Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-21. Published by the Egerváry Research Group, Pázmány P. sétány 1/C,
H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

On minimally k-rigid graphs

Viktória E. Kaszanitzky and Csaba Király

2012 November

On minimally k-rigid graphs

Viktória E. Kaszanitzky [⋆] and Csaba Király[⋆⋆]

Abstract

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ if $|V| \geq k + 1$ and after deleting at most $k - 1$ arbitrary vertices the resulting graph is generically rigid in $\mathbb{R}^d$. A k-rigid graph G is called *minimally k-rigid* if the omission of an arbitrary edge results in a graph that is not k-rigid. It was shown in [7] that the smallest possible number of edges is $2|V| - 1$ in a 2-rigid graph in $\mathbb{R}^2$. We generalize this result, provide an upper bound for the number of edges of minimally 2-rigid graphs (for any d) and give examples for minimally k-rigid graphs in higher dimensions.

1 Introduction

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ or shortly $[k, d]$-rigid if $|V| \geq k+1$ and for any $U \subseteq V$ with $|U| \leq k-1$ graph $G-U$ is generically rigid in $\mathbb{R}^d$. In this context we will call graphs that are rigid in $\mathbb{R}^d$ $[1, d]$-rigid. Every $[k, d]$-rigid graph is $[l, d]$-rigid by definition for $1 \leq l \leq k$. We remark that G is $[k, d]$-rigid if and only if the deletion of $k - 1$ arbitrary vertices results in a graph that is generically rigid in $\mathbb{R}^d$.

G is called *minimally $[k, d]$-rigid* if it is $[k, d]$-rigid but $G - e$ fails to be $[k, d]$-rigid for every $e \in E$. G is said to be *strongly minimally $[k, d]$-rigid* if it is minimally $[k, d]$-rigid and there is no minimally $[k, d]$-rigid graph with $|V|$ vertices and less than $|E|$ edges. If G is minimally $[k, d]$-rigid but not strongly minimally $[k, d]$-rigid then it is called *weakly minimally $[k, d]$-rigid*. Investigating the properties of $[k, d]$-rigid graphs is motivated by industrial applications, see [6, 8].

The following theorem gives a formula for the edge number of minimally rigid graphs.

Theorem 1.1 ([13]). *Let $G = (V, E)$ be minimally rigid in $\mathbb{R}^d$. If $|V| \geq d + 1$ then $|E| = d|V| - \binom{d+1}{2}$.* □

A natural question to ask is whether there is a similar formula for the edge number of minimally $[k, d]$-rigid graphs for $k \geq 2$. The answer is no (see Section 6.1), there

[⋆]Eötvös University, Pázmány Péter sétány 1/C Budapest, Hungary H-1117. E-mail: viktoria@cs.elte.hu

[⋆⋆]Eötvös University, Pázmány Péter sétány 1/C Budapest, Hungary H-1117. E-mail: csabi@cs.elte.hu

are minimally $[k,d]$-rigid graphs for $k \geq 2$ with different edge numbers, that is, the set of weakly minimally $[k,d]$-rigid graphs is not empty if $k \geq 2$.

To see a simple example consider the case $d = 1$. It is well known that G is rigid in $\mathbb{R}^1$ if and only if G is connected. Hence G is minimally $[k,1]$-rigid if and only if it is minimally k-connected. Since there are k-connected graphs with the same number of vertices and different number of edges for $k \geq 2$, weakly minimally $[k,1]$-rigid graphs exist for every $k \geq 2$.

It was shown in [7] that the smallest possible number of edges in a $[2,2]$-rigid graph is $2|V| - 1$. Later lower bounds were provided for the edge number of minimally $[k,d]$-rigid graphs in [6, 8, 9] for some other values of $[k,d]$.

The main result of the present paper is a lower bound for the number of edges of $[k,d]$-rigid graphs for every pair $[k,d]$ which is sharp for some values of k and d. We show that weakly minimally $[k,d]$-rigid graphs exist for every pair $[k,d]$. We also provide an upper bound for the number of edges of minimally $[k,d]$-rigid graphs for $k = 2$.

1.1 Notation

In this paper we use the basic definitions and theorems of rigidity theory. All of the non-introduced definitions and non-proved statements can be found in the book of Graver et al. [3]. $\mathcal{R}_d(G)$ denotes the d-dimensional generic rigidity matroid of G.

We shall also use some standard notation from graph theory. $\Delta(G)$ denotes the maximum degree in G. K_n is the complete graph with n vertices. C_n denotes the cycle on n vertices. We will use the notation $V(C_n) = \{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n\}$ where $v_{n+1} := v_1$. C_n^d is the dth power of C_n, or equivalently $E(C_n^d) = \{v_i v_j : i-d \leq j \leq i+d\}$ where $v_{n+i} := v_i$. P_n denotes the path on n vertices. We will use the notation $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. P_n^d is the dth power of P_n, or equivalently $E(P_n^d) = \{v_i v_j : \min\{1, i-d\} \leq j \leq \max\{n, i+d\}\}$.

2 Operations preserving rigidity

Constructive characterizations are useful tools in combinatorial rigidity. Even though we do not have a constructive characterization theorem for the class of rigid graphs for $d \geq 3$ it can be very useful to find operations that preserve rigidity. In this section we mention some of these operations.

The d-dimensional Henneberg-0 extension on G adds a new vertex and connects it to d distinct vertices of G. The d-dimensional Henneberg-1 extension deletes an edge $uw \in E$, adds a new vertex v and connects it to u, v and $d-1$ other vertices of G. The d-dimensional Henneberg-0 extension is also called *d-valent vertex addition*.

