Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2013-08. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Proof of Berge's path partition conjecture for $k \geq \lambda - 3$

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December 2013
Revised: August 2015

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Abstract

Let D be a digraph. A *path partition* of D is called k-optimal if the sum of the k-norms of its paths is minimal. The *k-norm* of a path P is $\min(|V(P)|, k)$. Berge's path partition conjecture claims that for every k-optimal path partition $\mathcal{P}$ there are k disjoint stable sets orthogonal to $\mathcal{P}$. For general digraphs the conjecture has been proven for $k = 1, 2, \lambda - 1, \lambda$, where λ is the length of a longest path in the digraph. In this paper we prove the conjecture for $\lambda - 2$ and $\lambda - 3$.

Keywords: directed graph; path partition; Berge's path partition conjecture

1 Introduction

Let $D = (V, A)$ be a digraph. A path partition of D is a set of disjoint (directed) paths $P_1, P_2, \ldots, P_m$ for which $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_m) = V$. Throughout the paper by path we always mean directed path and a single vertex is also considered to be a path. Let $\mathcal{P}$ be a path partition and $\mathcal{S}$ be a set of k disjoint stable sets. We say that $\mathcal{P}$ and $\mathcal{S}$ are *orthogonal* if each path P_i intersects as many of the k stable sets as possible, i.e. $\min(|V(P_i)|, k)$. The Greene-Kleitman theorem [1] has shown that if the digraph is acyclic and transitive (i.e. represents a partially ordered set), then for each positive integer k and for each path partition $\mathcal{P}$ minimizing $\sum \min(|V(P_i)|, k)$ there are k disjoint stable sets orthogonal to $\mathcal{P}$. In 1982, Berge made his conjecture claiming the same for all digraphs ([2]).

The conjecture is known to be true for acyclic digraphs ([3]) and for $k \geq \lambda - \sqrt{\lambda}$ (where λ is the cardinality of a longest path in D) for strongly connected digraphs ([4]). However, for general digraphs only four cases are known: $k = 1, 2, \lambda - 1, \lambda$ ([5],[6],[4],[7], [8]).

In this paper we introduce a new variation of the stability number and prove a min-max theorem which directly generalizes the Greene-Kleitman theorem for general

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December 2013. Revised: August 2015

directed graphs. We then use this result to prove the path partition cconjecture for $k \geq \lambda - 3$.

We use the following definitions and notations:

Definition 1.1. The *k-norm* of a path partition $\mathcal{P} = \{P_1, \dots, P_m\}$ is defined by:

$$|\mathcal{P}|_k = \sum \min(|V(P_i)|, k).$$

A path partition is *k-optimal* if its k-norm is minimal.

Definition 1.2. For a digraph D, $\pi_k(D)$ denotes $|\mathcal{P}|_k$ where $\mathcal{P}$ is a k-optimal path partition of D.

Definition 1.3. Let $\mathcal{P}$ be a path partition and $S^1, \dots, S^k$ disjoint stable sets. We say that $S^1, \dots, S^k$ are *orthogonal* to $\mathcal{P}$ if each path P of $\mathcal{P}$ intersects exactly $\min(|V(P)|, k)$ sets of $S^1, \dots, S^k$.

Remark 1.4. Let $\mathcal{P}$ be a path partition and $S^1, \dots, S^k$ k disjoint stable sets orthogonal to $\mathcal{P}$. Then we have $\sum |S^i| \geq |\mathcal{P}|_k$. Indeed,

$$\sum_i |S^i| = \sum_{P \in \mathcal{P}} \sum_i |V(P) \cap S^i| \geq \sum_{P \in \mathcal{P}} \min(|V(P)|, k) = |\mathcal{P}|_k$$

.

Definition 1.5. Let $\mathcal{P}$ be a path partition. We denote by $\mathcal{P}^{\leq k}$ the set of paths in $\mathcal{P}$ with cardinality at most k. Similarly we denote by $\mathcal{P}^{\geq k}$ the set of paths in $\mathcal{P}$ with cardinality at least k.

Conjecture 1.6 (Berge's path partition conjecture)**.** *Let D be a digraph and k a positive integer. Then for every k-optimal path partition $\mathcal{P}$ there are k disjoint stable sets orthogonal to $\mathcal{P}$.*

Finding a k-optimal path partition in general digraphs is NP-complete as $\pi_k(D) = k$ for any $k < n$ if and only if there is a Hamiltonian path in D. However, if we also allow cycles in our partition and thus consider path-cycle partitions, then finding a k-optimal path-cycle partition and k disjoint stable sets orthogonal to its paths can be done in polynomial time. In [8] E. Berger and I.B-A. Hartman gave a common proof for the $k = 1, \lambda - 1, \lambda$ cases by searching k-optimal path-cycle partitions in subdigraphs where a k-optimal path-cycle partition will contain no cycle and thus will be a path partition.

