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A note on bounded weighted graphic metric TSP

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Abstract

For metric TSP, Christofides' heuristic is still the best known approximation algorithm with its 3/2 guarantee. The set of graphic metrics and the set of (1,2)-metrics are some of the few classes known to admit a better-than-3/2 approximation algorithm. In this paper we investigate TSP for so-called β-bounded metrics and determine that, for any $\beta \geq 1$, the randomized approach of Oveis Gharan, Saberi and Singh [5] achieves a better-than-3/2 guarantee for β-bounded metrics. A metric space is called a β-bounded weighted graphic metric (β-bounded metric, for short) if it is realized by shortest path distances in a graph with edge-lengths between 1 and β. This result broadens the class of metric spaces with a better-than-3/2 approximation guarantee.

1 Introduction

In this note we consider the Traveling Salesman Problem for the special case when the metric space is β-bounded, and prove that for a fixed $\beta \geq 1$ there is a better-than-3/2-approximation algorithm for TSP in a β-bounded metric space.

Definition 1.1 (β-bounded weighted graphic metric, or β-bounded metric, for short)**.** Consider a ground set V, and a metric c on V. Let $\beta \geq 1$. Then c is called β-bounded if there is a connected graph $G = (V, E)$ with real-valued edge lengths $l : E \to [1, \beta]$ such that $c(u, v)$ is equal to the length of a shortest path between u and v with respect to l.

We remark that for a given metric space and a given $\beta \geq 1$ it is quite easy to check whether the metric space is β-bounded or not. If it is β-bounded, then a representation is given by the graph of edges obtained from pairs with distance between 1 and β. Actually, in a similar way we may also check whether a given metric can be scaled with a positive factor to obtain a β-bounded metric, that is, whether the metric can be represented by distances in a graph with edge length between some positive λ and $\beta\lambda$. Of course all approximation results in this note also apply for scaled β-bounded

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metric spaces. (Note that every metric space is equal to a scaled β-bounded metric space, when β is chosen large enough.)

The motivation is that for metric TSP, the best known approximation guarantee is 3/2 due to Christofides' heuristic [2], which has only been bettered for certain special classes of metrics. For graphic TSP – where the metric space is assumed to be the distance in an undirected graph – the first better-than-3/2-approximation is due to Mömke and Svensson [3] who proved a 1.461-approximation. Independently, and about at the same time Oveis Gharan, Saberi and Singh [5] proved a $3/2 - 10^{-50}$ approximation guarantee for graphic TSP, followed by Sebő, Vygen [6] with their 7/5-approximation. Mucha [4] improved the bound for Mömke and Svensson's approach to $13/9 \approx 1.444$. For $\{1, 2\}$-TSP – where every distance is 1 or 2 – the best known result is an 8/7-approximation, see Berman, Karpinski, [1]. For a great survey of recent developments on TSP, see Vygen [7].

The family of β-bounded metrics is a natural generalization of both the class of graphic metrics and $\{1, 2\}$-metrics. Also note that every metric space with distances between 1 and β is β-bounded.

A natural question to ask is whether $[1, 2]$-TSP – where every distance is a real number between 1 and 2 – also has a better than 3/2 ratio? For this question, the first idea would be to distort the metric space to obtain a $\{1, 2\}$-metric, and then apply the best known approximation for $\{1, 2\}$-TSP. A distortion of at most $\sqrt{2}$ is possible by rounding up distances to $\sqrt{2}$ or $2\sqrt{2}$, leading to an approximation guarantee of $\frac{8}{7}\sqrt{2} \approx 1.61$, thus this approach fails to improve on the 3/2 bound of Christofides. The question still remains, even for $[1, 2]$-TSP, whether there is a better-than-3/2 approximation or not? This question is answered by this paper, and actually, β-bounded TSP is shown to admit a better-than-3/2-approximation for any fixed α.

This paper heavily relies on results of Oveis Gharan, Saberi and Singh [5], who proved a $3/2 - 10^{-50}$ approximation guarantee for graphic TSP. We will use an algorithm that is essentially the same as the algorithm of Oveis Gharan, Saberi and Singh [5], and show that a similar analysis based on their Structure Theorem implies a ratio better than 3/2 for any fixed β. We remark that as β increases, the proved ratio tends to 3/2, and thus we only claim a better-than-3/2 ratio for any *fixed* β.

