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Abstract

Let $G = (V, E)$ be a graph and $u, v \in V$ be two distinct vertices. We give a necessary and sufficient condition for the existence of an infinitesimally rigid two-dimensional bar-and-joint framework (G, p), in which the positions of u and v coincide. We also determine the rank function of the corresponding modified generic rigidity matroid on ground-set E. The results lead to efficient algorithms for testing whether a graph has such a coincident realization with respect to a designated vertex pair and, more generally, for computing the rank of G in the matroid.

1 Introduction

A two-dimensional bar-and-joint *framework* (G, p) is a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^2$. We say that the framework (G, p) is a *realization* of the graph G in $\mathbb{R}^2$. The *rigidity matrix* of the framework is the matrix $R(G, p)$ of size $|E| \times 2|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the two columns corresponding to the vertices i and j contain the two coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of the rows of the rigidity matrix. The framework is said to be *independent* if the rows of $R(G, p)$ are linearly independent. A framework (G, p) is *generic* if the set of coordinates of the points $p(v)$, $v \in V$, is algebraically independent over the rationals. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the two-dimensional *rigidity matroid* $\mathcal{R}(G) = (E, r)$ of the graph G. We denote the rank of $\mathcal{R}(G)$ by $r(G)$.

A framework (G, p) in $\mathbb{R}^2$ is *infinitesimally rigid* if $\operatorname{rank} R(G, p) = 2|V| - 3$. This definition is motivated by the fact that if (G, p) is infinitesimally rigid then (G, p) is

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'rigid' in the sense that every continuous deformation of (G, p) which preserves the edge lengths $||p(u) - p(v)||$ for all $uv \in E$, must preserve the distances $||p(w) - p(x)||$ for all $w, x \in V$. We say that the graph G is *rigid* in $\mathbb{R}^2$ if $r(G) = 2|V| - 3$ holds. In this case every generic framework (G, p) in $\mathbb{R}^2$ is infinitesimally rigid and hence, by the above remark, is 'rigid'. $G = (V, E)$ is *minimally rigid* if it is rigid but $G - e$ is not rigid for every $e \in E$. See e.g. [3, 12] for more details on two- and higher-dimensional frameworks and rigidity matroids.

Independence in the two-dimensional rigidity matroid (and hence the family of rigid graphs) was characterized by Laman [6], who proved that the edge set F of a graph $H = (V, F)$ is independent in $\mathcal{R}(H)$ if and only if $i_H(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$, where $i_H(X)$ denotes the number of edges induced by X in H. The rank function was determined by Lovász and Yemini [7]. It remains a difficult open problem to characterize independence or rigidity in generic d-dimensional frameworks for all $d \geq 3$.

To verify the rigidity of (special families of) generic frameworks it is sometimes useful to consider non-generic realizations of graphs. For example, to prove a major conjecture of Tay and Whiteley [8], stating that a graph operation called X*-replacement* preserves rigidity in three-space, it could be useful to have a characterization of when a graph has an infinitesimally rigid realization in $\mathbb{R}^3$ in which the positions of four given vertices are coplanar, see [8, 9, 12].

Motivated by this connection, Jackson and Jordán [4] characterized when a graph has an infinitesimally rigid realization in $\mathbb{R}^2$ in which three given vertices are collinear. A set X of vertices in a minimally rigid graph G is *tight* if $i_G(X) = 2|X| - 3$. An *obstacle* for an ordered triple (x, y, z) of vertices is an ordered triple of tight sets (X, Y, Z) for which $X \cap Y = \{z\}$, $X \cap Z = \{y\}$, and $Y \cap Z = \{x\}$.

Theorem 1. *[4] Let $G = (V, E)$ be a minimally rigid graph and let $x, y, z \in V$ be distinct vertices. Then G has an infinitesimally rigid realization (G, p), in which $(p(x), p(y), p(z))$ are collinear if and only if G contains no obstacle for the triple (x, y, z).*

Watson [9] introduced the concept of flat realizations. He called a d-dimensional framework (G, p) U*-flat*, for some $U \subseteq V(G)$ with $2 \leq |U| \leq d + 1$, if the set $\{p(x) : x \in U\}$ is not affinely independent. He verified a number of results on U-flat realizations in $\mathbb{R}^3$ and formulated a conjecture for the existence of a two-dimensional U-flat realization. The special case when $|U| = 3$ is settled by Theorem 1 above. A slightly reformulated, but equivalent version of his conjecture for the case when $|U| = 2$ is as follows.

Conjecture 2. *[9, Conjecture 4.40] Let $G = (V, E)$ be a minimally rigid graph and $u, v \in V$ be two distinct vertices. Then there exists an infinitesimally rigid realization (G, p) of G in which $p(u) = p(v)$ if and only if*
(i) $uv \notin E$,
(ii) there is no $w \in V$ for which G contains an obstacle for $\{u, v, w\}$,
(iii) u and v have at most two common neighbours in G.

We have found a counterexample to Conjecture 2, see the graph of Figure 1.

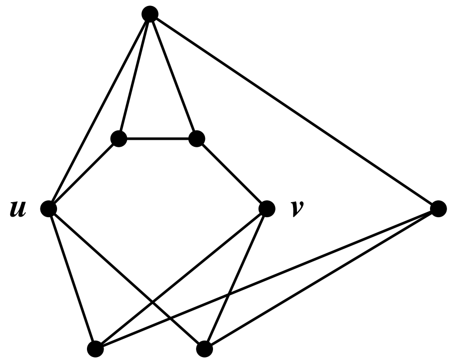

Figure 1: The graph G of this figure is minimally rigid and satisfies conditions (i)-(iii) of Conjecture 2 with respect to the designated vertex pair u, v. However, it does not have an infinitesimally rigid realization in which $p(u) = p(v)$. To see this observe that the existence of such a realization would imply that the graph obtained from G by contracting the vertex pair u, v is rigid - but this graph fails to satisfy this necessary condition.

