Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-19. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Globally linked pairs of vertices in rigid frameworks

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December 13, 2012

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Abstract

A 2-dimensional *framework* (G, p) is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^2$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^2$. Two realizations of G are *equivalent* if the corresponding edges in the two frameworks have the same length. A pair of vertices $\{u, v\}$ is *globally linked* in G if the distance between the points corresponding to u and v is the same in all pairs of equivalent generic realizations of G.

In this paper we extend our previous results on globally linked pairs and complete the characterization of globally linked pairs in minimally rigid graphs. We also show that the Henneberg 1-extension operation on a non-redundant edge preserves the property of being not globally linked, which can be used to identify globally linked pairs in broader families of graphs. We prove that if (G, p) is generic then the set of globally linked pairs does not change if we perturb the coordinates slightly. Finally, we investigate when we can choose a non-redundant edge e of G and then continuously deform a generic realization of $G - e$ to obtain equivalent generic realizations of G in which the distances between a given pair of vertices are different.

1 Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$.

We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such

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that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations (see [11], [33, Section 3.2]). It seems to be a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [26] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider *generic frameworks* i.e. frameworks in which there are no algebraic dependencies between the coordinates of the vertices.

It is known, see [33], that the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$ (and is a major open problem for $d \geq 3$). See Section 2 for more details.

A similar situation holds for global rigidity: the problem of characterizing when a generic framework is globally rigid in $\mathbb{R}^d$ has also been solved for $d = 1, 2$. A 1-dimensional generic framework (G, p) is globally rigid if and only if either G is the complete graph on two vertices or G is 2-connected. The characterization for $d = 2$ is as follows. We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all edges e of G.

Theorem 1.1. *[17] Let (G, p) be a 2-dimensional generic framework. Then (G, p) is globally rigid if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

It follows that the global rigidity of d-dimensional frameworks is a generic property when $d = 1, 2$. Gortler, Healy and Thurston [13] proved that this holds for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. As for rigidity, it is an important open problem to characterize globally rigid graphs when $d \geq 3$. Hendrickson [14] showed that redundant rigidity and $(d+1)$-connectivity are necessary conditions for all $d \geq 1$ (provided G has at least $d + 2$ vertices) but there are examples showing that these conditions are not sufficient when $d \geq 3$, see [5], [9].

We refer the reader to [12, 18, 33] for a detailed survey of rigid and globally rigid d-dimensional frameworks and their applications.

We will consider properties of 2-dimensional generic frameworks which are weaker than global rigidity. We assume henceforth that $d = 2$, unless specified otherwise. A pair of vertices $\{u, v\}$ in a framework (G, p) is *globally linked* in (G, p) if, in all equivalent frameworks (G, q), we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. The pair $\{u, v\}$ is *globally linked* in G if it is globally linked in all generic frameworks (G, p). Thus G is globally rigid if and only if all pairs of vertices of G are globally linked. Unlike global rigidity, however, 'global linkedness' is not a generic property in $\mathbb{R}^2$. Figures 1 and 2 give an example of a pair of vertices in a rigid graph G which is globally linked in one generic realization, but not in another.

We initiated the study of globally linked pairs in [19]. We next summarize the main results and conjectures from this paper.

The Henneberg *1-extension* operation [15] (on edge xy and vertex w) deletes an edge xy from a graph H and adds a new vertex z and new edges zx, zy, zw for some vertex $w \in V(H) - \{x, y\}$. We showed that the 1-extension operation preserves the property that a pair of vertices is globally linked *as long as $H - xy$ is rigid*.

Theorem 1.2. *[19] Let G, H be graphs such that G is obtained from H by a 1-extension on edge xy and vertex w. Suppose that $H - xy$ is rigid and that $\{u, v\}$ is globally linked in H. Then $\{u, v\}$ is globally linked in G.*

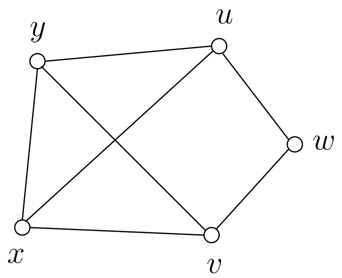

Figure 1: A realization (G, p) of a rigid graph G in $\mathbb{R}^2$. The pair $\{u, v\}$ is globally linked in (G, p).

Let $H = (V, E)$ be a graph and $x, y \in V$. We use $\kappa_H(x, y)$ to denote the maximum number of pairwise openly disjoint xy-paths in H. Note that if $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$.

Lemma 1.3. *[19] Let (G, p) be a generic framework, $x, y \in V(G)$, $xy \notin E(G)$, and suppose that $\kappa_G(x, y) \leq 2$. Then $\{x, y\}$ is not globally linked in (G, p).*

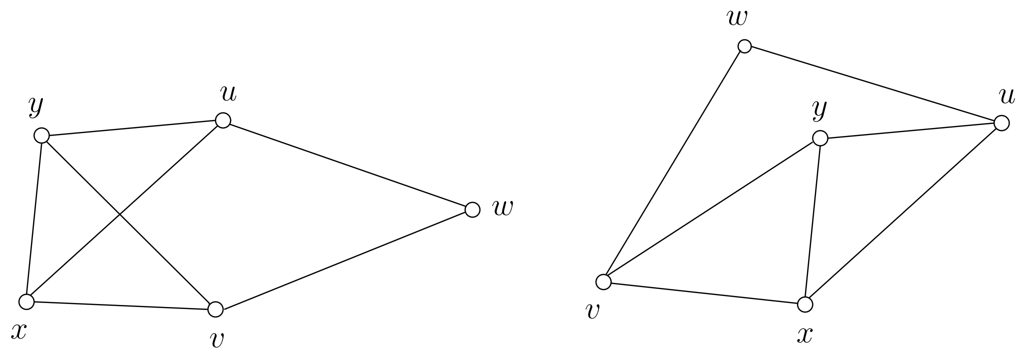

Figure 2: Two equivalent realizations of the rigid graph G of Figure 1, which show that the pair $\{u, v\}$ is not globally linked in G in $\mathbb{R}^2$.

We used Theorem 1.2 and Lemma 1.3 to characterize globally linked pairs for the family of M-connected graphs i.e graphs whose 2-dimensional rigidity matroid is connected (see Section 2 for formal definitions). This family lies strictly between the families of globally rigid graphs and redundantly rigid graphs.

Theorem 1.4. *[19] Let $G = (V, E)$ be an M-connected graph and $x, y \in V$. Then $\{x, y\}$ is globally linked in G if and only if $\kappa_G(x, y) \geq 3$.*

An *M-component* of a graph G is a maximal M-connected subgraph of G. Theorem 1.4 has the following immediate corollary.

Corollary 1.5. *[19] Let $G = (V, E)$ be a graph and $x, y \in V$. If either $xy \in E$, or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$, then $\{x, y\}$ is globally linked in G.*

We conjectured that the converse is also true.

Conjecture 1.6. *[19] The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ if and only if either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$.*

A *redundantly rigid component* of a graph G is a maximal redundantly rigid subgraph of G (see Section 2). We showed in [19] that Conjecture 1.6 is equivalent to two other conjectures concerning the redundantly rigid components of G.

Conjecture 1.7. *[19] Suppose that $\{x, y\}$ is a globally linked pair in a graph G. Then there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$.*

Conjecture 1.8. *[19] Let G be a graph. Suppose that there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$ and $\{x, y\}$ is globally linked in G. Then $\{x, y\}$ is globally linked in R.*

The Henneberg 0-*extension* operation on vertices x, y in a graph H adds a new vertex z and new edges xz, yz to H. We showed that the 0-extension operation preserves the property that a pair of vertices is not globally linked.

Lemma 1.9. *[19] If $\{u, v\}$ is not globally linked in H and G is a 0-extension of H then $\{u, v\}$ is not globally linked in G.*

The purpose of this paper is to extend the results of [19] in several directions. In Section 3 we prove that the 1-extension operation preserves the property that a pair of vertices is not globally linked whenever it is applied to a non-redundant edge in an arbitrary rigid graph. We use this to deduce that Conjecture 1.6 holds for minimally rigid graphs. (Since the M-connected components of a minimally rigid graph are all isomorphic to K_2 this is equivalent to showing that a pair of vertices in a minimally rigid graph is globally linked if and only if they are adjacent.) We consider frameworks with the property that all equivalent frameworks are infinitesimally rigid in Section 4. We show that for such a framework (G, p), the number of equivalent pairwise non-congruent frameworks does not increase if we make small perturbations to the positions of its vertices. This extends a result of Connelly and Whiteley [6] on infinitesimally rigid globally rigid frameworks. We deduce that, if G is minimally rigid or p is generic, then the set of globally linked pairs in (G, p) does not change if we make small perturbations to the positions of its vertices. In Section 5 we investigate when we can choose a non-redundant edge e in a graph G and then continuously deform a generic realization of $G - e$ to obtain equivalent generic realizations of G in which the distances between a given pair of vertices are different.

2 Preliminaries

In this section we summarize the definitions and results from rigidity theory that we shall use later.

2.1 The rigidity matroid

The rigidity matroid of a graph G is a matroid defined on the set of edges of G which reflects the rigidity properties of all generic realizations of G. We will need basic definitions and results on this matroid to define M-connected graphs.