Our approach is similar in that aspect but follows a different path. We prove a min-max theorem between the k-optimal path-cycle partitions and a variation of stability number. Then we use it on a suitable maximal acyclic subdigraph to prove that for a k-optimal path partition with each of its paths either not longer than $k+1$ vertices or not shorter than $\lambda - 1$ vertices, there are k disjoint stable sets orthogonal to it. As a special case of this result we will get that if $k \geq \lambda - 3$, then for any path partition we can either find k disjoint stable sets orthogonal to it or find a better path partition.

2 A min-max theorem

As stated above finding a k-optimal path partition is NP-hard. So instead we use a similar but easier to handle structure: the k-optimal path-cycle partitions.

On the other hand in remark 1.4 we have seen that if $S^1, S^2, \ldots, S^k$ are k disjoint stable sets orthogonal to a path partition $\mathcal{P}$ then $\sum |S^i| \geq |\mathcal{P}|_k$. Finding k disjoint stable sets such that the sum of their cardinality is at least m (where m is an input) is also NP-hard. So again we choose to replace it with a notion similar enough to be of use for us but easier to handle.

The advantage of this approach is that we can prove a min-max theorem for this easier to handle structures. In the next section we use this min-max theorem to prove our result for the original hard to handle structures.

Definition 2.1. The k-norm of a path-cycle partition $\mathcal{P}^c = \{P_1, \ldots, P_r, C_1, \ldots, C_t\}$ where P_i are paths and C_i are cycles is the following:

$$|\mathcal{P}^c|_k = \sum_{i=1}^{r} \min(|V(P_i)|, k).$$

A path-cycle partition is k-optimal if its k-norm is minimal.

Definition 2.2. Let D be a digraph. Then $\pi_k^c(D) = |\mathcal{P}^c|_k$ where $\mathcal{P}^c$ is a k-optimal path-cycle partition of D.

Let S be a stable set in the digraph $D = (V, A)$. We call a vertex set $S_{cut} \subseteq V$ an S-cut set (or just cut-set if S is unambiguous) if $S \cap S_{cut} = \emptyset$ and all directed paths from S to S contain a vertex from S_{cut} as an internal point – we say S_{cut} *cuts* every S to S dipath. The pair (S, S_{cut}) is called *stable-cut pair*. For a pair of stable set S and one of its cut-sets S_{cut}, we use the notation $\langle S, S_{cut} \rangle = |S| - |S_{cut}|$.

Definition 2.3. An (S, S_{cut}) stable-cut pair is *optimal* if $\langle S, S_{cut} \rangle$ is the largest possible.

Definition 2.4. Two stable-cut pairs (S^1, S^1_{cut}) and (S^2, S^2_{cut}) are *disjoint* if S^1 and S^2 are disjoint.

Definition 2.5. An $\mathcal{S} = \{(S^1, S^1_{cut}), \ldots, (S^k, S^k_{cut})\}$ is a *stable-cut k-family* if each (S^i, S^i_{cut}) is a stable-cut pair, and they are pairwise disjoint. $\mathcal{S}$ is *optimal* if $\sum \langle S^i, S^i_{cut} \rangle$ is the largest possible.

Definition 2.6. For an optimal stable-cut k-family $\mathcal{S} = \{(S^1, S^1_{cut}), \ldots, (S^k, S^k_{cut})\}$, we use the notation $\alpha_k^* = \sum \langle S^i, S^i_{cut} \rangle$.

Our main goal in this section is to prove that $\pi_k^c = \alpha_k^*$. For this we use the following lemma:

Lemma 2.7. *For every digraph $D = (V, A)$, $\pi_1^c = \alpha_1^*$.*

Proof. First we prove that $\pi_1^c \geq \alpha_1^*$. Let $\mathcal{P}^c$ be a path-cycle partition and (S, S_{cut}) be a stable-cut pair. Then for each path P in $\mathcal{P}^c$ we know that $|V(P) \cap S| \leq |V(P) \cap S_{cut}| + 1$. Indeed, let $V(P) \cap S = \{v_1, v_2, \ldots, v_t\}$ and suppose that they appear along the path P in this order. Let $P[v_i, v_{i+1}]$ denote the subpath of P between the vertices v_i, v_{i+1}. Then $P_i = P[v_i, v_{i+1}]$ is a path from S to S for all i, thus each P_i has to contain a vertex $w_i \in S_{cut}$. Since the paths P_i are internally disjoint, all there endvertices are from S and $S \cap S_{cut} = \emptyset$, combining all these together we get that we have $t-1$ different vertices w_i which proves our claim. Similarly we can prove that for each cycle C in $\mathcal{P}^c$ we have $|V(C) \cap S| \leq |V(C) \cap S_{cut}|$. Then we have

