Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2014-06. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Generic global rigidity of body-hinge frameworks

Tibor Jordán, Csaba Király, and
Shin-ichi Tanigawa

July 19, 2014

Generic global rigidity of body-hinge frameworks

Tibor Jordán*, Csaba Király**, and Shin-ichi Tanigawa***

Abstract

A d-dimensional body-hinge framework is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by a hinge, restricting the relative position of the corresponding bodies. The framework is said to be globally rigid if every other arrangement of the bodies and their hinges can be obtained by a congruence of the space. The combinatorial structure of a body-hinge framework can be encoded by a multigraph H, in which the vertices correspond to the bodies and the edges correspond to the hinges. We prove that a generic body-hinge realization of a multigraph H is globally rigid in $\mathbb{R}^d$, $d \geq 3$, if and only if $(\binom{d+1}{2} - 1)H - e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees for all edges e of $(\binom{d+1}{2} - 1)H$. (For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e.) This implies an affirmative answer to a conjecture of Connelly, Whiteley, and the first author.

We also consider bar-joint frameworks and show, for each $d \geq 3$, an infinite family of graphs satisfying Hendrickson's well-known necessary conditions for generic global rigidity in $\mathbb{R}^d$ (that is, $(d+1)$-connectivity and redundant rigidity) which are not generically globally rigid in $\mathbb{R}^d$. The existence of these families disproves a number of conjectures, due to Connelly, Connelly and Whiteley, and the third author, respectively.

1 Introduction

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

*Department of Operations Research, Eötvös University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `jordan@cs.elte.hu`

**Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `cskiraly@cs.elte.hu`

***Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan. e-mail: `tanigawa@kurims.kyoto-u.ac.jp`

We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid (i.e. fixed length) bar joining its end-points, then the framework is globally rigid if its bar lengths determine the realization up to congruence, and it is rigid if it has no non-trivial continuous deformations that preserve all bar lengths, see e.g. [27]. It is a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [17] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider *generic frameworks* i.e. frameworks in which there are no algebraic dependencies between the coordinates of the vertices.

It is known that the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic (see [27]). We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$, and is a major open problem for $d \geq 3$.

A similar situation holds for global rigidity. Gortler, Healy and Thurston [8] proved that the global rigidity of d-dimensional frameworks is a generic property for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. Hendrickson [9] proved two key necessary conditions for the global rigidity of a graph. We say that G is *redundantly rigid in* $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem 1.1 ([9]). *Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d + 1$ vertices, or G is*
(i) $(d + 1)$-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$.

He conjectured that the necessary conditions of Theorem 1.1 are also sufficient to imply the global rigidity of the graph in $\mathbb{R}^d$. It is indeed so for $d = 1, 2$. It is not hard to verifiy that a 1-dimensional generic framework (G, p) is globally rigid if and only if either G is the complete graph on two vertices or G is 2-connected. The characterization for $d = 2$ is as follows.

Theorem 1.2 ([10]). *Let G be a graph. Then G is globally rigid in $\mathbb{R}^2$ if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

However, there exist counterexamples to his conjecture for $d \geq 3$, and characterizing the globally rigid graphs in three-space and in higher dimensions remains another major open problem in rigidity theory.

In this paper, we shall give a combinatorial characterization of the d-dimensional global rigidity of a special class of graphs, called body-hinge graphs, which represent

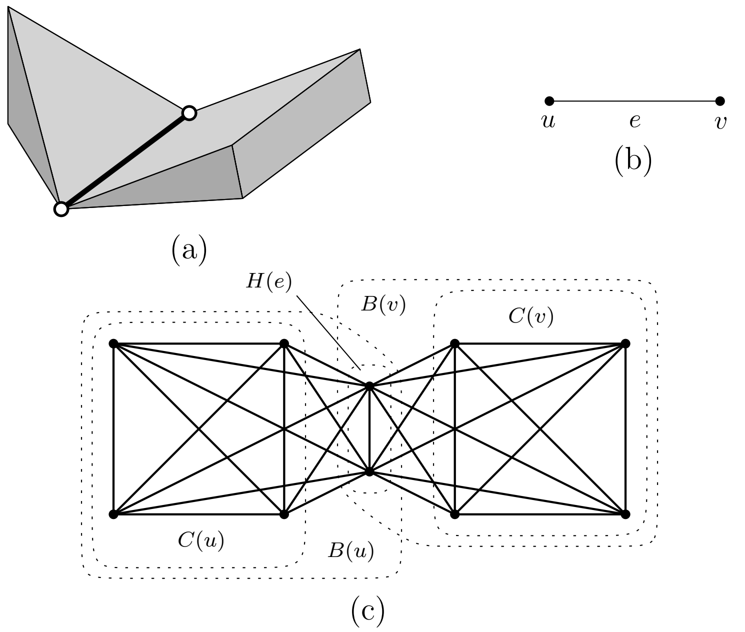


Figure 1: (a) A body-hinge framework in $\mathbb{R}^3$, (b) the underlying graph H, and (c) the body-hinge graph G_H.

generic body-hinge frameworks. A d-dimensional *body-hinge framework* is a structural model consisting of rigid bodies and hinges. Each hinge is a $(d-2)$-dimensional affine subspace that joins some pair of bodies. The bodies are free to move continuously in $\mathbb{R}^d$ subject to the constraint that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. The framework is rigid if every such motion preserves the distances between all pairs of points belonging to different rigid bodies, i.e. the motion extends to an isometry of $\mathbb{R}^d$. In the underlying graph of the framework the vertices correspond to the bodies and the edges correspond to the hinges. See Figure 1(a),(b). We can obtain an equivalent bar-joint framework by replacing each body by a bar-joint realization of a large enough complete graph in such a way that two bodies joined by a hinge share $d-1$ joints. The graph of such a bar-joint framework is a body-hinge graph. More precisely, for a multigraph H, the (d-dimensional) *body-hinge graph induced by* H, denoted by G_H, is obtained from H by replacing each vertex $v \in V$ by a complete graph $B(v)$ (the body of v) on $(d-1)d_H(v)+d+1$ vertices, in which $d+1$ vertices inducing a K_{d+1} form the *core* $C(v)$ of the body and the remaining vertices are partitioned into sets of $d-1$ vertices so that each set is assigned to one edge incident with v. (We use $d_H(v)$ to denote the degree of vertex v in H.) For each edge $e = uv$ the bodies $B(u)$ and $B(v)$ share the $d-1$ vertices assigned to e in these bodies. This set of $d-1$ vertices assigned to e, denoted by $H(e)$, is a *hinge* between the corresponding bodies. The cores of the bodies are parwise disjoint. See Figure 1(c).

