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Maximum negatable set in bipartite matching covered graphs

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Abstract

In an undirected graph $G = (V, E)$ with a conservative weighting $w : E \to \{1, -1\}$, the problem of determining the maximum number of edges whose weight can be changed from 1 to -1 retaining the conservativeness of the weighting is known to be NP-complete [1]. We show that this problem is polynomially tractable for bipartite matching-covered graphs. We also note that the directed version of this problem remains NP-complete for oriented bipartite matching-covered graphs.

1 Introduction

Let $G = (V, E)$ be an undirected graph. A weight function $w : E \to \{1, -1\}$ is called *conservative* if the sum of the edge weights along any circuit is non-negative. For directed graphs, we consider the sum of edge weights only along directed circuits. We say that (G, w) is a *conservative graph* if the edge weighting w is conservative. An edge (or an arc in directed graphs) e is negative if $w(e) = -1$, otherwise it is positive. A subset of positive edges or arcs is called a *negatable set* if changing the weight of the edges in F results in another conservative weighting. The problem of determining the maximum negatable set is known to be NP-complete for both undirected and directed graphs [1]. We show that it is polynomially tractable for bipartite matching-covered graphs. However, the directed case remains NP-complete even when restricted to oriented bipartite matching-covered graphs.

2 Join-covered graphs

A conservative graph (G, w) in which every edge belongs to a zero-weight circuit is said to be *join-covered*. A graph (G, w) is join-covered if and only if every edge lies in a minimum *T-join*, where T is the set of vertices incident with an odd number of negative edges. A graph is *matching-covered* if it is 2-connected and every edge belongs to a perfect matching. If w is a ± 1 weighting on a graph G such that the negative edges form a perfect matching, then (G, w) is join-covered if and only if G is matching-covered. Thus, join-covered graphs are a generalisation of matching covered graphs. Several important results on matching covered graphs have been extended to join-covered graphs [2–4]. The following result deals with bipartite join-covered graphs and shall be made use of subsequently.

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Lemma 2.1. *Let G be a 2-connected bipartite graph and w be a ± 1 conservative weighting on G. If every positive edge in (G, w) belongs to a zero-weight circuit, then it is join-covered.*

Proof: Let T be the set of vertices in (G, w) incident with an odd number of negative edges. Then the set of negative edges forms a minimum *T-join*. Consider a negative edge e. Since G is a bipartite graph there exist edge disjoint *T-cuts* such that each of these cuts has a unique negative edge, and every negative edge is contained in some cut [5]. Let f be a positive edge in the cut that contains e, the existence of f is guaranteed because G is 2-connected. In (G, w), every positive edge lies in a zero-weight circuit. Clearly, the zero-weight circuit containing f should also contain e. Thus it follows that every negative edge also belongs to some zero-weight circuit. Therefore (G, w) is join-covered. $\square$

The following two results from [2] and [5] respectively provide us useful information about distance between vertices in join-covered graphs.

Lemma 2.2. *Let (G, w) be a join-covered graph. Then $d(u, v) \leq 0$, for any two vertices u and v in (G, w).*

Lemma 2.3. *Let $G = (V_1, V_2; E)$ be a simple bipartite graph and w be a ± 1 conservative weighting on G. Suppose there exists a negative path between every two vertices of the same class V_i, then G is a tree and w is -1 everywhere.*

2.1 Contractions

Let (G, w) be a bipartite join-covered graph. Let v_1 and v_2 be two vertices in the same colour class V_i such that $d(v_1, v_2) = 0$. The existence of such vertices is guaranteed by Lemmas 2.2 and 2.3. We define a $contraction$ operation on G as $contracting$ the vertices v_1, v_2 to a vertex v and replacing resulting parallel edges with a single edge of weight +1 if all the parallel edges had a weight of +1 in G, otherwise -1. The other edges retain the same weight as in G.

If the resulting graph is 2-connected, we call it (H, w') and it is said to be a contraction of G. If v is a cut vertex, let the components of $G \backslash v$ be G_1 and G_2. We consider the contractions of G to be $H_1 = G_1 \cup \{v\}$ and the graph $H_2 = G_2 \cup \{v\}$. If one of them, say $G_2 = \{x\}$ is a singleton graph, then clearly $w(vx) = -1$, and H_2 is a graph with a single edge of negative weight. In this case, we consider the contraction of G to be H_1, and one negative edge is said to be $removed$. Else H_1 and H_2 are 2-connected. Any circuit in the contractions corresponds to either a circuit or to a path between v_1 and v_2 in (G, w). Therefore clearly, the contractions are also join-covered. From Lemma 2.3, note that there will always exist two vertices in V_i such that the distance between them is zero. Thus, we can repeatedly apply the contraction operation till we end up in a graph with two vertices and a single negative edge. And the original graph is said to be *fully contracted*. Therefore, the number of negative edges removed during the full contraction is the total number of negative edges in the original graph.

