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Kristóf Bérczi, Tamás Király, and
Yusuke Kobayashi

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Abstract

Edmonds' fundamental theorem on arborescences [4] characterizes the existence of k pairwise edge-disjoint arborescences with the same root in a directed graph. In [9], Lovász gave an elegant alternative proof which became the base of many extensions of Edmonds' result.

In this paper, we use a modification of Lovász' method to prove a theorem on covering intersecting bi-set families under matroid constraints. Our result can be considered as a common generalization of previous results on packing arborescences.

1 Introduction

Let $D = (V, A)$ be a directed graph (or digraph, for short). For disjoint sets $X, Y \subseteq V$ we say that Y is **reachable** from X if there is a directed path from a node of X to a node of Y. For some root-node $r \in V$, an **arborescence rooted at** r or an r-**arborescence** (U, F) is a directed tree in which each node in U is reachable from r. An arborescence is **spanning** if its node-set is V. We sometimes identify an arborescence (U, F) with its edge set F and say that F spans U. The node-set of an r-arborescence F is denoted by $V(F)$. An r-arborescence may consist of a single node and no edge but we always assume that an arborescence has at least one node. In that case, $V(F) = \{r\}$.

Let $\varrho(X)$ denote the in-degree of a set X. In [4], Edmonds proved the following.

Theorem 1.1 (Edmonds' disjoint arborescences theorem, weak form)**.** *Let $D = (V, A)$ be a digraph with $r \in V$. There are k pairwise edge-disjoint spanning r-arborescences in D if and only if*

$$\varrho(X) \geq k \quad \text{for all } \emptyset \neq X \subseteq V - r. \tag{1}$$

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest, Hungary. E-mail: `berkri@cs.elte.hu`.

**MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest, Hungary. E-mail: `tkiraly@cs.elte.hu`.

***University of Tokyo, Tokyo 113-8656, Japan. E-mail: `kobayashi@mist.i.u-tokyo.ac.jp`.

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In the same paper, Edmonds verified that the result remains true even if the spanning arborescences must be extensions of some initially given r-arborescences.

Theorem 1.2 (Edmonds' disjoint arborescences theorem, strong form)**.** *Let $D = (V, A)$ be a digraph with $r \in V$ and $\mathcal{F} = \{F'_1, \ldots, F'_k\}$ be a family of edge-disjoint –not necessarily spanning– arborescences rooted at r. There are pairwise edge-disjoint spanning r-arborescences $F_1, \ldots, F_k$ such that $F'_i \subseteq F_i$ if and only if*

$$\varrho(X) \geq p_{\mathcal{F}}(X) \quad \textit{for all } \emptyset \neq X \subseteq V - r,$$

where $p_{\mathcal{F}}(X) = |\{i : \ V(F'_i) \cap X = \emptyset\}|$.

Theorem 1.2 can be reformulated in terms of branchings. We call a collection of node-disjoint arborescences a **branching**. In other words, a branching (U, B) is a directed forest in which each node has degree at most one. The set of nodes having in-degree zero is called the **root-set** of the branching. For a digraph $D = (V, A)$ and $\emptyset \neq R \subseteq V$ a branching (V, B) is called a **spanning R-branching** if its root-set is exactly R.

Theorem 1.3 (Edmonds' disjoint branchings theorem)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a family of k non-empty subsets of V. There are k pairwise edge-disjoint spanning branchings of D with root-sets $R_1, \ldots, R_k$, respectively, if and only if*

$$\varrho(X) \geq p_{\mathcal{R}}(X) \quad \textit{for all } \emptyset \neq X \subseteq V,$$

where $p_{\mathcal{R}}(X)$ denotes the number of root-sets R_i disjoint from X.

It is easy to see that Theorems 1.2 and 1.3 are equivalent. Indeed, Theorem 1.2 follows from Theorem 1.3 by taking $R_i := V(F_i)$. The other direction can be shown by adding a new node r to the graph and taking arc sets $F_i := \{rv : \ v \in R_i\}$ as starting arborescences. Theorem 1.3 seems to be a genuine generalization of Theorem 1.1 as there is no known way to derive the former from the latter.

In [7], Kamiyama, Katoh and Takizawa found an extension of Edmonds' theorem for the case when not all nodes are reachable from the specified root nodes.

Theorem 1.4 (Kamiyama, Katoh and Takizawa)**.** *Let $D = (V, A)$ be a digraph and $r_1, \ldots, r_k \in V$ be a set of root nodes. Let U_i denote the set of nodes reachable from r_i in D. There are pairwise edge-disjoint arborescences $F_1, \ldots, F_k$ such that F_i is rooted at r_i and spans U_i if and only if*

$$\varrho(X) \geq p'(X) \quad \textit{for all } X \subseteq V,$$

where $p'(X) = |\{i : \ r_i \notin X, U_i \cap X \neq \emptyset\}|$.

It is worth mentioning that a reformulation of the theorem using root-sets is also true and is equivalent to the original one.

Theorem 1.5. *Let $D = (V, A)$ be a digraph and $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of root-sets. Let U_i denote the set of nodes reachable from R_i in D. There are pairwise edge-disjoint branchings $B_1, \dots, B_k$ such that B_i is an R_i-branching spanning U_i if and only if*

$$\varrho(X) \geq p'_{\mathcal{R}}(X) \quad \text{for all } X \subseteq V,$$

where $p'_{\mathcal{R}}(X) = |\{i : \ X \cap R_i = \emptyset, U_i \cap X \neq \emptyset\}|$.

For a digraph $D = (V, A)$, a set $U \subseteq V$ is called **convex** if there is no $v \in V \setminus U$ such that v is reachable from U and U is reachable from v. Fujishige [6] observed that the sets of reachable nodes U_i in the above theorem can be replaced by arbitrary convex sets.

Theorem 1.6 (Fujishige)**.** *Let $D = (V, A)$ be a digraph and $r_1, \dots, r_k \in V$ be a set of root nodes. Assume that $U_1, \dots, U_k$ are convex sets such that $r_i \in U_i$. There are pairwise edge-disjoint arborescences $F_1, \dots, F_k$ such that F_i is rooted at r_i and spans U_i if and only if*

$$\varrho(X) \geq p'(X) \quad \text{for all } X \subseteq V,$$

where $p'(X) = |\{i : \ r_i \notin X, U_i \cap X \neq \emptyset\}|$.

Clearly, Theorem 1.6 implies Theorem 1.4 as the set U of nodes reachable from a given node r is always convex. However, Cs. Király observed that the reverse implication also follows by a simple construction [8].

Recently, Durand de Gevigney, Nguyen and Szigeti [2] considered the problem of packing arborescences under matroid constraints. Let $D = (V, A)$ be a digraph and let t be a positive integer that specifies the number of arborescences to pack. The set of integers from 1 to t is denoted by $[t]$. We are also given a matroid $\mathcal{M} = ([t], r)$ with rank function r, and a mapping $\pi : [t] \to V$ that specifies the roots of the arborescences. The triple $(D, \mathcal{M}, \pi)$ is called an **$\mathcal{M}$-rooted digraph**. For a set $X \subseteq V$ we use $\pi^{-1}(X) = \{i \in [t] : \ \pi(i) \in X\}$.