Body-hinge frameworks (and body-hinge graphs) are extensively studied objects in rigidity theory with various applications. Among others, they can be used to investigate the flexibility of molecules, due to the fact that molecular conformations can be modeled by body-hinge frameworks with certain additional geometric constraints, see [14, 27]. Tay [21, 22] and Whiteley [25] characterized rigid d-dimensional body-hinge graphs in terms of their underlying multigraphs.

For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e.

Theorem 1.3 ([21, 22], [25]). *Let $H = (V, E)$ be a multigraph. Then the body-hinge graph G_H is rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Body-bar frameworks (and their graphs) are equally important in rigidity theory. Roughly speaking, they consist of full-dimensional rigid bodies connected by disjoint bars. Tay [20, 23] provided a characterization of rigid d-dimensional body-bar graphs and a recent result of Connelly, Jordán, and Whiteley [5] gives a combinatorial characterization of globally rigid body-bar graphs in $\mathbb{R}^d$, for all $d \geq 1$. In their paper they also conjectured a sufficient condition for the global rigidity of body-hinge graphs. We give an affirmative answer to their conjecture. Furthermore, we show that the conjectured sufficient condition is also necessary. Our main result is as follows.

Theorem 1.4. *Let $H = (V, E)$ be a multigraph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H - e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees for all edges e of $(\binom{d+1}{2} - 1)H$.*

Note that the two-dimensional characterization is slightly different, see Theorem 5.1.

1.1 Families of not globally rigid graphs

One of the important steps towards a possible characterization of global rigidity in higher dimensions is to identify new necessary or sufficient conditions for global rigidity. In particular, finding more counterexamples to Henderickson's conjecture is a challenging problem. We say that a graph G is an *H-graph* in $\mathbb{R}^d$ if it satisfies Hendrickson's necessary conditions in $\mathbb{R}^d$ (c.f. Theorem 1.1) but it is not globally rigid in $\mathbb{R}^d$. For $d = 3$, Connelly [2] showed that the complete bipartite graph $K_{5,5}$ is an H-graph. He presented H-graphs for all $d \geq 3$ as well. These H-graphs are all complete bipartite graphs on $\binom{d+2}{2}$ vertices. Frank and Jiang [7] found two more (bipartite) H-graphs in $\mathbb{R}^4$ and infinite families of H-graphs in $\mathbb{R}^d$ for $d \geq 5$. Some of their H-graphs in $\mathbb{R}^d$, $d \geq 5$, contain the complete graph K_{d+1} as a subgraph. We remark that a d-dimensional H-graph G can be turned into a $d+1$-dimensional H-graph by applying the coning operation, which adds a new vertex v to G along with all edges from v to the vertex set of G [6, 7].

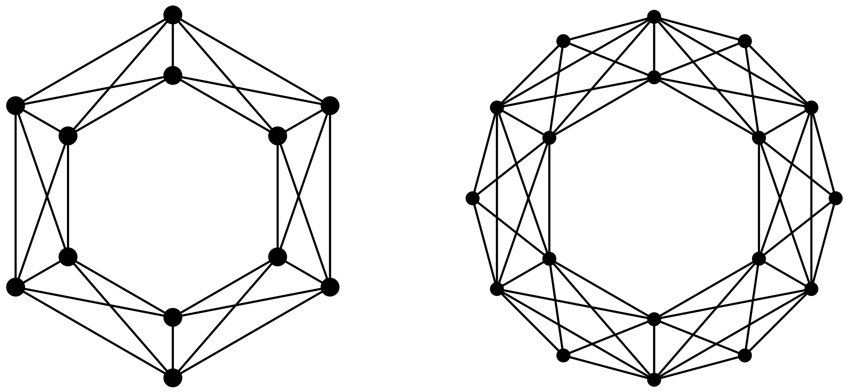
Connelly conjectured that $K_{5,5}$ is the only H-graph in $\mathbb{R}^3$ [4, 6]. Connelly and Whiteley [6] conjectured that there exist no H-graphs in $\mathbb{R}^d$ containing K_{d+1} as a subgraph. They also conjectured that the number of H-graphs is finite in $\mathbb{R}^d$, for all $d \geq 3$. Although the above mentioned examples [7] disproved the latter conjectures for $d \geq 5$, they remained open in the three- and four-dimensional cases. Tanigawa

[19] noted that every body-hinge graph which is rigid in $\mathbb{R}^d$ satisfies Hendrickson's necessary conditions and contains K_{d+1} as a subgraph. This motivated him to conjecture that a body-hinge graph is globally rigid in $\mathbb{R}^d$ if and only if it is rigid in $\mathbb{R}^d$. As a by-product of Theorem 1.4 we shall disprove (the remaining cases of) each of these conjectures by constructing various infinite families of H-graphs for all $d \geq 3$. Some of these families are in fact body-hinge graphs.

We close this section by analysing one of our 3-dimensional H-graphs which will also illustrate some of our arguments. The graph G of Figure 2a is 4-connected and minimally rigid in $\mathbb{R}^3$. Minimal rigidity can be verified by using some of the well-known inductive constructions or by using Theorem 1.3. Note that it can be obtained from the body-hinge graph induced by a six-cycle by deleting the cores of the bodies.

Theorem 1.1 implies that G is not globally rigid in $\mathbb{R}^3$. The graph $\hat{G}$ of Figure 2b is obtained from G by attaching a vertex of degree four to each of its six K_4 subgraphs. Thus $\hat{G}$ is 4-connected and redundantly rigid in $\mathbb{R}^3$. Since the new vertices are attached to complete subgraphs, the fact that G is not globally rigid implies that $\hat{G}$ is not globally rigid either in $\mathbb{R}^3$. We obtain that $\hat{G}$ is an H-graph in $\mathbb{R}^3$.