The main result in this note is that for any fixed β, the TSP for β-bounded metric spaces has a better-than-3/2-approximation algorithm. This applies to special cases as follows: $[1, 2]$-TSP has a better-than-3/2-approximation algorithm, or $[1, \beta]$-TSP for that matter. Here $[1, \beta]$-TSP means that the input metric space is assumed to have only distances between 1 and β.

2 Held-Karp LP and the Structure Theorem

Let $G = (V, E)$ denote a simple undirected graph. We consider the following LP known as the Held-Karp relaxation of the Traveling Salesman Problem, which has the property that its integer solutions are exactly the characteristic vectors of the

Hamiltonian cycles of G.

$$\begin{array}{lrl} \min & c(x) & \\ \text{for} & x(\delta(U)) \geq 2 & \text{if } \emptyset \neq U \subsetneq V \\ & x(\delta(v)) = 2 & \text{if } v \in V \\ & x(e) \geq 0 & \text{if } e \in E \end{array} \tag{1}$$

As every Hamiltonian cycle is a feasible solution of this LP, the LP optimum is a lower bound for TSP. For a given feasible solution x of the Held-Karp LP, the notion of a near-minimum cut and an odd cut is defined as follows – see [5] for further details.

Definition 2.1 ([5])**.** Let x be a feasible solution of (1), let δ be a (small) positive real number, and let $U \subseteq V$. The cut $(U, \overline{U})$ is called a $(1+\delta)$-*near-minimum cut* if $\sum_{e \in (U,\overline{U})} x(e) \leq 2\,(1+\delta)$. The cut $(U, \overline{U})$ is called an *odd cut* with respect to a spanning tree $F \subseteq E$, if $|F \cap \delta(U)|$ is odd. A cut is called an odd $(1+\delta)$-near-minimum cut if both of these properties hold.

Now let x be a feasible vector with respect to the Held-Karp LP. Note that $(1-\frac{1}{n})x$ is in the convex hull of spanning trees, and thus, $(1-\frac{1}{n})x$ may be decomposed as a convex combination of spanning trees. Also note that a convex combination may be regarded as a probabilistic distribution, and thus, there is a distribution μ over the set of spanning trees such that by taking a sample F (a spanning tree) of μ the following equation holds

$$\mathbb{P}(e \in F) = \left(1 - \frac{1}{n}\right) x_e \qquad \text{for all edges } e \in E. \tag{2}$$

This equation is just a simple reformulation of the property that $(1-\frac{1}{n})x$ is represented by a convex combination of spanning trees. The tools of probability theory provide useful insight based on this randomized model of sampling for such a distribution μ with a constraint on its marginals, and Oveis Gharan, Saberi and Singh [5] used the model of so-called maximum entropy distributions to obtain the Structure Theorem below.

Now consider a distribution μ over the set of spanning trees – at this point μ may be just an arbitrary distribution over the set of spanning trees. We introduce the following definition of a good edge, also see [5] for further details.

Definition 2.2. $e \in E$ is called a *good edge* for parameters δ, ρ if the probability that e is not contained in an odd $(1+\delta)$-near-minimum cut is at least ρ, that is,

$$\mathbb{P}(\nexists U \subseteq V \text{ such that } e \in \delta(U) \text{ and } U \text{ is odd and } (1+\delta)\text{-near-minimum}) \geq \rho.$$

The notion of good edges is motivated by the following result claiming that for every feasible solution x of the Held-Karp LP, either there is a large number of edges e with x_e close to one, or there is a large number of good edges. Here "large" means a bound determined by certain positive constants, and the theorem also claims that those constants may be chosen arbitrarily small.