Let (G, p) be a realization of a graph $G = (V, E)$. The *rigidity matrix* of the framework (G, p) is the matrix $R(G, p)$ of size $|E| \times 2|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the two columns corresponding to vertices v_i and v_j contain the two coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. See [33] for more details. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows of the rigidity matrix. Any two generic frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the *rigidity matroid* $\mathcal{R}(G) = (E, r)$ of the graph G. We denote the rank of $\mathcal{R}(G)$ by $r(G)$. Gluck characterized rigid graphs in terms of their rank.

Theorem 2.1. *[11] Let $G = (V, E)$ be a graph. Then G is rigid if and only if $r(G) = 2|V| - 3$.*

We say that a graph $G = (V, E)$ is *M-independent* if E is independent in $\mathcal{R}(G)$. Knowing when subgraphs of G are M-independent allows us to determine the rank of G. This can be accomplished using the following characterization of M-independent graphs due to Laman. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in G joining vertices in X.

Theorem 2.2. *[21] A graph $G = (V, E)$ is M-independent if and only if $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.*

A graph $G = (V, E)$ is *minimally rigid* if G is rigid, but $G - e$ is not rigid for all $e \in E$. Theorems 2.1 and 2.2 imply that G is minimally rigid if and only if G is M-independent and $|E| = 2|V| - 3$. Note that, if G is rigid, then the edge sets of the minimally rigid spanning subgraphs of G form the bases in the rigidity matroid of G.

A pair of vertices $\{u, v\}$ in a framework (G, p) is *linked* in (G, p) if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(w) - q(w)|| < \epsilon$ for all $w \in V$, then we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. Using Theorems 2.1 and 2.2, it can be seen that this is a generic property and that $\{u, v\}$ is linked in a generic framework (G, p) if and only if G has a rigid subgraph H with $\{u, v\} \subseteq V(H)$.

A compact characterization of all linked pairs can be deduced as follows. We define a *rigid component* of G to be a maximal rigid subgraph of G. It is well-known (see e.g. [17, Corollary 2.14]), that any two rigid components of G intersect in at most one vertex and hence that the edge sets of the rigid components of G partition the edges of G. Thus $\{u, v\}$ is linked in a generic framework (G, p) if and only if $\{u, v\} \subseteq V(H)$ for some rigid component H of G.

Recall the definitions of the 0- and 1-extension operations from Section 1. The basic result about 0-extensions is the following.

Lemma 2.3. *[32] Let G be a graph and let H be obtained from G by a 0-extension. Then H is minimally rigid if and only if G is minimally rigid.*

It is known that the 1-extension operation preserves rigidity [32]. We shall need the following lemma about the inverse operation of 1-extension on minimally rigid graphs.

Lemma 2.4. *[32] Let $G = (V, E)$ be a minimally rigid graph and let $v \in V$ be a vertex with $d(v) = 3$. Then v has two non-adjacent neighbours u, w such that the graph $H = G - v + uw$ is minimally rigid.*

By observing that a minimally rigid graph has a vertex of degree two or three, it follows that a graph is minimally rigid if and only if it can be constructed from an edge by a sequence of 0-extensions and 1-extensions.

2.2 M-connected graphs and M-components

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}(G)$. Using Theorem 2.2 we may deduce that G is an M-circuit if and only if $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$. Recall that a graph G is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. Note also that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{R}(G)$ i.e. an M-circuit of G.

Any two maximal redundantly rigid subgraphs of a graph $G = (V, E)$ can have at most one vertex in common, and hence are edge-disjoint (see [17]). Defining a *redundantly rigid component* of G to be either a maximal redundantly rigid subgraph of G, or a subgraph induced by an edge which belongs to no M-circuit of G, we deduce that the redundantly rigid components of G partition E. Since each redundantly rigid component is rigid, this partition is a refinement of the partition of E given by the rigid components of G. Note that the redundantly rigid components of G are induced subgraphs of G.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}(G)$ is connected. Thus M-circuits are examples of M-connected graphs. Another example is the complete bipartite graph $K_{3,m}$, which is M-connected for all $m \geq 4$. (This follows because $K_{3,4}$ is an M-circuit and any pair of edges of $K_{3,m}$ are contained in a copy of $K_{3,4}$.) The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. Since each M-component with at least two edges is redundantly rigid, the partition of E given by the M-components is a refinement of the partition given by the redundantly rigid components of G. Note that the M-components of G are induced subgraphs. For more examples and basic properties of M-circuits and M-connected graphs see [3, 17].

Note that the rigid components, redundantly rigid components, and M-components of a graph can all be determined in polynomial time, see for example [3].

2.3 Rigidity, infinitesimal rigidity, and flexes

In this subsection we consider d-dimensional frameworks for arbitrary $d \geq 1$. Let (G, p) be a d-dimensional framework. A *flexing* of the framework (G, p) is a continuous function $\pi : (-1, 1) \times V \to \mathbb{R}^d$ such that $\pi_0 = p$, and such that the frameworks (G, p) and (G, π_t) are equivalent for all $t \in (-1, 1)$, where $\pi_t : V \to \mathbb{R}^d$ by $\pi_t(v) = \pi(t, v)$ for all $v \in V$. The flexing π is *trivial* if the frameworks (G, p) and (G, π_t) are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial flexing. It is known [2, 11] that non-rigidity, flexibility and the existence of a non-trivial smooth flexing are all equivalent.

The first-order version of a flexing of the framework (G, p) is called an *infinitesimal motion*. This is an assignment of *infinitesimal velocities* to the vertices, $\tilde{p} : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(\tilde{p}(u) - \tilde{p}(v)) = 0 \text{ for all pairs } u, v \text{ with } uv \in E. \tag{1}$$

If π is a smooth flexing of (G, p), then $\dot{\pi}_0 := \frac{d\pi}{dt}|_{t=0}$ is an infinitesimal motion of (G, p). A *trivial infinitesimal motion* of (G, p) has the form $\tilde{p}(v) = Ap(v) + b$, for all $v \in V$, for some $d \times d$ antisymmetric matrix A and some $b \in \mathbb{R}^d$. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is *infinitesimally rigid*.

The set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{d|V|}$, given by the system of $|E|$ linear equations (1). The matrix of this system of linear equations is the *rigidity matrix* $R(G, p)$ of (G, p) defined earlier. The *rigidity map* for a graph $G = (V, E)$ is the map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$, given by

$$f_G(p) = (\ldots, ||p(u) - p(v)||^2, \ldots) .$$

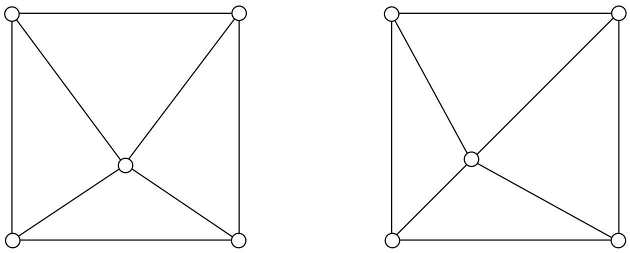

Figure 3: Two regular realizations of a graph G. The first one is globally rigid, but the second is not, since it can fold around one of the diagonals.

Note that the Jacobian of f_G at a point $p \in \mathbb{R}^{|V|d}$ is given by $2R(G,p)$.

Gluck [11] proved that if a framework (G,p) is infinitesimally rigid, then it is rigid. The converse of this is not true in general, but if we exclude certain 'degenerate' configurations, then rigidity and infinitesimal rigidity are equivalent. In order to establish this, let us recall some notions from differential topology. Given two smooth manifolds, M and N and a smooth map $f : M \to N$, we denote the derivative of f at some point $p \in M$ by $df|_p$, which is a linear map from T_pM, the tangent space of M at p, to $T_{f(p)}N$. Let k be the maximum rank of $df|_q$ over all $q \in M$. A point $p \in M$ is said to be a *regular point* of f, if rank $df|_p = k$, and a *critical point*, if rank $df|_p < k$. We say that a framework (G,p) is *regular*, if p is a regular point of f_G. Using the inverse function theorem, it can be shown (see for example [2, Proposition 2]) that if (G,p) is a regular framework, then there is a neighbourhood U_p of p, such that $f_G^{-1}(f_G(p)) \cap U_p$ is a manifold, whose tangent space at p is the kernel of df_p. This has the following corollary.

Theorem 2.5. *[2] Let (G,p) be a regular framework. If (G,p) is infinitesimally flexible, then it is flexible. Furthermore, if $\tilde{p}$ is a non-trivial infinitesimal motion of (G,p), then there is a non-trivial smooth flexing π of (G,p) such that $\dot{\pi}_0 = \tilde{p}$.*

Since the rank of the rigidity matrix for a given graph G is constant on the set of regular points of f_G and infinitesimal rigidity of a framework (G,p) depends only on the rank of $R(G,p)$ it follows that if a regular framework (G,p) is infinitesimally rigid, then all other regular frameworks (G,q) are infinitesimally rigid as well.

3 1-Extensions and globally linked pairs

In this section we prove that the 1-extension operation preserves the property that a pair of vertices is not globally linked assuming that it is performed on a non-redundant edge. By using this result we shall complete the characterization of globally linked pairs in minimally rigid graphs.