By using the same argument we can deduce that the body-hinge graph induced by the six-cycle is a 3-dimensional H-graph, too. Furthermore, as we shall see, we can construct an infinite family of H-graphs by replacing each vertex of the six-cycle by some multigraph M for which $5M$ contains 6 edge-disjoint spanning trees (and taking the induced body-hinge graph), see Section 4.2.



(a) A 4-connected minimally rigid graph G in $\mathbb{R}^3$.

(b) A 4-connected, redundantly rigid graph $\hat{G}$ in $\mathbb{R}^3$, which is not globally rigid in $\mathbb{R}^3$.

Figure 2: The construction of an H-graph in $\mathbb{R}^3$.

2 Preliminaries

Let $H = (V, E)$ be a multigraph. For a partition $\mathcal{P}$ of V let $e_H(\mathcal{P})$ denote the number of edges of H connecting distinct members of $\mathcal{P}$. We say that H is (m, ℓ)-

tree-connected, for some non-negative integers k, ℓ, if

$$e_H(\mathcal{P}) \geq m(t-1) + \ell \tag{1}$$

for all partitions $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$. A theorem of Nash-Williams [16] and Tutte [24] implies that H is (m, ℓ)-tree-connected if and only if $H - F$ contains m edge-disjoint spanning trees for all $F \subseteq E$ with $|F| \leq \ell$.

In what follows we shall be interested in multigraphs satisfying (1) with respect to parameters m and ℓ for which $m = \binom{d+1}{2}$, for some $d \geq 1$, and for some $\ell \in \{0, 1, 2\}$. We shall use a simpler terminology and say that an (m, ℓ)-tree-connected multigraph, where $\ell = 0$ (resp. $\ell = 1$, $\ell = 2$), is *m-tree-connected* (resp. *highly m-tree-connected*, *doubly highly m-tree-connected*).

To simplify the notation, D will denote the number $\binom{d+1}{2}$, where d (the dimension, in most cases) is clear from the context. For a multigraph H containing at least k copies of some edge e, we use ke to refer to k copies of e in H. The next lemma shows how a certain reduction step at some vertex with two neighbours preserves the tree-connectivity properties of the multigraph.

Lemma 2.1. *Let H be a multigraph and let v be a vertex of degree two in H with $N_H(v) = \{u, w\}$. Let $H_v = H - v + uw$ be obtained from H by removing v and adding a new edge uw. Suppose that $(D-1)H$ is highly D-tree-connected for some $d \geq 2$. Then $(D-1)H_v$ is highly D-tree-connected and $(D-1)H_v - 2(uw)$ is D-tree-connected.*

Proof. To prove the first statement, put $H' = (D-1)H_v$ and suppose, for a contradiction, that H' is not highly D-tree-connected. Then there is partition $\mathcal{P}' = \{X_1, X_2, ..., X_t\}$ of $V(H')$ with $t \geq 2$ for which $e_{H'}(\mathcal{P}') \leq D(t-1)$. If u and w belong to the same member, say P_1, then by adding v to P_1 we obtain a partition $\mathcal{P}$ of V with $e_{(D-1)H}(\mathcal{P}) = e_{H'}(\mathcal{P}') \leq D(t-1)$, contradicting the assumption that $(D-1)H$ is highly D-tree-connected. If u and v are in different members then by adding a new member $\{v\}$ to $\mathcal{P}'$ we obtain a partition $\mathcal{P}$ of V with $t+1$ members satisfying $e_{(D-1)H}(\mathcal{P}) = e_{H'}(\mathcal{P}') - (D-1) + 2(D-1) \leq D(t-1) + (D-1) = Dt - 1$, a contradiction.

The proof of the second statement is similar. □

The connectivity, edge-connectivity, and tree-connectivity parameters of H, $(D-1)H$, and G_H are related as follows. The proof of the next simple lemma is omitted.

Lemma 2.2. *(i) Suppose that $(D-1)H$ is D-tree-connected. Then H is 2-edge-connected.*
(ii) Suppose that H is k-edge-connected. Then G_H is $(d-1)k$-connected.
(iii) Suppose that $(D-1)H$ is D-tree-connected for some $d \geq 3$. Then G_H is $(d+1)$-connected.
(iv) H is 3-edge-connected if and only if $2H$ is doubly highly 3-tree-connected.

3 Truncated body-hinge graphs and skeletons

We shall consider graphs obtained from a body-hinge graph G_H, induced by some multigraph H, by deleting one vertex of the hinge set $H(e)$, for some $e \in E(H)$. In this graph that we denote by $G_{(H,e)}$, the two bodies associated with the endvertices of e share only $d-2$ vertices.

The following lemma is implicit in [19, Theorem 5.2]. The proof is given in the Appendix for completeness.

Lemma 3.1. *Let $H = (V, E)$ be a multigraph and $e \in E(H)$. Suppose that $(D-1)H - 2e$ is D-tree-connected. Then $G_{(H,e)}$ is rigid in $\mathbb{R}^d$.*

Let $H = (V, E)$ be a multigraph. The *skeleton* S_H of the induced body-hinge graph G_H is obtained by deleting the cores $C(v)$ for all $v \in V$. The rigidity, global rigidity, and connectivity properties of G_H and S_H are the same in the following sense.

Lemma 3.2. *Let H be a multigraph with minimum degree at least two and let $d \geq 3$ be an integer. Then G_H is globally rigid in $\mathbb{R}^d$ (rigid in $\mathbb{R}^d$, $(d+1)$-connected, resp.) if and only if S_H is globally rigid in $\mathbb{R}^d$ (rigid in $\mathbb{R}^d$, $(d+1)$-connected, resp.).*

Proof. The lemma follows by observing that G_H is obtained from S_H by iteratively attaching complete subgraphs to complete subgraphs of size at least $2(d-1) \geq d+1$. □

The *skeleton* of $G_{(H,e)}$ is defined in the same manner as the graph obtained from $G_{(H,e)}$ by deleting the cores $C(v)$ for all $v \in V$.