Theorem 2.1 ([10]). *If G is rigid in $\mathbb{R}^d$ and G' is the graph that we get from G by a d-dimensional Henneberg-0 or Henneberg-1 extension then G' is rigid in $\mathbb{R}^d$.* □

As d-dimensional Henneberg extensions are used when we are in $\mathbb{R}^d$, we will simply call them Henneberg extensions if d is clear from context. For $d = 2$ the following stronger statement holds:

Theorem 2.2 ([10]). *G is minimally rigid in $\mathbb{R}^2$ if and only if it can be built up from the graph K_2 by a sequence of Henneberg-0 and Henneberg-1 extension.* □

If $G = (V, E)$ is minimally rigid in $\mathbb{R}^3$ then $|E| = 3|V| - 6$ by Theorem 1.1. Hence a minimally rigid graph in $\mathbb{R}^3$ does not necessarily have a vertex with degree 3 or 4. Thus for proving a 3-dimensional version of Theorem 2.2 one would need an operation that results in adding a vertex with degree 5. One such operation is the *3-dimensional X-replacement* which deletes two non-adjacent edges $e = ab$ and $f = cd$ of G, chooses $w \in V$ different from a, b, c, d, adds a new vertex v and connects it to a, b, c, d, w. It is not known whether the X-replacement preserves rigidity in $\mathbb{R}^3$.

Conjecture 2.3 ([4]). *Let G be rigid in $\mathbb{R}^3$ and let G' be the result of a 3-dimensional X-replacement applied to G. Then G' is rigid in $\mathbb{R}^3$.*

Conjecture 2.3 has been proved for some special cases of the 3-dimensional X-replacement (see [10, 12] for examples). We will use the special case when a, b, w form a triangle in G and we will call this version of the operation *$\underline{\Delta}$-X-replacement*. It is a folklore that the $\underline{\Delta}$-X-replacement preserves independence. We did not find the proof of the following lemma in the literature and we include the sketch of its proof for completeness.

Lemma 2.4. *Let G be rigid in $\mathbb{R}^3$ and let G' be the result of a $\underline{\Delta}$-X-replacement applied to G. Then G' is rigid in $\mathbb{R}^3$.*

Proof. (Sketch) Suppose for simplicity that G is minimally rigid in $\mathbb{R}^3$. Let (G, p) be a generic realization of G. Let S be the plane that contains $p(a), p(b), p(w)$ and let ℓ be the line of $p(c), p(d)$. Put $p(v) = S \cap \ell$ and $G_0 = (V + v, E + \{va, vb, vc\})$. By Theorem 2.1 framework (G_0, p) is rigid.

Now we have to construct framework (G', p) from (G_0, p) by replacing edges ab and cd with vw and vd, respectively. We shall also prove that (G', p) is rigid. First add vw, let $G_1 = G_0 + vw$. There is a circuit in (G_1, p) which is the K_4 induced by v, a, b, w. (Note that points $p(v), p(a), p(b), p(w)$ lie on a plane.) Thus with notation $G_1 - ab = G_2$ framework (G_2, p) is independent. Using a similar argument it is not difficult to show that replacing cd with vd preserves independence. □

It was shown in [7] that every strongly minimally $[2, 2]$-rigid graph can be built up from a suitable base graph using Henneberg-1 extensions. The author also showed that 3-valent vertex addition preserves minimal $[2, 2]$-rigidity under certain conditions.

There is a two-dimensional version of the X-replacement which is known to preserve rigidity in $\mathbb{R}^2$ [1]. The *2-dimensional X-replacement* deletes two non-adjacent edges $e = ab$ and $f = cd$ of G, adds a new vertex v and connects it to a, b, c, d. It was observed in [9] that the 2-dimensional X-replacement preserves minimally $[2, 2]$-rigidity in specific cases. Summers, Yu and Anderson conjectured that the 3-valent vertex addition and the 2-dimensional X-replacement operations are sufficient to build up every weakly minimally $[2, 2]$-rigid graph with at least nine vertices.

Conjecture 2.5 ([8, 9]). *Let G(V, E) be a minimally $[2, 2]$-rigid graph with at least nine vertices. Then there exists either (a) a degree 4 vertex on which a reverse X-replacement operation can be performed to obtain a weakly minimally $[2, 2]$-rigid graph or (b) there exists a degree three vertex on which a reverse 3-valent vertex addition can be performed to obtain a weakly minimally $[2, 2]$-rigid graph.*

We will disprove this conjecture by constructing weakly minimally $[2, 2]$-rigid graphs on n vertices that does not have such a vertex, where n can be arbitrarily large.

3 On the number of edges in $[k, d]$-rigid graphs

First we present some results that apply to every dimension.

3.1 Lower bound for the number of edges

It was known that every $[2, 2]$-rigid graph has at least $2|V| - 1$ edges, see [7]. In [6] Motevallian et al. gave a lower bound for the edge number of $[k, 2]$-rigid graphs. We improve their results and extend it to every d. In Sections 4 and 5 we show that this lower bound is sharp for some values of $[k, d]$.

Theorem 3.1. *If a graph $G = (V, E)$ is $[k, d]$-rigid with $|V| \geq d^2 + d + k$ then*

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d. \qquad (1)$$

Proof. Observe that if a graph $H = (V', E')$ is $[1, d]$-rigid with $|V'| \geq d^2 + d$ then $\Delta(H) \geq 2d$. (To see this suppose that $\Delta(H) \leq 2d - 1$. Then $|E'| \leq |V'|d - \frac{|V'|}{2} < |V'|d - \binom{d+1}{2}$ which contradicts Theorem 1.1.) Let $v_1, v_2, ..., v_{k-1} \in V$ be such that $d_{G-\{v_1 \dots v_{\ell-1}\}}(v_\ell) = \Delta(G_\ell)$ for every $1 \leq \ell \leq k-1$ where $G_1 = G$ and $G_\ell = G - \{v_1 \dots v_{\ell-1}\}$. As G_k is $[1, d]$-rigid,

$$|E(G_k)| \geq d(|V| - (k-1)) - \binom{d+1}{2} = d|V| - \binom{d+1}{2} - (k-1)d$$

by Theorem 1.1. Using this inequality, we have

$$|E| \geq d|V| - \binom{d+1}{2} - (k-1)d + (|E| - |E(G_k)|).$$

G_ℓ is $[1, d]$-rigid with $|V(G_\ell)| = |V| - \ell + 1 \geq d^2 + d$ hence $\Delta(G_\ell) \geq 2d$ for every $1 \leq \ell \leq k$. This implies that $|E| - |E(G_k)| \geq (k-1)2d$. Thus $|E| \geq d|V| - \binom{d+1}{2} + (k-1)d$ as we claimed. □

3.2 Upper bound for $k = 2$

In this section we give an upper bound for the number of edges of minimally $[2, d]$-rigid graphs.