Given a field $K \subseteq \mathbb{C}$ we use $\overline{K}$ to denote the algebraic closure of K. We say that a point $P = (x,y) \in \mathbb{C}^2$ is *generic over* K, if the set $\{x,y\}$ is algebraically independent over K. To prove the framework extension result of this section, we need a lemma concerning polynomials whose zeros are algebraically dependent over K. For

a polynomial $f \in K[X_1, X_2]$, we denote the set of zeros of f in K^2 by $V(f, K)$. We will use the following facts concerning two polynomials $f, g \in K[X_1, X_2]$.
Fact 1: if g is irreducible over an algebraically closed subfield of $\mathbb{C}$ then g is irreducible over $\mathbb{C}$, see [16, page 76, Corollary to Theorem IV].
Fact 2: if $V(f, K) \cap V(g, K)$ is infinite then f and g have a non-trivial common factor in $K[X_1, X_2]$, see [10, Chapter 1, Proposition 2].

Lemma 3.1. *Let L be an algebraically closed subfield of $\mathbb{C}$ and $K = L \cap \mathbb{R}$. Suppose that $g \in K[X_1, X_2]$ is irreducible over K. Then g is irreducible over $\mathbb{R}$.*

Proof. Let $g = g_1 g_2 ... g_m$ be the factorization of g into irreducible factors over L. Then $g = g_1 g_2 ... g_m$ is also the factorization of g into irreducible factors over $\mathbb{C}$ by Fact 1. Now suppose that $g = h_1 h_2$ is a non-trivial factorization of g over $\mathbb{R}$. Relabeling if necessary and using the fact that $\mathbb{C}[X_1, X_2]$ is a unique factorization domain we have $h_1 = g_1 g_2 ... g_s$ and $h_2 = g_{s+1} g_{s+2} ... g_m$ for some $1 \leq s \leq m - 1$. This implies that $h_1, h_2 \in L[X_1, X_2]$. Since we also have $h_1, h_2 \in \mathbb{R}[X_1, X_2]$ we get $h_1, h_2 \in K[X_1, X_2]$. This contradicts the irreducibility of g over K. •

Lemma 3.2. *Let L be a countable algebraically closed subfield of $\mathbb{C}$, $K = L \cap \mathbb{R}$, and $f \in \mathbb{R}[X_1, X_2]$ be irreducible over $\mathbb{R}$. Suppose that $V(f, \mathbb{R})$ is uncountable and each $(x_1, x_2) \in V(f, \mathbb{R})$ is algebraically dependent over K. Then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda f \in K[X_1, X_2]$.*

Proof. Since each $(x_1, x_2) \in V(f, \mathbb{R})$ is algebraically dependent over K, each $(x_1, x_2) \in V(f, \mathbb{R})$ is a root of an irreducible polynomial in $K[X_1, X_2]$. Since $K[X_1, X_2]$ is countable and $V(f, \mathbb{R})$ is uncountable there exists an irreducible polynomial $g \in K[X_1, X_2]$ such that $V(f, \mathbb{R}) \cap V(g, \mathbb{R})$ is uncountable. Since L is algebraically closed, Lemma 3.1 implies that g is irreducible over $\mathbb{R}$. Since $V(f, \mathbb{R}) \cap V(g, \mathbb{R})$ is infinite, Fact 2 implies that f and g have a non-trivial common factor in $\mathbb{R}[X_1, X_2]$. Since they are both irreducible over $\mathbb{R}$, we have $f = \lambda g$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. •

A framework (G, p) is *quasi-generic* if it is congruent to a generic framework. It is in *standard position with respect to two vertices* (v_1, v_2) if $p(v_1)$ lies at the origin and $p(v_2)$ lies on the second coordinate axis. We may use a translation and a rotation to transform every framework to a congruent framework which is in standard position with respect to any two given vertices. The next result determines what happens when we apply such a transformation to a quasi-generic framework.

Lemma 3.3. *[19, Lemma 3.5] Let (G, p) be a realisation of a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$. Let (G, q) be a congruent realisation which is in standard position with respect to (v_1, v_2). Suppose $q(v_i) = (x_i, y_i)$ for $1 \leq i \leq n$ (so $x_1 = y_2 = x_2 = 0$). Then (G, p) is quasi-generic if and only if $\{y_2, x_3, y_3, \ldots, x_n, y_n\}$ is algebraically independent over $\mathbb{Q}$.*

The following rather technical lemma is fundamental to our proof that constructing a 1-extension by deleting a non-redundant edge from a graph H preserves the property that two given vertices of H are not globally linked.

Lemma 3.4. *Let $G = (V, E)$ be a graph and v be a vertex of G of degree three with neighbour set $\{u, w, z\}$. Let $(G - v, p)$ and $(G - v, q)$ be equivalent frameworks which are in standard position with respect to (u, w). Suppose that p is quasi-generic, $q(u)$, $q(w)$ and $q(z)$ are not collinear, and $||q(u) - q(w)||^2 \notin \overline{\mathbb{Q}(p)}$. Then there are equivalent frameworks (G, p^*) and (G, q^*) where p^* is quasi-generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = q$.*

Proof. Let $L = \overline{\mathbb{Q}(p)}$ and $K = L \cap \mathbb{R}$. We have $p(u) = (0, 0)$, $p(w) = (0, p_3)$, $p(z) = (p_4, p_5)$, $q(u) = (0, 0)$, $q(w) = (0, q_3)$ and $q(z) = (q_4, q_5)$. Since $q(u)$, $q(w)$ and $q(z)$ are not collinear, $q_3 \neq 0 \neq q_4$. Moreover $q_3^2 = ||q(u) - q(w)||^2 \notin K$. By reflecting the configuration q on the second coordinate axis, if necessary, we may assume that $q_4 \neq p_4$. By Lemma 3.3, $\{p_3, p_4, p_5\}$ is algebraically independent over $\mathbb{Q}$.

We call a point $(p_1, p_2) \in \mathbb{R}^2$ *feasible*, if there exists a point $(q_1, q_2) \in \mathbb{R}^2$, such that the extended frameworks (G, p^*) and (G, q^*) are equivalent, where $p^*|_{V-v} = p$, $q^*|_{V-v} = q$, $p^*(v) = (p_1, p_2)$ and $q^*(v) = (q_1, q_2)$. We will prove the lemma by finding a feasible point that is generic over K and then applying Lemma 3.3.

The set of feasible points can be described by the following system of equations:

$$q_1^2 + q_2^2 = p_1^2 + p_2^2 \tag{2}$$

$$q_1^2 + (q_2 - q_3)^2 = p_1^2 + (p_2 - p_3)^2 \tag{3}$$

$$(q_1 - q_4)^2 + (q_2 - q_5)^2 = (p_1 - p_4)^2 + (p_2 - p_5)^2 . \tag{4}$$

Equations (2) and (3) give

$$q_2 = \frac{q_3^2 - p_3^2 + 2p_2p_3}{2q_3}. \tag{5}$$

Equations (2), (4) and (5) now give

$$q_1 = \frac{q_4^2 + q_5^2 - p_4^2 - p_5^2 + 2p_1p_4 + 2p_2p_5 - q_5(\frac{q_3^2 - p_3^2 + 2p_2p_3}{q_3})}{2q_4} . \tag{6}$$

We can use (5) and (6) to eliminate q_2, q_1 from (2) and obtain

$$4q_3^2q_4^2(q_1^2 + q_2^2 - p_1^2 - p_2^2) = a_{11}p_1^2 + a_{22}p_2^2 + a_{12}p_1p_2 + a_1p_1 + a_2p_2 + a_0 = 0 \tag{7}$$

where

$$\begin{aligned}
a_{11} &= 4q_3^2(p_4^2 - q_4^2) \\
a_{22} &= 4q_4^2(p_3^2 - q_3^2) + 4(q_3p_5 - p_3q_5)^2 \\
a_{12} &= 8p_4q_3(q_3p_5 - p_3q_5) \\
a_1 &= 4p_4q_3(q_3(r + s) - q_5(q_3^2 - p_3^2)) \\
a_2 &= 4(q_3p_5 - p_3q_5)(q_3(r + s) - q_5(q_3^2 - p_3^2)) + 4p_3q_4^2(q_3^2 - p_3^2) \\
a_0 &= (q_3(r + s) - q_5(q_3^2 - p_3^2))^2 + q_4^2(q_3^2 - p_3^2)^2
\end{aligned}$$

taking $r = q_4^2 - p_4^2$ and $s = q_5^2 - p_5^2$. Note that any real solution (p_1, p_2) to (7) gives a real solution (q_1, q_2) to (5) and (6). Thus the set of feasible points is the set of points lying on the conic $f = 0$ where

$$f = a_{11}X_1^2 + a_{22}X_2^2 + a_{12}X_1X_2 + a_1X_1 + a_2X_2 + a_0 \in \mathbb{R}[X_1, X_2].$$

Note that since $q_3^2 \notin K$ we have $q_3^2 \neq p_3^2$, and since $q_4 \neq 0$, this gives $a_0 > 0$.