4 Globally rigid body-hinge graphs

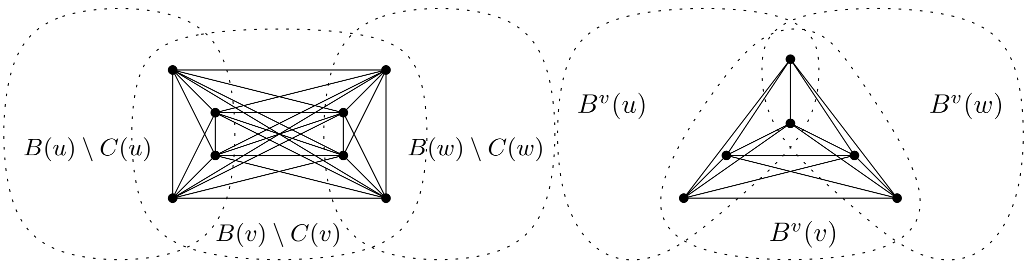
In this section we prove our main result. We shall use the following sufficient conditions for global rigidity. For some graph G and $X \subseteq V(G)$, the graph $G + K(X)$ is obtained from G by adding all edges uv, where u and v are non-adjacent vertices of X in G.

Theorem 4.1 ([19])**.** *Let $G = (V, E)$ be a graph and let $x \in V$. Suppose that $G - x$ is rigid in $\mathbb{R}^d$ and $G - x + K(N_G(x))$ is globally rigid in $\mathbb{R}^d$. Then G is globally rigid in $\mathbb{R}^d$.*

We say that G is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$.

Theorem 4.2 ([19])**.** *If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.*

We shall also rely on the well-known vertex splitting operation. Let G be a graph, let $v_1 \in V$, let $v_1v_2, ..., v_1v_d$ be $d-1$ designated edges incident with v_1, and let $v_1v_{d+1}, ..., v_1v_{d+k_1}$ and $v_1v_{d+k_1+1}, ..., v_1v_{d+k_1+k_2}$ be a bipartition of the remaining edges incident with v_1. The (d-dimensional) *vertex splitting* operation at v_1 removes the edges $v_1v_{d+1}, ..., v_1v_{d+k_1}$, adds a new vertex v_0, and adds the new edges $v_0v_1, v_0v_2, ..., v_0v_d, v_0v_{d+1}, ..., v_0v_{d+k_1}$. Whiteley [26] proved that this operation preserves the rigidity of graphs in $\mathbb{R}^d$. The vertex splitting operation is *non-trivial* if $k_1 \geq 1$ and $k_2 \geq 1$ hold. The new edge v_0v_1 is called the *bridging edge* in the resulting graph. The following result is due to Connelly [4, Section 11].



(a) S_H (b) S_H^v

Figure 3: The construction of S_H^v from S_H in $\mathbb{R}^5$. The figure shows the bodies corresponding to vertex v with $N_H(v) = \{u, w\}$ of the underlying multigraph H before and after the edge contractions.

Theorem 4.3 ([4])**.** *Let G be globally rigid in $\mathbb{R}^d$ and let G' be obtained from G by a non-trivial vertex splitting operation. If the bridging edge e is redundant (i.e. $G' - e$ is rigid in $\mathbb{R}^d$) then G' is also globally rigid in $\mathbb{R}^d$.*

4.1 The main result

Before the proof of our main result, we prove one more lemma that we need only in the higher dimensional cases, when $d \geq 4$. To verify the three-dimensional case the next lemma and the preceeding discussion can be skipped.

Let $H = (V, E)$ be a multigraph and suppose that H contains a vertex v of degree two with $N_H(v) = \{u, w\}$. Let $H(uv) = \{x_1, \ldots, x_{d-1}\}$ and $H(vw) = \{y_1, \ldots, y_{d-1}\}$ denote the hinge sets associated with uv and vw in the corresponding d-dimensional skeleton S_H. If $d = 3$ then we simply put $S_H^v = S_H$. Otherwise, when $d \geq 4$, we denote by S_H^v the graph obtained from S_H by contracting the edges $x_i y_i$ for all $3 \leq i \leq d-1$. See Figure 3. This operation changes the bodies of u, v, w and results in deformed hinge sets associated with edges uv and vw. We shall use $B^v(a)$ and $H^v(e)$ to denote the bodies and hinges in S_H^v associated with the vertices and edges of H, respectively. Thus $|B^v(u) \cap B^v(w)| = d - 3$ and $B^v(v)$ induces a complete graph K_{d+1}.

We recall another basic notion of rigidity theory which will be used in the proof of the next lemma. Let (G, p) be a d-dimensional framework with $G = (V, E)$. An *infinitesimal motion* of (G, p) is a map $m : V \to \mathbb{R}^d$ satisfying

$$\langle p(u) - p(v), m(u) - m(v) \rangle = 0 \text{ for all edges } uv \in E. \tag{2}$$

A *trivial infinitesimal motion* of (G, p) is a map for which $m(v) = Sp(v) + t$ holds for all $v \in V$, for some $d \times d$ skew-symmetric matrix S and some $t \in \mathbb{R}^d$. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is *infinitesimally rigid* if it has only trivial infinitesimal motions.

Lemma 4.4. *Let $H = (V, E)$ be a multigraph and v be a vertex of degree two in H. Suppose that $d \geq 3$. Then S_H is globally rigid in $\mathbb{R}^d$ if and only if S_H^v is globally rigid in $\mathbb{R}^d$.*

Proof. We shall prove the "only if" direction. The other direction can be proved in a similar fashion. As above, let $N_H(v) = \{u, w\}$, $H(uv) = \{x_1, \ldots, x_{d-1}\}$, and $H(vw) = \{y_1, \ldots, y_{d-1}\}$. Denote the vertex obtained by the contraction of edge $x_i y_i$ by z_i, for $3 \leq i \leq d-1$.

Consider a generic realization (S_H, p) of S_H in $\mathbb{R}^d$. Then the intersection L of the two affine subspaces spanned by the hinge sets $H(uv)$ and $H(vw)$, respectively, is a $(d-4)$-dimensional affine subspace of $\mathbb{R}^d$. Take $d-3$ points $a_3, \ldots, a_{d-1}$ in such a way that their affine span is equal to L and define $p' : V(S_H^v) \to \mathbb{R}^d$ so that $p'(j) = p(j)$ for all $j \in V(S_H^v)$ with $j \neq z_i$ and by putting $p'(z_i) = a_i$ for $3 \leq i \leq d-1$. It follows that the affine subspaces spanned by $H(e)$ and $H^v(e)$ are the same for all $e \in E(H)$. Thus, informally speaking, the frameworks (S_H, p) and (S_H^v, p') give rise to the same body-hinge structure, since their underlying multigraphs as well as the corresponding hinge subspaces are all the same.