Theorem 3.2. *Let $G = (V, E)$ be a minimally $[2, d]$-rigid graph. Then*

$$|E| \leq 2d|V| - 3\binom{d+1}{2}.$$

Proof. As G is $[2, d]$-rigid, it is also $[1, d]$-rigid, thus it has a minimally $[1, d]$-rigid subgraph H that has exactly $d|V| - \binom{d+1}{2}$ edges. Now, we count the edges in $E - E(H)$. For a vertex $v \in V$, let E_v denote the set of edges in $E - E(H)$ for which $G - v - e$ is not $[2, d]$-rigid. By the minimality of G, $\bigcup_{v \in V} E_v = E - E(H)$. As H is minimally rigid, the graph $H - v$ is independent in $\mathcal{R}_d(H - v)$ for any $v \in V$. By our assumption, $G - v$ is rigid for every $v \in V$ hence there is a set of edges $F_v \subseteq E(G - v)$ for which $(H - v) + F_v$ is minimally rigid. Since $|E(H - v)| = d|V| - \binom{d+1}{2} - d_H(v)$, we have $|F_v| = d_H(V) - d$. (Note that $d_H(v) \geq d$ as H is $[1, d]$-rigid.) The existence of F_v ensures that $G - e - v$ is rigid for every $e \in (E - E(H)) - F_v$. Hence $E_v \subseteq F_v$ thus $|E_v| \leq d_H(v) - d$. Therefore,

$$|E| = |E(H)| + \left|\bigcup_{v \in V} E_v\right| \leq |E(H)| + \sum_{v \in V}(d_H(v) - d) = 3|E(H)| - d|V| = 2d|V| - 3\binom{d+1}{2}$$

which completes the proof. □

The upper bound given in Theorem 3.2 is $4|V| - 9$ for $d = 2$. The number of edges of graph $W_{t,2}^2$ (to be defined in Section 6.1) is $3|V| - 7$ and this is the minimally $[2, 2]$-rigid graph with the highest number of edges that we know of, see [8, 9]. Hence it remains open if there are examples for minimally $[2, 2]$-rigid graphs with more edges or the bound given in Theorem 3.2 can be improved.

4 Strongly minimally $[2, d]$-rigid graphs

In this section we consider the case $k = 2$. We show that the lower bound given in Theorem 3.1 is sharp for $k = 2$ in any dimension and we disprove Conjecture 2.5.

Consider the graph C_n^d and its subgraph L_d induced by vertices $v_{n-d+1}, \ldots, v_n$. (Note that L_d is isomorphic to K_d.) $H_{n,2}^d = C_n^d - E(L_d)$ denotes the graph we get from C_n^d after deleting the edge set of L_d. First we prove that $H_{n,2}^d$ is $[2, d]$-rigid.

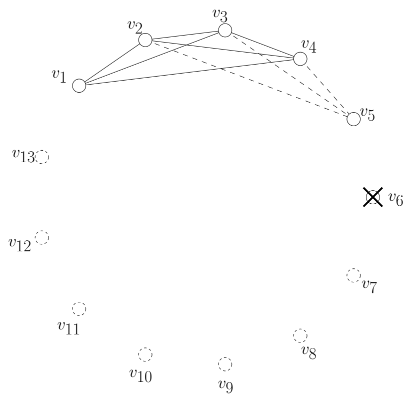
Lemma 4.1. *$H_{n,2}^d$ is $[2, d]$-rigid if $n \geq 3d$.*

Proof. Let $v_i \in V(H_{n,2}^d)$ be arbitrary. We will prove that $H_{n,2}^d - v_i$ is $[1, d]$-rigid by constructing it from a subgraph isomorphic to K_d using (d-dimensional) Henneberg-0 and Henneberg-1-extensions.

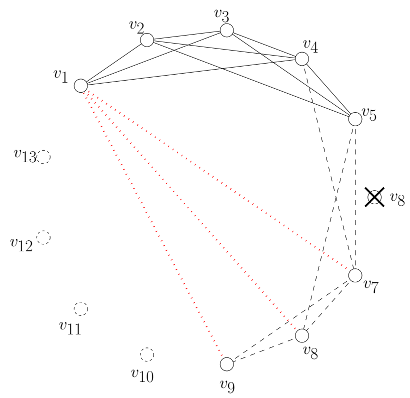
First suppose that $v_i \notin V(L_d)$. For simplicity, we can assume that $\lfloor \frac{n-d+1}{2} \rfloor \leq i \leq n - d$. Since $n \geq 3d$ we have $i \geq d + 1$. Vertices $v_1, \ldots, v_d$ induce a subgraph isomorphic to K_d hence we can add $v_{d+1}, \ldots, v_{i-1}$ in this order using Henneberg-0 extensions which connect v_j to vertices $v_{j-d+1}, \ldots, v_{j-1}$ for every $d + 1 \leq j \leq i - 1$. Therefore $v_1, \ldots, v_{i-1}$ induce a $[1, d]$-rigid subgraph.

Now we will add vertices $v_{i+1}, \ldots, v_{i+d}$ in this order using Henneberg-0 extensions. If $j \leq n - d$ then the extension connects v_j to vertices $v_{j-d}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}$ and to v_1. Note that $v_j v_1$ is not an edge of $H^d_{n,2} - v_i$ if $j \leq n - d$. We will apply Henneberg-1 extensions on these extra edges. If $j > n - d$ then it will be connected to $v_{j-d}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-d}$ and to $v_1, \ldots, v_{d-n+j}$ all of which are edges of $H^d_{n,2} - v_i$.

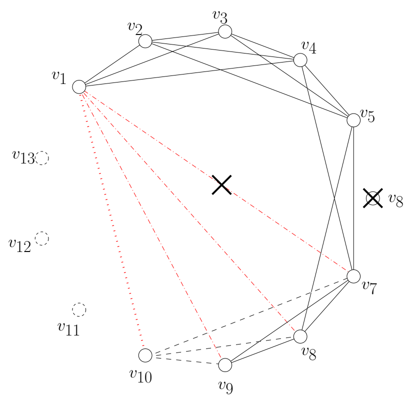
From now on we will use Henneberg-1 extensions only for adding vertices $v_{i+d+1}, \ldots, v_n$ in this order. When adding v_j for $j \leq n - d$ we apply the Henneberg-1 extension on edge $v_{j-d}v_1$ that connects v_j to $v_{j-d+1}, \ldots, v_{j-1}$. In this case we remove the extra edge $v_{j-d}v_1$ and add a new one $v_j v_1$. If $j > n - d$ then similarly we apply the Henneberg-1 extension on edge $v_{j-d}v_1$ but we connect v_j to $v_{j-d}, \ldots, v_{n-d}$ and to $v_2, \ldots, v_{d-n+j}$ and all of these edges are present in $H^d_{n,2} - v_i$. In this case the number of extra edges decreased by one.