Our main result is a characterization for the existence of a two-dimensional U-flat realization for a given graph G and $U \subseteq V(G)$ with $|U| = 2$, which completes the solution of the two-dimensional flatness problem.

We need the following definitions. Let $G = (V, E)$ be a graph and let $u, v \in V$ be two distinct vertices of G. A realization (G, p) is called *uv-coincident* if $p(u) = p(v)$ holds. A uv-coincident realization is *uv-generic* if the set of coordinates of the points $\{p(z) : z \in V - v\}$ is algebraically independent over the rationals. Any two uv-coincident uv-generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the two-dimensional *uv-rigidity matroid* $\mathcal{R}_{uv}(G) = (E, r_{uv})$ of the graph G. We denote the rank of $\mathcal{R}_{uv}(G)$ by $r_{uv}(G)$. We say that the graph G is *uv-rigid* in $\mathbb{R}^2$ if $r_{uv}(G) = 2|V| - 3$ holds. A set $F \subseteq E$ is said to be *uv-independent* if F is independent in $\mathcal{R}_{uv}(G)$. The graph G is said to be *minimally uv-rigid* if G is uv-rigid and E is uv-independent.

The structure of the paper is as follows:

(i) we introduce a new count matroid $\mathcal{M}_{uv}(G)$ on the edge set of G, describe its rank function, and show that uv-independence implies independence in $\mathcal{M}_{uv}(G)$ (Section 2),

(ii) we give a Henneberg-type inductive construction for minimally uv-rigid graphs and show that $\mathcal{M}_{uv}(G)$ is in fact isomorphic to $\mathcal{R}_{uv}(G)$. In addition, we prove that G is uv-rigid if and only if the deletion of the edge uv (if it exists in G) and the contraction of the pair u, v both give rise to rigid graphs (Section 3),

(iii) we give a different, obstacle-based characterization of minimally uv-rigid graphs (Section 4).

We close this section with some definitions. Let $G = (V, E)$ be a graph. For some $X \subseteq V$ let $G[X]$ denote the subgraph of G induced by X and let $E_G(X)$ be the set of edges of $G[X]$. Thus $i_G(X) = |E_G(X)|$. For a family $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$, where

$S_i \subseteq V$ for all $i = 1, \ldots, k$, we define $E_G(\mathcal{S}) = \cup_{i=1}^{k} E_G(S_i)$ and put $i_G(\mathcal{S}) = |E_G(\mathcal{S})|$. We also define $cov(\mathcal{S}) = \{(x, y) : x, y \in V, \{x, y\} \subseteq S_i \text{ for some } 1 \leq i \leq k\}$. We say that $\mathcal{S}$ *covers* $F \subseteq E$ if $F \subseteq cov(\mathcal{S})$. A system $\mathcal{K} = \{\mathcal{S}_1, ..., \mathcal{S}_l\}$ is a *cover* of F if $F \subseteq \cup_{i=1}^{l} cov(\mathcal{S}_i)$. The degree of a vertex w is denoted by $d_G(w)$. We let $N_G(w) = \{z \in V : wz \in E\}$ denote the *neighbours* of w in G. We may omit the subscripts referring to G if the graph is clear from the context.

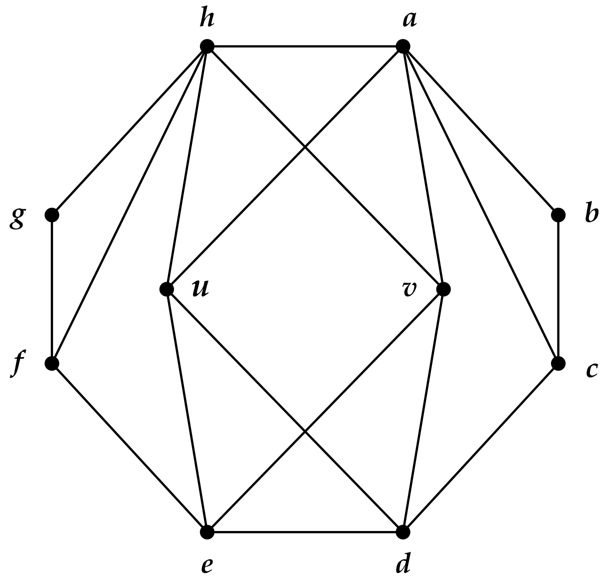

Figure 2: A rigid but not uv-rigid graph $G = (V, E)$ with $|V| = 10$. Consider the cover $\mathcal{K} = \{\{\{u, v, a, h\}, \{u, v, e, d\}\}, \{a, b, c\}, \{c, d\}, \{e, f\}, \{f, g, h\}\}$ of E. Its value equals 16, which is less than $2|V| - 3 = 17$, showing that G is not uv-rigid.

2 The count matroid

Let $G = (V, E)$ be a graph and $u, v \in V$ be two distinct vertices of G. Let $\mathcal{H} = \{H_1, ..., H_k\}$ be a family with $H_i \subseteq V$, $1 \leq i \leq k$. We say that $\mathcal{H}$ is *uv-compatible* if $u, v \in H_i$ and $|H_i| \geq 3$ hold for all $1 \leq i \leq k$. We define the *value* of subsets of V of size at least two and of uv-compatible families as follows. For $H \subseteq V$ with $|H| \geq 2$ and $H \neq \{u, v\}$ we let

$$val(H) = 2|H| - 3,$$

and put $val(\{u, v\}) = 0$. For a uv-compatible family $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ we let

$$val(\mathcal{H}) = \sum_{i=1}^{k} (2|H_i| - 3) - 2(k - 1).$$

Note that if $\mathcal{H} = \{H\}$ is a uv-compatible family containing only one set then the two definitions are compatible, i.e. $val(\mathcal{H}) = val(H)$ holds.