$$\langle S, S_{cut} \rangle = |S| - |S_{cut}| = \sum_{\texttt{path } P \in \mathcal{P}^c} |V(P) \cap S| - |V(P) \cap S_{cut}| \; +$$

$$+ \sum_{\texttt{cycle } C \in \mathcal{P}^c} |V(C) \cap S| - |V(C) \cap S_{cut}| \leq \sum_{\texttt{path } P \in \mathcal{P}^c} 1 \; + \sum_{\texttt{cycle } C \in \mathcal{P}^c} 0 =$$

$$= |\mathcal{P}^c|_1.$$

To see the other direction we construct a path-cycle partition $\mathcal{P}^c$ and a stable-cut pair (S, S_{cut}) such that $|\mathcal{P}^c|_1 = \langle S, S_{cut} \rangle$.

For the digraph $D = (V, A)$ we make a bipartite graph $G = (S \cup T, E)$ in the following way:

$$S = \{v' \;:\; v \in V\}, \qquad T = \{v'' \;:\; v \in V\}, \qquad E = \{(u', v'') \;:\; (u, v) \in A\}$$

A matching M in G corresponds to a path-cycle partition of D with $|V| - |M|$ paths and vice versa. From Kőnig's theorem we know that in a bipartite graph the size of a maximum matching equals the size of a minimum vertex cover. Let Z be a min vertex cover in G and $Z_1 = \{v \in V \;:\; v' \in Z\}$, $Z_2 = \{v \in V \;:\; v'' \in Z\}$. Now all arcs in D have either their tails in Z_1 or their heads in Z_2, so $S = V \setminus (Z_1 \cup Z_2)$ is a stable set. Let P be a path from S to S. Let a be the first arc along P whose tail is in Z_1. This arc exists because P ends in a vertex of S. Moreover this arc is not the first arc of P. Hence, by definition the arc preceding a in P has its head in Z_2, and so the tail of a is in $Z_1 \cap Z_2$. So $S_{cut} = Z_1 \cap Z_2$ is an S-cut. For the pair of (S, S_{cut}) we have $\langle S, S_{cut} \rangle = |V| - |Z_1 \cup Z_2| - |Z_1 \cap Z_2| = |V| - |Z_1| - |Z_2| = |V| - |Z| = |V| - |M|$, where M is a maximum matching. Thus $\alpha_1^* \geq \langle S, S_{cut} \rangle = \pi_1^c$. □

Remark 2.8. If the digraph D is acyclic then any 1-optimal path-cycle partition is a path partition. If the digraph is transitive then for any optimal stable-cut pair (S, S_{cut}) the S_{cut} has to be empty. Thus if the digraph is acyclic and transitive (so it is the digraph of a partially ordered set) then we have Dilworth's theorem. Thus Lemma 2.7 is a generalization of Dilworth's theorem. In the same way, the min-max theorem $\pi_k^c = \alpha_k^*$ generalizes the Greene-Kleitman theorem.

Theorem 2.9. *For every digraph $D = (V, A)$, $\pi_k^c = \alpha_k^*$*

Proof. We use the same idea M. Saks used to prove the Greene-Kleitman theorem in [9]. To a digraph D and a positive integer k, we associate the digraph D^k defined by

$$V_k = \{(v,i) \ : \ v \in V, \ \ 1 \leq i \leq k\}$$

$$A_k = \{\big((u,i),(v,i)\big) \ : \ (u,v) \in A\} \cup \{\big((u,i),(u,j)\big) \ : \ i < j\}.$$

For a vertex subset X in D^k we denote by $X\big|_i$ the vertex set $X \cap \{(v,i) : v \in V\}$.

We prove that $\alpha^*_k(D) = \alpha^*_1(D^k) = \pi^c_1(D^k) = \pi^c_k(D)$.

1. $\alpha^*_k(D) \geq \alpha^*_1(D^k)$: If S is stable in D^k and S_{cut} is a cut-set of S, then $(S\big|_i, S_{cut}\big|_i)$ for $1 \leq i \leq k$ are k pairwise disjoint stable sets with their cut-set. Thus if (S, S_{cut}) is an optimal stable-cut pair of D^k then we have $\alpha^*_k(D) \geq \sum\langle S\big|_i, S_{cut}\big|_i\rangle = \langle S, S_{cut}\rangle = \alpha^*_1(D^k)$.