Now consider (G, w) to be a bipartite join-covered graph such that distance between any

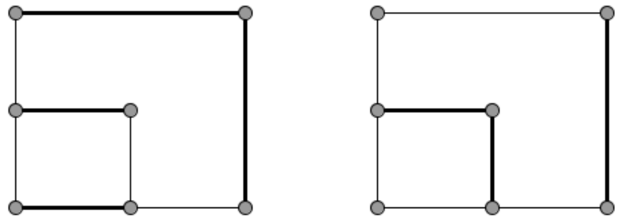

Figure 1: Join covered weightings with different number of negative edges

two vertices in the same colour class is zero. Let (H, w') be a contraction of G obtained by contracting some two vertices v_1, v_2 to v.

Lemma 2.4. *For any two vertices x, y in the same vertex class as v in (H, w'), $d_{w'}(x, y) =$ 0.*

Proof: Let v be the contracted vertex in H. Any path from x to y in H is either a path in G as well, or is a union of two disjoint paths from x and y to v_1 and v_2. Therefore, the path cannot have negative weight as it contradicts the property that the distance between any two vertices in the same colour class of G is zero. But (H, w') is join-covered. Therefore, $d_{w'}(x, y) = 0$. □

Suppose w_1 and w_2 are two ± 1 conservative weightings of a graph G, such that (G, w_1) and (G, w_2) are join-covered. It can be easily seen that the two weightings need not have an equal number of negative edges, as can be seen from Figure 1. The thicker edges represent negative edges, and the others positive edges. However, we show that if G is a bipartite matching-covered graph, all weightings w such that (G, w) is join-covered shall have equal number of negative edges.

Lemma 2.5. *Let G be a bipartite matching-covered graph, and w_1 , w_2 be two ± 1 weightings such that (G, w_1) and (G, w_2) are join-covered. Then (G, w_1) and (G, w_2) have an equal number of negative edges.*

Proof: Let w be a weighting on G such that the negative edges form a perfect matching. G is matching-covered, so (G, w) is join-covered. Moreover, the distance between any two vertices in the same colour class is zero. Let w' be another ± 1 edge-weighting such that (G, w') is join-covered. Clearly it suffices to show that (G, w) and (G, w') have equal number of negative edges. As noted above, we *fully contract* (G, w') by contracting suitable vertices belonging to the same class at each step. Now, note that the distance between all pairs of vertices belonging to the same class in (G, w) is zero. From Lemma 2.4, all the pairs of vertices will have zero distance between them in the subsequent contractions as well. And thus each contraction operation applied in the *full contraction* of (G, w') can be applied to (G, w) as well. Therefore, since (G, w) and (G, w') can be *fully contracted* identically, (G, w) and (G, w') have an equal number of negative edges. □

Let G be a bipartite graph and w be a ± 1 conservative weighting. Let F_m be a maximal negatable set, and w_m be the conservative weighting obtained by changing the weights of edges in F_m to -1.

Lemma 2.6. (G, w_m) *is join-covered.*

Proof: (G, w_m) is conservative. For any edge $e \in E(G)$, let $C(e)$ denote the minimum weight of a cycle containing e in (G, w_m). If for all edges e, $C(e)$ is zero, we are done. Suppose there is a positive edge e such that $C(e)$ is non-zero. Since (G, w_m) is a bipartite conservative graph, $C(e) \geq 2$. Then, the weight of e could be changed from +1 to -1 without violating the conservativeness. This is a contradiction to the maximality of F_m. Hence, every positive edge belongs to a zero-weight circuit in (G, w_m) and from Lemma 2.1, it follows that (G, w_m) is join-covered. □

Given a conservative graph, a maximal negatable set can be constructed in polynomial time. For a bipartite matching-covered graph, (G, w_m) is join-covered, and therefore from Lemma 2.5, the following result is immediate.

