *Theorem 6*.

Proof Suppose that the statement is not true for every such digraph and take a counterexample D with the least possible number of vertices.

Suppose there is a maximum stable set A and a circuit C in D for which $A \cap C = \emptyset$. Then we use Knuth's theorem with that circuit C which gives us a partition of D into the smaller digraphs $D_1, \dots, D_r$ and the arcs between them: $e_1, \dots, e_r$. Because each of C's arcs is critical, r must be at least 2. Another important observation is that since $A \cap C = \emptyset$, thus $A_i = A \cap D_i$ is a maximum stable set for D_i for each i. It comes from the fact that the only arcs connecting the D_is are those of C, so since $A \cap C = \emptyset$ there is no element of A adjacent to them. Thus if we replace the A_i for one specific i by another stable set in D_i then we will still have a stable set. So if there would be an A_i^* stable set in D_i bigger than A_i, then we could replace A_i with A_i^* and get a bigger stable set in D than A. The property of A_i being a maximum stable set for D_i (thus $|A_i| \geq 1$) and C being disjoint from A_i also shows that each D_i must have at least two vertices.

So if we have a maximum stable set and a circuit which are disjoint then we can decompose the digraph into smaller digraphs in such a way that it will satisfy the first two properties so taking a valid decomposition for each of these smaller digraphs separately (which we have as our counterexample was minimal) gives us a valid decomposition for the original digraph.

So we can assume that each maximum stable set and each circuit have at least one common vertex.

Now we show using the lemma below that if there is a circuit with at least 4 vertices then contracting it into one vertex will decrease the stability number of the digraph.

Lemma 2.3. *Let $D = (V, A)$ be a digraph and $V_1 \subset V$ such that $\alpha(D[V_1]) \geq 2$. Then contracting V_1 into a point v^* does not decrease the stability number if and only if there is a maximum stable set A in D which is disjoint of V_1.*

Proof Let D' be the digraph we get from contracting V_1 into the point v^*. Suppose that there is a maximum stable set A' in D' such that it contains v^*. Then we can replace v^* with a maximum stable subset of V_1 and get a stable set A in D which is strictly bigger than A'. So if $\alpha(D') = \alpha(D)$ then there can not be a maximum stable set of D' which contains v^* hence all maximum stable sets of D' are maximum stable sets in D so there is a maximum stable set A in D which is disjoint of V_1.

□

So if we have a circuit C with at least 4 vertices in D then we contract it into one vertex (lets call this vertex v) and the digraph $D^* = (V^*, E^*)$ we get from it will have a stability number strictly less than $\alpha(D)$. Since D^* has less vertices than D and there are still no non-critical arcs so it has a valid decomposition $V_1^*, \ldots, V_r^*$ in which the class V_1^* is containing the vertex v. Let D_1^* be the subdigraph of D^* spanned by V_1^*. If we contract the whole V_1^* in D^* into one vertex (lets call it vertex d) then the stability number decreases by $\alpha(D_1^*)$ or by $\alpha(D_1^*)-1$ because a maximum stable set intersects V_1^* in $\alpha(D_1^*)$ vertices (that is seen from the second property of the valid decomposition that $\alpha(D_1^*)+\ldots+\alpha(D_r^*) = \alpha(D^*)$). Let D^{**} be the digraph we get from the contraction of V_1^* into a new vertex d. If the decrease is $\alpha(D_1^*)$ then we take a valid decomposition of D^{**}. Let V_1^{**} be the class that contains d and D_1^{**} the subdigraph that it spans. Blowing V_1^* up from d will increase the stability number of D_1^{**} by at most $\alpha(D_1^*)$ and it must increase it that much because the contraction decreased the stability number of the whole graph by $\alpha(D_1^*)$ and $\alpha(D_1^{**}) + \ldots + \alpha(D_r^{**}) = \alpha(D^{**}) = \alpha(D^*) - \alpha(D_1^*)$. For the number of handles the blowing up of V_1^* will also give an at most $\alpha(D_1^*)$ increase because that much handles was enough to cover V_1^*.

So we get a new valid decomposition of V^* which has a strictly bigger class containing v than V_1^*. So if we suppose that the $V_1^*, \ldots, V_r^*$ is a valid decomposition in which $v \in V_1^*$ is as large as it can be then by contracting V_1^* the stability number must decrease by $\alpha(D_1^*)-1$. That means that a maximum stable set A^{**} in D^{**} must contain d (we use again that $\alpha(D_1^*) + \ldots + \alpha(D_r^*) = \alpha(D^*)$) so there is a maximum stable set A^* in D^* for which $A \setminus V_1^*$ does not have a neighbour in V_1^*. This means that when we blow the circuit C back from the vertex v then it increase the stability number of D_1^* just as much as it increases the stability number of D^*. So we get a decomposition of V which satisfies the first two properties of a valid decomposition and as the blowing back of a circuit adds just one new handle and since the blowing up of the circuit C increased the stability number of the class it belongs so the third property is satisfied as well.