This equivalence is not entirely obvious in our bar-and-joint setting. The fact that these structures are indeed equivalent as far as global and infinitesimal rigidity are concerned can be made precise as follows.

Let (S_H^v, q') be a realization of S_H^v which is equivalent to (S_H^v, p'). Then for each $i \in V(H)$ there is an isometry f_i such that $q'(x) = f_i(p'(x))$ for each $x \in B^v(i)$. Note that for any edge $e = ij$ we have $f_i(s) = f_j(s)$ for all points s that belong to the affine span of $p'(H^v(e))$. Hence, if we define a realization (S_H, q) such that $q(x) = f_i(p(x))$ for $x \in B(i)$ for each $i \in V(H)$, then q is well-defined. Note that (S_H^v, p') is congruent to (S_H^v, q') if and only if the isometries f_i are the same for all $i \in V(H)$, which is equivalent to saying that (S_H, p) is congruent to (S_H, q). Thus (S_H^v, p') is globally rigid if (and only if) (S_H, p) is globally rigid.

Similarly, one can check that (S_H^v, p') is infinitesimally rigid if (S_H, p) is infinitesimally rigid. Let $m' : V(S_H^v) \to \mathbb{R}^d$ be an infinitesimal motion of (S_H^v, p'). Since each rigid body affinely spans $\mathbb{R}^d$, for each $i \in V(H)$, there is a skew-symmetric matrix S_i and a vector $t_i \in \mathbb{R}^d$ such that $m'(x) = S_i p'(x) + t_i$ for $x \in B(i)$. Note that, for edge $e = ij$, we have $S_i s + t_i = S_j s + t_j$ for all points s that belong to the affine span of $p'(H^v(e))$. Hence, if we define an infinitesimal motion m of (S_H, p) such that $m(x) = S_i p(x) + t_i$ for $x \in B(i)$, for each $i \in V(H)$, then m is well-defined. Note that m is trivial if and only if the pairs (S_i, t_i) are the same for all $i \in V(H)$, which is equivalent to saying that m' is trivial.

The realization (S_H, p) is globally rigid and generic. Hence it is infinitesimally rigid. Thus the above arguments imply that (S_H^v, p'), which may not be generic, is also globally rigid and infinitesimally rigid. By a theorem by Connelly and Whiteley [6, Corollary 14], this implies that S_H^v is globally rigid, as required. □

We note that the "if" direction in Lemma 4.4 also follows from Theorem 4.3 by observing that S_H can be obtained from S_H^v by a sequence of vertex splitting operations and adding edges in such a way that in each iteration the bridging edge is part of complete subgraph K_{d+2} (and hence it is redundant) in the resulting graph.

We are now ready to prove our main result (Theorem 1.4) that we restate here for convenience.

Theorem 4.5. *Let $H = (V, E)$ be a multigraph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(D-1)H$ is highly D-tree-connected.*

Proof. To prove necessity, suppose, for a contradiction, that G_H is globally rigid in $\mathbb{R}^d$ but $(D-1)H$ is not highly D-tree-connected. This implies that there is a partition $\mathcal{P}$ of V with $t \geq 2$ members satisfying $e_{(D-1)H}(\mathcal{P}) \leq D(t-1)$. By adding sufficiently many new edges inside the non-singleton partition classes to make their induced body-hinge graphs globally rigid and then contracting each of them into one vertex we may suppose that each partition member is a single vertex. This in turn implies that

$$(D-1)|E(H)| = D|V(H)| - D. \tag{3}$$

Hence there is a vertex v of degree two in H.

By Lemmas 3.2 and 4.4, S_H^v is globally rigid. Call an edge a hinge-edge if it is induced by some hinge set. Remove non-hinge edges from each body of S_H^v so that the sparsified skeleton, denoted by T_H, spans a minimally rigid graph on the vertex set of each body of S_H^v. This can be done, since each hinge set induces a small complete graph K_{d-1} and the hinge sets are pairwise disjoint (except in the body of v, which induces K_{d+1} and hence is already minimally rigid) and hence they can be extended to a spanning minimally rigid graph within each body. It is clear that T_H is rigid. We claim that T_H is minimally rigid. This follows by counting the edges and using (3):

$$\begin{aligned}
|E(T_H)| &= \sum_{u \in V(H)\setminus\{v\}} \left(d[(d-1)d_H(u)] - \binom{d+1}{2} \right) + \binom{d+1}{2} - \binom{d-1}{2}|E(H)| \\
&= \sum_{u \in V(H)} \left(d(d-1)d_H(u) - \binom{d+1}{2} \right) - d(d-3) - \binom{d-1}{2}|E(H)| \\
&= (d-1)\left(\frac{3}{2}d + 1\right)|E(H)| - \binom{d+1}{2}|V(H)| - d(d-3) \\
&= d((d-1)|E(H)| - (d-3)) - \binom{d+1}{2} \\
&= d|V(S_H^v)| - D \\
&= d|V(T_H)| - D.
\end{aligned}$$

It follows that the edges of K_{d+1} induced by $B(v)$ are M-bridges (i.e. edges that belong to all rigid spanning subgraphs) in S_H^v. So S_H^v is not redundantly rigid. We can conclude, by using Theorem 1.1 and Lemma 4.4, that S_H is not globally rigid. By Lemma 3.2 this implies that G_H is not globally rigid either, a contradiction. This proves necessity.