The value of a system $\mathcal{K} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_l\}$ of set families (which may consist of uv-compatible families as well as subsets of V) is defined by $val(\mathcal{K}) = \sum_{i=1}^{l} val(\mathcal{H}_i)$.

The next lemmas will enable us to consider uv-compatible families of special types in the main proof of this section.

Lemma 3. *Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a uv-compatible family. If $|H_i \cap H_j| \geq 3$ for some pair $1 \leq i < j \leq k$, then there is a uv-compatible family $\mathcal{H}'$ with $cov(\mathcal{H}) \subseteq cov(\mathcal{H}')$ for which $val(\mathcal{H}') \leq val(\mathcal{H}) - 1$.*

Proof. We may assume that $i = k-1$ and $j = k$. Let $\mathcal{H}' = \{H_1, \dots, H_{k-2}, (H_{k-1} \cup H_k)\}$. Then

$$val(\mathcal{H}) = \sum_{l=1}^{k}(2|H_l| - 3) - 2(k-1) =$$

$$= \sum_{l=1}^{k-2}(2|H_l| - 3) - 2((k-1)-1) + (2|H_{k-1}| - 3) + (2|H_k| - 3) - 2 =$$

$$= \sum_{l=1}^{k-2}(2|H_l| - 3) + (2|H_{k-1} \cup H_k| - 3) + 2((k-1)-1) + (2|H_{k-1} \cap H_k| - 3) - 2 \geq val(\mathcal{H}') + 1.$$

Clearly, we have $cov(\mathcal{H}) \subseteq cov(\mathcal{H}')$. □

Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. We say that G is *uv-sparse* if for all $H \subseteq V$ with $|H| \geq 2$ we have $i_G(H) \leq val(H)$ and for all uv-compatible families $\mathcal{H}$ we have $i_G(\mathcal{H}) \leq val(\mathcal{H})$. Note that if G is uv-sparse then $uv \notin E$ must hold. A set $H \subseteq V$ of vertices with $|H| \geq 2$ (resp. a uv-compatible family $\mathcal{H} = \{H_1, \dots, H_k\}$) is called *tight* if $i_G(H) = val(H)$ (resp. $i_G(\mathcal{H}) = val(\mathcal{H})$) holds.

Lemma 4. *Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a uv-compatible family with $|H_i \cap H_j| = 2$ for all $1 \leq i < j \leq k$, and let $Y \subseteq V$ be a set of vertices with $|Y \cap \{u, v\}| \leq 1$ and $|Y \cap H_i| \geq 2$ for some $1 \leq i \leq k$. Then there is a uv-compatible family $\mathcal{H}'$ with $cov(\mathcal{H}) \cup cov(Y) \subseteq cov(\mathcal{H}')$ for which $val(\mathcal{H}') \leq val(\mathcal{H}) + val(Y)$ holds. Furthermore, if G is uv-sparse and $\mathcal{H}$ and Y are both tight then $\mathcal{H}'$ is also tight.*

Proof. By renumbering the sets of $\mathcal{H}$, if necessary, we may assume that $|Y \cap H_i| \geq 2$ if $i \geq j$, for some $j \leq k$, and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq j-1$. Let $X = Y \cup \cup_{i=j}^{k} H_i$ and $\mathcal{H}' = \{H_1, \dots, H_{j-1}, X\}$. Then we have $cov(\mathcal{H}) + cov(Y) \subseteq cov(\mathcal{H}')$ and

$$val(\mathcal{H}) + val(Y) = \sum_{i=1}^{k}(2|H_i| - 3) - 2(k-1) + (2|Y| - 3) =$$

$$= \sum_{i=1}^{j-1}(2|H_i| - 3) - 2(j-1) + \sum_{i=j}^{k}(2|H_i| - 3) - 2(k-j) + (2|Y| - 3) =$$

$$= \sum_{i=1}^{j-1}(2|H_i| - 3) + (2|X| - 3) - 2(j-1) + 4(k-j) - 3(k-j+1) +$$

$$+2\sum_{i=j}^{k}|Y\cap H_i|-2(k-j)-2|Y\cap\{u,v\}|(k-j)\geq$$

$$\geq val(\mathcal{H}')+\sum_{i=j}^{k}val(Y\cap H_i).$$

Now suppose that $\mathcal{H}$ and Y are tight. Then we have

$$i(\mathcal{H}')+\sum_{i=j}^{k}i(Y\cap H_i)\geq i(\mathcal{H})+i(Y)=val(\mathcal{H})+val(Y)\geq$$

$$\geq val(\mathcal{H}')+\sum_{i=j}^{k}val(Y\cap H_i)\geq i(\mathcal{H}')+\sum_{i=j}^{k}i(Y\cap H_i),$$

where the first inequality follows from the fact that edges spanned by $\mathcal{H}$ or Y are spanned by $\mathcal{H}'$ and if some edge is spanned by both $\mathcal{H}$ and Y then it is spanned by $Y\cap H_i$ for some i. The first equality holds because $\mathcal{H}$ and Y are tight, and the second inequality holds by our calculations above. The last inequality holds because G is uv-sparse. Hence equality must hold everywhere, which implies that $\mathcal{H}'$ is also tight. □