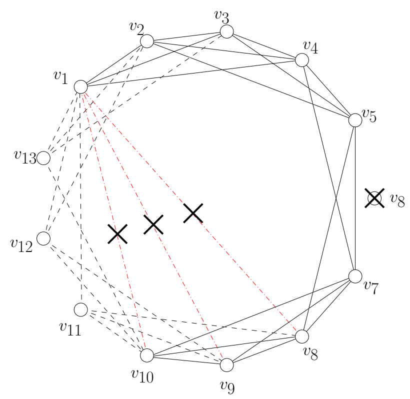
(a) Add v_5 with a Henneberg-0-extension.



(b) Add v_7, v_8, v_9 with Henneberg-0-extensions by adding the extra (dotted red) edges $v_j v_1$ for $7 \leq j \leq 9$.



(c) Add v_{10} with Henneberg-1-extension on the (crossed red) extra edge $v_7 v_1$ by adding the extra (dotted red) edge $v_{10} v_1$



(d) Add v_{11}, v_{12}, v_{13} with Henneberg-1-extensions on the (red) edges $v_8 v_1, v_9 v_1, v_{10} v_1$, respectively.

Figure 1: Building up $C^3_{13} - E(L_3) - v_5$ using Henneberg operations.

If $v \in V(L_d)$, then it is easy to see that $H^d_{n,2}$ has a subgraph that can be built up using Henneberg-0-extensions only (we first build up the subgraph induced by vertices of $H^d_{n,2}$ and then we add the nodes in $V(L_d) - v$). □

If $G = (V, E)$ is $[2, d]$-rigid then $|E| \geq d|V| - \binom{d+1}{2} + d = d|V| - \binom{d}{2}$ if $|V| \geq d^2 + d + 2$ by Theorem 3.1. $|E(H^d_{n,2})| = dn - \binom{d}{2}$ since C^d_n has dn edges if $n \geq 2d + 1$ and the deleted edges form a complete subgraph with d vertices. Hence by Lemma 4.1 we get the mail result of this section:

Theorem 4.2. *If $G = (V, E)$ is a strongly minimally $[2, d]$-rigid graph with $|V| \geq d^2 + d + 2$ then $|E| = d|V| - \binom{d}{2}$.*

5 Strongly minimally $[3, 3]$-rigid graphs

In this section we show that the lower bound given in Theorem 3.1 is sharp when $k = d = 3$.

Lemma 5.1. *C^3_n is $[3, 3]$-rigid if $n \geq 9$.*

Proof. Let $v_i, v_j \in V(C^3_n)$ be arbitrary. We will prove that $C^3_n - \{v_i, v_j\}$ is $[1, 3]$-rigid by constructing it from a subgraph isomorphic to K_4 using 3-dimensional Henneberg-0 and Henneberg-1-extensions and $\underline{\Delta}$-X-replacements.

We can assume that $j = n$ and $i \geq \lceil \frac{n+1}{2} \rceil$. $n \geq 9$ hence $i \geq 5$ and as in proof of Lemma 4.1 it can be seen easily that the subgraph induced by $v_1, \ldots, v_{i-1}$ is rigid.

Let $\ell = n - i - 1$. We have to perform ℓ more extension to add the remaining vertices. We split the proof into two cases depending on ℓ.

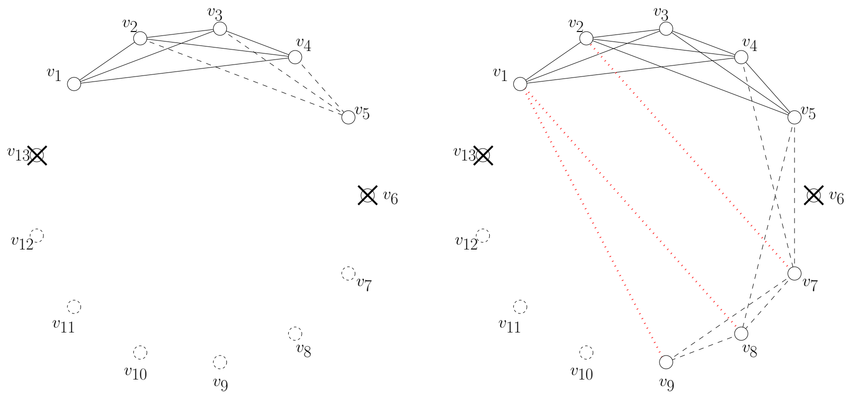
If $1 \leq \ell \leq 3$, we add v_{i+1} and connect it to v_1, v_{i-2}, v_{i-1}. If $\ell \geq 2$ then we add v_{i+2} and connect it to v_1, v_{i-1}, v_{i+1}. If $\ell = 3$ then we can add v_{i+3} performing a Henneberg-1 extension on edge $v_{i+1}v_1$ and connecting v_{i+3} to v_{i+2} and v_2.

If $\ell \geq 4$ then we will need a $\underline{\Delta}$-X-replacement on edges v_2v_{n-3}, v_1v_{n-4}. In this case we will add vertices $v_{i+1}, v_{i+2}, v_{i+3}$ by Henneberg-0 extensions, $v_{i+4}, \ldots, v_{n-2}$ by Henneberg-1 extensions. We will perform these operations such that after adding v_{n-2} edges $v_2v_{n-3}, v_1v_{n-2}, v_1v_{n-4}, v_{n-2}v_{n-4}$ will be present in the resulting graph.