2. $\alpha^*_1(D^k) = \pi^c_1(D^k)$: This was proved in Lemma 2.7

3. $\pi^c_1(D^k) \geq \pi^c_k(D)$: Let $\mathcal{P}^c$ be a path-cycle partition of D^k. Then $\mathcal{P}'^c = \mathcal{P}^c\big|_k$ is a path-cycle partition of D. We prove that by choosing a suitable 1-optimal $\mathcal{P}^c$ of D^k we get $|\mathcal{P}^c|_1 \geq |\mathcal{P}'^c|_k$. We use the following notations:

$$\mathrm{ini}(\mathcal{P}^c) = \{\text{the first vertex from each path in } \mathcal{P}^c\}.$$
$$\mathrm{ter}(\mathcal{P}^c) = \{\text{the last vertex from each path in } \mathcal{P}^c\}.$$

Let $\mathrm{ter_i}(\mathcal{P}^c) = \{\mathrm{v} \in \mathrm{V} \ : \ (\mathrm{v},\mathrm{i}) \in \mathrm{ter}(\mathcal{P}^c)\}$. Given a path-cycle partition $\mathcal{P}^c$ we say that the vertex v *covers* u if the arc (u,v) is an arc used in $\mathcal{P}^c$.

We call $\mathcal{P}^c$ *suitable* if for every vertex $(u,k) \in ter_k(\mathcal{P}^c)$ we have either

(a) all the vertices $(u,i) \in ter_i(\mathcal{P}^c)$ for $i = 1, \ldots, k$, or

(b) the vertex (u,k) covers the vertex (u,i) for some i. Let $ter'_k(\mathcal{P}^c)$ denote these vertices and $ter^*_k(\mathcal{P}^c)$ denote the rest in $ter_k(\mathcal{P}^c)$.

Suppose that $\mathcal{P}^c$ is suitable. Then $|\mathcal{P}^c|_1$ = number of paths $= |\mathrm{ter}(\mathcal{P}^c)| = \sum|\mathrm{ter_i}(\mathcal{P}^c)| \geq \mathrm{k} \cdot |\mathrm{ter^*_k}(\mathcal{P}^c)| + |\mathrm{ter'_k}(\mathcal{P})| \geq |\mathcal{P}'^c|_\mathrm{k}$.

So it is enough to prove that there is a 1-optimal suitable path-cycle partition of D^k. Let $\mathcal{P}^c$ be any 1-optimal path-cycle partition. If $(u,i) \in \mathrm{ter}(\mathcal{P}^c)$ and $\big((u,i)(v,j)\big) \in A_k$, then the following operation is called a *swap*: delete the arc in $\mathcal{P}^c$ which enters (v,j) and add the arc $\big((u,i)(v,j)\big)$. A swap does not increase the number of paths so the new path partition remains 1-optimal. Thus if we can find a suitable path-cycle partition by using swaps starting from $\mathcal{P}^c$ then we are done.

If there is a vertex $(u, i+1) \in \mathrm{ter}(\mathcal{P}^c)$ for which the vertex (u,i) is covered by a vertex (v,j), then there are two cases:

- the vertex $(u,k) \in ter_k$ and it does not cover any vertex (u,i), $i = 1,\dots,k-1$,
- otherwise.

In the first case we can use a swap to have the vertex $(v, j+1)$ cover the vertex $(u, i+1)$. Indeed, the only case where we could not use such a swap is when $j = k$ but since $i < k$ $(i+1 \le k)$ this means that for $\big((u,i),(v,k)\big)$ to be an arc in D^k we need $u = v$, thus the vertex (u,k) covers the vertex (u,i) which we supposed not to be the case.

In the second case we do not need to do anything as this case was allowed in the definition of suitable path-cycle partition.

Let

$$x_m = \big|\{(u,m) \,:\, (u,m) \text{ is covered by } (v,n) \text{ and } (u,m{+}1) \text{ is covered by } (v,n{+}1)\}\big|.$$

Lets examine what happens to x_m, $m = 1,\dots,k-1$ when we use a swap. Suppose that we have a vertex $(u, i+1) \in \text{ter}(\mathcal{P}^c)$ for which the vertex (u,i) is covered by a vertex (v,j). Using a swap on such a vertex $(u,i+1)$ increases x_i and does not decrease all x_m where $m < i$. So by applying a swap we increase $(x_1,\dots,x_{k-1})$ according to the lexicographical order and thus after a finite number of steps we stop. This means that the first case does not occur anymore in our path-cycle partition and thus it is suitable.