Lemma 5. *Let $\mathcal{H}=\{H_1,\dots,H_k\}$ be a uv-compatible family with $|H_i\cap H_j|=2$ for all $1\leq i<j\leq k$, and let $Y\subseteq V$ be a set of vertices with $Y\cap\{u,v\}=\emptyset$ and $|Y\cap H_i|\leq 1$ for all $1\leq i\leq k$, for which $|Y\cap H_i|=|Y\cap H_j|=1$ for some pair $1\leq i<j\leq k$. Then there is a uv-compatible family $\mathcal{H}'$ with $cov(\mathcal{H})\cup cov(Y)\subseteq cov(\mathcal{H}')$ for which $val(\mathcal{H}')=val(\mathcal{H})+val(Y)$. Furthermore, if G is uv-sparse and $\mathcal{H}$ and Y are both tight then $\mathcal{H}'$ is also tight.*

Proof. We may assume that $i=k-1$ and $j=k$. Let $\mathcal{H}'=\{H_1,\dots,H_{k-2},(H_{k-1}\cup H_k\cup Y)\}$. Then

$$val(\mathcal{H})+val(Y)=\sum_{i=1}^{k}(2|H_i|-3)-2(k-1)+(2|Y|-3)=$$

$$=\sum_{i=1}^{k-2}(2|H_i|-3)-2((k-1)-1)-2+(2|H_{k-1}|-3)+(2|H_k|-3)+(2|Y|-3)=$$

$$=\sum_{i=1}^{k-2}(2|H_i|-3)-2((k-1)-1)+(2(|H_{k-1}|+|H_k|+|Y|)-3)-8=$$

$$=\sum_{i=1}^{k-2}(2|H_i|-3)+(2|H_{k-1}\cup H_k\cup Y|-3)-2((k-1)-1)=val(\mathcal{H}').$$

Clearly, we have $cov(\mathcal{H})\cup cov(Y)\subseteq cov(\mathcal{H}')$.

Now suppose that G is uv-sparse and $\mathcal{H}$ and Y are tight. Then we have

$$i(\mathcal{H}) + i(Y) = val(\mathcal{H}) + val(Y) = val(\mathcal{H}') \geq i(\mathcal{H}') \geq i(\mathcal{H}) + i(Y)$$

where the last inequality follows since $|Y \cap H_{k-1}| = |Y \cap H_k| = 1$, $|H_{k-1} \cap H_k| = 2$, and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold everywhere, which implies that $\mathcal{H}'$ is also tight. □

Lemma 6. *Let $G = (V, E)$ be uv-sparse and let $X, Y \subseteq V$ be tight sets in G with $|X \cap Y| \geq 2$ and $X \neq \{u, v\} \neq Y$. Then $X \cap Y \neq \{u, v\}$ and $X \cup Y$ and $X \cap Y$ are also tight.*

Proof. If $X \cap Y \neq \{u, v\}$ then the lemma follows as in [4, Lemma 2.3]. Otherwise we obtain $i(\{u, v\}) = 1$, which contradicts the fact that G is uv-sparse. □

Lemma 7. *Let $G = (V, E)$ be uv-sparse and suppose that there is a tight uv-compatible family in G. Then there is a unique tight uv-compatible family $\mathcal{H}_{max}$ in G for which $cov(\mathcal{H}) \subseteq cov(\mathcal{H}_{max})$ for all tight uv-compatible families $\mathcal{H}$ of G.*

Proof. It follows from Lemma 3 that if $\mathcal{H} = \{X_1, X_2, \ldots, X_k\}$ is a tight uv-compatible family in G then $X_i \cap X_j = \{u, v\}$ holds for all $1 \leq i < j \leq k$. Now consider a pair $\mathcal{H}_1 = \{X_1, X_2, \ldots, X_k\}$ and $\mathcal{H}_2 = \{Y_1, Y_2, \ldots, Y_l\}$ of tight uv-compatible families. Let $\mathcal{F} = (V, \mathcal{E})$ be a hypergraph where $\mathcal{E} = \{X_i - \{u, v\} : 1 \leq i \leq k\} \cup \{Y_j - \{u, v\} : 1 \leq j \leq l\}$ and let $C_1 = (V_1, \mathcal{E}_1), \ldots, C_t = (V_t, \mathcal{E}_t)$ be the connected components of $\mathcal{F}$. We define the following families:

$$\mathcal{H}_\cup = \{H_s : H_s = (\cup_{(X_i - \{u,v\}) \in \mathcal{E}_s} X_i) \cup (\cup_{(Y_j - \{u,v\}) \in \mathcal{E}_s} Y_j) \text{ for } 1 \leq s \leq t\}$$

$$\mathcal{H}_\cap = \{Z \subseteq V : |Z| \geq 3, \exists 1 \leq i \leq k, 1 \leq j \leq l \text{ such that } X_i \cap Y_j = Z\}$$