To prove sufficiency, suppose that $(D-1)H$ is highly D-tree-connected. The proof is by induction on $|V|$. The statement is trivial for $|V| = 1$. If $(D-1)H$ is doubly highly D-tree-connected then we can use Lemma 3.1 to deduce that G_H is vertex-redundantly rigid in $\mathbb{R}^d$ and then it follows from Theorem 4.2 that G_H is globally rigid in $\mathbb{R}^d$, as required. Thus we may suppose that there is a partition $\mathcal{P} = \{P_1, ..., P_t\}$ of V with $t \geq 2$ and $(D-1)e(\mathcal{P}) = D(|\mathcal{P}| - 1) + 1$. We can also assume that $H[P_i]$ is highly

D-tree-connected for all i (for otherwise we could refine $\mathcal{P}$ by an appropriate partition of P_i). By induction, $G_{H[P_i]}$ is globally rigid for all i. If one partition class, say P_1, is not a singleton then consider H' obtained from H by contracting P_1 to a vertex v_1. Then $(D-1)H'$ is highly D-tree-connected and hence, by induction, its body-hinge graph is globally rigid. So we may suppose that each partition class is a singleton and hence $(D-1)|E| = D|V| - D + 1$. Thus there is vertex in H with $d_H(v) = 2$.

By Lemmas 3.2 and 4.4, it suffices to show that S_H^v is globally rigid. Let $N_H(v) = \{u, w\}$. Let $H(uv) \setminus B^v(v) = \{x_1, x_2\}$ and $H(vw) \setminus B^v(v) = \{y_1, y_2\}$ be the two pairs of hinge-vertices not involved in the contractions when constructing S_H^v from S_H. Notice that x_1y_1 is incident to $d-1$ triangles in S_H^v, and hence the contraction of x_1y_1 corresponds to the inverse operation of vertex splitting. Let S' be the graph obtained from S_H^v by contracting x_1y_1. Then observe that $S' - x_2y_2$ is the skeleton of $G_{(H_v,uw)}$, where $H_v = H - v + uw$. Since H is highly D-tree-connected, Lemma 2.1 implies that $(D-1)H_v - 2(uw)$ is D-tree-connected. This in turn implies that $S' - x_2y_2$ is rigid by Lemma 3.1. It follows from Whiteley's vertex-splitting theorem that $S_H^v - x_2y_2$ is rigid, and x_iy_j is redundant in S_H^v for any $i = 1, 2$ and $j = 1, 2$ by symmetry.

Since $S' - x_2y_2$ is rigid, $S' - x_2$ is clearly rigid (since x_2 does not belong to any of the "hinge sets" in $S' - x_2y_2$). Observe that $S' - x_2 + K(N_{S'}(x_2))$ is the skeleton of G_{H_v}. Since $(D-1)H_v$ is highly D-tree-connected with $|V(H_v)| < |V(H)|$, $S' - x_2 + K(N_{S'}(x_2))$ is globally rigid by induction. Therefore by Theorem 4.1 S' is globally rigid.

Since S_H is constructed from S' by a vertex splitting operation and x_1y_1 is redundant in S_H^v, we can apply Theorem 4.3 to conclude that S_H is globally rigid. $\square$

4.2 Infinite families of H-graphs

By using the necessary condition of Theorem 1.4 we can easily construct infinite families of d-dimensional H-graphs for all $d \geq 3$. Let H be a multigraph for which $(D-1)H$ is D-tree-connected but not highly D-tree-connected, or equivalently, for which $(D-1)H$ contains D edge-disjoint spanning trees and at the same time has a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$ satisfying

$$e_H(\mathcal{P}) = \frac{D(t-1)}{D-1}.$$

For example we may obtain such multigraphs H from a cycle of length D by replacing each vertex by any subgraph H' for which $(D-1)H'$ contains D edge-disjoint spanning trees.

Then G_H is redundantly rigid (it follows by Theorem 1.3 and the fact that each edge belongs to a large enough complete subgraph), and $(d+1)$-connected (by Lemma 2.2(iii)), but not globally rigid in $\mathbb{R}^d$ (by Theorem 1.4). Thus it is a d-dimensional body-hinge graph which is also an H-graph.

It is possible to construct several other examples, including families which are not body-hinge graphs. For example, as noted earlier, the cone of an H-graph is also an H-graph. Another way to create examples is to take a body-hinge H-graph and then replace one (or more) of its bodies by a globally rigid graph, keeping the same vertices of attachment.

5 Globally rigid body-hinge graphs in two dimensions

Theorem 1.2 gives a complete description of globally rigid graphs in $\mathbb{R}^2$. We can use this result to characterize those multigraphs H that induce globally rigid body-hinge graphs in two dimensions. It turns out that the necessary and sufficient condition is different from that of the higher dimensional version in Theorem 1.4.

Theorem 5.1. *Let H be a multigraph and let G_H be the two-dimensional body-hinge graph induced by H. Then G_H is globally rigid in $\mathbb{R}^2$ if and only if H is 3-edge-connected.*

Proof. Suppose that G_H is globally rigid. Then G_H is 3-connected, and hence H is 3-edge-connected. Conversely, suppose that H is 3-edge-connected. By Lemma 2.2(iv) $2H$ is (doubly) highly 3-tree-connected. By Theorem 1.3 we conclude that G_H is redundantly rigid, too. Therefore it is globally rigid by Theorem 1.2. □

Note that Theorem 5.1 can also be deduced from Lemma 3.1 and Theorem 4.2. If H is 3-regular then S_H is a so-called *combinatorial zeolite* in the plane. A different proof for this special case was given in [13].

The reader may wonder why the construction of higher dimensional H-graphs does not work for $d = 2$. The key property of each H-graph G given above for $d \geq 3$ is that it is constructed from a rigid and $(d+1)$-connected skeleton S, in which each M-bridge belongs to a subgraph isomorphic to K_{d+1}, by attaching complete graphs to these complete subgraphs. By attaching all but one of these complete graphs to S, we can obtain a rigid and $(d+1)$-connected graph G' in which there is a single K_{d+1}-subgraph which contains all M-bridges (then G is obtained from G' by attaching one complete graph, making all edges of G redundant).

However, a graph G' with these properties does not exist in $\mathbb{R}^2$.