Theorem 2.3 (Structure Theorem – Oveis Gharan, Saberi, Singh, [5]). *Let $\varepsilon_0 > 0$. Then there are constants $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \rho, \gamma, \delta < \varepsilon_0$ so that the following assertion holds. There is a randomized polynomial time algorithm with the input of a feasible solution x of the Held-Karp relaxation that generates samples from a distribution μ over the set of all spanning trees so that the following condition holds:*

$$\mathbb{P}(e \in F) \leq (1 + \varepsilon_3)\left(1 - \frac{1}{n}\right) x_e \qquad \text{for all edges } e \in E, \tag{3}$$

and so that for this distribution μ at least one of the following two cases holds:

1. **(Case 1 – "abundance of good edges")** *$x(E_{good}) \geq \varepsilon_1 n$, where E_{good} denotes the set of good edges for parameters ρ, δ.*

2. **(Case 2 – "x is nearly integral")** *There are at least $(1 - \varepsilon_2)n$ edges with x value greater than $1 - \gamma$.*

Oveis Gharan, Saberi, Singh, [5] proved this Structure Theorem by using properties of a so-called maximum entropy distribution, with the side constraint of (2). Inequality (3) comes from the fact that a polynomial-time algorithm exists to generate samples approximately, that is samples from from another distribution μ that is a good approximate of the maximum entropy distribution. Inequality (3) describes the kind of approximation. Here, instead of relying on the deep understanding of the maximum entropy distribution, we rather put this property of the approximate maximum entropy distribution μ into the Structure Theorem itself, and thus avoid the discussion of maximum entropy.

In the Structure Theorem, Case 1 vaguely says that the set of good edges retains a positive fraction of the x-weight of all edges, or as [5] formulates, "there is an abundance of good edges". Case 2 says that nearly all edges are nearly integral, which we may also put as x is "nearly integral".

3 The randomized algorithm

The sketch of the algorithm is the following. For a given fixed constant $0 < \beta$, we choose constants $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \rho, \gamma, \delta$ to satisfy needs of Theorem 2.3, and make sure that $\frac{4}{3(1-\gamma)} + \beta(2\varepsilon_2 + 2\gamma) < 3/2$ holds. With the left hand side we can get arbitrarily close to $4/3$, so we only need the constants to be small enough. This is to make sure a certain portion of the analysis implies a strong enough bound.

3.1. ALGORITHM: Approximation algorithm for the TSP with β-bounded metrics.
Input: A $G_0 = (V, E_0)$ graph with a β-bounded metric on the vertex set V.
Output: Hamiltonian circuit on V.

1. Let x be an optimum solution of the Held-Karp relaxation.

2. (a) Sample a spanning tree F from distribution μ of the Structure Theorem.
   (b) Let T_F be the set of odd degree vertices in (V, F).

(c) Let J_F be a a minimal weight T_F-join, for which $F \cup J_F$ is a connected Euler-subgraph. Shortcut to obtain a Hamiltonian cycle H_1.

3. (a) Let I be the set of $(1-\gamma)$-nearly integral edges. Add a minimum cost set of edges to I to get a connected spanning subgraph F' containing I. Let F be a minimum weight spanning tree in F'. Let T_F the set of odd degree vertices in (V, F).

   (b) Let J_F be a a minimal weight T_F-join, for which $F \cup J_F$ is a connected Euler-subgraph. Shortcut to obtain a Hamiltonian cycle H_2.

4. Choose the cheapest of H_1 and H_2.

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4 Analysis of the approximation

The analysis of the algorithm follows the lines of Oveis Gharan, Saberi, Singh, [5], with only a little more detail here and there. In a few places of their analysis, [5] bounded the optimum value by n, or the LP optimum by $\frac{3}{2}n$, which we cannot do here, but those bounds can be replaced by some similar ones with β taken also into account. The first step is to bound the cost of the spanning tree by comparing it to the LP optimum.

Lemma 4.1. *The expected cost of spanning tree F is at most $(1+\varepsilon_3)c(x)$.*

Proof. $\mathbb{P}(e \in F) \leq (1+\varepsilon_3)(1-\frac{1}{n})x_e \leq (1+\varepsilon_3)x_e$, thus

$$\mathrm{E}[c(F)] = \sum_F \mathbb{P}(F) \sum_{e \in F} c(e) = \sum_{e \in E} c(e)\mathbb{P}(e \in F) \leq (1+\varepsilon_3) \sum_{e \in E} c(e)x_e = (1+\varepsilon_3)c(x).$$

□

The next two lemmas will be used to compare the cost of the T-join against the LP optimum, which is done by showing that a given vector belongs to the convex hull of T-joins.

Lemma 4.2. *Let*

$$y_e = \begin{cases} \frac{x_e}{2} & \text{if } e \text{ is in at least one odd } (1+\delta)\text{-near-minimum cut} \\ \frac{x_e}{2(1+\delta)} & \text{otherwise} \end{cases}$$

Then y is a feasible vector in the T-join polytope.