It is easy to see that $\mathcal{H}_\cup$ and $\mathcal{H}_\cup$ are both uv-compatible. For convenience we rename the families as $\mathcal{H}_\cup = \{A_1, \ldots, A_p\}$ and $\mathcal{H}_\cap = \{B_1, \ldots, B_q\}$. By using that $X_i \cap X_j = Y_{i'} \cap Y_{j'} = \{u, v\}$ we obtain $p + q \geq k + l$. We also have $i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap)$, since the family $\mathcal{H}_\cup$ spans all the edges spanned by $\mathcal{H}_1$ or $\mathcal{H}_2$ and $\mathcal{H}_\cap$ spans all the edges spanned by both $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus

$$\sum_{i=1}^{k}(2|X_i| - 3) - 2(k-1) + \sum_{j=1}^{l}(2|Y_j| - 3) - 2(l-1) = val(\mathcal{H}_1) + val(\mathcal{H}_2) =$$

$$= i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap) \leq val(\mathcal{H}_\cup) + val(\mathcal{H}_\cap) =$$

$$= \sum_{s=1}^{p}(2|A_s| - 3) - 2(p-1) + \sum_{t=1}^{q}(2|B_t| - 3) - 2(q-1) =$$

$$= \sum_{s=1}^{p} 2(|A_s| - 2) - (p-2) + \sum_{t=1}^{q} 2(|B_t| - 2) - (q-2) \leq$$

$$\leq \sum_{i=1}^{k} 2(|X_i| - 2) - (k-2) + \sum_{j=1}^{l} 2(|Y_j| - 2) - (l-2) =$$

$$= \sum_{i=1}^{k}(2|X_i| - 3) - 2(k-1) + \sum_{j=1}^{l}(2|Y_j| - 3) - 2(l-1),$$

where the last inequality follows from $\sum_{k=1}^{p}(|A_k|-2)+\sum_{l=1}^{q}(|B_l|-2) = \sum_{i=1}^{k}(|X_i|-2)+\sum_{j=1}^{l}(|Y_j|-2)$ and $p+q \geq k+l$. Hence we can deduce that $\mathcal{H}_\cup$ and $\mathcal{H}_\cap$ are both tight. Clearly, we have $cov(\mathcal{H}_1) \cup cov(\mathcal{H}_2) \subseteq cov(\mathcal{H}_\cup)$. Thus the lemma follows by choosing the tight uv-compatible family $\mathcal{H}_{\max}$ of G for which $cov(\mathcal{H}_{\max})$ is maximal. □

2.1 The matroid and its rank function

Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. In this subsection we prove that the family

$$\mathcal{I}_G = \{F : F \subseteq E, H = (V, F) \text{ is } uv\text{-sparse}\} \tag{1}$$

is a family of independent sets of a matroid on ground-set E. We shall also characterize the rank function of this matroid. We need the following definition.

Let $\mathcal{H} = \{X_1, \ldots, X_t\}$ be a uv-compatible family and let $H_1, \ldots, H_k$ be subsets of V of size at least two. We say that the system $\mathcal{K} = \{H_1, \ldots, H_k\}$ is *thin* if
(i) $|H_i \cap H_j| \leq 1$ for all pairs $1 \leq i, j \leq k$.
The system $\mathcal{L} = \{\mathcal{H}, H_1, \ldots, H_k\}$ is *thin* if (i) holds and
(ii) $X_i \cap X_j = \{u, v\}$ for all pairs $1 \leq i, j \leq t$, and
(iii) $|H_i \cap \cup_{j=1}^{t} X_j| \leq 1$ for all $1 \leq i \leq k$.

Theorem 8. *Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. Then $\mathcal{M}_{uv}(G) = (E, \mathcal{I}_G)$ is a matroid on ground-set E, where $\mathcal{I}_G$ is defined by (1). The rank of a set $E' \subseteq E$ in $\mathcal{M}_{uv}(G)$ is equal to*

$$\min\{val(\mathcal{K}) : \mathcal{K} \text{ is a thin cover of } E'\}.$$

Proof. Let $\mathcal{I} = \mathcal{I}_G$, let $E' \subseteq E$ and let $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq val(\mathcal{K})$ for all covers $\mathcal{K}$ of E'. We shall prove that there is a (thin) cover $\mathcal{K}$ of E' with $|F| = val(\mathcal{K})$, from which the theorem will follow.

Let $J = (V, F)$ denote the subgraph induced by the edge set F. First suppose that there is no tight uv-compatible family in J and consider the following cover of F:

$$\mathcal{K}_1 = \{H_1, H_2, \ldots, H_k\},$$

where $H_1, H_2, \ldots, H_k$ are the maximal tight sets in J. Every edge $f \in F$ induces a tight set in J, hence $\mathcal{K}_1$ is indeed a cover of F. It is thin by Lemma 6. Thus

$$|F| = \sum_{j=1}^{k} |E_J(H_j)| = \sum_{j=1}^{k}(2|H_j| - 3) = val(\mathcal{K}_1)$$

follows. We claim that $\mathcal{K}_1$ is a cover of E'. To see this consider an edge $ab = e \in E' - F$. Since F is maximal subset of E' in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. By our assumption there is no tight uv-compatible family in J, and hence there must be a tight set X in J with $a, b \in X$. Hence $X \subseteq H_i$ for some $1 \leq i \leq k$ which implies that $\mathcal{K}_1$ covers e, too.

Next suppose that there is a tight uv-compatible family in J and consider the following cover of F:

$$\mathcal{K}_2 = \{\mathcal{H}_{max}, H_1, H_2, \ldots, H_k\},$$

where $\mathcal{H}_{max} = \{X_1, X_2, \ldots, X_l\}$ is the uv-compatible family of G for which $cov(\mathcal{H}_{max})$ is maximal (c.f. Lemma 7) and $H_1, H_2, \ldots, H_k$ are maximal tight sets of $J' = (V, F - E(\mathcal{H}_{\max}))$. It is easy to see that $\mathcal{K}_2$ is indeed a cover of F. By Lemmas 3, 4, 5 and 6 the cover $\mathcal{K}_2$ is thin, and hence

$$|F| = \sum_{i=1}^{l} |E_J(X_i)| + \sum_{j=1}^{k} |E_J(H_j)| = \sum_{i=1}^{l} (2|X_i| - 3) - 2(l-1) + \sum_{j=1}^{k} (2|H_i| - 3) = val(\mathcal{K}_2).$$

We claim that $\mathcal{K}_2$ is a cover of E'. As above, let $ab = e \in E' - F$ be an edge. By the maximality of F we have $F + e \notin \mathcal{I}$. Thus either there is a tight set $X \subseteq V$ in J with $a, b \in X$ or there is a tight uv-compatible family $\mathcal{H}' = \{Y_1, \ldots, Y_t\}$ in J with $a, b \in Y_i$ for some $1 \leq i \leq t$.