Theorem 2.7. *Determining the maximum negatable set is polynomially tractable in a bipartite matching-covered graph.*

3 The directed case

Given a directed graph $G = (V, A)$, with a conservative ± 1 weighting w of the arcs , a negatable set of arcs is a set of positive arcs such that changing their weight to -1 retains the conservativeness of (G, w). In [1], the problem of determining the maximum size of a negatable set in directed graphs is shown to be NP-complete via reduction to the maximum size of a stable set problem. The proof presented is very similar. We extend their argument slightly and show that determining the maximum negatable set remains NP-complete even when restricted to oriented bipartite matching-covered graphs.

The following NP-completeness result follows directly from the results that cubic bridgeless graphs are matching-covered and the minimum vertex cover problem is NP-complete for cubic bridgeless graphs.

Proposition 3.1. *The problem of determining the maximum stable set in a matching-covered graph is NP-complete.*

Theorem 3.2. *The problem of determining the maximum size of a negatable set in an oriented bipartite matching covered graph is NP-complete.*

Proof: Let $G = (V, E)$ be an undirected bipartite matching covered graph. We construct an oriented bipartite graph $D = (V', A)$ from G. We replace each vertex v in G, by an arc $v'v''$ and each edge $e = uv$ by the arcs shown in Figure 3. Let D_e denote the subgraph of D formed by these arcs. D_e is matching covered and $D_e - \{u', u'', v', v''\}$ is perfectly matchable. And now since G is matching covered, this implies that D is an oriented bipartite matching covered graph. We define a weighting w on the arcs of D as shown. It is easily seen that the weighting is conservative. Also, if a is a negatable arc in D, then $a = v'v''$ for

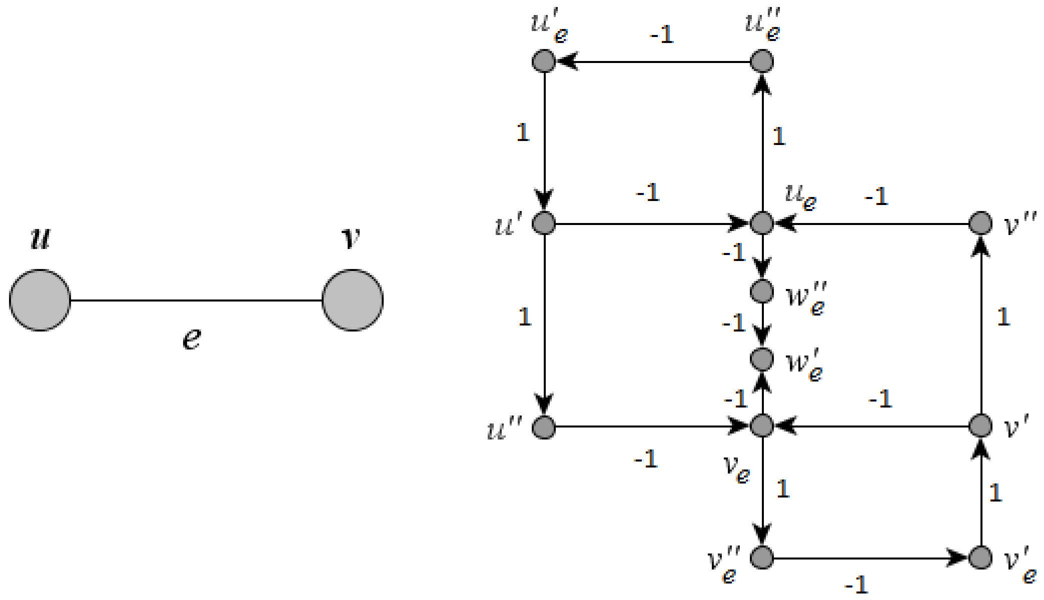

Figure 2: Constructing D_e that replaces an edge e of G

some vertex v of G.

Let $N \subseteq A$ be a negatable set of arcs. Let $S = \{v \in V : v'v'' \in N\}$. For u, v in S, if $uv \in E$, then we obtain a contradiction because the weight of the cycle formed by the edges $\{u'u'', u''v_e, v_ev''_e, v''_ev'_e, v'_ev', v'v'', v''u_e, u_eu''_e, u''_eu'_e, u'_eu'\}$ is negative.

Now consider a stable set $S \subseteq E$. Then it is easy to see that $N = \{v'v' : v' \in S\}$ is a negatable set by the same arguments as in [1]. Thus the maximum size of a stable set in G is equal to the size of maximum negatable set in D. Determining the maximum stable set in G is NP-complete. Therefore, it is NP-complete to determine the maximum size of a negatable set in an oriented bipartite matching covered graph.

□

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