Claim 3.5. *The conic $f = 0$ is not empty and is not a single point.*

Proof. Since $q_4 \neq p_4$, the conic contains the point

$$A = \left(\frac{p_3 + q_3}{2}, \frac{r + s - (p_3 + q_3)(p_5 - q_5)}{2(p_4 - q_4)} \right)$$

and hence is not empty. If $q_4 \neq -p_4$ then it also contains the point

$$B = \left(\frac{p_3 - q_3}{2}, \frac{r + s - (p_3 - q_3)(p_5 + q_5)}{2(p_4 + q_4)} \right),$$

and $A \neq B$ since $q_3 \neq 0$. Hence we may suppose that $q_4 = -p_4$. In this case $a_{11} = 0$. Thus $f \neq (b_1 X_1 - b_2)^2 + (b_3 X_2 - b_4)^2$ for all $b_1, b_2, b_3, b_4 \in \mathbb{R}$ with $b_1 \neq 0 \neq b_3$ so the conic cannot be a single point. •

Let us suppose indirectly, that:

$$\text{no point on the conic } f = 0 \text{ is generic over } K. \tag{8}$$

Since this conic is not empty, and is not a single point, it follows from the classification of conics that it is either an ellipse, a parabola, a hyperbola or the union of two lines. Applying Lemma 3.2 to the irreducible components of f we deduce that there is a $\lambda \in \mathbb{R} \setminus \{0\}$, such that $\lambda f \in K[X_1, X_2]$. Hence

$$\{\lambda a_{11}, \lambda a_{12}, \lambda a_{22}, \lambda a_1, \lambda a_2, \lambda a_0\} \subset K. \tag{9}$$

Claim 3.6. *$q_4^2 = p_4^2$ and $a_{11} = 0$.*

Proof. Suppose that $q_4^2 \neq p_4^2$. Consider the following two polynomials $g, h \in \mathbb{R}[X]$:

$$\begin{aligned} g &= a_{12}^2(p_3^2 a_{12}^2 - 4a_1^2)X^3 + 8p_4^2 a_{11}\left[p_3^2 a_{22} a_{12}^2 + 2a_1^2(a_{11} - a_{22}) - 2a_{12}^2 a_0\right] X^2 \\ &\quad + 16 p_4^4 a_{11}^2 \left[4(a_{11} - a_{22})a_0 + p_3^2 a_{22}^2 + a_1^2\right] X + 64 p_4^6 a_{11}^3 a_0, \\ h &= (4p_4^2 q_4^2 a_{11}^2 - 4 s p_4^2 a_{22} a_{11} - s^2 a_{12}^2)X - 4p_3^2 p_4^2 q_4^2 a_{11}^2. \end{aligned}$$

Since $q_3^2 \neq 0$ and $q_4^2 \neq p_4^2$, $a_{11} \neq 0$. The fact that p_3, p_4, q_4, a_0 are non-zero now implies that the constant terms of both g and h are non-zero and hence neither h nor g is identically zero. Substituting all coefficients with their appropriate expressions we see that $g(q_4^2 - p_4^2) = 0$ and $h(q_3^2) = 0$. We have $\lambda^4 g \in K[X]$ by (9). Since $g(q_4^2 - p_4^2) = 0$, $q_4^2 - p_4^2 \in \overline{\mathbb{Q}(p)}$ and hence $q_4 \in \overline{\mathbb{Q}(p)}$. Thus $q_4 \in K$. This and (9) imply that $\lambda^2 h \in K[X]$ and we may use a similar argument to deduce that $q_3^2 \in K$, which is a contradiction. Hence $q_4^2 = p_4^2$ and $a_{11} = 0$. •

Claim 3.7. $a_{12} \neq 0$.

Proof. Suppose $a_{12} = 0$. Then $q_3 p_5 - p_3 q_5 = 0$, $a_{22} = 4p_4^2(p_3^2 - q_3^2)$, and $a_{22} \neq 0$ since $q_3^2 \neq p_3^2$. Consider the polynomial

$$g = p_5(p_3 - p_5)a_{22}X - p_3^2 p_4 a_1 \in \mathbb{R}[X].$$

Since $a_{22} \neq 0$, g is not identically zero. On the other hand $g(q_3^2) = 0$. We may now use (9) to deduce that $\lambda g \in K[X]$ and the argument of Claim 3.6 gives $q_3^2 \in K$, which is a contradiction. •

Claim 3.8. *Either $q_5 \in K$ or there is $\mu \in K$ such that $q_5 = \mu q_3$.*

Proof. Suppose $2a_1 + p_3 a_{12} \neq 0$. Substituting all coefficients with their appropriate expressions we see that

$$\lambda\left(\left[2p_4(a_2 + p_3 a_{22}) - p_5(2a_1 + p_3 a_{12})\right] q_3 + p_3(2a_1 + p_3 a_{12})q_5\right) = 0.$$

We may now use (9) to deduce that $q_5 = \mu q_3$ for some $\mu \in K$.

Hence we may suppose that

$$2a_1 + p_3 a_{12} = 8p_4 q_3(q_5^2 - q_3 q_5 - p_5^2 + p_3 p_5) = 0,$$

and thus $q_5^2 - q_3 q_5 = p_5^2 - p_3 p_5$. In this case q_5^2 is a zero of the following polynomial $g \in \mathbb{R}[X]$:

$$\begin{aligned} g = \; & \left[((p_3 - p_5)^2 - p_4^2)a_{12} + 2p_4(p_3 - p_5)a_{22}\right] X^2 + \\ & \left[(p_5(p_3 - p_5)(p_5^2 - p_4^2 - 2p_3 p_5) + p_3^2 p_4^2)a_{12} + 2p_5 p_4 (p_3 - p_5)^2 a_{22}\right] X + \\ & p_5^2(p_3 - p_5)^2((p_5^2 - p_4^2)a_{12} - 2p_5 p_4 a_{22}). \end{aligned}$$

We may now use (9) and the argument of Claim 3.6 to deduce that $\lambda g \in K[X]$, and hence that $q_5 \in K$, as long as g is not identically zero. Let us suppose indirectly, that $g = 0$. Equating the coefficient of X^2 and the constant term of g to zero gives the following system of linear equations for a_{12}, a_{22}:

$$\begin{array}{rcrcl} [(p_3 - p_5)^2 - p_4^2]\, a_{12} & + & 2p_4(p_3 - p_5)\, a_{22} & = & 0 \\ (p_5^2 - p_4^2)\, a_{12} & - & 2p_5 p_4\, a_{22} & = & 0\,. \end{array}$$

Since $a_{12} \neq 0$ by Claim 3.7, the determinant of this system, which is a non-zero polynomial in p_3, p_4, p_5 with integer coefficients, must be zero. This contradicts the fact that $\{p_3, p_4, p_5\}$ is algebraically independent over $\mathbb{Q}$. •

We can now complete the proof of the lemma. Consider the following polynomial $g \in \mathbb{R}[X, Y]$:

$$g = \left[(p_5^2 - p_4^2)a_{12} - 2p_5 p_4 a_{22}\right] X^2 + 2p_3(p_4 a_{22} - p_5 a_{12})XY + p_3^2 a_{12} Y^2 + p_3^3 p_4^2 a_{12}.$$

We have $g(q_3, q_5) = 0$.

Suppose $q_5 \in K$. Let $h = g(X, q_5)$. Then h is not identically zero since its constant term, $p_3^2 a_{12}(q_5^2 + p_4^2) \neq 0$. On the other hand $h(q_3) = g(q_3, q_5) = 0$. We may use (9) to deduce that $\lambda h \in K[X]$ and then use the argument of Claim 3.6 to deduce that $q_3 \in K$, which is a contradiction. Thus $q_5 \notin K$.

By Claim 3.8, $q_3 = \mu q_5$ for some $\mu \in K$. Let $h' = g(X, \mu X)$. Then h' is not identically zero since its constant term, $p_3^2 p_4^2 a_{12} \neq 0$. On the other hand $h'(q_3) = g(q_3, q_5) = 0$. We may use (9) to deduce that $\lambda h' \in K[X]$. The argument of Claim 3.6 then gives $q_3 \in K$, which is a contradiction.

The only way out of this contradiction is that our assumption (8) must be false. Hence some point (p_1, p_2) on the conic $f = 0$ is generic over K. This gives us the required quasi-generic realisation (G, p^*) by Lemma 3.3. •

We can use Lemma 3.4 to show that, if G is obtained by performing a 1-extension on a non-redundant edge, then the end-vertices of this edge are not globally linked in G.

Theorem 3.9. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose the $H - uw$ is not rigid. Then $\{u, w\}$ is not globally linked in G.*

Proof. Let (H, p) be a quasi-generic framework which is in standard position with respect to (u, w). Since (H, p) is infinitesimally rigid, but $(H - uw, p)$ is not infinitesimally rigid, there is an infinitesimal motion $\tilde{p}$ of $(H - uw, p)$, such that

$$(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0.$$

Theorem 2.5 gives a smooth flexing $\pi : (-1, 1) \times V \to \mathbb{R}^2$ of the framework $(H - uw, p)$ such that $\dot{\pi}_0 = \tilde{p}$.