In the latter case Lemma 7 implies that $cov(\mathcal{H}') \subseteq cov(\mathcal{H}_{max})$ and hence e is covered by $\mathcal{K}_2$. In the former case, when $a, b \in X$ for some tight set X in J we have two possibilities. First suppose that $|X \cap \cup_{i=1}^{l} X_i| \geq 2$. Then we can deduce that $X \subseteq X_i$ for some $1 \leq i \leq l$ by using Lemma 4 or 5 and the maximality of $\mathcal{H}_{max}$, which implies that $\mathcal{K}_2$ covers e. Next suppose that $|X \cap \cup_{i=1}^{l} X_i| \leq 1$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq H_i$ for some $1 \leq i \leq k$, since every edge of J' induces a tight set and every tight set is contained in a maximal tight set. Hence e is covered by $\mathcal{K}_2$, as claimed. □

2.2 Independence

Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Let G_{uv} denote the graph obtained from G by contracting the vertex pair u, v into a new vertex z_{uv} (and deleting the resulting loops and parallel copies of edges). Given a realization (G_{uv}, p_{uv}) of G_{uv}, we obtain a uv-coincident realization (G, p) of G by putting $p(u) = p(v) = p_{uv}(z)$ and $p(x) = p_{uv}(x)$ for all $x \in V - \{u, v\}$. Furthermore, each vector in the kernel of $R(G_{uv}, p_{uv})$ determines a vector in the kernel of $R(G, p)$ in a natural way. It follows that

$$dimKerR(G, p) \geq dimKerR(G_{uv}, p_{uv}). \quad (2)$$

We can use this fact to prove that uv-independence implies independence in $\mathcal{M}_{uv}(G)$. The reverse implication will be verified in the next section.

Lemma 9. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. If G is uv-independent then E is independent in $\mathcal{M}_{uv}(G)$.*

Proof. Let (G,p) be an independent uv-coincident realization of G. Independence implies that $i(H) \leq val(H)$ holds for all $H \subseteq V$ with $|H| \geq 2$. Since $p(u) = p(v)$, $uv \notin E$ follows.

Let $\mathcal{H} = \{X_1, \dots, X_k\}$ be a uv-compatible family and consider the subgraph $F = (\cup_{i=1}^k X_i, \cup_{i=1}^k E(X_i))$. By contracting the vertex pair u, v in F we obtain the graph F_{uv}, in which $\mathcal{H}_{uv} = \{X_1/\{u,v\}, \dots, X_k/\{u,v\}\}$ is a cover. Thus $r(F_{uv}) \leq \sum_{i=1}^k (2(|X_i| - 1) - 3)$. This bound and (2) imply that $dimKerR(F,p) \geq dimKerR(F_{uv}, p_{uv}) \geq 2(|\cup_{i=1}^k X_i| - 1) - \sum_{i=1}^k (2|X_i| - 5)$. Since (G,p) is uv-independent, we have

$$i_F(\mathcal{H}) = |F| \leq 2\left|\bigcup_{i=1}^k X_i\right| - \left(2\left(|\bigcup_{i=1}^k X_i| - 1\right) - \sum_{i=1}^k (2|X_i| - 5)\right) =$$

$$\sum_{i=1}^k (2|X_i| - 3) - 2(k-1) = val(\mathcal{H}).$$

Thus E is independent in $\mathcal{M}_{uv}(G)$, as claimed. □

3 Inductive constructions

The (two-dimensional versions of) the well-known Henneberg operations are as follows. Let $G = (V, E)$ be a graph. The 0-*extension* operation (on a pair of distinct vertices $a, b \in V$) adds a new vertex z and two edges za, zb to G. The *1-extension* operation (on edge $ab \in E$ and vertex $c \in V - \{a, b\}$) deletes the edge ab, adds a new vertex z and edges za, zb, zc.

We shall need the following specialized versions. Let $u, v \in V$ be two distinct vertices. The *0-uv-extension* operation is a 0-extension on a pair a, b with $\{a, b\} \neq \{u, v\}$. The *1-uv-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u, v\}$ is not a subset of $\{a, b, c\}$. The inverse operations are called *0-uv-reduction* and *1-uv-reduction*, respectively.

The Henneberg operations preserve independence in the two-dimensional rigidity matroid, see e.g. [12, Lemma 2.1.3, Theorem 2.2.2]. The same arguments can be used to verify the next lemma.