We call π **$\mathcal{M}$-independent** if $\pi^{-1}(v)$ is independent in $\mathcal{M}$ for each $v \in V$. The digraph is **$\mathcal{M}$-connected** if

$$\varrho(X) \geq r([t]) - r(\pi^{-1}(X)) \quad \text{for all } \emptyset \neq X \subseteq V.$$

An **$\mathcal{M}$-basic packing of arborescences** is a collection of pairwise edge-disjoint arborescences $F_1, \dots, F_t$ such that F_i is rooted at $\pi(i)$ and $\{i : \ v \in V(F_i)\}$ forms a base of $\mathcal{M}$ for each $v \in V$. The following was proved in [2].

Theorem 1.7 (Gevigney, Nguyen and Szigeti)**.** *Let $(D, \mathcal{M}, \pi)$ be an $\mathcal{M}$-rooted digraph. There exists an $\mathcal{M}$-basic packing of arborescences if and only if π is $\mathcal{M}$-independent and D is $\mathcal{M}$-connected.*

It is easy to derive Theorem 1.3 from this. For a family $\mathcal{R} = \{R_1, \dots, R_k\}$ of root-sets, let $t = |R_1| + \dots + |R_k|$, and let f be a bijection from the multiset $R_1 + \dots + R_k$ to $[t]$. We define M as the partition matroid with classes $f(R_j)$ $(j = 1, \dots, k)$, and let $\pi(i) = f^{-1}(i)$ $(i = 1, \dots, t)$. It can be seen easily that the existence of an $\mathcal{M}$-basic packing of arborescences in $(D, \mathcal{M}, \pi)$ corresponds to a pairwise edge-disjoint packing of branchings in D with root-sets $R_1, \dots, R_k$. Thus Theorem 1.7 generalizes the strong form of Edmonds' theorem.

It is a natural question to find a common generalization of Theorems 1.4 and 1.7. Let $D = (V, A)$ be a digraph, t a positive integer, and $\mathcal{M} = ([t], r)$ be a matroid. Using the notation of [8], the set of nodes from which a given set $X \subseteq V$ is reachable (including the nodes of X themselves) is denoted by P_X. If an $\mathcal{M}$-rooted digraph $(D, \mathcal{M}, \pi)$ is given, we call a set of pairwise edge-disjoint arborescences $F_1, \dots, F_t$ a **maximal $\mathcal{M}$-independent packing of arborescences** if F_i is a $\pi(i)$-arborescence, $\{i : v \in V(F_i)\}$ is independent and $r(\{i : v \in V(F_i)\}) = r(\pi^{-1}(P_v))$ for $v \in V$.

Cs. Király [8] proved the following extension of Theorem 1.7.

Theorem 1.8 (Cs. Király)**.** *Let $(D, \mathcal{M}, \pi)$ be an $\mathcal{M}$-rooted digraph. There exists a maximal $\mathcal{M}$-independent packing of arborescences if and only if π is $\mathcal{M}$-independent and*

$$\varrho(X) \geq r(\pi^{-1}(P_X)) - r(\pi^{-1}(X)) \quad \text{for all } X \subseteq V.$$

There is another line of results which extends Edmonds' theorems in a different direction. Frank observed that Edmonds' weak theorem can be reformulated in terms of covering intersecting set families and thus gave an abstract extension of Edmonds' results [5]. Given a directed graph $D = (V, A)$, a family $\mathcal{F} \subseteq 2^V$ of subsets of V is called **intersecting** if $X, Y \in \mathcal{F}$ and $X \cap Y \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$. We say that an arc $a \in A$ **covers** a set $X \in \mathcal{F}$ if a enters X, that is, the tail of a is outside of X while the head of A is inside X. A subset of edges $A' \subseteq A$ **covers** an intersecting family $\mathcal{F}$ if each member of $\mathcal{F}$ is covered by at least one arc from A'.

Theorem 1.9 (Frank)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F} \subseteq 2^V$ be an intersecting family. Then there are pairwise disjoint arc-sets $A_1, \dots, A_k$ such that A_i covers $\mathcal{F}$ for $i = 1, \dots, k$ if and only if*

$$\varrho(X) \geq k \quad \text{for all } X \in \mathcal{F}.$$

By choosing $\mathcal{F} = 2^{V-r} - \emptyset$, we immediately obtain the weak form of Edmonds' disjoint arborescences theorem. However, a weakness of Frank's result is that it does not imply the strong form. This was overcome in [10] by Szegő, who introduced the notion of mixed intersection property. Given a digraph $D = (V, A)$ and k intersecting families $\mathcal{F}_1, \dots, \mathcal{F}_k \subseteq 2^V$, we say that these families satisfy the **mixed intersection property** if $X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X \cap Y \neq \emptyset$ implies $X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$.

Theorem 1.10 (Szegő)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \dots, \mathcal{F}_k \subseteq 2^V$ be intersecting families satisfying the mixed intersection property. Then there are pairwise disjoint arc sets $A_1, \dots, A_k \subseteq A$ such that A_i covers $\mathcal{F}_i$ if and only if*

$$\varrho(X) \geq p(X) \quad \text{for all } X \subseteq V,$$

where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

Although Szegő's theorem provides a common extension of Edmonds' and Frank's results, it does not seem to easily imply the result of Kamiyama et al. The problem with the natural choice $\mathcal{F}_i = 2^{U_i - r_i} - \emptyset$ $(i = 1, \ldots, k)$ is that an arbitrary edge set A_i covering $\mathcal{F}_i$ does not necessarily contain an r_i-arborescence spanning U_i. Indeed, it may happen that a set in $2^{U_i - r_i} - \emptyset$ is covered in A_i by an edge which has a tail outside of U_i and hence can not be added to such an arborescence.

To circumvent this problem, a bi-set counterpart of Szegő's theorem was proved in [1]. Given a digraph $D = (V, A)$, a **bi-set** is a pair $X = (X_I, X_O)$ such that $X_I \subseteq X_O \subseteq V$ where X_I and X_O are called the **inner** and the **outer set** of X, respectively. We will identify a bi-set $X = (X_O, X_I)$ for which $X_O = X_I$ with the simple set X_I and hence the following notation can be also interpreted for sets. The set of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** and **union** of bi-sets can be defined in a straightforward manner: for bi-sets X and Y, we define $X \cap Y = (X_I \cap Y_I, X_O \cap Y_O)$ and $X \cup Y = (X_I \cup Y_I, X_O \cup Y_O)$. An edge $a \in A$ **enters** or **covers** a bi-set X if its head is in X_I and its tail is outside X_O. A subset of edges $A' \subseteq A$ **covers** a bi-set family $\mathcal{F}$ if each member of $\mathcal{F}$ is covered by at least one arc from A'. The set of arcs entering a bi-set X is denoted by $\Delta^{in}(X)$, while the **number of arcs entering** X is denoted by $\varrho(X)$. An arc is **contained** in bi-set X if its tail is in X_O and its head is in X_I. We say that $X \subseteq Y$ if $X_I \subseteq Y_I$ and $X_O \subseteq Y_O$. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$. A family $\mathcal{F}$ of bi-sets is called **intersecting** if $X, Y \in \mathcal{F}, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$.