4. $\pi_k^c(D) \ge \alpha_k^*(D)$: Let $\mathcal{P}^c$ be a path-cycle partition of D and $\big((S^1,S^1_{cut}),\dots,(S^k,S^k_{cut})\big)$ a stable-cut k-family. Let $P \in \mathcal{P}^c$ be a path. Then for each i we have $|V(P)\cap S^i| - 1 \le |V(P)\cap S^i_{cut}|$ otherwise there would be an uncut path from S^i to S^i. Similarly, if $C \in \mathcal{P}^c$ is a cycle, then $|V(C)\cap S^i| \le |V(C)\cap S^i_{cut}|$. So we get

$$\begin{aligned}|\mathcal{P}^c|_k &= \sum_{P\in\mathcal{P}^c \text{ is a path}} \min(|V(P)|,k) + \sum_{C\in\mathcal{P}^c \text{ is a cycle}} 0 \\ &\ge \sum_{P\in\mathcal{P}^c}\sum_i \Big(|V(P)\cap S^i| - |V(P)\cap S^i_{cut}|\Big) = \sum_i |S^i| - |S^i_{cut}|.\end{aligned}$$

Thus we have $\pi_k^c(D) \ge \alpha_k^*(D) \ge \alpha_1^*(D^k) = \pi_1^c(D^k) \ge \pi_k^c(D)$. □

Remark 2.10. From the proof we also got $\alpha_k^*(D) = \alpha_1^*(D^k)$. Furthermore, if (S, S_{cut}) is an optimal stable-cut pair in D^k then $\mathcal{S} = \big((S^1,S^1_{cut}),\dots,(S^k,S^k_{cut})\big)$, where $S^i = S\big|_i$, $S^i_{cut} = S_{cut}\big|_i$, is an optimal stable-cut k-family in D. We will use this observation later on.

In the proof above we can observe that if $\mathcal{P}^c$ is a k-optimal path-cycle partition and $\mathcal{S} = \{(S^1,S^1_{cut}),\dots,(S^k,S^k_{cut})\}$ is an optimal stable-cut k-family, then their intersection must satisfy some properties.

Definition 2.11. Let $\mathcal{P}^c$ be a path-cycle partition and $\mathcal{S} = \{(S^1, S^1_{cut}), \dots, (S^k, S^k_{cut})\}$ be a stable-cut k-family. We call them *orthogonal* if:

1. Each path P in $\mathcal{P}^c$ intersects exactly $\min(|V(P)|, k)$ of the $S^1, \dots, S^k$.

2. For each path P in $\mathcal{P}^c$ and i we have $|V(P) \cap S^i_{cut}| = \max(|V(P) \cap S^i| - 1, 0)$ and moreover vertices of S^i and S^i_{cut} alternate along P, beginning and ending with a vertex of S^i.

3. For each cycle C in $\mathcal{P}^c$ and i we have $|V(C) \cap S^i| = |V(C) \cap S^i_{cut}|$.

Corollary 2.12. *Let $\mathcal{P}^c$ be a path-cycle partition and $\mathcal{S} = \big\{(S^1, S^1_{cut}), \dots, (S^k, S^k_{cut})\big\}$ a stable-cut k-family. Then $\mathcal{P}^c$ and $\mathcal{S}$ are respectively k-optimal and optimal if and only if they are orthogonal.*

3 Application to Berge's conjecture

If D is acyclic, then we are done since any optimal stable-cut k-family is orthogonal to every k-optimal path partition and so if we take just the stable sets without the cut-sets, those will be orthogonal to every k-optimal path partition.

Our strategy to tackle Berge's path partition conjecture for general digraphs will take this logic a bit further: for a given k-optimal path partition of D we will build a suitable maximal acyclic subdigraph D' of D, then find a suitable optimal stable-cut k-family of D', and use it to construct k disjoint stable sets orthogonal to $\mathcal{P}$.

As it can be guessed from this rough sketch, we will use the notion of maximal acyclic subdigraph more than we would want to write down the whole expression so we will use an abbreviation:

Abbreviation We call a maximal acyclic subdigraph MASD for simplicity.