Lemma 10. *Let $G = (V, E)$ be an uv-independent graph and suppose that G' is obtained from G by a 0-uv-extension or a 1-uv-extension. Then G' is uv-independent.*

Lemma 11. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that $|E| = 2|V| - 3$, E is independent in $\mathcal{M}_{uv}(G)$, and $d(a) \geq 3$ for all $a \in V - \{u, v\}$. Then either $G = K_4 - uv$ or there is a vertex $z \in V - \{u, v\}$ with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$.*

Proof. For a contradiction suppose that for all $z \in V - \{u, v\}$ with $d(z) = 3$ we have $z \in N(u) \cap N(v)$ and let m denote the number of vertices of degree three in $N(u) \cap N(v)$. We may assume that $m \leq d(u) \leq d(v)$. By our assumptions we have

$$4|V| - 6 = 2|E| = \sum d(v) \geq d(u) + d(v) + 3m + 4(|V| - m - 2)$$

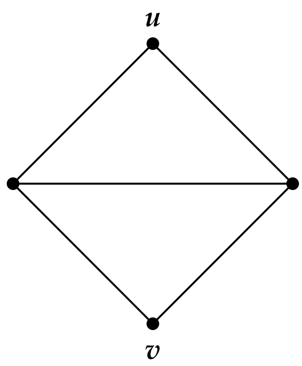

Figure 3: The graph $K_4 - uv$.

$$= 4|V| - m + d(u) + d(v) - 8 \geq 4|V| + d(v) - 8,$$

which implies that $m = d(u) = d(v) = 2$ must hold. Let $N(u) \cap N(v) = \{a, b\}$. Then either $ab \in E$ and hence $G = K_4 - uv$ or $U = V - \{u, v, a, b\}$ is non-empty and $i(U) \geq 2|U| - 1$ holds, contradicting the fact that E is independent in $\mathcal{M}_{uv}(G)$. □

Lemma 12. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that E is independent in $\mathcal{M}_{uv}(G)$ and let $z \in V - \{u, v\}$ be a vertex with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$. Then there is a 1-reduction at z which leads to a graph G' which is independent in $\mathcal{M}_{uv}(G')$.*

Proof. Let $F = \{ab \notin E : a, b \in N(z)\}$, let $G_1 = G - z + F$ and $G_2 = G + F$. For a contradiction suppose that $r_{uv}(G_1) \leq r_{uv}(G) - 3$. Consider a base B_1 of $\mathcal{M}_{uv}(G_1)$ which contains the triangle on $N(z)$ and let B_2 be a base of $\mathcal{M}_{uv}(G_2)$ with $B_1 \subseteq B_2$. Since K_4 is a circuit of $\mathcal{M}_{uv}(G_2)$, we have $r_{uv}(G_2) \leq r_{uv}(G_1) + 2$. Thus $r_{uv}(G) \leq r_{uv}(G_2) \leq r_{uv}(G) - 1$, a contradiction. □

Theorem 13. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-independent if and only if E is independent in $\mathcal{M}_{uv}(G)$.*

Proof. Necessity follows from Lemma 9. Now suppose that E is independent in $\mathcal{M}_{uv}(G)$. We prove that G is uv-independent by induction on $|V|$. By extending E to a base of $\mathcal{M}_{uv}(G)$, if necessary, we may assume that $|E| = 2|V| - 3$ holds. If $|V| \leq 4$ then we must have $G = K_4 - uv$, which is uv-independent. Thus we may assume that $|V| \geq 5$.

First suppose that there is a vertex $w \in V - \{u, v\}$ with $d(w) = 2$. Let $N(w) = \{a, b\}$. Clearly, $a \neq b$ holds. If $\{a, b\} = \{u, v\}$ then let $\mathcal{H} = \{\{u, v, w\}, \{V - w\}\}$. We have

$$2|V| - 3 = |E| = i_E(\mathcal{H}) \leq val(\mathcal{H}) = 2 \cdot 3 - 3 + 2(|V| - 1) - 3 - 2 = 2|V| - 4,$$

a contradiction. Hence $\{a, b\} \neq \{u, v\}$, which implies that the 0-uv-reduction operation can be applied at w to obtain a graph $G' = (V - w, E')$ that is independent in the matroid $\mathcal{M}_{uv}(G')$ and satisfies $|E'| = 2|V-w|-3$. By induction, G' is uv-independent. Now Lemma 10 implies that G is uv-independent.

Next suppose that there is no vertex of degree two in G. By Lemmas 11 and 12 we may apply the 1-uv-reduction operation at some vertex z of degree three to obtain a graph $G' = (V - w, E')$ that is independent in the matroid $\mathcal{M}_{uv}(G')$ and satisfies $|E'| = 2|V - w| - 3$. By induction G' is uv-independent. Lemma 10 implies that G is uv-independent. This completes the proof. □

As a by-product of the proof of Theorem 13 we obtain the following corollary.

Theorem 14. *Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$ and let $u, v \in V$ be distinct vertices. Then G is uv-independent if and only if G can be obtained from $K_4 - uv$ by a sequence of 0-uv-extensions and 1-uv-extensions.*

3.1 Main result

Theorem 15. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-rigid if and only if $G - uv$ and G_{uv} are both rigid.*

Proof. Necessity follows from the fact that an infinitesimally rigid uv-coincident realization of G gives rise to an infinitesimally rigid realization of $G - uv$ as well as G_{uv}, by (2).

To prove sufficiency, suppose, for a contradiction, that $G - uv$ and G_{uv} are both rigid but G is not uv-rigid. By Theorems 8 and 13 this implies that there is a thin cover $\mathcal{K}$ of $G - uv$ with $val(\mathcal{K}) \leq 2|V| - 4$. If $\mathcal{K}$ consists of subsets of V only then $r(G - uv) \leq 2|V| - 4$ follows, which contradicts the fact that $G - uv$ is rigid.