A **bi-set function** is a function $p : \mathcal{P}_2 \to \mathbb{R}$. A bi-set function p is called **fully supermodular** (respectively, **intersecting supermodular**) if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

for $X, Y \in \mathcal{P}_2$ (respectively, for intersecting $X, Y \in \mathcal{P}_2$). If the reverse inequality holds, we call p **fully submodular**. A basic example for a submodular bi-set function is the in-degree function ϱ. We call p **positively intersecting supermodular** or **positively intersecting submodular** if the corresponding inequality holds whenever X and Y are intersecting and $p(X), p(Y) > 0$.

We say that the bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ satisfy the **mixed intersection property** if $X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$. The following theorem extends the result of Szegő to bi-set families.

Theorem 1.11 (Bérczi and Frank)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting bi-set families satisfying the mixed intersection property. Then there are pairwise disjoint arc-sets $A_1, \ldots, A_k \subseteq A$ such that A_i covers $\mathcal{F}_i$ if and only if*

$$\varrho(X) \geq p_2(X) \quad \text{for all } X \in \mathcal{P}_2,$$

where $p_2(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

Up to this point, we have two completely different generalizations of Edmonds' theorems. Theorem 1.8 is an extension of previous results about packing arborescences and characterizes the existence of maximal $\mathcal{M}$-independent arborescence packings. Meanwhile, Theorem 1.11 states the existence of disjoint covers of intersecting bi-set families and gives an abstract generalization of previous results. Our motivation was to find a common extension of these two results (for an overview of known results see Figure 1).

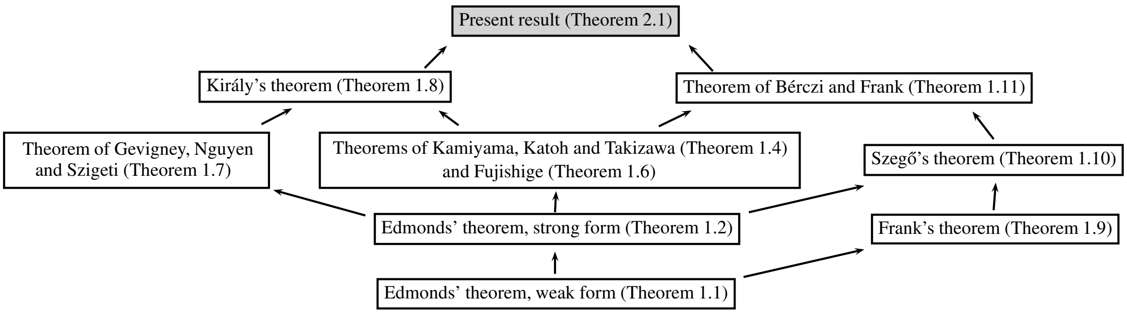

Figure 1: Overview of previous results

The rest of the paper is organized as follows. In Section 2, we prove our main result on covering intersecting bi-set families under matroid constraints by following the mains steps of Lovász's approach. In Section 3, we present some observations that follow from our proof. We also show how the proof implies Theorem 1.8. Finally, we propose some open problems related to arborescence packings in Section 4.

2 Main theorem

2.1 Statement

Let $\mathcal{M} = ([t], r)$ be a matroid. The **closure** of $I \subseteq [t]$ is denoted by $\texttt{Span}(I)$, that is, $\texttt{Span}(I) = \{i : \ r(I + i) = r(I)\}$. A set $I \subseteq [t]$ is called **closed** if $\texttt{Span}(I) = I$. Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting, $\mathcal{G}_1, \dots, \mathcal{G}_t$ be arbitrary bi-set families over $\mathcal{P}_2(V) = \mathcal{P}_2$. In what follows, we consider the case when $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$. We denote the set of bi-sets appearing in at least one of the $\mathcal{F}_i$'s by $\mathcal{F} = \bigcup \mathcal{F}_i$.

For a bi-set $X \in \mathcal{P}_2$, let

$$I_X = \{i : \ X \in \mathcal{F}_i\},$$
$$J_X = \{i : \ X \in \mathcal{G}_i\}.$$

By $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for each $i \in [t]$, $I_X \cap J_X = \emptyset$ for each bi-set $X \in \mathcal{P}_2$. We introduce the following bi-set function defined on $\mathcal{P}_2$:

$$p(X) = \begin{cases} r(I_X \cup J_X) - r(J_X) & \text{if } X \in \mathcal{F}, \\ 0 & \text{otherwise.} \end{cases}$$

A bi-set X is said to be **active** if $p(X) > 0$, which is equivalent to $I_X \setminus \texttt{Span}(J_X) \neq \emptyset$. We denote the set $I_X \setminus \texttt{Span}(J_X)$ by I_X^{act} and say that X is **active for** i if $i \in I_X^{act}$. A bi-set X is called **tight** if $\varrho(X) = p(X) > 0$ and $I_X \neq [t]$. A bi-set is **tight for** i if it is tight and $i \notin I_X^{act}$. Note that both active and tight bi-sets are contained in $\mathcal{F}$.

We say that $\mathcal{F}_1, \dots, \mathcal{F}_t, \mathcal{G}_1, \dots, \mathcal{G}_t$ satisfy the **active intersection property** if

(AIP) $X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X_I \cap Y_I \neq \emptyset, i \in I_X^{act} \Rightarrow i \in I_{X\cap Y}^{act}$.

Here we state our main result; the proof is left to Sections 2.2 and 2.3.

Theorem 2.1. *Let $\mathcal{M} = ([t], r)$ be a matroid and $D = (V, A)$ a digraph. Let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting, $\mathcal{G}_1, \dots, \mathcal{G}_t$ arbitrary bi-set families over $\mathcal{P}_2$, satisfying the active intersection property and $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ ($i = 1, \dots, t$). Assume that*

(a) $I_{X'} \cup J_{X'} = I_X \cup J_X$ for $X, X' \in \mathcal{F}, X' \subseteq X$,

(b) $\varrho(X) \geq r(I_X \cup J_X) - r(J_X)$ for $X \in \mathcal{F}$.

Then there are pairwise disjoint arc-sets $A_1, \dots, A_t \subseteq A$ such that

$$r(J_X \cup \{i \in I_X : \ \varrho_{A_i}(X) \geq 1\}) = r(J_X \cup I_X) \tag{2}$$

for each $X \in \mathcal{F}$.

2.2 Preliminaries for the proof

To prove Theorem 2.1, we will use the following easy but important observation.

Proposition 2.2. *If $X, X' \in \mathcal{F}$ and $X' \subseteq X$, then $J_{X'} \subseteq \texttt{Span}(J_X)$.*

Proof. Condition (a) impies that $I_{X'} \cup J_{X'} = I_X \cup J_X$. If X is not active, then $\texttt{Span}(J_X) \supseteq I_X \cup J_X \supseteq J_{X'}$ so we are done. If X is active, then by the active intersection property for X and X' we have $I_{X'} \setminus \texttt{Span}(J_{X'}) \supseteq I_X \setminus \texttt{Span}(J_X)$, which, together with $I_{X'} \cup J_{X'} = I_X \cup J_X$, implies that $J_{X'} \subseteq \texttt{Span}(J_X)$. □

From the active intersection property and (a) we can derive the positively intersecting supermodularity of p.