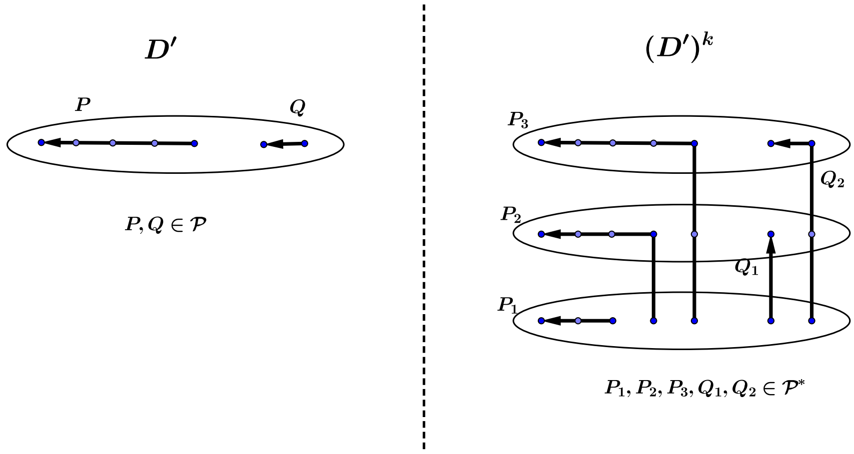
First let us take a look at what will be a suitable stable-cut k-family. We will prove that for an acyclic digraph we can assume that the vertices of the cut-sets are only on paths with at least $k + 2$ vertices.

Lemma 3.1. *Let D' be an acyclic digraph. Then there is an optimal stable-cut k-family with the following property: for any k-optimal path partition $\mathcal{P}$*

$$\bigcup_i S^i_{cut} \subset \bigcup_{P \in \mathcal{P}^{\geq k+2}} V(P).$$

Proof. Let $\mathcal{P}$ be a k-optimal path partition in D'. We make $(D')^k$ just like in the proof of Theorem 2.9 but now we generate a 1-optimal path partition in $(D')^k$ from the k-optimal path partition in D'.

Let $P = (v_1, v_2, \dots, v_t)$ be a path from $\mathcal{P}$. Then we have a path partition of the vertex subset $\{(v_j, i) \ : \ v_j \in V(P), i = 1, \dots, k\} \subseteq V_k$ with $\min(t, k)$ paths (see the figure below):



More formally:

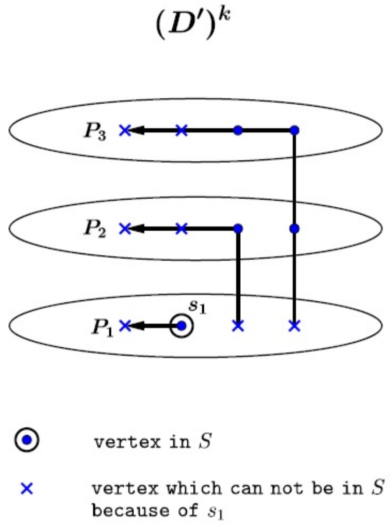
- if $t < k$ then $P_i = \big((v_{t+1-i}, 1), \dots, (v_{t+1-i}, k-t+i), (v_{t+2-i}, k-t+i), \dots, (v_k, k-t+i)\big)$
- if $t \geq k$ then $P_i\big((v_{t+1-k}, 1), \dots, (v_{t+1-k}, i), (v_{t+2-k}, i), \dots, (v_k, i)\big)$

Let $\mathcal{P}^*$ denote this path partition of $(D')^k$. Because $\pi_1((D')^k) = \pi_k(D')$ and $|\mathcal{P}|_k = |\mathcal{P}^*|_1$, $\mathcal{P}^*$ is 1-optimal in $(D')^k$. Now let (S, S_{cut}) be an optimal stable-cut pair in $(D')^k$. In the proof of Theorem 2.9 we have seen that if (S, S_{cut}) is an optimal stable-cut pair in $(D')^k$ then
$\mathcal{S} = \big((S^1, S^1_{cut}), \dots, (S^k, S^k_{cut})\big)$, where $S^i = S\big|_i$, $S^i_{cut} = S_{cut}\big|_i$ is an optimal stable-cut k-family in D'. Thus (S, S_{cut}) is orthogonal to $\mathcal{P}^*$ (in $(D')^k$) and $\mathcal{S}$ is orthogonal to $\mathcal{P}$ (in D'). We use these two orthogonalities to prove that in case of $|V(P)| \leq k+1$ we have $V(P) \cap S^i_{cut} = \emptyset$, for all $i = 1, \dots, k$.

Let $P \in \mathcal{P}^{\leq k+1}$ and $P_1, \dots, P_{\min(k, |V(P)|)}$ be the paths in $(D')^k$ generated from P. If $|V(P)| \leq k$, then because $\mathcal{S}$ and $\mathcal{P}$ are orthogonal we know that $|V(P) \cap S^i| \leq 1$ for any i so $|V(P) \cap S^i_{cut}| = 0$ for all i. Thus we can assume that $|V(P)| = k+1$. Then P_1 is a two-vertices long path in $(D')^k\big|_1$ and so $V(P_1) \cap S = \{s_1\} = (u, 1)$ and $V(P_1) \cap S_{cut} = \emptyset$ otherwise there would also be at least two vertices from S and at least one from S_{cut} on P_1 and so that would be at least three vertices in total.