Hence $\mathcal{K} = \{\mathcal{H}, H_1, \ldots, H_k\}$, where $\mathcal{H} = \{X_1, \ldots, X_l\}$ is a uv-compatible family. Contract the vertex pair u, v in G into a new vertex z_{uv}. This leads to a graph G_{uv} and a cover

$$\mathcal{K}' = \{X'_1, \ldots, X'_l, H_1, \ldots, H_k\}$$

of G_{uv}, where X'_j is obtained from X_j by replacing u, v by z_{uv}, for $1 \leq j \leq l$. Then we obtain

$$\sum_{i=1}^{k}(2|H_i| - 3) + \sum_{j=1}^{l}(2|X'_j| - 3) = \sum_{i=1}^{k}(2|H_i| - 3) +$$

$$+ \sum_{j=1}^{l}(2|X_j| - 3) - 2l = val(\mathcal{K}) - 2 \leq 2|V| - 4 - 2 = 2(|V| - 1) - 4,$$

which implies that G_{uv} is not rigid, a contradiction. This completes the proof. □

A similar proof can be used to verify the following more general result:

Theorem 16. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then $r_{uv}(G) = \min\{r(G - uv), r(G_{uv}) + 2\}$.*

Theorems 15 and 16 show that the polynomial-time algorithms for computing the rank of a graph in the two-dimensional rigidity matroid (see e.g. [1]) can be used to test whether G is uv-rigid, or more generally, to compute $r_{uv}(G)$.

4 An obstacle for minimal uv-rigidity

We may also obtain a characterization of minimally uv-rigid graphs which is similar to the obstacle-based characterization for the collinear problem given in Theorem 1.

Theorem 17. *Let $G = (V, E)$ be a minimally rigid graph and let $u, v \in V$ be distinct vertices. Suppose that $uv \notin E$. Then the following statements are equivalent:*
(i) G is uv-rigid,
(ii) there is no subgraph $G' = (V', E')$ of G with $\{u, v\} \subseteq V'$ and $|E'| = 2|V'| - (3+s)$ such that $G' - \{u, v\}$ has at least $s + 2$ components, for $s = 0$ or $s = 1$.

Proof. First suppose that there is a subgraph $G' = (V', E')$ of G with $|E'| = 2|V'| - (3 + s)$ for which $G' - \{u, v\}$ has at least $s + 2$ components, for $s = 0$ or $s = 1$. Let $G_1 = (E_1, V_1), \ldots, G_t = (E_t, V_t)$ be the components of $G - \{u, v\}$. Consider the following cover of G:

$$\mathcal{K} = \{\{V_i \cup \{u, v\} : 1 \leq i \leq t\}\} \cup \{\{v_p, v_q\} : v_p v_q \in E - E'\}.$$

Since $t \geq s + 2$, we obtain

$$r_{uv}(E) \leq \sum_{i=1}^{t}(2|V_i + \{u, v\}| - 3) - 2(t - 1) + |E - E'| = \sum_{i=1}^{t} 2|V_i| - t + 2 + |E - E'| =$$

$$= 2|(\bigcup_{i=1}^{t} V_i) \cup \{u, v\}| - (t + 2) + |E - E'| \leq 2|V'| - (s + 4) + |E - E'| < |E|.$$

Thus G is not uv-independent (and hence not uv-rigid) by Lemma 9. Hence (i) implies (ii).

Next suppose that G is not uv-rigid. Then, by Theorems 8 and 13, there is a thin cover $\mathcal{K}_0$ of G with $val(\mathcal{K}_0) \leq 2|V| - 4$. Since G is rigid, $\mathcal{K}_0 = \{\mathcal{H}, H_1, \ldots, H_k\}$, where $\mathcal{H} = \{X_1, \ldots, X_l\}$ is a uv-compatible family with $l \geq 2$. Since $\mathcal{K}_0$ is thin, the set $\{u, v\}$ separates the subgraph $G' = (V', E')$, where $V' = V(\mathcal{H})$ and $E' = E(\mathcal{H}) = E(V')$.

We claim that by choosing $\mathcal{K}_0$ so that the number of its members is maximized, we have $i(H_i) = 2|H_i| - 3$ for all $1 \leq i \leq k$ and $i(X_i) \geq 2|X_i| - 4$ for all $1 \leq j \leq l$. The claim follows by observing that we can replace a set H_i or X_j violating these counts by the pairs of end-vertices of the edges it covers to obtain another cover with the same or smaller value. (If $X_j \in \mathcal{H}$ then we also remove X_j from the uv-compatible family.) Furthermore, since G is independent and $uv \notin E$, there can be at most one $X_i \in \mathcal{H}$ with $E(X_i) = 2|X_i| - 3$, c.f. Lemma 6.

If there is a $X_i \in \mathcal{H}$ with $E(X_i) = 2|X_i| - 3$ then it is easy to see that we have $|E'| = 2|V'| - 3$. Since $l \geq 2$, $G' - \{u, v\}$ has at least two components.

If $E(X_i) = 2|X_i| - 4$ for all $1 \leq i \leq l$ then we have $|E'| = 2|V'| - 4$ and $l \geq 3$. To see the latter inequality suppose that $l = 2$ and take the cover $\mathcal{K}_3 = \{H_1, \ldots, H_k\} \cup \{\{n_a, n_b\} : n_a n_b \in E(X_1)\} \cup \{\{n_a, n_b\} : n_a n_b \in E(X_2)\}$. We have $val(\mathcal{K}_3) = val(\mathcal{K}_0) < 2|V| - 3$. Since there is no uv-compatible family in $\mathcal{K}_3$, this contradicts the fact that G is rigid. Hence $l \geq 3$, as claimed, which implies that $G' - \{u, v\}$ has at least three components. Thus (ii) implies (i). □

Finally we remark that it may be interesting to see whether our results imply that if G is minimally rigid on at least four vertices then there is a pair u, v for which G is uv-rigid, c.f. [4, Corollary 4.4].

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