Lemma 5.2. *Let $G = (V, E)$ be a rigid, but not redundantly rigid graph in $\mathbb{R}^2$ and suppose that all M-bridges of G are edges of the same triangle in G. Then G is not 3-connected.*

Proof. Suppose, for a contradiction, that G is 3-connected. Let $a, b, c \in V$ be the vertices of the triangle that contains all M-bridges of G. We may assume that $e = ab$ is an M-bridge. Let G' be the graph obtained from G by removing edge ab and attaching two $K_4's$ along the edges bc, ca, respectively. It is easy to see that G' is not rigid. On the other hand, it is nearly 3-connected (that is, it can be made 3-connected by adding one edge) and that each edge of G' is redundant. Hence, by [12, Theorem 5.1], G' is M-connected (that is, its rigidity matroid is connected, see [10]). This in turn implies that G' is rigid (see e.g. [10, Lemma 3.1]), a contradiction. □

6 Concluding remarks

Theorem 1.4 gives rise to a polynomial time algorithm to determine whether a body-hinge graph is globally rigid in $\mathbb{R}^d$. This follows from the fact that, as we noted earlier, a multigraph H is highly m-tree-connected if and only if $H - e$ contains m edge-disjoint spanning trees for all $e \in E(H)$. Thus efficient tree-packing algorithms can be used to test whether a given multigraph is highly m-tree-connected. We refer the reader to [18, Chapter 51] for a complexity survey of tree packing algorithms. A recent algorithm, focusing on high m-tree-connectivity, can be found in [15].

Another algorithmic observation is that one can easily test whether a given graph G is a body-hinge graph: the vertices of G with a non-complete neighbour set are the candidates for being the hinge vertices. If G is a body-hinge graph then these vertices are partitioned into classes in such a way that two vertices u, v are in the same class if and only if $uv \in E$ and $N(u) - v = N(v) - u$. This partition, if it exists, can be used to check whether G is indeed a body-hinge graph and if yes, to determine the underlying multigraph H.

We did not directly refer to Hendrickson's $(d + 1)$-connectivity condition of Theorem 1.1 in our characterization for $d \geq 3$ in Theorem 1.4. This is because $(d + 1)$-vertex-connectivity follows for 'free' for body-hinge graphs G_H when $(D - 1)H$ is highly D-tree-connected, provided $d \geq 3$. Another related observation, which follows from Theorems 1.4 and 5.1, is that if G_H is $2d$-connected then it is globally rigid in $\mathbb{R}^d$. Thus the general conjecture, saying that sufficiently high connectivity implies global rigidity in $\mathbb{R}^d$ for all $d \geq 1$ [5], holds for body-hinge graphs.

7 Acknowledgements

This work was supported by the Hungarian Scientific Research Fund grant no. K81472, and the National Development Agency of Hungary, grant no. K 109240, based on a source from the Research and Technology Innovation Fund.

8 Appendix

In this section, we provide a proof of Lemma 3.1 that we restate here for convenience.

Lemma 8.1. *Let $H = (V, E)$ be a multigraph and $e \in E(H)$. Suppose that $(D - 1)H - 2e$ is D-tree-connected. Then $G_{(H,e)}$ is rigid in $\mathbb{R}^d$.*

Proof. To prove that $G_{(H,e)}$ is rigid in $\mathbb{R}^d$, it suffices to show that it has an infinitesimally rigid d-dimensional realization, see e.g. [27]. We shall prove that such a realization exists. Recall that $G_{(H,e)}$ is obtained by removing one vertex from $H(e)$. The set of the remaining $d - 2$ vertices in $H(e)$ is denoted by $H'(e)$.

By the assumption of the lemma $(\binom{d+1}{2} - 1)(H - e) + (\binom{d+1}{2} - 3)e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees $T_{i,j}, 1 \leq i < j \leq d + 1$. We must have at least three spanning trees containing no copies of e. Thus, by relabelling the trees, if necessary, we may assume that

$$\left(\left(\binom{d+1}{2}-3\right)e\right)\cap(T_{d-1,d}\cup T_{d-1,d+1}\cup T_{d,d+1})=\emptyset. \tag{4}$$

Let $e_1,\dots,e_d$ be the standard basis of $\mathbb{R}^d$. It will be convenient to denote the origin of $\mathbb{R}^d$ by e_{d+1}. We shall consider a realization of $G_{(H,e)}$ by defining $p: V(G_{(H,e)})\to\mathbb{R}^d$ as follows. For each $v\in V(H)$, we take a realization of the core $C(v)$ such that $\{p(v')\mid v'\in C(v)\}=\{e_i\mid 1\le i\le d+1\}$. For each $f\in E(H)\setminus\{e\}$, there is at least one pair (k,l) of indices such that $((\binom{d+1}{2}-1)f)\cap T_{k,l})=\emptyset$. Hence one can define the realization of the hinge $H(f)$ such that $\{p(v'')\mid v''\in H(f)\}=\{e_i\mid 1\le i\le d+1, i\neq k,l\}$. Also, for e, we shall define the realization of $H'(e)$ such that $\{p(v'')\mid v''\in H'(e)\}=\{e_i\mid 1\le i\le d-2\}$.

We shall show that $(G_{(H,e)},p)$ is infinitesimally rigid in $\mathbb{R}^d$. To this end, let us take an infinitesimal motion $m: V(G_{(H,e)})\to\mathbb{R}^d$ of $(G_{(H,e)},p)$. Since $\{p(v')\mid v'\in C(v)\}$ affinely spans $\mathbb{R}^d$ and $C(v)$ induces a complete graph, for each $v\in V(H)$, there is a $d\times d$ skew-symmetric matrix S_v and $t_v\in\mathbb{R}^d$ such that $m(v')=S_vp(v')+t_v$ for every $v'\in C(v)$ and $m(v'')=S_vp(v'')+t_v$ for every v'' adjacent to $C(v)$.

For $1\le i<j\le d+1$, consider any edge $f=uv\in T_{i,j}$. If $f\neq e$, by the definition of p, there is at one vertex $v''\in H(f)$ such that $p(v'')$ is either e_i or e_j. Similarly, if $f=e$, there is $v''\in H'(e)$ such that $p(v'')$ is either e_i or e_j by (4). Therefore, one can take a vertex v' from $C(v)$ such that $\{p(v''),p(v')\}=\{e_i,e_j\}$.