Proposition 2.3. *Function p is positively intersecting supermodular.*

Proof. Let X, Y be intersecting bi-sets with $p(X) > 0$, $p(Y) > 0$. Both X and Y are active and so $X \cap Y \in \mathcal{F}$ by (AIP). Note that Proposition 2.2 implies $J_{X\cap Y} \subseteq \texttt{Span}(J_X) \cap \texttt{Span}(J_Y)$.

Assume first that $X \cup Y \in \mathcal{F}$. As the $\mathcal{F}_i$'s are intersecting, $I_{X\cup Y} \supseteq I_X \cap I_Y$ holds. This, together with (a), implies $J_{X\cup Y} \subseteq J_X \cup J_Y$. Thus we have

$$\begin{aligned} r(J_X) + r(J_Y) &= r(\texttt{Span}(J_X)) + r(\texttt{Span}(J_Y)) \\ &\geq r(\texttt{Span}(J_X) \cap \texttt{Span}(J_Y)) + r(\texttt{Span}(J_X) \cup \texttt{Span}(J_Y)) \\ &\geq r(J_{X\cap Y}) + r(J_{X\cup Y}). \end{aligned}$$

Using this and (a) we get

$$\begin{aligned} p(X) + p(Y) &= r(I_X \cup J_X) - r(J_X) + r(I_Y \cup J_Y) - r(J_Y) \\ &\leq r(I_{X\cap Y} \cup J_{X\cap Y}) - r(J_{X\cap Y}) + r(I_{X\cup Y} \cup J_{X\cup Y}) - r(J_{X\cup Y}) \\ &= p(X \cap Y) + p(X \cup Y). \end{aligned}$$

Now assume that $X \cup Y \notin \mathcal{F}$. By (a), $I_X \cup J_X = I_{X\cap Y} \cup J_{X\cap Y} = I_Y \cup J_Y$; let q denote the rank of this set. As the $\mathcal{F}_i$'s are intersecting, we have $I_X \cap I_Y = \emptyset$, which implies $I_X \subseteq J_Y$ and $I_Y \subseteq J_X$. Hence we have

$$\begin{aligned} p(X) + p(Y) &= 2q - r(J_X) - r(J_Y) \\ &= 2q - r(\mathtt{Span}(J_X)) - r(\mathtt{Span}(J_Y)) \\ &\leq 2q - r(\mathtt{Span}(J_X) \cap \mathtt{Span}(J_Y)) - r(\mathtt{Span}(J_X) \cup \mathtt{Span}(J_Y)) \\ &\leq q - r(\mathtt{Span}(J_X) \cap \mathtt{Span}(J_Y)) \\ &\leq q - r(J_{X\cap Y}) \\ &= p(X \cap Y) + p(X \cup Y). \end{aligned}$$

□

2.3 Proof of Theorem 2.1

The basis of the proof is the following reduction step.

Definition 2.4 (Reduction on (X, a, i))**.** Assume that $X \in \mathcal{F}$, $i \in I_X^{act}$ and $a \in \Delta^{in}(X)$ are such that a is contained in none of the bi-sets active for i and enters no bi-set tight for i. Then the triple (X, a, i) is called **reducible**. A **reduction on** (X, a, i) consists of

- adding a to A_i and removing it from A;
- adding bi-sets in $\mathcal{F}_i$ covered by a to $\mathcal{G}_i$;
- deleting bi-sets covered by a from $\mathcal{F}_i$.

We usually denote the bi-set families obtained from $\mathcal{F}_i$ and $\mathcal{G}_i$ after a reduction on (X, a, i) by $\mathcal{F}_i'$ and $\mathcal{G}_i'$.

The next lemma shows the main advantage of the reduction procedure.

Lemma 2.5. *A reduction step on (X, a, i) preserves the conditions of the theorem.*

Proof. Let $\mathcal{F}'$ denote the family of bi-sets in $\mathcal{F}$ that have not been removed, and let ϱ' be the in-degree function in the digraph obtained by deleting arc a. $\mathcal{F}_i'$ is intersecting because if $U \in \mathcal{F}_i$ and $W \in \mathcal{F}_i$ are not covered by a, then neither are their intersection $U \cap W$ and their union $U \cup W$.

The validity of the active intersection property can be seen as follows. Assume that $U \in \mathcal{F}_i'$ with $i \in I_U^{act}$ and $W \in \mathcal{F}_j$ for some j. As only $\mathcal{F}_i$ changes during this step, it suffices to show that if U and W are intersecting then $i \in I_{U\cap W}^{act}$. If this does not hold then a covers $U \cap W$ but does not cover U, hence it is contained in U. However, we assumed that a is not contained in a bi-set active for i, a contradiction.

Note that by deleting bi-sets covered by a from $\mathcal{F}_i$ and adding them to $\mathcal{G}_i$ the union $I_X \cup J_X$ does not change for any bi-set $X \in \mathcal{F}$, hence (a) remains valid. Finally, $\varrho'(Z) \geq r(I_Z \cup J_Z) - r(J_Z)$ holds for each $Z \in \mathcal{F}'$ as we assumed that a enters no bi-set that is tight for i. □

We prove the theorem by induction on $\sum_{Z\in\mathcal{F}} p(Z)$. If $\mathcal{F}=\emptyset$ or $p(Z)=0$ for each $Z\in\mathcal{F}$ then (2) is clearly satisfied, and we are done. Otherwise take an inclusion-wise maximal bi-set X in $\mathcal{F}$ with $p(X)\geq 1$. From now on, our aim is to show that there exists a triple (X,a,i) satisfying the conditions of Definition 2.4. This would prove the theorem by induction.

From the maximal choice of X one can derive the following.

Proposition 2.6. *There is no bi-set Z intersecting X such that Z is active for each $i\in I_X^{act}$ and $X\cup Z$ is strictly larger than X.*

Proof. As $I_X^{act}\cap I_Z^{act}\neq\emptyset$ and the $\mathcal{F}_i$'s are intersecting, $X\cup Z\in\mathcal{F}$. Moreover, $I_X^{act}\subseteq I_Z^{act}$ so $I_X\setminus \texttt{Span}(J_X)\subseteq I_{X\cup Z}$. By (a), $I_X\cup J_X=I_{X\cup Z}\cup J_{X\cup Z}$, so $J_{X\cup Z}=(I_X\cup J_X)\setminus I_{X\cup Z}\subseteq \texttt{Span}(J_X)$. Hence we have

$$\begin{aligned} p(X\cup Z) &= r(I_{X\cup Z}\cup J_{X\cup Z})-r(J_{X\cup Z})\\ &\geq r(I_X\cup J_X)-r(J_X)\\ &\geq 1, \end{aligned}$$

contradicting the choice of X. □

In what follows, we distinguish two cases.

Case 1: There is a tight bi-set $Y\in\mathcal{F}$ intersecting X with $I_X^{act}\setminus I_Y^{act}\neq\emptyset$.

Let $Y\in\mathcal{F}$ be inclusion-wise minimal among these bi-sets.

Proposition 2.7. *There is an arc in $\Delta^{in}(X\cap Y)\setminus\Delta^{in}(Y)$.*

Proof. Take any index $i\in I_X^{act}\setminus I_Y^{act}$. As X is active for i, (AIP) implies that $X\cap Y$ is also active for i, thus $i\in I_{X\cap Y}^{act}=I_{X\cap Y}\setminus \texttt{Span}(J_{X\cap Y})$. Condition (a) shows that $i\in I_Y\cup J_Y$. But Y is tight for i, so $i\in \texttt{Span}(J_Y)\setminus \texttt{Span}(J_{X\cap Y})$. This, together with Proposition 2.2 gives $r(J_Y)>r(J_{X\cap Y})$.