Because of the orthogonality between (S, S_{cut}) and $\mathcal{P}^*$ we know that the vertices from S and S_{cut} alternate along the paths of $\mathcal{P}^*$, starting with a vertex from S and also ending with a vertex from S. Specially, $ini(P_i) \notin S_{cut}$ for $i = 2, \dots, \min(|V(P)|, k)$. We have also seen that $V(P_1) \cap S_{cut} = \emptyset$ so this means that $S^1_{cut} = \emptyset$. Since $\mathcal{S}$ and $\mathcal{P}$ are orthogonal we have $|V(P) \cap S^1| = |V(P) \cap S^1_{cut}| + 1 = 1$. Thus $S^1 = \{u\}$, where $s_1 = (u, 1)$.

Also, by going along P_1 then going up using the arc between $(v, 1)$ and (v, i), we have an uncut path from $s_1 = (u, 1)$ to each vertex (v, i), where v is on the subpath of P from u to $ter(P)$. Thus, those vertices can not be from S. See the figure below which vertices can not be in S because of s_1:



Now we can continue in the same way. Because of the orthogonality between $\mathcal{S}$ and $\mathcal{P}$ we know that the vertices from S^i and S^i_{cut}, $i = 2, \dots, k$ are alternating along the path P with a vertex from S being the first and last one. This means that those vertices (v, i) where v is on the subpath of P from u to $ter(P)$ are also not in S_{cut}. So we can disregard those vertices. This leaves at most two vertices in P_2 which can be in S or S_{cut} so by continuing with the same logic we can prove for $i = 2, \dots, \min(|V(P)|, k)$ that the paths P_i contain no vertices from S_{cut}. □

Corollary 3.2. *Let D be a digraph. If there is a k-optimal path partition $\mathcal{P}$ of D with no path longer than $k+1$ vertices, then there exist k disjoint stable sets orthogonal to $\mathcal{P}$.*

Proof. Let D' be any MASD of D containing $\mathcal{P}$. Since the length of any path is at most $k+1$, we know from Lemma 3.1 that there is an optimal stable-cut k-family with empty cut-sets. We claim that $S^1, \dots, S^k$ will be stable sets even in D. Indeed, suppose that is not the case, there is a (u, v) arc between two vertices from the same S^i. Then the arc (u, v) was not in D' and since D' is a MASD this is only possible if there is a path from v to u in D'. But then there should be a vertex from S^i_{cut} along the path which is a contradiction. □

We have seen so far that for short paths with length no more than $k+1$ vertices it does not matter which MASD we use as long as it contains $\mathcal{P}$. We will use this to choose a suitable MASD to deal with long paths at the same time. The following theorem is our main result:

Theorem 3.3. *Let $\mathcal{P}$ be a k-optimal path partition such that $\mathcal{P} = \mathcal{P}^{\leq k+1} \cup \mathcal{P}^{\geq \lambda-1}$ (so there is no $P \in \mathcal{P}$ with $k+2 \leq |V(P)| \leq \lambda - 2$). Then there are k disjoint stable sets orthogonal to $\mathcal{P}$.*

Proof. We will construct a MASD D' from the path partition $\mathcal{P}$. For this we split the vertex set V into two classes:

$$V_1 = \bigcup_{P \in \mathcal{P}^{\leq k+1}} V(P) \qquad\qquad V_2 = \bigcup_{P \in \mathcal{P}^{\geq \lambda-1}} V(P)$$

We start with no arcs in D' and will greedily build up D' into a MASD through the following steps:

1. First we add all the arcs of $\mathcal{P}$ to D'.

2. We add as many arcs induced by V_1 to D' as possible without making a cycle.

3. We add all arcs leaving V_1 to D'

4. We complete the subdigraph into a MASD.

Since $\mathcal{P}$ is k-optimal in D, it will be also k-optimal in D'. From Lemma 3.1 we know that there is an $\mathcal{S} = \big((S^1, S^1_{cut}), \dots, (S^k, S^k_{cut})\big)$ optimal stable-cut k-family such that $\bigcup S^i_{cut} \subset V_2$. Now we define

$$S^i_{first} = \bigcup_{P \in \mathcal{P}} (\text{ the first vertex along the path } P \text{ that is in } S^i).$$

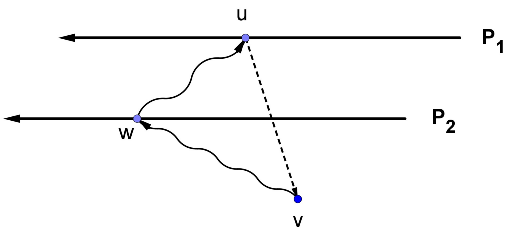
We will prove that S^i_{first} is a stable set in D for all i. Then $S^1_{first}, \dots, S^k_{first}$ will be k disjoint stable sets orthogonal to $\mathcal{P}$.