Let us assume $p(v'')=e_i$ and $p(v')=e_j$. The constraint (2) of edge $v'v''$ implies

$$\begin{aligned}0&=\langle p(v'')-p(v'),m(v'')-m(v')\rangle\\&=\langle p(v'')-p(v'),S_up(v'')+t_u-S_vp(v')-t_v\rangle\\&=\langle e_i-e_j,S_ue_i+t_u-S_ve_j-t_v\rangle\\&=-e_j^\top S_ue_i-e_i^\top S_ve_j+\langle e_i-e_j,t_u-t_v\rangle.\end{aligned}\tag{5}$$

This equation follows even when $p(v'')=e_j$ and $p(v')=e_i$ by changing the role of u and v.

For $j=d+1$, since $e_{d+1}=0$, (5) implies

$$\langle e_i,t_u-t_v\rangle=0 \qquad \text{for } 1\le i\le d \text{ and } uv\in T_{i,d+1}.$$

This implies $t_a=t_b$ for any pair $a,b\in V(H)$ since $T_{i,d+1}$ is a spanning tree. Therefore, using the skew-symmetry of S_v, (5) becomes

$$e_i^\top S_ve_j=e_i^\top S_ue_j \qquad \text{for } 1\le i<j\le d \text{ and } uv\in T_{i,j}.$$

As above, this implies $S_a=S_b$ for any pair $a,b\in V(H)$. Thus m is trivial and hence $(G_{(H,e)},p)$ is infinitesimally rigid in $\mathbb{R}^d$, as claimed. As we remarked above, we can now deduce that $G_{(H,e)}$ is rigid in $\mathbb{R}^d$. □

References

[1] T. G. Abbot: Generalizations of Kempe's universality theorem, MSc thesis, MIT (2008). http://web.mit.edu/tabbott/www/papers/mthesis.pdf

[2] R. Connelly: On generic global rigidity, in *Applied Geometry and Discrete Mathematics*, DIMACS Ser. Discrete Math, Theoret. Comput. Sci. 4, AMS, 1991, pp 147-155.

[3] R. Connelly: Generic global rigidity, *Discrete Comp. Geometry* 33 (2005), pp 549-563.

[4] R. Connelly: Questions, Conjectures and Remarks on Globally Rigid Tensegrities, manuscript, November 2009.

[5] R. Connelly, T. Jordán, and W. Whiteley: Generic global rigidity of body-bar frameworks, *J. Combinatorial Theory, Ser. B.*, Vol. 103, Issue 6, November 2013, pp. 689-705.

[6] R. Connelly and W. Whiteley: Global rigidity: the effect of coning, *Discrete Comp. Geometry*, Volume 43, Number 4, 717-735 (2010).

[7] S. Frank and J. Jiang: New classes of counterexamples to Hendrickson's global rigidity conjecture, *Discrete Comp. Geometry*, Volume 45, Number 3, 574-591 (2011).

[8] S. Gortler, A. Healy, and D. Thurston: Characterizing generic global rigidity, *American Journal of Mathematics*, Volume 132, Number 4, August 2010, pp. 897-939.

[9] B. Hendrickson: Conditions for unique graph realizations, *SIAM J. Comput* 21 (1992), pp 65-84.

[10] B. Jackson and T. Jordán: Connected rigidity matroids and unique realization graphs, *J. Combin. Theory Ser. B* 94, 2005, pp 1-29.

[11] B. Jackson and T. Jordán: Graph theoretic techniques in the analysis of uniquely localizable sensor networks, in: *Localization algorithms and strategies for wireless sensor networks*, G. Mao, B. Fidan (eds), IGI Global, 2009, pp. 146-173.

[12] B. Jackson, T. Jordán, and Z. Szabadka: Globally linked pairs of vertices in equivalent realizations of graphs, *Discrete Comp. Geometry*, Vol. 35, 493-512, 2006.

[13] T. Jordán: Generically globally rigid zeolites in the plane, *Information Proc. Letters*, Vol. 110, Issues 18-19, 2010, Pages 841-844.

[14] N. Katoh and S. Tanigawa: A proof of the molecular conjecture. *Discrete Comput. Geom.*, 45:647–700, 2011.

[15] C. Király: Algorithms for finding a rooted $(k, 1)$-edge-connected orientation, *Discrete Applied Mathematics* 166 (2014) 263-268.

[16] C.St.J.A. Nash-Williams: Edge disjoint spanning trees of finite graphs, *J. London Math. Soc.* 36 (1961), 445-450.

[17] J.B. Saxe: Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[18] A. Schrijver: *Combinatorial Optimization*, Springer 2003.

[19] S. Tanigawa: Sufficient conditions for the global rigidity of graphs, `arXiv:1403.3742v1`

[20] T-S. Tay: Rigidity of multi-graphs I, linking rigid bodies in n-space, *J. Combin. Theory Ser. B* 36 (1984), 95–112.

[21] T-S. Tay: Linking $(n-2)$-dimensional panels in n-space II:$(n-2, 2)$-frameworks and body and hinge structures. *Graphs and Combinatorics*, 5(1):245–273, 1989.

[22] T. Tay: Linking $(n-2)$-dimensional panels in n-space I:$(k-1, k)$-graphs and $(k-1, k)$-frames. *Graphs and Combinatorics*, 7(3):289–304, 1991.

[23] T-S. Tay: Henneberg's method for bar and body frameworks. *Structural Topology* 17 (1991) 53-58.

[24] W.T. Tutte: On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* 36 (1961) 221-230.

[25] W. Whiteley: The union of matroids and the rigidity of frameworks. *SIAM J. Discrete Math.*, 1(2):237–255, 1988.

[26] W. Whiteley: Vertex splitting in isostatic frameworks, *Structural Topology* 16 (1991), pp 23-30.

[27] W. Whiteley: Some matroids from discrete applied geometry, in Matroid theory (J.E. Bonin, J.G. Oxley and B. Servatius eds., Seattle, WA, 1995), Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.

[28] W. Whiteley: Rigidity and scene analysis, *in* J. E. Goodman and J. O'Rourke (eds.), *Handbook of Discrete and Computational Geometry,* 2nd ed., CRC Press, Boca Raton, 2004, pp. 1327-1354.

[29] W. Whiteley: Counting out to the flexibility of molecules, *Physical Biology* 2 (2005) S116-S126.