The above yields

$$\begin{aligned} \varrho(Y) &= p(Y)\\ &= r(J_Y\cup I_Y)-r(J_Y)\\ &< r(J_{X\cap Y}\cup I_{X\cap Y})-r(J_{X\cap Y})\\ &\leq \varrho(X\cap Y), \end{aligned}$$

therefore there is an arc $a\in\Delta^{in}(X\cap Y)\setminus\Delta^{in}(Y)$. □

Choose an arc a provided by Proposition 2.7.

Proposition 2.8. *There is no bi-set W such that $a\in\Delta^{in}(W)$ and W is tight for some index in $I_X^{act}\setminus I_Y^{act}$.*

Proof. Suppose for contradiction that there is a bi-set W that is tight for some $i \in I_X^{act} \setminus I_Y^{act}$. Since a enters W, it is a tight bi-set that intersects Y. By Proposition 2.3 we have

$$\begin{aligned}\varrho(Y) + \varrho(W) &= p(Y) + p(W) \\ &\leq p(Y \cap W) + p(Y \cup W) \\ &\leq \varrho(Y \cap W) + \varrho(Y \cup W).\end{aligned}$$

As the in-degree function ϱ is submodular, we have equality throughout which has two important consequences. First, $p(X \cap Y) = \varrho(X \cap Y)$ and also $\varrho(X \cap Y) \geq 1$ due to arc a. On the other hand, the proof of Proposition 2.3 shows that necessarily $r(J_{Y \cap W}) = r(\texttt{Span}(J_Y) \cap \texttt{Span}(J_W))$ which together with Proposition 2.2 imply $\texttt{Span}(J_{Y \cap W}) = \texttt{Span}(J_Y) \cap \texttt{Span}(J_W)$. We know from (a) that $I_Y \cup J_Y = I_{Y \cap W} \cup J_{Y \cap W} = I_W \cup J_W$. These together give $I_{Y \cap W} \setminus \texttt{Span}(J_{Y \cap W}) = (I_Y \setminus \texttt{Span}(J_Y)) \cup (I_W \setminus \texttt{Span}(J_W))$, i.e. $I_{Y \cap W}^{act} = I_Y^{act} \cup I_W^{act}$. We assumed that there is an index i in $I_X^{act} \setminus (I_Y^{act} \cup I_W^{act})$, so there is an index $i \in I_X^{act}$ for which $Y \cap W$ is tight, contradicting the choice of Y. □

Proposition 2.9. *There is an index $i \in I_X^{act}$ such that (X, a, i) is reducible.*

Proof. We know by Proposition 2.8 that a does not enter any bi-set tight for any $i \in I_X^{act} \setminus I_Y^{act}$. Suppose that for each $i \in I_X^{act} \setminus I_Y^{act}$ there is a bi-set Z_i active for i containing a. Then, by (AIP), $Z' = \bigcap Z_i$ is a bi-set which is active for each $i \in I_X^{act} \setminus I_Y^{act}$. On the other hand, Y is active for each $i \in I_Y^{act}$, so $Z = Y \cap Z'$ is active for each $i \in I_X^{act}$, Z intersects X and $X \cup Z$ is strictly larger than X, contradicting Proposition 2.6. □

We have proved the existence of a reducible triple (X, a, i), and we are done.

Case 2: There is no tight bi-set $Y \in \mathcal{F}$ with $I_X^{act} \setminus I_Y^{act} \neq \emptyset$ intersecting X.

Take an arbitrary arc a entering X. Suppose that for each $i \in I_X^{act}$ there is a bi-set Z_i which is active for i and contains a. By (AIP), $Z = \bigcap Z_i$ is active for each $i \in I_X^{act}$, Z intersects X and $X \cup Z$ is strictly larger than X, contradicting Proposition 2.6. Hence there is an index $i \in I_X^{act}$ such that (X, a, i) is reducible, thus concluding the proof.

3 Consequences of the proof

From the proof in the previous section we can derive the following strengthening of Theorem 2.1.

Corollary 3.1. *The arc-sets $A_1, \ldots, A_t$ provided by the proof of Theorem 2.1 satisfy the following for each minimal $M \in \mathcal{F}$:*

(i) $\varrho_{A_i}(M) \leq 1$ *for* $1 \leq i \leq t$.

(ii) If J_M is independent, then $J_M \cup \{i : \ \varrho_{A_i}(M) \geq 1\}$ is independent.

Proof. Consider the addition of an arc a to an arc set A_p. According to the proof, a enters a bi-set $X \in \mathcal{F}_p$ which is actually maximal and is active for p. If M is a minimal set in $\mathcal{F}$ entered by a then $M \subseteq X$. If M was already covered by an arc $a' \in A_p$ at this point then a' is contained in X as X is active for p, contradicting the choice of a'. Thus (i) indeed holds. Furthermore, the rank of $J_M \cup \{i : \varrho_{A_i}(M) \geq 1\}$ increases by 1 due to the addition of index p, since M is active for p. Therefore at the end of the algorithm $J_M \cup \{i : \varrho_{A_i}(M) \geq 1\}$ is independent unless J_M was itself dependent. □

As was mentioned earlier, Theorem 2.1 is a common generalization of previous results on packing arborescences. First we show that it is an extension of Theorem 1.11.

Theorem 3.2 (Bérczi and Frank). *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting bi-set families satisfying the mixed intersection property. Then there are pairwise disjoint arc-sets $A_1, \dots, A_k$ such that A_i covers $\mathcal{F}_i$ if and only if*

$$\varrho(X) \geq p_2(X) \quad \textit{for all } X \in \mathcal{P}_2, \tag{3}$$

where $p_2(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

Proof. The 'only if' part can be seen easily, hence it suffices to prove the 'if' part.

Assume that (3) holds. Let $\mathcal{G}_i = \{X \in \mathcal{P}_2 : X \notin \mathcal{F}_i\}$. Then $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ and $I_X \cup J_X = [k]$ for each bi-set $X \in \mathcal{P}_2$, hence condition (a) of Theorem 2.1 is satisfied.

Let $\mathcal{M} = ([k], r)$ be the free matroid on $[k]$. For a bi-set $X \in \mathcal{P}_2$ we have $r(I_X \cup J_X) - r(J_X) = |\{i : X \in \mathcal{F}_i\}| = p_2(X)$, thus condition (b) of Theorem 2.1 holds by (3). Note that $I_X^{act} = I_X$ in this case and $r(S) = |S|$ for each $S \subseteq [k]$. Moreover, the mixed intersection property of the $\mathcal{F}_i$'s imply that the bi-set families $\mathcal{F}_1, \dots, \mathcal{F}_k, \mathcal{G}_1, \dots, \mathcal{G}_k$ altogether satisfy the active intersection property.