But that's a contradiction so we can't have a circuit with length of at least 4 in our minimal counterexample.

So we must suppose that all circuits in D have a length of at most 3. For such a digraph we use Gallai's conjecture proven by Bessy and Thomassé which states that there is set of at most $\alpha(D)$ circuits in D covering all of the vertices of D.

Lemma 2.4. *The connected components of the union of these circuits form a valid decomposition of D^*.*

Proof Being "built" by circuits all components must be strongly connected and have at least two vertices. As each circuit used has at most 3 vertices, so each component can have a stability number at most as many as the number of circuits used to span it. There was at most $\alpha(D^*)$ circuits used in total so each component has its stability number equal to the number of circuits used to span it and the sum of their stability numbers is $\alpha(D^*)$. And using again that each circle has a length of at most 3, the circuits spanning a component gives the handles of the component.

□

So we proved that if all arcs are critical in the digraph D than it has a valid decomposition.

□□

Now we will generalize this result a bit by allowing some non-critical arcs to be in the digraph.

Theorem 2.5. *If D is a digraph in which all non-critical arcs are spanned by a circuit C then D has a valid decomposition.*

proof The first thing to notice is that we can suppose that the digraph does not have a decomposition into at least two classes, because in that case we could just handle the smaller subdigraphs spanned by the classes by possibly adding some new arcs to them if necessary to have them a circuit which spans all the non-critical arcs.

Definition We will call such digraphs (which do not have a decomposition into at least two classes) *non-separable.*

Notation Let D be a digraph then $f(D) = \min\{\ k\ :\ \text{D has a spanning } k\text{-handle}\}$.
Let D be a non-separable digraph and let C be such a circuit that every non-critical arc is spanned by C. Let D' be the digraph we get by contracting C into a point v^*. We will prove that then either $\alpha(D') < \alpha(D)$ or $f(D') \geq f(D)$ or $f(D') < \alpha(D)$.

First we will see that this is enough to prove the theorem. The digraph D' does not have any non-critical arcs so by *Theorem 6* we know that D' has a valid decomposition. As in the proof of *Theorem 6* we take the valid decomposition of D' in which the class containing v^* is the biggest possible. Because D is non-separable this decomposition must have only one class so $f(D') \leq \alpha(D')$. From this we can see that if $\alpha(D') < \alpha(D)$ or $f(D') \geq f(D)$ then $f(D) \leq \alpha(D)$. If $f(D') < \alpha(D)$ then we are also ready since $f(D) \leq f(D') + 1$.

Now we will prove that $\alpha(D') < \alpha(D)$ or $f(D') \geq f(D)$ or $f(D') < \alpha(D)$. Suppose that $f(D') < f(D)$ and $\alpha(D') < \alpha(D)$. For this we will use the following easy remark:

Remark $f(D') < f(D) \Leftrightarrow$ there is an $f(D)$-handle in D which contains C

Let us take an $f(D)$-handle in D which contains C and lets call it H. This H is a minimal strong spanning subdigraph so all its arcs are critical in H. So we can use again Knuth's theorem to decompose H into at least two smaller strong digraph joined together by C. And also because H is a strong spanning digraph it must contain all of the critical arcs of D. So the only arcs that are not in H are the chords of C. So if there is a maximum stable set A which doesn't have a common vertex with C then by the same argument as presented in the proof of *Theorem 2.1* we get that D is separable which is a contradiction. The lack of such maximum stable set means that in D' either all of the maximum stable sets contain v^* or $\alpha(D') < \alpha(D)$. We supposed in the beginning that it is not the latter so it must be the former. Then adding a new vertex x to the digraph and the arcs between v^* and x in both direction we get a digraph D'' for which $\alpha(D'') = \alpha(D')$ and $f(D'') = f(D') + 1$. By the same argument as for D' we can conclude that $f(D'') \leq \alpha(D'') = \alpha(D')$.
With this we proved that one of the above three cases is always true and with this we finished the proof of *Theorem 2.5*.

□□

From the above arguments it is seen that the main difficulty here that we can't usually guarantee that there is a "good" circuit whose contraction helps us. So the next idea is that instead of the stability number we will use the *clique cover number* $\overline{\chi}(G)$: for a graph G the clique cover number is the minimum number of cliques needed to cover all the vertices of G. Now we will use this graph parameter instead of the stability number.