Let $\sigma : \mathbb{Z} \to \{1, 2\}$ be a function with $\sigma(t) := 2$ if $t \equiv \ell - 2 \pmod 3$ and $\sigma(t) := 1$ otherwise. We add v_{i+1} with Henneberg-0-extension that connects it to $v_{i-2}, v_{i-1}, v_{\sigma(1)}$. Then add v_{i+2} with a Henneberg-0-extension that connects it to $v_{i-1}, v_{i+1}, v_{\sigma(2)}$. Next, we add v_{i+3} with a Henneberg-0-extension that connects it to $v_{i-1}, v_{i-2}, v_{\sigma(3)}$. Then we add v_{i+m} for $4 \leq m \leq \ell - 1$ in sequence with Henneberg-1 extension on $v_{i+m-3}v_{\sigma(m-3)}$ that connects it to v_{i+m-2}, v_{i+m-1}. Finally, we add v_{n-1} with a $\underline{\Delta}$-X-replacement on edges v_2v_{n-3}, v_1v_{n-4} as $v_{n-2}v_1v_{n-1}$ is a triangle. □

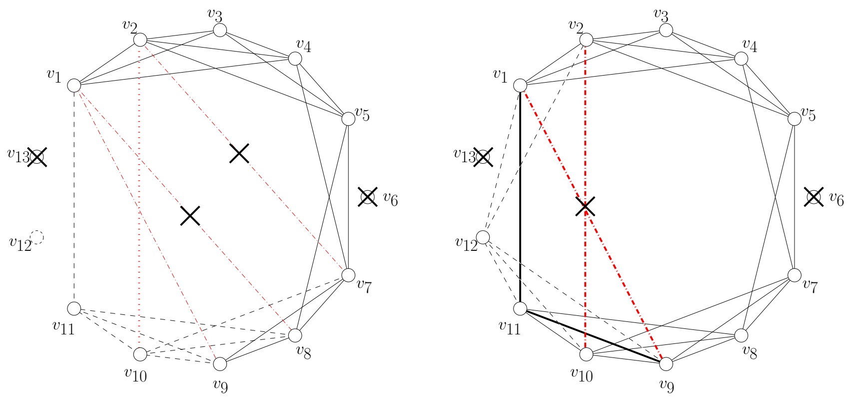
We have proved that C^3_n is $[3, 3]$-rigid and clearly C^3_n has $3n$ edges if $n \geq 7$. This together with Theorem 3.1 gives the following:

Theorem 5.2. *If $G = (V, E)$ is a strongly minimally $[3, 3]$-rigid graph with $|V| \geq 9$ then $|E| = 3|V|$.*



(a) Add u_5 with a Henneberg-0 extension.

(b) Add v_7, v_8, v_9 with Henneberg-0 extensions by adding extra edges $v_{i+m}v_{\sigma(i)}$.



(c) Add v_{10} and v_{11} with Henneberg-1 extensions on edges v_7v_2, v_8v_1, respectively. By performing the first of these extensions we create the extra edge $v_{10}v_2$.

(d) Add v_{12} with $\underline{\Delta}$-X-replacement on edges v_2v_{10}, v_1v_9. (Note that $v_1v_9v_{11}$ is a triangle.)

Figure 2: Building up $C_{12}^3 - \{u, v\}$.

6 Higher dimensions revisited

Recall that L_d denotes the complete subgraph of C_n^d spanned by vertices $v_{n-d+1}, \dots, v_n$. Let L'_d denote the graph that we get from L_d by deleting the Hamiltonian cycle that consist of edges v_iv_{i+1} for $n - d + 1 \le i \le n - 1$ and $v_{n-d+1}v_n$. Note that L'_3 is the empty graph on three vertices. Lemma 5.1 states that $C_n^d - L'_d$ is strongly minimally $[3, 3]$-rigid.

$|E(C_n^d - L'_d)| = dn - \binom{d}{2} + d = dn - \binom{d+1}{2} + 2d$ which motivates the second part of the following conjecture:

Conjecture 6.1. *The lower bound given in Theorem 3.1 is sharp for $k = 3$ for any $d \ge 3$. Moreover, $C_n^d - L'_d$ is a strongly minimally $[3, d]$-rigid if n is sufficiently large.*

It remains open if the lower bound given in Theorem 3.1 is tight for some pairs $[k, d]$ different from $[2, d]$ and $[3, 3]$. This question seems to be more complicated for larger values of k and d as there are just a few operations known that preserve rigidity in higher dimensions. Furthermore it was shown in [6] that the lower bound given in Theorem 3.1 is not tight for $k = 3$ and $d = 2$, a strongly minimally $[3, 2]$-rigid graph on at least 6 vertices has $2|V| + 2$ edges. Following their idea, the lower bound given in Theorem 3.1 can also be improved if the right-hand side of (1) is larger than $d|V|$ because in this case $\Delta(G) \geq 2d + 1$ holds.

6.1 Examples for minimally $[k, d]$-rigid graphs

The question whether weakly minimally $[k, d]$-rigid graphs exist for every pair (k, d) can still be solved without knowing the edge count of strongly minimally $[k, d]$-rigid graphs. There are examples for weakly minimally $[2, 2]$-rigid graphs in [7, 8, 9] but the existence of weakly minimally $[k, d]$-rigid graphs for other values of k and d was open so far. In this section we will give examples for minimally $[k, d]$-rigid graphs with the same number of vertices but with different number of edges. Such a pair of graphs shows that the graph with the larger number of edges has to be weakly minimally $[k, d]$-rigid.

First we generalize an example from [8, 9]. In the following lemma, P^0_n denotes the empty graph on n vertices.

Lemma 6.2. *Let t, k and d be three positive integers with $t \geq kd + 1$. Then there exists a minimally $[k, d]$-rigid graph with $t + k$ vertices and $(d + k - 1)t - \binom{d}{2}$ edges.*

Proof. Let the graph $W^d_{t,k}$ consist of P^{d-1}_t (with vertex set $\{v_1, \ldots, v_t\}$) and k additional vertices $s_1, ..., s_k$ each of which is connected to all vertices of P^{d-1}_t (see Figure 3). We first prove that $W^d_{t,k}$ is $[k, d]$-rigid.