Let $u, v \in S^i_{first}$. Suppose that (u, v) is an arc in D (and not an arc in D' otherwise u and v could not be in the same S^i).

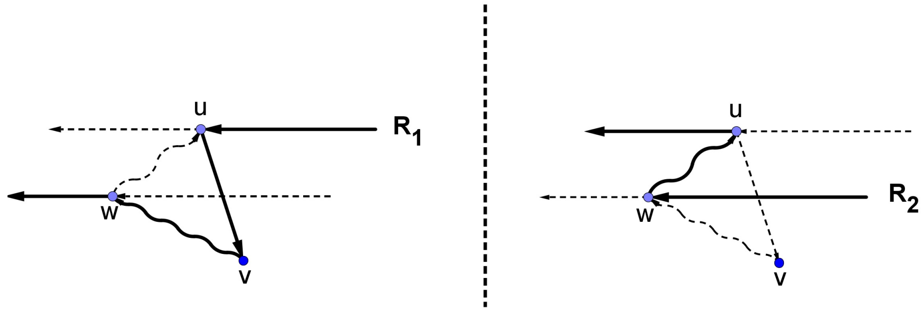
1. If $u, v \in V_1$, then the only reason why we did not add (u, v) to D' is that there was already a path from v to u inside V_1 (see the second step of building D'), but because $S^i_{cut} \cap V_1 = \emptyset$ it would mean an uncut S^i to S^i path which is a contradiction.

2. If $u \in V_1$ and $v \in V_2$, then the arc (u, v) is in D' (see the third step of building D') so u and v can not be in the same S^i.

3. If $u \in V_2$, then there must be a path Q from v to u in D' with a $w \in S^i_{cut}$ as an internal vertex. Both u and w are in V_2 (the latter because $S^i_{cut} \subset V_2$) so they are respectively on $P_1, P_2 \in \mathcal{P}^{\geq \lambda-1}$. Moreover, $P_1 \neq P_2$. Indeed, since $\mathcal{S}$ and $\mathcal{P}$ are orthogonal, the vertices of S^i and S^i_{cut} along P, beginning with a vertex of S^i. Because of this, if $P_1 = P_2$ then w would be after u along P_1. But that would mean that $P_1 \cup Q$ contains a cycle, which is impossible as D' is acyclic). Thus $P_1 \neq P_2$. The vertex v can possibly be on P_2, but not on P_1 as we took only one element of S^i from all paths in $\mathcal{P}$. On the figure below we choose not to place v on P_2 but that is of no importance in the proof and v could be on P_2 somewhere before w (but not after w as it would mean a cycle in D').



Now if we take a look at the figure we can see that from the arc (u, v) and from the parts of P_1, P_2 and Q we can glue together two new paths: R_1 and R_2.



Thus $2\lambda \geq |V(R_1)| + |V(R_2)| \geq |V(P_1)| + |V(P_2)| + 3$. But $P_1, P_2 \in \mathcal{P}^{\geq \lambda - 1}$, so $|V(P_1)| + |V(P_2)| + 3 \geq 2\lambda + 1$, which is a contradiction.

□

In the above proof we used the k-optimality of $\mathcal{P}$ only to ensure that $\mathcal{P}$ will be k-optimal in D'. If $k \geq \lambda - 3$, then any path partition of D will satisfy the conditions of Theorem 3.3, so the following corollaries hold:

Corollary 3.4. *Given a digraph D, an integer $k \geq \lambda - 3$, and a path partition $\mathcal{P}$ of D, there are either k disjoint stable sets orthogonal to $\mathcal{P}$ or a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}|_k < |\mathcal{P}|_k$.*

Corollary 3.5. *Berge's path partition conjecture holds for $k \geq \lambda - 3$.*

4 Acknowledgement

Many thanks are due to Tamás Király for a lot of helpful advices and corrections. I thank the two anonymous reviewers whose suggestions helped to improve and clarify this manuscript.

The work was supported by the National Development Agency of Hungary (grant no. K 109240), based on a source from the Research and Technology Innovation Fund.

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