By the above, Theorem 2.1 ensures the existence of pairwise disjoint arc-sets such that

$$\begin{aligned}|J_X| + |\{i : X \in \mathcal{F}_i, \ \varrho_{A_i}(X) \geq 1\}| &= r(J_X \cup \{i \in I_X^{act} : \varrho_{A_i}(X) \geq 1\}) \\ &= r(I_X \cup J_X) \\ &= |J_X| + |\{i : X \in \mathcal{F}_i\}|.\end{aligned}$$

That is, $\mathcal{F}_i$ is covered by A_i, thus concluding the proof. □

Now we show that the proof of Theorem 2.1 also implies Theorem 1.8.

Theorem 3.3 (Cs. Király). *Let $(D, \mathcal{M}, \pi)$ be an $\mathcal{M}$-rooted digraph. There exists a maximal $\mathcal{M}$-independent packing of arborescences if and only if π is $\mathcal{M}$-independent and*

$$\varrho(X) \geq r(\pi^{-1}(P_X)) - r(\pi^{-1}(X)) \quad \textit{for all } X \subseteq V. \tag{4}$$

Proof. The 'only if' part can be seen easily, hence it suffices to show the 'if' part. Assume that $\mathcal{M} = ([t], r)$ with $\pi(i) = r_i$, π is $\mathcal{M}$-independent and that (4) holds. Let U_i be the set of nodes reachable from r_i in D. The U_i's define a partition of

V into **atoms**: two nodes u and v belong to the same atom if there is no U_i with $|\{u,v\} \cap U_i| = 1$. We call a subset of an atom a **subatom**. Let

$$\begin{aligned}\bar{\mathcal{P}}_2 = \{X \in \mathcal{P}_2 :\ & X_I \text{ is a nonempty subatom,} \\ & X_O \setminus X_I \text{ does not intersect the atom containing } X_I\}.\end{aligned}$$

Define the bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_t, \mathcal{G}_1, \ldots, \mathcal{G}_t$ as follows:

$$\begin{aligned}\mathcal{F}_i &= \{X \in \bar{\mathcal{P}}_2 :\ X_I \subseteq U_i - r_i,\ X_O \cap U_i = X_I\}, \\ \mathcal{G}_i &= \{X \in \bar{\mathcal{P}}_2 :\ X_I \subseteq U_i,\ X \notin \mathcal{F}_i\}.\end{aligned}$$

Note that bi-sets of form (W, W) where W is a subatom are contained in $\mathcal{F} = \bigcup \mathcal{F}_i$. In this case we use I_W and J_W for denoting the corresponding sets of indices. Clearly, at the beginning $I_W = \{i :\ W \subseteq U_i,\ r_i \notin W\}$ and $J_W = \{i :\ W \subseteq U_i,\ r_i \in W\}$.

It is easy to check that the $\mathcal{F}_i$'s are intersecting bi-set families. For a bi-set $X \in \mathcal{F}$ the index sets I_X and J_X are

$$\begin{aligned}I_X &= \{i :\ X_I \subseteq U_i - r_i,\ X_O \cap U_i = X_I\}, \text{ and} \\ J_X &= \{i :\ X_I \subseteq U_i,\ X \notin \mathcal{F}_i\}.\end{aligned}$$

By the above,

$$I_X \cup J_X = \{i :\ X_I \subseteq U_i\},$$

so the condition (a) in Theorem 2.1 is clearly satisfied. One can easily see that $J_{X'} \subseteq J_X$ if $X, X' \in \mathcal{F}, X' \subseteq X$, so (AIP) holds too.

We claim that the condition (b) in Theorem 2.1 is also satisfied. Indeed, for a given bi-set $X \in \mathcal{F}$ let Y be the bi-set with $Y_I = X_I$ and $Y_O \setminus Y_I = V \setminus (\bigcup_{i \notin J_X} U_i)$. Then $I_Y = I_X, J_Y = J_X$, and $\Delta^{in}(Y) \subseteq \Delta^{in}(X)$ (although $X_O \subseteq Y_O$ not necessarily holds). Note that each arc entering Y_O enters Y_I, hence we have

$$\begin{aligned}r(I_X \cup J_X) - r(J_X) &= r(I_Y \cup J_Y) - r(J_Y) \\ &= r(\pi^{-1}(P_{Y_O})) - r(\pi^{-1}(Y_O)) \\ &\leq \varrho(Y_O) = \varrho(Y) \leq \varrho(X),\end{aligned}$$

proving (b).

The above means that the conditions of Theorem 2.1 hold, hence there are pairwise disjoint arc-sets $A_1, \ldots, A_t$ such that

$$r(J_X \cup \{i :\ \varrho_{A_i}(X) \geq 1\}) = r(I_X \cup J_X) \tag{5}$$

for each $X \in \mathcal{F}$. We claim that these arc-sets can be chosen such that A_i is an arborescence rooted at r_i. For proving this, we slightly modify the steps of the proof of Theorem 2.1.

Consider a general step of the proof and assume that the sets of arcs that were already added to $A_1, \ldots, A_t$ form arborescences. At this point the bi-set families are already modified; we denote them by $\mathcal{F}'_1, \ldots, \mathcal{F}'_t, \mathcal{G}'_1, \ldots, \mathcal{G}'_t$. Let $\mathcal{F}' = \bigcup \mathcal{F}'_i$. For a bi-set $X \in \mathcal{P}_2$ we still use I_X, J_X for denoting the corresponding sets of indices, that is,

$$I_X = \{i : \ X_I \subseteq U_i \setminus V(A_i), \ X_O \cap U_i = X_I\}, \text{ and}$$
$$J_X = \{i : \ X_I \subseteq U_i, \ X \notin \mathcal{F}'_i\}.$$

Specially, for a subatom $W \subseteq V$ we have $I_W = \{i : \ W \subseteq U_i, \ W \cap V(A_i) = \emptyset\}$ and $J_W = \{i : \ W \cap V(A_i) \neq \emptyset\}$.

Proposition 3.4. *If $V(A_i) \cap X_O \neq \emptyset$ for a bi-set $X \in \mathcal{F}$, then X is not active for i.*

Proof. Since X_I is a subatom, either $X_I \cap U_i = \emptyset$ or $X_I \subseteq U_i$ holds. We consider the following three cases.

- If $X_I \cap U_i = \emptyset$, then $i \notin I_X \cup J_X$.
- If $X_I \subseteq U_i$ and $V(A_i) \cap X_I \neq \emptyset$, then $i \in J_X$.
- If $X_I \subseteq U_i$ and $V(A_i) \cap X_I = \emptyset$, then $X_O \cap U_i \neq X_I$, which implies that $i \in J_X$.

□

Take an active bi-set X that is inclusion-wise maximal among those for which $|I_X \cup J_X|$ is minimal. Let $uv = a \in \Delta^{in}(X)$.

Proposition 3.5. *The node u is reached in some A_j such that $j \in I_X^{act}$.*

Proof. Observe that because of the arc uv we have $I_u \cup J_u = I_X \cup J_X$.

<u>Case 1.</u> Assume that u and v are in the same atom. Consider bi-set $X' = (X_I + u, X_O + u)$. Note that $I_X \cup J_X = I_{X'} \cup J_{X'} = I_u \cup J_u$ and we have $I_{X'} = I_X \setminus J_u$ and $J_{X'} = J_X \cup J_u$. If there is an index $j \in I_X^{act} \cap J_u$ then u is reached in A_j, and we are done. Otherwise $J_u \setminus \texttt{Span}(J_X) = \emptyset$, implying $I_{X'}^{act} = I_X^{act} \neq \emptyset$, contradicting the choice of X.