For $k = 1$, we need to show that $W^d_{t,1}$ is minimally $[1, d]$-rigid. As $t \geq d + 1$, s_1 and $v_1, \ldots, v_d$ form a complete graph with $d + 1$ vertices. Starting with this subgraph, $W^d_{t,1}$ can be built up by adding vertices $v_{d+1}, \ldots, v_t$ with Henneberg-0 extensions. This proves case $k = 1$.

Assume that $k \geq 2$. First, we show that $W^d_{t,k} - \{u_1, \ldots, u_{k-1}\}$ is $[1, d]$-rigid if $\{u_1, \ldots, u_{k-1}\} \subseteq \{v_1, \ldots, v_t\}$. As $t \geq kd + 1$ there should be some integer $1 \leq j \leq n - d + 1$ such that $\{v_j, \ldots, v_{j+d-1}\} \cap \{u_1, \ldots, u_{k-1}\} = \emptyset$. Starting with the complete subgraph spanned by $\{s_1, v_j, \ldots, v_{j+d-1}\}$ we can build up a subgraph of $W^d_{t,k} - \{u_1, ..., u_{k-1}\}$ up by Henneberg-0 extensions. First we add $s_2, ...s_k$ one after one and then the vertices that are not deleted from $\{v_{j+d}, \ldots, v_t, v_{j-1}, \ldots, v_1\}$ in this order.

Next observe that $W^d_{t,k} - s_i$ is isomorphic to $W^d_{t,k-1}$ for any $i \in \{1, ..., k\}$. Thus by induction, $W^d_{t,k} - \{s_i, u_1, \ldots, u_{k-2}\}$ is $[1, d]$-rigid for every $i \in \{1, ..., k\}$ and $\{u_1, \ldots, u_{k-2}\} \subseteq \{v_1, \ldots, v_t\}$. So far we proved that $W^d_{t,k}$ is $[k, d]$-rigid.

Moreover, as subgraphs, $\{W^d_{t,k} - s_i : i \in \{1, ..., k\}\}$ cover all edges of $W^d_{t,k}$ and by induction these subgraphs are all minimally $[k - 1, d]$-rigid graphs, $W^d_{t,k}$ is minimally $[k, d]$-rigid.

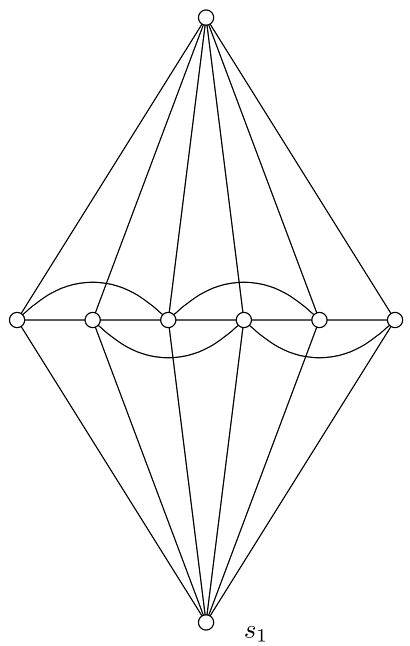


Figure 3: $W^3_{6,2}$.

Clearly $|V(W^d_{t,k})| = t+k$. $|E(W^d_{t,k})| = |E(P^{d-1}_t)| + kt = (d+k-1)t - \binom{d}{2}$ if $t \geq kd+1$ since in this case $|E(P^{d-1}_t)| = (d-1)t - \binom{d}{2}$. This completes the proof. □

The *cone graph* of G is the graph that arises from G by adding a new vertex s and edges sv for every $v \in V$. The operation that creates the cone graph of G is called *coning*. The following claim states that one can construct $[k,d]$-rigid graphs by coning $[k-1,d]$-rigid graphs. However these examples will not necessarily be minimal but by omitting some of their edges one can achieve minimality.

Claim 6.3. *Let $k \geq 2$ and $d \geq 1$ integers. Let $G = (V,E)$ be a $[k-1,d]$-rigid graph and let $H = (V+s, E')$ be the cone graph of G. Then H is $[k,d]$-rigid.*

Proof. We need to show that after omitting $k-1$ vertices H remains $[1,d]$-rigid. If s is omitted, then we are done by the $[k-1,d]$-rigidity of G. Otherwise, let $u_1, ..., u_{k-1}$ be the omitted vertices. $G - \{u_1, ..., u_{k-2}\}$ is $[1,d]$-rigid and s is connected to every neighbor of v_{k-1}. Hence $H - \{u_1, ..., u_{k-1}\}$ has a subgraph isomorphic to the $[1,d]$-rigid graph $G - \{u_1, ..., u_{k-2}\}$ showing that it is $[1,d]$-rigid. □

Let $H^d_{n,i}$ denote the cone graph of $H^d_{n,(i-1)}$ for $i \geq 3$. (For the definition of $H^d_{n,2}$ see Section 4.) By Claim 6.3 and Lemma 4.1, we get the following:

Corollary 6.4. *Let t, d and k be three positive integers such that $t \geq 3d$ and $k \geq 2$. Then there exists a minimally $[k,d]$-rigid graph $H^d_{t,k}$ with $t+k-2$ vertices and at most $(d+k-2)t - \binom{d}{2} + \binom{k-2}{2}$ edges.*

We shall also use Claim 6.3 in the proof of the following lemma.

Lemma 6.5. *Let $t \geq 2$, $k \geq 1$ and $d \geq 3$ be three integers. There exists a minimally $[k,d]$-rigid graph with $t+k+d-2$ vertices and $(d+k-1)t+\binom{k+d-2}{2}-1$ edges.*

Proof. Define graph M_t^{k+d-2} as follows. Take the disjoint union of a path P_t (on vertex set $\{v_1,\dots,v_t\}$) and a complete graph K_{k+d-2} (on vertex set $\{w_1,\dots,w_{k+d-2}\}$) and add edges v_iw_j for every pair $1 \leq i \leq t, 1 \leq j \leq k+d-2$ (see Figure 4).

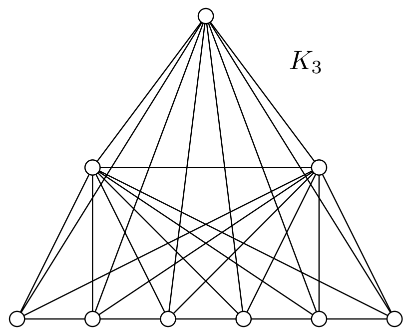


Figure 4: M_6^3.