Suppose that G is the 1-extension of H with a new vertex v with neighbour set $\{u, w, z\}$. Since p is quasi-generic, $p(u)$, $p(w)$ and $p(z)$ are not collinear. Since π is continuous, we can choose $t_1 > 0$ such that $\pi_{t_1}(u)$, $\pi_{t_1}(w)$ and $\pi_{t_1}(z)$ are not collinear for all $0 < t < t_1$. Let

$$f(t) = ||\pi_t(u) - \pi_t(w)||^2.$$

Then $\frac{df}{dt}|_{t=0} = 2(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0$. Since $\overline{\mathbb{Q}(p)}$ is countable, it follows that $f(t_2) \notin \overline{\mathbb{Q}(p)}$ for some $0 < t_2 < t_1$. In particular,

$$||\pi_{t_2}(u) - \pi_{t_2}(w)|| \neq ||p(u) - p(w)||.$$

Let $(G - v, q)$ be a framework which is congruent to $(G - v, \pi_{t_2})$ and in standard position with respect to (u, w). Applying Lemma 3.4 to $(G - v, p)$ and $(G - v, q)$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is quasi-generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = \pi_{t_2}$. This gives

$$||q^*(u) - q^*(w)|| = ||\pi_{t_2}(u) - \pi_{t_2}(w)|| \neq ||p(u) - p(w)|| = ||p^*(u) - p^*(w)||.$$

Hence $\{u, w\}$ is not globally linked in G. •

We next use Lemma 3.4 to prove a counterpart of Theorem 1.2.

Theorem 3.10. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

Proof. Since $\{x, y\}$ is not globally linked in H, there are equivalent frameworks (H, p_1) and (H, p_2) in standard position with respect to (u, w) and such that p_1 is quasi-generic and

$$||p_1(x) - p_1(y)|| \neq ||p_2(x) - p_2(y)||.$$

Since H is rigid and (H, p_1) is quasi-generic, [19, Corollary 3.7] implies that (H, p_2) is quasi-generic. Hence (H, p_2) is infinitesimally rigid and $(H - uw, p_2)$ is not infinitesimally rigid. It follows that there is an infinitesimal motion $\tilde{p}$ of $(H - uw, p_2)$ such that

$$(p_2(u) - p_2(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0.$$

Theorem 2.5 gives a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p_2)$ such that $\dot{\pi}_0 = \tilde{p}$.

Suppose that G is the 1-extension of H with a new vertex v with neighbour set $\{u, w, z\}$. Since p_2 is quasi-generic, $p_2(u)$, $p_2(w)$ and $p_2(z)$ are not collinear. Since π is continuous, we may choose $t_1 > 0$ such that $\pi_t(u)$, $\pi_t(w)$ and $\pi_t(z)$ are not collinear and $||\pi_t(x) - \pi_t(y)|| \neq ||p_1(x) - p_1(y)||$ for all $0 < t < t_1$. Let

$$f(t) = ||\pi_t(u) - \pi_t(w)||^2.$$

Then $\frac{df}{dt}|_{t=0} = 2(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0$. Since $\overline{\mathbb{Q}(p)}$ is countable, it follows that $f(t_2) \notin \overline{\mathbb{Q}(p)}$ for some $0 < t_2 < t_1$.

Let $(G - v, q)$ be a framework which is congruent to $(G - v, \pi_{t_2})$ and in standard position with respect to (u, w). Applying Lemma 3.4 to $(G - v, p_1)$ and $(G - v, q)$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is quasi-generic, $p^*|_{V-v} = p_1$ and $q^*|_{V-v} = q$. Therefore

$$||q^*(x) - q^*(y)|| = ||\pi_{t_2}(x) - \pi_{t_2}(y)|| \neq ||p_1(x) - p_1(y)|| = ||p^*(x) - p^*(y)||.$$

Hence $\{x, y\}$ is not globally linked in G. •

We can use Theorems 3.9 and 3.10 to deduce that Conjectures 1.7 and 1.8 hold for graphs with at most one non-trivial redundantly rigid component.

Theorem 3.11. *Let $G = (V, E)$ be a rigid graph, $u, v \in V$, and $R = (U, F)$ be a redundantly rigid component of G. Suppose that $G - e$ is not rigid for all $e \in E - F$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$ or $\{u, v\}$ is globally linked in R.*

Proof. Sufficiency is clear so we need only prove necessity. Suppose $\{u, v\}$ is globally linked in G. If $U = V$ then $G = R$ and the result is trivially true. Hence we may suppose that $U \neq V$.

We first show that there exists either a vertex of $V - U$ of degree two in G or at least three vertices of $V - U$ of degree three in G. Since G is rigid every vertex of G

has degree at least two. We have $|E - F| = r(G) - r(R) = 2|V - U|$. If $|V - U| = 1$ this implies that the unique vertex of $V - U$ has degree two. Hence we may suppose that $|V - U| \geq 2$. The rigidity of G now implies that there are at least three edges between U and $V - U$. Hence

$$\sum_{x \in V-U} d_G(x) \leq 2|E - F| - 3 = 4|V - U| - 3.$$

The assertion about vertices of degree two or three in $V - U$ now follows.

Suppose there exists $x \in V - U$ with $d(x) = 2$. It is not difficult to see that x is only globally linked to its neighbours in G. Hence the theorem holds if $x \in \{u, v\}$ and we may suppose that this is not the case. Let $H = G - x$. Then (H, R) satisfies the hypotheses of the theorem. The result now follows by applying induction and Lemma 1.9.

Hence we may assume that there are at least three vertices of $V - U$ of degree three. Choose $x \in V - U$ with $d(x) = 3$ and $x \notin \{u, v\}$. By Lemma 2.4 there is a pair y, z of neighbors of x for which $H = G - x + yz$ is rigid. The rigidity of H implies that $\{y, z\} \not\subseteq U$ and that (H, R) satisfies the hypotheses of the theorem. The result now follows by applying induction and Theorem 3.10 when $\{y, z\} \neq \{u, v\}$, and by Theorem 3.9 when $\{y, z\} = \{u, v\}$. •

Conjectures 1.7 and 1.8 follow for a (not necessarily rigid) graph G with at most one non-trivial redundantly rigid component by applying Theorem 3.11 to the rigid components of G (and using the fact that pairs of vertices belonging to different rigid components are not globally linked).

The special case of Theorem 3.11 when G has no non-trivial redundantly rigid components characterises globally linked pairs in minimally rigid graphs.

Corollary 3.12. *Let $G = (V, E)$ be a minimally rigid graph and $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$.*

Suppose we apply a 1-extension on a non-redundant edge xy of a rigid graph H. Then Theorem 3.9 implies that $\{x, y\}$ is not globally linked in the resulting graph G. On the other hand, Conjecture 1.6 would imply that this is the only pair of globally linked vertices of H which is not globally linked in G.

Conjecture 3.13. *Suppose G is a 1-extension on a non-redundant edge xy of a rigid graph H and $\{u, v\} \neq \{x, y\}$ is globally linked in H. Then $\{u, v\}$ is globally linked in G.*

4 Neighbourhood stability

In this section we obtain analogues of the following result of R. Connelly and W. Whiteley for globally linked pairs.

Theorem 4.1. *[7, Theorem 13] Given a framework (G, p) which is globally rigid and infinitesimally rigid in $\mathbb{R}^d$, there is an open neighborhood U of p such that for all $q \in U$ the framework (G, q) is globally rigid and infinitesimally rigid.*

We will concentrate on the 2-dimensional case.[1] We first show that the global-linkedness of two vertices is preserved in an open neighbourhood of (G, p) as long as G is minimally rigid and all equivalent framework are infinitesimally rigid. We then prove that the same conclusion holds when (G, p) is rigid and generic. We will need the following well known 'averaging' result, see for example the proof of [7, Theorem 13].

Lemma 4.2. *Suppose that (G, p) and (G, q) are equivalent but non-congruent frameworks. Then $p - q$ is a non-trivial infinitesimal motion of $(G, p + q)$.*

We say that an infinitesimally rigid framework (G, p) is *regular valued* if all equivalent frameworks are infinitesimally rigid.[2] It is known that an infinitesimally rigid, regular valued framework (G, p) has only finitely many equivalent and pairwise non-congruent realisations.[3] We denote this number by $r(G, p)$. This parameter is related to global linkedness by the fact that two vertices u, v are globally linked in an infinitesimally rigid, regular valued framework (G, p) if and only if $r(G, p) = r(G + uv, p)$. Our first result shows that, for such a framework (G, p), $r(G, p)$ does not increase in some open neighbourhood of p.

Theorem 4.3. *Suppose that (G, p) is an infinitesimally rigid, regular valued framework. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G, q) is an infinitesimally rigid, regular valued framework with $r(G, q) \leq r(G, p)$.*

Proof. The theorem is trivially true if G has at most two vertices. Hence we may suppose that $|V(G)| \geq 3$.

We first show that there exists an open neighbourhood U of p such that (G, q) is an infinitesimally rigid, regular valued framework for all $q \in U$. Suppose not. Then there exists a sequence of realisations (G, p^k) with $p^k \to p$, and such that (G, p^k) is not infinitesimally rigid and regular valued for all k. Since (G, q) is infinitesimally rigid for q close enough to p, we may suppose that (G, p^k) is infinitesimally rigid and not regular valued for all k. Hence (G, p^k) has an equivalent realisation (G, q^k) which is not infinitesimally rigid. By compactness q^k has has a convergent subsequence $q^m \to q$. Since q^m is equivalent to p^m and $p^m \to p$, (G, q) is equivalent to (G, p). Since (G, p) is regular valued, (G, q) is infinitesimally rigid. This contradicts the fact that $q^m \to q$ and (G, q^m) is not infinitesimally rigid for all m.