<u>Case 2.</u> If u and v lie in different atoms, then consider bi-sets $X' = (u, X_O \setminus X_I + u)$ and $X'' = (X_I, X_O + u)$. The definition of the U_i's and $uv \in A$ imply that there is no index i with $u \in U_i, v \notin U_i$ but there is an index i_0 such that $v \in U_{i_0}$ but $u \notin U_{i_0}$. That is, $i_0 \in (I_X \cup J_X) \setminus (I_{X'} \cup J_{X'})$. Recall that $X \in \bar{\mathcal{P}}_2$. These together imply that $I_{X'} = I_X \cap I_u, J_{X'} \subseteq J_X \cup J_u$ and $I_{X''} = I_X \setminus (I_u \cup J_u), J_{X''} = J_X \cup I_u \cup J_u$.

Case 2.1 If there is an index $j \in I_X^{act} \cap J_u$ then u is reached in A_j, and we are done.

Case 2.2 If $I_X^{act} \cap I_u \neq \emptyset$ then $I_X^{act} \cap I_u \subseteq \texttt{Span}(J_X \cup J_u)$ as otherwise X' is active, contradicting the choice of X. This is only possible if $J_u \setminus \texttt{Span}(J_X) \neq \emptyset$ as otherwise $I_X^{act} \cap I_u \subseteq I_{X'}^{act}$, a contradiction. That is, there is an index $j \in I_X^{act} \cap J_u$ thus u is reached in A_j and we are done.

Case 2.3 If $I_X^{act} \cap (I_u \cup J_u) = \emptyset$ then $I_X^{act} \subseteq \texttt{Span}(J_X \cup I_u \cup J_u)$ as otherwise X'' is active, contradicting the choice of X. This is only possible if $(I_u \cup J_u) \setminus \texttt{Span}(J_X) \neq \emptyset$ as otherwise $I_{X''}^{act} = I_X^{act}$, a contradiction. That is, there is an index $j \in (I_u \cup J_u) \cap I_X^{act}$, contradicting the assumption of this case.

□

Assume now that there is a tight bi-set $Y \in \mathcal{F}'$ intersecting X with $I_X^{act} \setminus I_Y^{act} \neq \emptyset$ and choose a minimal one. By Proposition 2.7, there is an arc $uv = a \in \Delta^{in}(X)$ contained in Y. By Proposition 3.5, u is reached in some A_j such that $j \in I_X^{act}$. By Proposition 3.4, we have $v \notin V(A_j)$, $j \in I_X^{act} \setminus I_Y^{act}$ and that a is not contained in a bi-set Z active for j.

If there is no such tight bi-set then choose an arc $uv = a \in \Delta^{in}(X)$ arbitrarily. By Proposition 3.5, u is reached in some A_j such that $j \in I_X^{act}$ but uv is not contained in a bi-set active for j.

By the above, (X, a, j) is a reducible triple. Moreover, $u \in V(A_j), v \notin V(A_j)$, so we can extend the jth arborescence by adding a to it.

At the end of the algorithm we get arborescences $A_1, \dots, A_t$ satisfying (5). We claim that these arborescences give a maximal $\mathcal{M}$-independent packing of arborescences. Indeed, the construction is such that A_i is rooted at $\pi(i)$. As π is supposed to be $\mathcal{M}$-independent, J_v is independent for each $v \in V$. By Corollary 3.1, $J_v \cup \{i : \varrho_{A_i}(v) \geq 1\} = \{s_i : v \in V(A_i)\}$ is independent and by (5),

$$r(\{i : v \in V(A_i)\}) = r(J_v \cup \{i : \varrho_{A_i}(v) \geq 1\}) = r(I_v \cup J_v) = \pi^{-1}(P_v),$$

thus proving the theorem. □

4 Open questions

In this section we propose generalizations of Theorems 1.7 and 1.8. First of all, we reformulate these theorems in terms of directed graphs with a matroid given on the set of edges leaving a root-node.

Let $D = (V + r_0, A \cup S)$ be a directed graph where $\Delta^{out}(r_0) = S = \{a_1, \dots, a_t\}$. For an r_0-arborescence $F \subseteq A \cup S$ and node $v \in V$ the unique $r_0 - v$ path in F is denoted by $F(r_0, v)$ and the index of the first arc of $F(r_0, v)$ is denoted by i_v^F. In other words, $i_v^F = j$ if and only if $F(r_0, v) \cap S = a_j$. For a subset $X \subseteq V$ we use $P_X = \{i :$ there is a path from r_0 to X through $a_i\}$.

Theorem 1.7 can be reformulated as follows.

Theorem 4.1. *Let $D = (V + r_0, A \cup S)$ be a digraph and $\Delta^{out}(r_0) = S = \{a_1, \dots, a_t\}$. Assume that $\mathcal{M} = ([t], r)$ is a matroid with rank $r([t]) = k$. There exist t pairwise edge-disjoint r_0-arborescences $F_1, \dots, F_t$ such that $a_i \in F_i$ and $\{i : v \in V(F_i)\}$ forms a base of $\mathcal{M}$ for each $v \in V$ if and only if*

$$\varrho_A(X) \geq k - r(\Delta_S^{in}(X)) \quad \text{for all } \emptyset \neq X \subseteq V.$$

A direct extension of Edmonds' theorem would be the following.

Conjecture 4.2. *Let $D = (V + r_0, A \cup S)$ be a digraph and $\Delta^{out}(r_0) = S = \{a_1, \dots, a_t\}$. Assume that $\mathcal{M} = ([t], r)$ is a matroid with rank $r([t]) = k$. There* *exist k pairwise edge-disjoint spanning r_0-arborescences $F_1, \dots, F_k$ such that $\{i_v^{F_j} : j = 1, \dots, k\}$ forms a base of $\mathcal{M}$ for each $v \in V$ if and only if*

$$\varrho_A(X) \geq k - r(\Delta_S^{in}(X)) \quad \textit{for all } \emptyset \neq X \subseteq V. \tag{6}$$

It can be verified that, if true, Conjecture 4.2 implies Theorem 4.1. Also, it is worth mentioning the following analogy. In case of Edmonds' weak theorem, (1) is equivalent to the rooted k-edge-connectivity of D, that is, when there are k pairwise edge-disjoint directed paths from r to v for each $v \in V - r$. In Conjecture 4.2, (6) is equivalent to a matroid analogue of rooted connectivity. We call a set of directed $r_0 - v$ paths **$\mathcal{M}$-independent** if they are pairwise edge-disjoint and the first edge of these paths form an independent set of $\mathcal{M}$. We show that (6) is equivalent to the existence of k $\mathcal{M}$-independent $r_0 - v$ paths. Indeed, fix an arbitrary node $v_0 \in V$. Let $\mathcal{M}_1 = \mathcal{M}$, while consider a set of arcs $I \subseteq S$ to be independent in $\mathcal{M}_2$ if there exist $|I|$ edge-disjoint paths from r_0 to v_0 using all arcs in I (note that $\mathcal{M}_2$ is a gammoid). By Edmonds' matroid intersection theorem [3],

$$\text{there exist } k \ \mathcal{M}\text{-independent } r_0 - v_0 \text{ paths}$$
$$\Updownarrow$$
$$\mathcal{M}_1 \text{ and } \mathcal{M}_2 \text{ have a common independent set of size } k$$
$$\Updownarrow$$
$$\min_{B \text{ is closed in } \mathcal{M}_2} \{r_1(S \setminus B) + r_2(B)\} \geq k.$$