First we show that M_t^{k+d-2} is minimally $[k,d]$-rigid. If $k=1$ then $v_1, v_2, w_1, \dots, w_{k+d-2}$ form a complete subgraph with $d+1$ vertices. Starting with this subgraph, M_t^{d-1} can be built up by adding $v_3,\dots,v_t$ with Henneberg-0 extensions.

For $k \geq 2$ graph M_t^{k+d-2} is $[k,d]$-rigid by induction and Claim 6.3. Moreover, $M_t^{k+d-2} - w_j$ is isomorphic to $M_t^{(k-1)+d-2}$ for any $1 \leq j \leq k+d-2$ that is minimally $[k-1,d]$-rigid by induction. As $d \geq 3$ these subgraphs cover M_t^{k+d-2} showing the minimality.

Clearly, $|V(M_t^{k+d-2})| = t+k+d-2$ and $|E(M_t^{k+d-2})| = (t-1)+\binom{k+d-2}{2}+(k+d-2)t = (d+k-1)t+\binom{k+d-2}{2}-1$. $\square$

Let k,d,n be integers such that $k \geq 2$, $d \geq 3$ and $n \geq k(d+1)+1$. Put $t_1 = n-k$ and $t_2 = n-k-d+2$. With this notation $n = |V(W_{t_1,k}^d)| = |V(M_{t_2}^{k+d-2})|$. We will prove that $|E(W_{t_1,k}^d)| < |E(M_{t_2}^{k+d-2})|$ which shows that $M_{t_2}^{k+d-2}$ is weakly minimally $[k,d]$-rigid. By Lemmas 6.2 and 6.5, we have to prove that

$$(d+k-1)(n-k)-\binom{d}{2} < (d+k-1)(n-k-d+2)+\binom{k+d-2}{2}-1.$$

By subtracting $(d+k-1)(n-k)-\binom{d}{2}$ from each side, we get

$$0 < \frac{d(d-1)}{2}+(d+k-1)(-d+2)+\frac{(k+d-2)(k+d-3)}{2}-1,$$

that is,

$$0 < \frac{k^2-k}{2}$$

that holds for $k \geq 2$.

Now, let k, d, n be positive integers such that $k \geq 2$ and $n \geq \max\{k(d+1)+1, 3d+k-2, 3k+2d-4+\binom{k-2}{2}\}$. Put $t_0 = n-k+2$.With this notation $n = |H^d_{t_0,k}| = |V(W^d_{t_1,k})|$. We will prove that $|E(H^d_{t_0,k})| < |E(W^d_{t_1,k})|$ which shows that $W^d_{t_1,k}$ is weakly minimally $[k, d]$-rigid. By Lemma 6.2 and Corollary 6.4, it is enough to prove that

$$(d+k-2)(n-k+2) - \binom{d}{2} + \binom{k-2}{2} < (d+k-1)(n-k) - \binom{d}{2}$$

By subtracting $(d+k-2)(n-k+2) - \binom{d}{2} + \binom{k-2}{2}$ from each side, we get

$$0 < n - 2d - 3k + 4 - \binom{k-2}{2}$$

that holds because of the choice of n.

We have proved the following theorem:

Theorem 6.6. *Let d and k be positive integers with $k \geq 2$. Then there are weakly minimally $[k, d]$-rigid graphs, that is, there are minimally $[k, d]$-rigid graphs that are not strongly minimally $[k, d]$-rigid.*

7 A counterexample for Conjecture 2.5

In this section we disprove Conjecture 2.5 by constructing minimally [2,2]-rigid graphs that do not have a vertex at which the reverse degree 3 vertex addition or the reverse X-replacement can be performed. To give such an example we will need the following simple observation.

Claim 7.1. *Let $G = (V, E)$ be a graph. Suppose $v \in V$ with $d(v) = 4$ is contained in a K_4 subgraph of G. Then every possible reverse X-replacement at v creates a parallel pair of edges.*

We define an operation called *K_4-extension* that preserves $[2, 2]$-rigidity although the resulting graph may not be minimally $[2, 2]$-rigid. Let $G = (V, E)$ be a graph with $|V| \geq 4$, and let $v_1, v_2, v_3, v_4 \in V$ be four distinct vertices. The K_4-extension adds four new vertices u_1, u_2, u_3, u_4 to G, connects v_i to u_i for every $1 \leq i \leq 4$ and u_k to u_l for every pair $1 \leq k, l \leq 4$.

Claim 7.2. *If $G = (V, E)$ is $[2, 2]$-rigid then $G' = (V', E')$ obtained by a K_4-extension is also $[2, 2]$-rigid. Furthermore $G' - e$ is not $[2, 2]$-rigid for any $e \in E' - E$.*

Proof. Clearly, $G' - v$ is rigid for any $v \in V'$.

Consider the graph $G' - e$ for some $e \in E' - E$. Let $u_i \in V' - V$ be such that e is not incident to u_i. We claim that $G'' = G' - u_i - e$ is not rigid. G'' consist of G and a set of three vertices that is incident to five edges only. Hence there are only $2|V|-3+5 = 2|V'|-4$ independent edges in G'' thus G'' is not rigid as we claimed. □

Now let $G_0 = (V_0, E_0)$ be a $[2,2]$-rigid graph with $V_0 \geq 4$. Apply some K_4-extensions to vertices of V_0, let the resulting graph be $G_1 = (V_1, E_1)$ (see Figure 5). Suppose that every vertex in V_0 is incident to at least five edges from $E_1 - E_0$. After the extensions delete edges from E_1 (if necessary) to obtain a minimally $[2,2]$-rigid graph $G_2 = (V_1, E_2)$. By Claim 7.1 deleting any edge from $E_1 - E_0$ results in a graph that is not $[2,2]$-rigid hence the minimum degree in G_2 is four and all the degree four vertices are in $V_1 - V_0$. Clearly we cannot perform the reverse degree 3 vertex addition in G_2. But every vertex of $V_1 - V_0$ is contained in a K_4 subgraph of G_2 and by Claim 7.1 every reverse X-replacement on one of these vertices creates a parallel pair of edges. Thus no reverse X-replacement operation preserves minimal $[2,2]$-rigidity of G_2 which disproves Conjecture 2.5.