Proof. We verify inequalities of a linear programming description of the T-join polytope. $0 \leq y_e$ holds trivially. We also need to verify the odd cut inequality. x is a feasible solution of the Held-Karp LP, hence the value of x on every cut is at least 2. If a cut is odd but not $(1+\delta)$-near-minimum then

$$\sum_{e \in (S, \overline{S})} y_e = \frac{1}{2(1+\delta)} \sum_{e \in (S, \overline{S})} x_e \geq \frac{1}{2(1+\delta)} \cdot 2(1+\delta) = 1.$$

If a cut is $(1+\delta)$-near-minimum then

$$\sum_{e\in(S,\overline{S})} y_e = \frac{1}{2}\sum_{e\in(S,\overline{S})} x_e \geq 1.$$

□

Lemma 4.3. *Let*

$$y_e = \begin{cases} \frac{x_e}{3(1-\gamma)} & \textit{if } e \in F\cap I \\ 1 & \textit{if } e\in F-I \\ x_e & \textit{otherwise} \end{cases}$$

Then y is a feasible vector in the T-join polytope.

Proof. We check the inequalities defining the T-join LP. $0 \leq y_e$ holds trivially. Consider a T-odd cut $(S,\overline{S})$. Since T is the set of odd degree vertices of F, $(S,\overline{S})$ contains an odd number of edges of F. If there is an edge $e \in (F\setminus I)\cap(S,\overline{S})$, then $y_e = 1$, and we are done. Otherwise $(S,\overline{S})\cap F \subseteq I$, and thus, $|F\cap I\cap(S,\overline{S})|$ is odd. Then

$$y(\delta(U)) \geq y(\delta(U)\setminus(F\cap I)) \geq x(\delta(U)\setminus(F\cap I)) \geq x(\delta(U)) - 1 \geq 1$$

if $|F\cap I\cap(S,\overline{S})| = 1$, and

$$y(\delta(U)) \geq y(\delta(U)\cap(F\cap I)) \geq 3\frac{1}{3(1-\gamma)}(1-\gamma) = 1$$

if $|F\cap I\cap(S,\overline{S})| = 1$. □

The final lemma is one that needs to be taken care of in more detail than in [5], because there they bounded $c(F\setminus I)$ with the number of nodes, but here we need to compare with the LP optimum.

Lemma 4.4. *$|F\setminus I| \leq (\varepsilon_2+\gamma)n$.*

Proof. Since F is a spanning tree, $|F| = n-1$. By assuming Case 2 we get that $|I| \geq (1-\varepsilon_2)n$. Furthermore, note that I consists of paths, and cycles of length at least $1/\gamma$. F contains all edges of I except for just one edge from each cycle, thus $|F\cap I| \geq (1-\gamma)|I| \geq (1-\gamma)(1-\varepsilon_2)n$. We get that $|F\setminus I| = |F| - |F\cap I| \leq n-(1-\gamma)(1-\varepsilon_2)n \leq (\varepsilon_2+\gamma)n$. □

Proof. (Proof of the approximation guarantee.) First let us assume that Case 1 of the Structure Theorem holds. The cost of Hamiltonian cycle H_1 is bounded by

$$\mathrm{E}[c(H_1)] \leq \mathrm{E}[c(F\cup J_F)] \leq \mathrm{E}[c(F)] + \mathrm{E}[c(y)].$$

We already have a good bound for $\mathrm{E}[c(F)]$ based on Lemma 4.1. Next, we prove a bound for $\mathrm{E}[c(y)]$. (Below, onmc stands for odd near minimum cut.)