Let B be a closed set in $\mathcal{M}_2$ attaining the minimum. Take an arbitrary set $X \subseteq V$ with $v_0 \in X$ and $\varrho_{A \cup B}(X) = r_2(B)$ (note that there must be a set like that due to the definition of $\mathcal{M}_2$). Each arc in $S \setminus B$ has to enter X as B is supposed to be closed. If $B \cap \Delta^{in}(X) \neq \emptyset$ then $B' = B \setminus \Delta^{in}(X)$ gives $r_1(S \setminus B') + r_2(B') \leq r_1(S \setminus B) + r_2(B)$ and $\varrho_{A \cup B'}(X) = \varrho_A(X) = r_2(B')$. Thus

$$\varrho_A(X) + r(\Delta_S^{in}(X)) \geq k \quad \text{for all } v_0 \in X \subseteq V$$
$$\Downarrow$$
$$\min_{B \text{ is closed in } \mathcal{M}_2} \{r_1(S \setminus B) + r_2(B)\} \geq k.$$

Now let X be an arbitrary set $X \subseteq V$ with $v_0 \in X$. Define $B = S \setminus \Delta_S^{in}(X)$. Then $r_1(S \setminus B) = r(\Delta_S^{in}(X))$ while $r_2(B) \leq \varrho_A(X)$, hence

$$\min_{B \text{ is closed in } \mathcal{M}_2} \{r_1(S \setminus B) + r_2(B)\} \geq k$$
$$\Downarrow$$
$$\varrho_A(X) + r(\Delta_S^{in}(X)) \geq k \quad \text{for all } v_0 \in X \subseteq V.$$

These together show the equivalence of the two conditions.

A similar reformulation of Theorem 1.8 is as follows.

Theorem 4.3. *Let $D = (V + r_0, A \cup S)$ be a digraph and $\Delta^{out}(r_0) = S = \{a_1, \ldots, a_t\}$. Assume that $\mathcal{M} = ([t], r)$ is a matroid. There exist t pairwise edge-disjoint r_0-arborescences $F_1, \ldots, F_t$ such that $a_i \in F_i$, $\{i : v \in V(F_i)\}$ is independent in $\mathcal{M}$ and $r(\{i : v \in V(F_i)\}) = r(P_v)$ for each $v \in V$ if and only if*

$$\varrho_A(X) \geq r(P_X) - r(\Delta_S^{in}(X)) \quad \text{for all } \emptyset \neq X \subseteq V.$$

A strengthening of Conjecture 4.2 would then be the following.

Conjecture 4.4. *Let $D = (V + r_0, A)$ be a digraph and $\Delta^{out}(r_0) = S = \{a_1, \ldots, a_t\}$. Assume that $\mathcal{M} = ([t], r)$ is a matroid with rank $r([t]) = k$. There exist k pairwise edge-disjoint r_0-arborescences $F_1, \ldots, F_k$ such that $\{i_v^{F_j} : v \in V(F_j)\}$ is independent in $\mathcal{M}$ and $r(\{i_v^{F_j} : v \in V(F_j)\}) = r(P_v)$ for each $v \in V$ if and only if*

$$\varrho_A(X) \geq r(P_X) - r(\Delta_S^{in}(X)) \quad \text{for all } \emptyset \neq X \subseteq V.$$

Conjectures 4.2 and 4.4 basically state that the t arborescences appearing in Theorems 4.1 and 4.3 can be choosen in a very special way: they can be divided into k groups such that the node-sets of arborescences corresponding to the same group are disjoint apart from the root-node.

Another possible strengthening of these theorems would be to put some restrictions on the node sets spanned by the arborescences in question. A natural idea is to formulate such a restriction in terms of convexity introduced in Theorem 1.6. We only mention here a variant of Theorem 4.3.

Conjecture 4.5. *Let $D = (V + r_0, A \cup S)$ be a digraph and $\Delta^{out}(r_0) = S = \{a_1, \ldots, a_t\}$. Assume that $\mathcal{M} = ([t], r)$ is a matroid with rank $r([t]) = k$. There exist t pairwise edge-disjoint r_0-arborescences $F_1, \ldots, F_t$ such that $V(F_i)$ is a convex set and $a_i \in F_i$ for $i = 1, \ldots, t$, $\{a_i : v \in V(F_i)\}$ is independent in $\mathcal{M}$ and $r(\{i : v \in V(F_i)\}) = r(P_v)$ for each $v \in V$ if and only if*

$$\varrho_A(X) \geq r(P_X) - r(\Delta_S^{in}(X)) \quad \text{for all } \emptyset \neq X \subseteq V.$$

Besides the validity of the above conjectures, their connection to Theorem 2.1 is also of interest.

Finally, we show that a further extension of Conjecture 4.4 to root-sets results an NP-complete problem.

Problem 4.6. Let $D = (V, A)$ be a digraph and $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a list of root-sets. Assume that $\sum |R_i| = t$ and $\mathcal{M} = ([t], r)$ is a matroid with $r([t]) = k$. Let f be a bijection from the multiset $R_1 + \cdots + R_k$ to $[t]$. Assume that $B \subseteq A$ is a branching with root-set R_i and $v \in V(B)$. If r_0 is the root of the arborescence in B containing v then let $i_v^B = f(r_0)$. Now let $P_v = \{f(u) : u \in R_1 + \cdots + R_k, \ v \text{ is reachable from } u\}$.

Decide the existence of k pairwise edge-disjoint branchings $B_1, \ldots, B_k$ such that B_i is rooted at R_i, $\{i_v^{B_j} : v \in V(B_j)\}$ is independent in $\mathcal{M}$ and $r(\{i_v^{B_j} : v \in V(B_j)\}) = r(P_v)$ for each $v \in V$.

Theorem 4.7. *Problem 4.6 is NP-complete.*

Proof. Let R be a ground set and $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be partition matroids on R with $r_i(R) = k$ for $i = 1, 2, 3$. It is known that deciding the existence of a common base of three partition matroids is NP-complete.

Let $R_1, \dots, R_k$ denote the partition classes of $\mathcal{M}_1$, that is, a set $X \subseteq R$ is independent in $\mathcal{M}_1$ if and only if $|X \cap R_i| \leq 1$ for each $i = 1, \dots, k$. As partition matroids are special cases of gammoids, there is a digraph $D = (R + T, A)$ such that R is a stable set with $\varrho(R) = 0$ and a set $X \subseteq R$ is independent in $\mathcal{M}_2$ if and only if there exist $|X|$ directed paths – node-disjoint apart from v_0 – from X to $v_0 \in T$.

For each $v \in T - v_0$, add k directed arcs going from v_0 to v to the digraph. Then the existence of k pairwise edge-disjoint branchings $B_1, \dots, B_k$ satisfying the conditions of Problem 4.6 with choice $\mathcal{M} = \mathcal{M}_3$ is equivalent to the existence of a common base of the $\mathcal{M}_i$'s. $\square$

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