We next show that there exists an open neighbourhood U of p such that $r(G, q) \leq r(G, p)$ for all $q \in U$. Suppose not. Then, by the previous paragraph, there exists a sequence of infinitesimally rigid, regular valued realisations (G, p^k) with $p^k \to p$ and $r(G, p^k) > r(G, p)$ for all $k \geq 1$. Since (G, p) is infinitesimally rigid we have

[1] We believe that our results extend to the d-dimensional case but the proofs become more complicated because of their reliance on 'special position' arguments. In particular we would need a d-dimensional version of Lemma 3.3.

[2] This is equivalent to saying that $f_G(p)$ is a *regular value* of the rigidity map of G i.e. q is a regular point of f_G for all $q \in f_G^{-1}(f_G(p))$.

[3] Since $f_G(p)$ is a regular value of f_G, $f_G^{-1}(f_G(p))$ is a 0-dimensional manifold. Compactness and the fact that (G, q) is infinitesimally rigid for all $q \in f_G^{-1}(f_G(p))$ now tells us that $f_G^{-1}(f_G(p))$ is finite.

$p(u) \neq p(v)$ for some edge uv of G. Let $S = \{(G, p_1), (G, p_2), \ldots, (G, p_s)\}$ be the set of all equivalent realisations which are in standard position with respect to (u, v). Since (G, p) is infinitesimally rigid and regular valued, each of the (G, p_i) is infinitesimally rigid and hence, in particular, does not have all its vertices on a line. This implies that each congruence class of (G, p) will be represented exactly four times in S and hence $s = 4r(G, p)$. Since $r(G, p^k) > r(G, p)$ for each $k \geq 1$, we may choose a set $\{q_1^k, q_2^k, \ldots, q_{s+1}^k\}$ of realisations which are equivalent to (G, p^k) and are in standard position with respect to u, v . By compactness there exist convergent subsequences $q_i^m \to q_i$ for all $1 \leq i \leq s+1$. Since q_i^m is equivalent to p^m and $p^m \to p$, each q_i is equivalent to p. Hence $q_i = p_j$ for some $1 \leq j \leq s$. By the pigeon hole principle, we may choose two sequences q_1^m, q_2^m say, converging to the same realisation, (G, p_1) say, of G. By Lemma 4.2, $q_1^m - q_2^m$ is a non-trivial infinitesimal motion of $(G, q_1^m + q_2^m)$, and hence $(G, q_1^m + q_2^m)$ it is not infinitesimally rigid. Since $q_1^m + q_2^m \to 2p_1$, $(G, 2p_1)$ is not infinitesimally rigid. This implies that (G, p_1) is not infinitesimally rigid and contradicts the hypothesis that all equivalent realisations of (G, p) are infinitesimally rigid. •

Note that Theorem 4.3 generalises (the 2-dimensional version of) Theorem 4.1 since an infinitesimally rigid, globally rigid framework (G, p) is regular valued and has $r(G, p) = 1$.

We can have $r(G, q) < r(G, p)$ for any framework (G, p) satisfying the hypotheses of Theorem 4.3 and q arbitrarily close to p. Consider for example the realisation (G, p) of a wheel in which the central vertex and two nonconsecutive rim vertices are collinear, see Figure 3. Then $r(G, p) = 2$ but $r(G, q) = 1$ for all generic q. We will show, however, that $r(G, p)$ is constant in some open neighbourhood of p if either G is minimally rigid or p is generic.

Theorem 4.4. *Suppose that (G, p) is an infinitesimally rigid, regular valued realisation of a minimally rigid graph $G = (V, E)$. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G, q) is infinitesimally rigid, regular valued, and has $r(G, q) = r(G, p)$.*

Proof. By Theorem 4.3, it will suffice to show that an arbitrary sequence of infinitesimally rigid realisations (G, p^k) with $p^k \to p$ has $r(G, p^k) \geq r(G, p)$ for k large enough. Since (G, p) is infinitesimally rigid we have $p(u) \neq p(v)$ for some $uv \in E$. Let $\{p_1, p_2, \ldots, p_s\}$ be the set of all realisations in standard position with respect to (u, v) which are equivalent to (G, p). Then $s = 4r(G, p)$ as in the proof of Theorem 4.3. By the inverse function theorem, we may choose neighbourhoods U_i of p_i and W of $f_G(p)$ such that f_G maps U_i diffeomorphically onto W for all $1 \leq i \leq s$. We may also assume that $p = p_1$ and hence that $p^k \in U_1$ for k large enough, say $k \geq K$. Then $f_G(p^k) \in W$ and hence there exists $p_i^k \in U_i$ with $f_G(p_i^k) = f_G(p^k)$ for all $1 \leq i \leq s$ and all $k \geq K$. This implies that there are at least s distinct realisations (G, p_i^k) in standard position with respect to (u, v) which are equivalent to (G, p^k). Hence $r(G, p^k) \geq r(G, p)$ for $k \geq K$. •

Corollary 4.5. *Suppose that (G, p) is an infinitesimally rigid, regular valued realisation of a minimally rigid graph $G = (V, E)$. Then there exists an open neighbourhood*

U of p such that, for all $u,v \in V$ and all $q \in U$, $\{u,v\}$ is globally linked in (G,q) if and only if $\{u,v\}$ is globally linked in (G,p).

Proof. This follows immediately from Theorem 4.4 and the fact that, for any infinitesimally rigid, regular valued realisation (G,q), $\{u,v\}$ is globally linked in (G,q) if and only if $r(G,q) = r(G+uv,q)$. •

We next show that $r(G,p)$ remains constant in an open neighbourhood of p for any rigid graph G when p is generic. Our proof uses the Tarski-Seidenberg theorem on semi-algebraic sets. A subset S of $\mathbb{R}^n$ is *semi-algebraic over* $\mathbb{Q}$ if it can be expressed as a finite union of sets of the form

$$\{x \in \mathbb{R}^n : P_i(x) = 0 \text{ for } 1 \leq i \leq s \text{ and } Q_j(x) > 0 \text{ for } 1 \leq j \leq t\},$$

where $P_i \in \mathbb{Q}[X_1,\ldots,X_n]$ for $1 \leq i \leq s$, and $Q_j \in \mathbb{Q}[X_1,\ldots,X_n]$ for $1 \leq j \leq t$. [4]

Theorem 4.6. *[31] Let $S \subseteq \mathbb{R}^{n+k}$ be semi-algebraic over $\mathbb{Q}$ and $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection onto the first n coordinates. Then $\pi(S)$ is semi-algebraic over $\mathbb{Q}$.*

Theorem 4.7. *Suppose that $G = (V,E)$ is rigid and (G,p) is generic. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G,q) is infinitesimally rigid, regular valued, and has $r(G,p) = r(G,q)$.*

Proof. The hypothesis that (G,p) is generic implies that $f_G(p)$ is a regular value of f_G by [19, Corollary 3.7]. Hence (G,q) is infinitesimally rigid and regular valued for q in some open neighbourhood of p by Theorem 4.3.

Let $V = \{v_1,v_2,\ldots,v_n\}$, $|E| = m$ and suppose that $v_1v_2 \in E$. Choose a realisation (G,p_1) which is congruent to p and is in standard position with respect to (v_1,v_2). We will consider the set of all realisations of G which are in standard position with respect to (v_1,v_2). For any such realisation (G,q) the first three coordinates of q are zero. We will abuse notation and consider $q \in \mathbb{R}^{2n-3}$. Similarly we will consider the rigidity map f_G to be a map from $\mathbb{R}^{2n-3}$ to $\mathbb{R}^m$.

Let $s = 4r(G,p)$, and let S be the set of all s-tuples of vectors $(q_1,q_2,\ldots,q_s)$ where $q_i \in \mathbb{R}^{2n-3}$ and $\{(G,q_1),(G,q_2),\ldots,(G,q_s)\}$ is a set of distinct pairwise equivalent realisations of G in standard position with respect to (v_1,v_2). Then $S \subseteq \mathbb{R}^{s(2n-3)}$ and we may represent S as

$$S = \{(q_1,q_2,\ldots,q_s) \,:\, q_i \in \mathbb{R}^{2n-3},\ f_G(q_i) = f_G(q_1),\ q_i \neq q_j \text{ for } 1 \leq i \neq j \leq s\}.$$

Hence S is semi-algebraic over $\mathbb{Q}$. Let

$$S_1 = \{q_1 \in \mathbb{R}^{2n-3} \,:\, (q_1,q_2,\ldots q_s) \in S \text{ for some } q_2,q_3,\ldots,q_s \in \mathbb{R}^{2n-3}\}$$

be the projection of S onto the first $2n-3$ coordinates. Then S_1 is the set of all $q \in \mathbb{R}^{2n-3}$ such that (G,q) has s distinct pairwise equivalent realisations in standard position with respect to (v_1,v_2). Thus $p_1 \in S_1$.

[4] The usual definition for a semi-algebraic set uses polynomials with coefficients in $\mathbb{R}$, or more generally in a real closed field. The fact that the Tarski-Seidenberg Theorem holds for semi-algebraic sets over $\mathbb{Q}$ follows from the original papers [31, 27].