$$\begin{aligned}
\mathrm{E}[c(y)] &= \sum_F \mathbb{P}(F) \sum_{e\in E} y_e c(e) = \\
&= \sum_F \mathbb{P}(F) \left[\sum_{e\in E} \frac{x_e}{2} c(e) - \sum_{e: e\notin \text{onmc}} x_e \left(\frac{1}{2} - \frac{1}{2(1+\delta)} \right) c(e) \right] = \\
&= \frac{c(x)}{2} - \sum_{e\in E} x_e \left(\frac{1}{2} - \frac{1}{2(1+\delta)} \right) c(e) \mathbb{P}\Big(e \notin \text{ onmc}\Big) \leq \\
&\leq \frac{c(x)}{2} - \sum_{e\in E_{good}} x_e \left(\frac{1}{2} - \frac{1}{2(1+\delta)} \right) c(e) \mathbb{P}\Big(e \notin \text{ onmc}\Big) \leq \\
&\leq \frac{c(x)}{2} - \frac{\delta}{2(1+\delta)} \sum_{e\in E_{good}} x_e c(e) \rho = \\
&= \frac{c(x)}{2} - \frac{\delta\rho}{2(1+\delta)} \sum_{e\in E_{good}} x_e c(e) \leq \\
&\leq \frac{c(x)}{2} - \frac{\delta\rho}{2(1+\delta)} x(E_{good}) \leq \\
&\leq \frac{c(x)}{2} - \frac{\delta\rho}{2(1+\delta)} \varepsilon_1 n \leq \\
&\leq \frac{c(x)}{2} - \frac{\delta\rho}{2(1+\delta)} \varepsilon_1 \frac{c(x)}{2\beta} \leq \\
&\leq \frac{c(x)}{2} - \frac{\delta\rho}{2(1+\delta)} \varepsilon_1 \frac{c(x)}{2\beta} = c(x) \left(\frac{1}{2} - \frac{\delta\rho\varepsilon_1}{4(1+\delta)\beta} \right).
\end{aligned}$$

The first inequality holds because $E_{good} \subseteq E$. The second inequality holds because of the definition of a good edge. The third inequality holds because of $c(e) \geq 1$. The fourth inequality follows from $n \geq \frac{c(x)}{2\beta}$, which is true because of the double tree argument, and the fact that edges in E have cost no more than β.

Thus we obtain the following bound for the expected cost of a T-join J_F.

$$\mathrm{E}[c(J_F)] \leq c(x) \left(\frac{1}{2} - \frac{\delta\rho\varepsilon_1}{4(1+\delta)\beta} \right).$$

By adding two of the above inequalities, we get that the expected cost of the algorithm's solution is bounded by a factor strictly less than 3/2.

$$\mathrm{E}[c(F \cup J_F)] \leq c(x) \left(\frac{3}{2} - \frac{\delta\rho\varepsilon_1}{4(1+\delta)\beta} \right).$$

Next we assume that Case 2 holds in the Structure Theorem. We can bound the cost of Hamiltonian cycle H_2 by

$$\mathrm{E}[c(H_2)] \leq \mathrm{E}[c(F \cup J_F)] \leq \mathrm{E}[c(F)] + \mathrm{E}[c(y)],$$

and

$$\begin{aligned}c(F)+c(y) = c(I)+c(F\setminus I)+\frac{c(x|_I)}{3(1-\gamma)}+c(F\setminus I)+c(x|_{E\setminus F}) \leq \\ \leq \frac{4c(x|_I)}{3(1-\gamma)}+2c(F\setminus I)+c(x|_{E\setminus F}) \leq \\ \leq \frac{4c(x|_I)}{3(1-\gamma)}+2\beta n(\varepsilon_2+\gamma)+c(x|_{E\setminus I}) \leq \\ \leq \frac{4c(x)}{3(1-\gamma)}+2\beta c(x)(\varepsilon_2+\gamma).\end{aligned}$$

The first equation follows by substituting the definition of y. The first inequality follows from $x(e) \geq 1-\gamma$ for all $e \in I$, implying $c(I) \leq c(x|_I)/(1-\gamma)$. The second inequality follows from $c(F\setminus I) \leq \beta|F\setminus I| \leq \beta n(\varepsilon_2+\gamma)$ which follows from Lemma 4.4. The third inequality follows from $c(x|_{E\setminus I}) \leq \frac{4c(x|_{E\setminus I})}{3(1-\gamma)}$ and $n \leq c(x)$. We obtain the following bound on the cost of Hamiltonian cycle H_2, assuming Case 2 holds:

$$c(H_2) \leq c(x)\left(\frac{4}{3(1-\gamma)}+2\beta(\varepsilon_2+\gamma)\right).$$

By choosing the constants small enough – which is possible by the Structure Theorem – the coefficient on the right hand side becomes smaller than 3/2. □

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