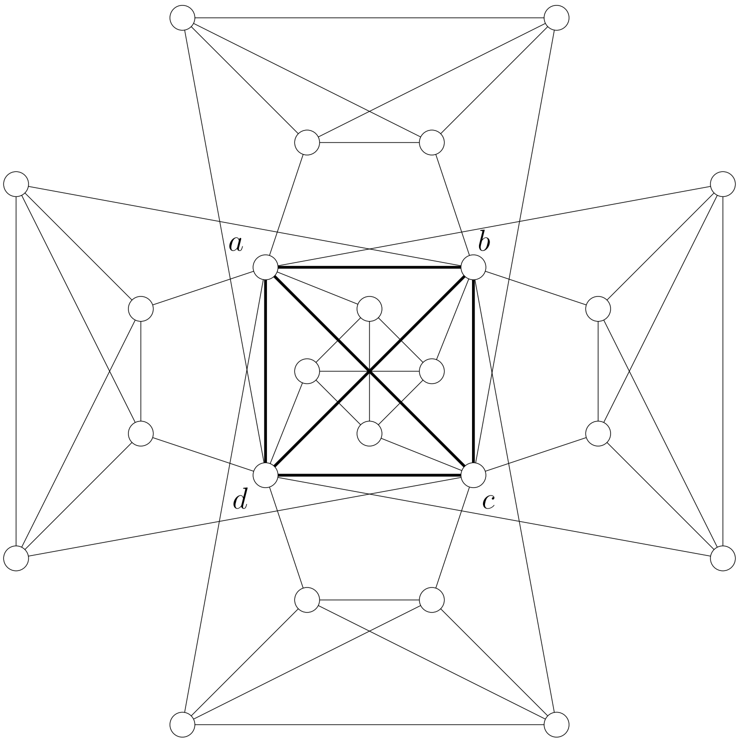


Figure 5: A counterexample G_c for Conjecture 2.5 that we get by performing five K_4-extensions on the subgraph induced by vertices a, b, c, d. Clearly, K_4 is minimally $[2,2]$-rigid hence G_c is $[2,2]$-rigid by Claim 7.2. It can be easily seen that deleting any of the edges bc, cd, db from graph $G_c - a$ results in a flexible graph. By symmetry the deletion of any edge of the starting graph results in a graph that is not $[2,2]$-rigid. This implies that G_c is minimally $[2,2]$-rigid.

Remark 7.3. We also remark that for any positive integer t graph G_1 can be constructed such that every vertex in V_0 is incident to at least t edges from $E_1 - E_0$. Hence G_2 has vertices of degree four and the rest of its vertices has degree at least t. Since t can be arbitrarily large this example shows that it may be difficult to find a constructive characterization that only uses operations that add low-degree vertices.

8 Concluding remarks

The results presented in this paper are about the edge numbers of minimally $[k,d]$-rigid graphs. Similar questions were asked about minimally globally $[k,d]$-rigid graphs in [8] where $G=(V,E)$ is globally $[k,d]$-rigid if $|V|\geq k+1$ and after deleting at most $k-1$ arbitrary vertices the resulting graph is globally rigid in $\mathbb{R}^d$.

Other version of the problem is $[k,d]$-edge rigidity (and global $[k,d]$-edge rigidity) where instead of at most $k-1$ vertices we delete at most $k-1$ edges of the graph. Proving similar results on these variants of the problem considered is a possible direction of future research.

A different direction is to characterize inductively the class of graphs mentioned above for some values of $[k,d]$ which seems to be an interesting and difficult open question.

Acknowledgments

The authors received a grant (no. CK 80124) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund. Research was supported by the MTA-ELTE Egerváry Research Group.

The authors thank Zsuzsanna Jankó and János Geleji for the inspiring discussions and Tibor Jordán for posing the interesting questions solved in this paper.

References

[1] L. Berenchtein, L. Chavez, and W. Whiteley, Inductive constructions for 2-rigidity: bases and circuits via tree partitions. *Manuscript*, York University, Toronto, 2002.

[2] R. Connelly and W. Whiteley, Global Rigidity: The effect of coning. *Discrete Comput. Geom.* 43: pp 717–735, 2010.

[3] J. Graver, B. Servatius, and H. Servatius, *Combinatorial Rigidity.* AMS Graduate Studies in Mathematics Vol. 2, 1993.

[4] J. Graver, *A Combinatorial Approach to Infinitesimal Rigidity.* Technical Report ♯10, Department of Mathematics, Syracuse University, August 1984.

[5] G. Laman, On graphs and rigidity of plane skeletal structures. *J. Engineering Math.* 4: pp 331–340, 1970.

[6] S.A. Motevallian, C. Yu, and B.D.O. Anderson, Multi-agent rigid formations: A study of robustness to the loss of multiple agents. In: *Proceedings of the 50th IEEE CDC-ECE*, pp 3602–3607, 2011.

[7] B. Servatius, Birigidity in the plane. *SIAM J. Discrete Math* 8: pp 582–589, 1989.

[8] T.H. Summers, C. Yu, and B.D.O. Anderson, Robustness to Agent Loss in Vehicle Formations and Sensor Networks. In: *Proceedings of the 47th IEEE CDC*, pp 1193–1199, 2008.

[9] T.H. Summers, C. Yu, and B.D.O. Anderson, Addressing agent loss in vehicle formations and sensor networks. *International Journal of Robust and Nonlinear Control*, 19(15): pp 1673–1696, 2009.

[10] T-S. Tay and W. Whiteley, Generating Isostatic Frameworks. *Structural Topology* 11: pp 21–69, 1985.

[11] W. Whiteley, Cones, infinity and one-story buildings, *Structural Topology* 8: pp 53–70, 1983.

[12] W. Whiteley, Rigidity and scene analysis. In: (J.E. Goodman and J. O'Rourke ed.) *Handbook of Discrete and Computational Geometry* , CRC-Press, 1997.

[13] W. Whiteley, Some matroids from discrete applied geometry. In: (J. Bonin, J. Oxley, and B. Servatius ed.) *Matroid theory*, pp 171–311, Contemp. Math., 197, AMS, Providence, 1996.