Proposition 2.6. *Let $D = (V, A)$ be a strongly connected digraph with the underlying graph G. Then there is a partition of V into $r \geq 1$ class such that:*

1. *Each class V_i of the partition span a strongly connected D_i digraph and $|V_i| \geq 2$ and underlying graph G_i,*
2. $\overline{\chi}(G_1) + \ldots + \overline{\chi}(G_r) = \overline{\chi}(G)$,
3. *Each D_i has a spanning k_i-handle with $k_i \leq \overline{\chi}(G_i)$.*

Proof Let $K_1, \ldots, K_{\overline{\chi}(D)}$ be a minimum clique cover of D with not necessarily disjoint cliques. For all K_i there is a circuit C_i such that $K_i \subseteq V(C_i)$. If for all $i \neq j$ $K_i \cap V(C_j) = \emptyset$ then the $K_1, \ldots, K_{\overline{\chi}(D)}$ is a valid decomposition of D with each C_i being a 1-handle. So lets suppose that there is a C_i which intersects some cliques. We contract C_i into a new point v^* and we add v^* to every clique K_j the C_i intersects. Now in this new digraph we have a (not necessarily minimum) clique cover with strictly less cliques than before. We do this step till we get a clique cover $K_1, \ldots, K_l$ and a circuit cover $C_1, \ldots, C_l$ for which $K_i = V(C_i)$ for all i. Now we blow back each circuit we contracted before thus we get some disjoint k_i-handle spanning D. Since each component spanned by a k_i-handle is the union of at least k_i of the original $K_1, \ldots, K_{\overline{\chi}(D)}$ so it is indeed a partition we wanted.

□

The above proposition is related to *Conjecture 2* in a special case: if the underlying graph is perfect then $\overline{\chi}(G_i) = \alpha(D_i)$ for all i. In fact since $\alpha(G_i) \leq \overline{\chi}(G_i)$ for each i and $\overline{\chi}(G_1) + \ldots + \overline{\chi}(G_r) = \overline{\chi}(G)$ it is enough to suppose that $\alpha(G) = \overline{\chi}(G)$ and $\overline{\chi}(G_i) = \alpha(D_i)$ will automatically hold even if G is not perfect. So we get:

Theorem 2.7. *Every strong digraph with an underlying graph G satisfying the property $\alpha(G) = \overline{\chi}(G)$ has a valid decomposition.*

Finally we will prove that Conjecture 2 also holds for the case $\alpha = 2$ for which Chen and Manalastas proved that *Conjecture 1* is true the .

Theorem 2.8. *Every digraph with stability number 2 has a valid decomposition*

Proof The only case where the Chen-Manalastas theorem differs from the *Conjecture 2* for $\alpha = 2$ is when D is spanned by two disjoint circuits C_1, C_2 and $\alpha(D[C_1]) = 2$. Then let D' be the digraph we get from contracting C_1 into a v^*. If D' is a tournament then we are ready. So we can suppose that $\alpha(D') = \alpha(D) = 2$. Then v^* is not in any stable set of cardinality 2 in D' in other words v^* is connected with every vertex of D'. But $V(D') = V(C_2) \cup \{v^*\}$ so we can extend C_2 to cover $V(D')$ and we are ready.

□□

3 Acknowledgement

The work was supported by the National Development Agency of Hungary (grant no. CK 80124), based on a source from the Research and Technology Innovation Fund.

References

[1] C.C. Chen and P. Manalastas, Jr., Every finite strongly connected digraph of stability 2 has a Hamiltonian path, *Discrete Math. 44 (1983) p. 243-250.*

[2] S. Bessy and S. Thomassé, Spanning a strong digraph by α circuits: A proof of Gallai's conjecture, *Combinatorica, 27 (2007), p. 659-667.*

[3] S. Bessy, S. Thomassé, Every strong digraph has a spanning strong subgraph with at most $n + 2\alpha - 2$ arcs, *J. Combin. Theory Ser. B, 87 (2003), p. 289-299.*

[4] S. Thomassé, Covering a strong digraph by $\alpha - 1$ disjoint paths: a proof of Las Vergnas' conjecture, *J. Combin. Theory Ser. B, 83 (2001), p. 331-333.*

[5] T. Gallai and A.N. Milgram, Verallgemeinerung eines graphentheoretischen Satzes von Rédei, *Acta Sci. Math. (Szeged) 21 1960 181-186.*

[6] D. E. Knuth, Wheels within wheels, *Journal of Combinatorial Theory, Series B 16. (1974) p. 42-46*