By Theorem 4.6, S_1 is semi-algebraic over $\mathbb{Q}$. Since p is generic, the (non-zero) coordinates of p_1 are algebraically independent over $\mathbb{Q}$ by Lemma 3.3. Hence $P(p_1) \neq 0$ for all $P \in \mathbb{Q}[X_1, \ldots, X_{2n-3}]$. Since $p_1 \in S_1$ and S_1 is semi-algebraic over $\mathbb{Q}$, we must have $p_1 \in S_2$ for some $S_2 \subseteq S_1$ of the form

$$S_2 = \{q \in \mathbb{R}^{2n-3} : Q_j(q) > 0 \text{ for } 1 \leq j \leq t\},$$

where $Q_j \in \mathbb{Q}[X_1, \ldots, X_{2n-3}]$ for $1 \leq j \leq t$. We may choose an open neighbourhood U of p_1 in $\mathbb{R}^{2n-3}$ such that $Q_j(q) > 0$ for all $q \in U$ and all $1 \leq j \leq t$. Then $q \in S_2 \subseteq S_1$ for all $q \in U$. By the first paragraph of the proof, we may choose U small enough so that (G, q) is infinitesimally rigid and regular valued for all $q \in U$. Then $r(G, q) \geq s/4 = r(G, p)$ for all $q \in U$. Theorem 4.3 now implies that there exists a possibly even smaller open neighbourhood U' of p_1 in $\mathbb{R}^{2n-3}$ with $r(G, q) = r(G, p)$ for all $q \in U'$. We can now complete the proof by choosing an open neighbourhood U'' of p in $\mathbb{R}^{2n}$ such that, for each $q \in U''$, (G, q) is congruent to (G, q_1) for some $q_1 \in U'$. •

Corollary 4.8. *Suppose that $G = (V, E)$ is rigid and (G, p) is generic. Then there exists an open neighbourhood U of p such that, for all $u, v \in V$ and all $q \in U$, $\{u, v\}$ is globally linked in (G, p) if and only if $\{u, v\}$ is globally linked in (G, q).*

Proof. This follows immediately from Theorem 4.7 and the fact that, for any infinitesimally rigid, regular valued realisation (G, q), $\{u, v\}$ is globally linked in (G, q) if and only if $r(G, q) = r(G + uv, q)$. •

The realisation (G, p) of a wheel in which the central vertex and two nonconsecutive rim vertices are collinear (see Figure 3) shows that Corollaries 4.5 and 4.8 become false if we remove the respective hypothesis that G is minimally rigid and p is generic. The problem is that there are pairs of vertices which are not globally linked in (G, p) but are globally linked in (G, q) for q arbitrarily close to p. We have no examples, however, in which the property of being globally linked is not preserved in some open neighbourhood.

Conjecture 4.9. *Suppose that $\{u, v\}$ is a globally linked pair of vertices in an infinitesimally rigid, regular valued framework (G, p). Then there exists an open neighbourhood U of p such that $\{u, v\}$ is globally linked in (G, q) for all $q \in U$.*

Conjecture 4.9 would follow from:

Conjecture 4.10. *Let (G, p) be an infinitesimally rigid, regular valued framework and u, v be vertices of G. Then there exists an open neighbourhood U of p such that, for all $q \in U$, $r(G, p) - r(G + uv, p) \geq r(G, q) - r(G + uv, q)$.*

5 Finding equivalent realizations by flexing

In this section we describe a possible approach to verifying Conjecture 1.7 which is analogous to that used by Hendrickson [14] to show that redundant rigidity is a necessary condition for global rigidity. We need to show that if two vertices u, v are not contained in the same redundantly rigid component of a rigid graph G then they are not globally linked.[5] The idea is to find an edge $e = wx$ in G such that u, v do not belong to the same rigid component of $G - e$. We then choose a flexing of a generic realisation $(G - e, p)$ to find another realisation $(G - e, q)$ with the properties that $\|q(w) - q(x)\| = \|p(w) - p(x)\|$ and $\|q(u) - q(v)\| \neq \|p(u) - p(v)\|$. The equivalent realisations (G, p) and (G, q) will then certify that that $\{u, v\}$ is not globally linked in G. The first step in this approach is to show that we can find a suitable edge e.

Lemma 5.1. *Let $G = (V, E)$ be a rigid graph and $u, v \in V$ with $uv \notin E$. Then $\{u, v\}$ is contained in a redundantly rigid component of G if and only if $\{u, v\}$ is contained in a rigid component of $G - e$ for all $e \in E$.*

Proof. We first prove necessity. Suppose u, v is contained in a redundantly rigid component H of G. Then $H \neq K_2$ and so $H - e$ is a rigid subgraph of G for all $e \in E$. Hence u, v is contained in a rigid component of $G - e$ for all $e \in E$.

We next prove sufficiency. Suppose u, v is not contained in a redundantly rigid component of G. Then G is not redundantly rigid so at least one edge of G is an M-bridge. Let $F = \{e_1, e_2, ..., e_m\}$ be the set of M-bridges of G. The rigid components of $G - F$ are exactly the non-trivial redundantly rigid components of G (since if the union of a set of redundantly rigid graphs is rigid and contains no M-bridges then it must be redundantly rigid). Thus u, v is not contained in a rigid component of $G - F$. Let H' be a maximal M-independent subgraph of $G - F$. Note that the vertex sets of the rigid components of $G - F$ and H' are the same and $H' + F$ is an M-independent (and rigid) spanning subgraph of G.

Let F' be a maximal proper subset of F for which u, v is not contained in a rigid component of $H' + F'$. If $F - F' = \{f\}$ then we are done by choosing $e = f$. This follows from the fact that u, v is not contained in a rigid component of $H' - f$ and hence is not contained in a rigid component of $G - f$ as well. So we may suppose that we have two distinct edges $f_1, f_2 \in F - F'$. By the maximality of F' there is a rigid subgraph $G_i = (V_i, E_i)$ of $H' + F' + f_i$ which contains u and v, for $i = 1, 2$. Since $H' + F$ is M-independent, these subgraphs are induced subgraphs of $H' + F$ and we must have $f_1, f_2 \notin G_1 \cap G_2$. Then $G_1 \cap G_2$ is a rigid subgraph of $H' + F'$ which contains u and v. This contradicts the choice of F'. •

Our next result implies that the 'flexing approach' to showing that $\{u, v\}$ is not globally linked in G works when $G + uv$ is an M-circuit.

Lemma 5.2. *Let (C, p_0) be a quasi-generic realisation of an M-circuit $C = (V, E)$, $e_1 = uv$ and $e_2 = wx$ be edges of C and $H = C - \{e_1, e_2\}$. Let $\mathcal{F}$ be the set of all frameworks which can be obtained by a flexing of (H, p_0). Then there exists $(H, p_1) \in \mathcal{F}$ with $\|p_0(u) - p_0(v)\| \neq \|p_1(u) - p_1(v)\|$ and $\|p_0(w) - p_0(x)\| = \|p_1(w) - p_1(x)\|$.*

[5] It is straightforward to reduce Conjecture 1.7 to rigid graphs since pairs of vertices which do not belong to the same rigid component of a graph cannot be globally linked.

Proof. We suppose that all realisations of H considered are in standard position with respect to (w, x). For each such realisation (H, q) we suppress the (zero) coordinates of q corresponding to $q(w)$ and the first coordinate of $q(x)$ and consider $q \in \mathbb{R}^{2|V|-3}$.

Let
$$S = \{p \in \mathbb{R}^{2|V|-3} : (H, p) \in R\}.$$
Let $F : \mathbb{R}^{2|V|-3} \to \mathbb{R}$ be given by $F(q) = \|q(u) - q(v)\|^2$ (in the corresponding realisation (H, q)) and let f be the restriction of F to S. We can also view the rigidity maps f_{H+e_1}, f_H as maps on S. Note that the rigidity map f_{H+e_1} is obtained from f_H by adding an extra coordinate corresponding to f i.e. the length of the edge e in the realisation of $H + e_1$.

We can adapt the proof technique of [14] to show that S is a 1-dimensional manifold diffeomorphic to a circle. For each $p \in S$, [20, Lemma 3.4] gives

rank $df|_p =$ rank $df_{H+e_1}|_p -$ rank $df_H|_p =$ rank $R(H + e_1, p) -$ rank $R(H, p)$.

Thus, for every generic point $p \in S$, we have rank $df|_p = 1$ so p is a regular value of f.

Choose a direction for traversing S and let p_1 be the first point after p_0 we reach when traversing S which satisfies $\|p_0(w) - p_0(x)\| = \|p_1(w) - p_1(x)\|$. We will show that $\|p_0(u)-p_0(v)\| \neq \|p_1(u)-p_1(v)\|$. Suppose to the contrary that $\|p_0(u)-p_0(v)\| = \|p_1(u) - p_1(v)\|$. Then (C, p_0) is equivalent to (C, p_1).

We first consider the case when C is 3-connected. Then C is globally rigid by [3] so (C, p_0) is congruent to (C, p_1). Since (C, p_0) and (C, p_1) are in standard position (C, p_1) is a reflection of (C, p_0) in the line through $p_0(w), p_0(x)$. Let $a : [0, 1] \to S$ be the smooth path from p_0 to p_1 induced by the diffeomorphism from S to the circle, and let $b : [0, 1] \to S$ be obtained by putting $b(t) = \overline{a(t)}$ for all $0 \le t \le 1$, where $\overline{a(t)}$ is the reflection of $a(t)$ in the line through $p_0(w), p_0(x)$. Then b is a smooth path in S from p_1 to p_0. Furthermore, we claim that a and b do not have the same image in S. For suppose to the contrary that a and b traverse some path P in S in opposite directions. Then by the intermediate value theorem there is some $t \in [0, 1]$ with $a(t) = b(t)$. This implies that $(H, a(t))$ has all vertices on the line through $p_0(w), p_0(x)$ which is impossible since (H, p), and hence also $(H, a(t))$, has $2|V(H)| - 4$ algebraically independent edge-lengths. It follows that a and b trace out two paths that together form the entire manifold S. We can choose $t_1, t_2 \in [0, 1]$ with $f(a(t_1)) < f(p_0)$ and $f(a(t_2)) > f(p_0)$. Now the intermediate value theorem gives some t between t_1 and t_2 with $f(a(t)) = f(p_0)$. This contradicts the choice of p_1.

We next consider the case when C is not 3-connected. Let $C_1C_2 \dots C_m$ be the path in the cleavage unit tree of C with $e_1 \in E(C_1)$ and $e_2 \in E(C_m)$. (We refer the reader to [17, Section 3] for more details on cleavage unit trees of M-circuits.) Let $C' = C_1 \oplus_2 C_2 \oplus_2 \dots \oplus_2 C_m$. If $C' \neq C$ then we can apply induction to C'. Hence we may assume that $C' = C$. We will proceed by adapting the proof of the case when C is 3-connected.

Let $V(C_i) \cap V(C_{i+1}) = \{x_i, w_i\}$ for $1 \le i < m$. For each $p \in S$, let $\ell_m(p)$ be the line through $p(w), p(x)$, and $\ell_i(p)$ be the line through $p(w_i), p(x_i)$ for $1 \le i < m$. Let $\theta_i(C, p)$ be the realisation of C obtained from (C, p) by reflecting $C_1, C_2, \dots, C_i$ in the line through $\ell_i(p)$. For each $S = \{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, m\}$, let $\theta_S(C, p)$ be the realisation of C obtained recursively from (C, p) by applying θ_{i_s} to $\theta_{S-i_s}(C, p)$, and taking $\theta_S(C, p) = (C, p)$ when $S = \emptyset$.[6] Since (C, p_1) is equivalent to (C, p_0), it follows from the proof of [19, Theorem 8.2] that $(C, p_1) = \theta_S(C, p_0)$ for some $\emptyset \neq S \subseteq \{1, 2, \ldots, m\}$. Let $a : [0, 1] \to S$ be the smooth path from p_0 to p_1 induced by the diffeomorphism from S to the circle, and let $b : [0, 1] \to S$ be obtained by putting $b(t) = \overline{a(t)}$ for all $0 \leq t \leq 1$, where $(C, \overline{a(t)}) = \theta_S(C, a(t))$. Then b is a smooth path in S from p_1 to p_0. Furthermore, we claim that a and b do not have the same image in S. For suppose to the contrary that a and b traverse some path P in S in opposite directions. Then by the intermediate value theorem there is some $t \in [0, 1]$ with $a(t) = b(t) = p_2$, say. But this implies that $(C, p_2) = \theta_S(C, p_2)$, and in particular $(C_1, p_2) = \theta_S(C_1, p_2)$. Since the action of θ_S on (C, p_2) is a non-empty sequence of reflections through lines with algebraically independent slopes it is either a rotation or a reflection. Since (C_1, p_2) remains fixed under this action all vertices of (C_1, p_2) must lie on the same line. This is impossible since $|V(C_1)| \geq 4$, and (H, p), and hence also (H, p_2), have $2|V(H)| - 4$ algebraically independent edge-lengths. It follows that a and b trace out two paths that together form the entire manifold S. We can choose $t_1, t_2 \in [0, 1]$ with $f(a(t_1)) < f(p_0)$ and $f(a(t_2)) > f(p_0)$. Now the intermediate value theorem gives some t between t_1 and t_2 with $f(a(t)) = f(p_0)$. This contradicts the choice of p_1. •

Lemma 5.2 gives the following strengthening of Corollary 3.12 for a special family of minimally rigid graphs. We say that a pair of vertices $\{u, v\}$ is *globally loose* in a graph G if $\{u, v\}$ is not globally linked in all generic realisations of G.

Corollary 5.3. *Suppose G is minimally rigid and $G + uv$ is an M-circuit for two non-adjacent vertices u, v of G. Then $\{u, v\}$ is globally loose.*

The special case of Corollary 5.3, when $G + uv$ is a 3-connected M-circuit, follows from [19, Theorem 7.1]. The example in Figure 1 shows that the stronger conclusion, that $\{u, v\}$ is not globally linked in all generic realisations of G, may not hold when $G + uv$ is not an M-circuit. On the other hand, we can try to apply Lemma 5.2 to an arbitrary rigid graph as follows.

Given a framework (G, p) let $\mathcal{F}(G, p)$ be the set of all frameworks which can be obtained by a flexing of (G, p). We refer to $\mathcal{F}(G, p)$ as the *flex* of (G, p).

Suppose that G and H are minimally rigid graphs with at least three vertices and $H \subseteq G$. Let e be an edge of H and (G, p) be a realisation of G. We say that $(G - e, p)$ is *free* for $H - e$ if, for every $(H - e, q_0) \in \mathcal{F}(H - e, p|_H)$, there exists a $(G - e, q) \in \mathcal{F}(G - e, p)$ such that $q_0 = q|_H$. Intuitively $(G - e, p)$ is free for $H - e$ if the edges of $E(G) \setminus E(H)$ put no restriction on the flex of $(H - e, p|_H)$. We conjecture that such realisations always exist.

Conjecture 5.4. *Let G be a minimally rigid graph, H be a minimally rigid subgraph of G with at least three vertices and e be an edge of H. Then there exists a generic realisation (G, p) of G such that $(G - e, p)$ is* free *for $H - e$.*

[6]It can be shown that $\theta_i(\theta_j(C, p)) = \theta_j(\theta_i(C, p))$ and hence $\theta_S(C, p)$ is independent of the ordering of the elements of S. We will not use this fact in our proof.

We can use Lemmas 5.1 and 5.2 to show that Conjecture 1.7 would follow from Conjecture 5.4.

Lemma 5.5. *Suppose Conjecture 5.4 is true. Let $G = (V, E)$ be a rigid graph, and $u, v \in V$ be such that $\{u, v\}$ is not contained in any redundantly rigid component of G. Then $\{u, v\}$ is not globally linked in G.*

Proof. Let $e_1 = uv$ and let C be an M-circuit of $G+e_1$ containing e_1. By Lemma 5.1 we can find an edge $e_2 = wx$ such that $\{u, v\}$ is not contained in any rigid component of $G - e_2$. Then $e_2 \in E(C)$. Let G' be a minimally rigid spanning subgraph of G which contains $C - e_1$. By Conjecture 5.4, there exists a generic realisation (G, p) of G such that $(G' - e_2, p)$ is free for $C - e_1 - e_2$ By applying Lemma 5.2 to C, we may deduce that there exists $q \in \mathcal{F}(G' - e, p)$ such that $\|p(u) - p(v)\| \neq \|q(u) - q(v)\|$ and $\|p(w) - p(x)\| = \|q(w) - q(x)\|$. Since the distances between all pairs of vertices in the same rigid component of $G' - e_2$ remain constant for all $(G' - e_2, q) \in \mathcal{F}(G' - e_2, p)$, (G, q) is equivalent to (G, p). Since $\|p(u) - p(v)\| \neq \|q(u) - q(v)\|$, $\{u, v\}$ is not globally linked in (G, p). •

5.1 Closing Remark

It is not difficult to show that if H is a minimally rigid subgraph of a minimally rigid graph G, then G can be obtained from H by a sequence of Henneberg extensions, see for example [17]. This fact encouraged us to try to prove Conjecture 5.4 recursively. Let $H = H_0, H_1, \ldots, H_s = G$ be a sequence of minimally rigid graphs with the property that H_i is a Henneberg extension of H_{i-1} for all $1 \leq i \leq s$ and let e be an edge of H. We could assume inductively that there exists a generic realisation $(H_{s-1} - e, p_{s-1})$ which is free for $H - e$ and try to extend it to a realisation $(H_s - e, p_s)$ which is free for $H - e$. A similar idea was outlined previously by Owen and Power [24, Problem 2]. It can be shown that $(H_{s-1} - e, p_{s-1})$ can be extended to a realisation $(H_s - e, p_s)$ which is free for $H - e$ when H_s is a 0-extension of H_{s-1}. We conjectured that the same should hold for 1-extensions at a workshop on rigidity held at BIRS (Banff, Canada) in 2012. Herman and Brigitte Servatius subsequently constructed an infinite family of counterexamples.

Lemma 5.6. *[28] There exist minimally rigid graphs H, K, L with $H \subset K$ and $H \subset L$ such that L is a 1-extension of K, e is an edge of H, $(K - e, p_0)$ is free for $H - e$ for some generic p_0, and $(L - e, p)$ is not free for $H - e$ for all generic p with $p|_K = p_0$.*

6 Acknowledgements

Section 3 is based on an earlier manuscript of one of the authors [30]. We thank Viet-Hang Nguyen for pointing out an error in [30] and for several useful discussions. We also thank Balázs Csikós and András Szűcs for interesting discussions on the differential topology aspects of our problems.

This work was supported by the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, the Hungarian Scientific Research Fund grant no. K81472, and the National Development Agency of Hungary, grant no. CK 80124, based on a source from the Research and Technology Innovation Fund.

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