Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2014-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Edge-Disjoint Paths Problem in Highly Connected, Infinite Graphs

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October 2014
Revised December 2014

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Abstract

We construct for all $k \in \mathbb{N}$ a k-edge-connected digraph D with $x_1, y_1, x_2, y_2 \in V(D)$ such that there are no edge-disjoint $x_1 \to y_1$ and $x_2 \to y_2$ paths. We also prove that contrary to the directed case, for undirected graphs, $(2n-1)$-edge-connectivity implies the linkability of arbitrary n terminal pairs with edge-disjoint paths.

1 Introduction

1.1 Basic notions

In this paper by "path" we mean a finite, simple path (which is directed in the context of digraps). As usual, we define a path of a digraph $D = (V, A)$ as a finite sequence $v_0, \ldots, v_n$ of vertices of D. If there are more than one edges from v_i to v_{i+1} for some $i < n$ then it is not specified which edge is used by the path, so we use this kind of definition only if it does not matter. An $u \to v$ path is a path with starting-point u and endpoint v. Its length is the number of its edges. For $U \subseteq V$ let $\mathsf{span}_D(U)$ be the set of those edges of D whose starting- and endpoints are contained in U and let $D[U] \overset{\text{def}}{=} (U, \mathsf{span}_D(U))$. If it is clear what digraph we talk about, then we omit the subscripts.

In the edge-disjoint paths problem (from now on EPP) we have a (di)graph $G = (V, E)$ and terminal pairs $x_1, y_1, \ldots, x_m, y_m \in V$. We want to find pairwise edge-disjoint paths in G to connect the terminal pairs. If G is directed then we also demand the paths to be directed and go in the $x_i \to y_i$ direction.

If G is a (directed) graph and T is a set of (ordered) pairs of its vertices with possibly multiplicity then we denote by $\mathrm{EPP}(G, T)$ the EPP determined by G and T, and we call it satisfiable iff the desired paths exist.

1.2 Background and Motivation

Several highly nontrivial sufficient conditions are known for the satisfiability of $\mathrm{EPP}(G, T)$ in the finite case (about these and about computational complexity results one can find a survey in [4]). Let us see the edge-connectivity based conditions.

Theorem 1.1. *EPP(D,T) is satisfiable if D is a $|T|$-edge-connected finite digraph.*

Theorem 1.2. *EPP(G,T) is satisfiable if G is a $(2|T|-1)$-edge-connected finite graph.*

These theorems are immediate consequences of the following theorems of W. Mader. Theorem 1.1 is also derivable easily from Edmonds' Branching theorem (see [5]).

Theorem 1.3 (W. Mader [1])**.** *Let $D=(V,A)$ be a $k+1$-edge-connected, finite digraph and $x,y\in V$. Then there is an $x\to y$ path P such that $(V,A\setminus A(P))$ is k-edge-connected.*

Theorem 1.4 (W. Mader [2])**.** *Let $G=(V,E)$ be a $k+2$-edge-connected, finite graph and $x,y\in V$. Then there is a path P between x and y such that $(V,E\setminus E(P))$ is k-edge-connected.*

The following theorem of R. Aharoni and C. Thomassen showed that the finiteness conditions of the graphs cannot be omitted in the theorems of W. Mader above and in Edmonds' Branching theorem. (In the case of Edmonds' Branching theorem the finiteness condition can be weaken to disallow forward-infinite paths, see [6].)

Theorem 1.5 (R. Aharoni, C. Thomassen [3])**.** *For all $k\in\mathbb{N}$ there is an infinite graph $G=(V,E)$ and $x,y\in V$ such that E has a k-edge-connected orientation but for each path P between x and y the graph $G=(V,E\setminus E(P))$ is not connected.*

This motivated us to investigate the connection between satisfiability of EPP and edge-connectivity in infinite (di)graphs.

2 Main results

In the undirected case we will give an elementary proof for that Theorem 1.2 remains true without the finiteness-condition for G. In the directed case case we will show that Theorem 1.1 without the finiteness-condition for D becomes "very false": there is no $k\in\mathbb{N}$ such that k-edge-connectivity would imply the satisfiability even for two terminal pairs.

2.1 Directed case

Our result is that for arbitrary $k\in\mathbb{N}$ there exists a k-edge-connected infinite digraph D and a two-element set of terminal pairs T such that EPP(D,T) is unsatisfiable. In fact, our terminal pairs will be reverses of each other.

Theorem 2.1. *For all $k\in\mathbb{N}$ there exists a k-edge-connected digraph without back and forth edge-disjoint paths between a certain vertex pair.*

Proof. Let $k \geq 3$ be fixed, $I = \{0, \ldots, 2k-1\}$, $I_e = \{i \in I : i \text{ is even }\}$, $I_o = I \setminus I_e$. Denote by I^* the set of finite sequences from I. Let the vertex-set of the digraph be $V = \{s_\mu, t_\mu : \mu \in I^*\}$, where if μ is the empty sequence we write simply s, t and we denote the concatenation of sequences by writing them successively. Let the edge-set A be the following. There are k edges in both directions for all $\mu \in I^*$ between the following pairs: $\{s_\mu, t_{\mu 1}\}$, $\{s_{\mu i}, t_{\mu(i+2)}\}$ $(i = 0, \ldots, 2k-3)$, $\{s_{\mu(2k-2)}, t_\mu\}$. Simple directed edges are $(s_\mu, t_{\mu 0}), (t_{\mu i}, s_{\mu(i+1)})_{i \in I_e}, (s_{\mu i}, t_{\mu(i+1)})_{i \in I_o \setminus \{2k-1\}}, (s_{\mu(2k-1)}, t_\mu)$ for all $\mu \in I^*$. Finally $D \overset{\text{def}}{=} (V, A)$ (see figure 1).

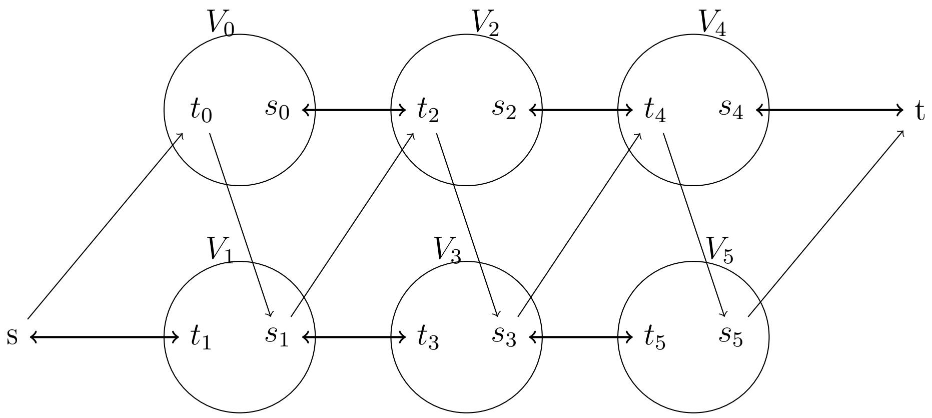

Figure 1: The digraph D in the case $k = 3$. Thick, two-headed arrows stand for k parallel edges in both directions. The (just partially drawn) $D[V_i]$-s are isomorphic to the whole D by Proposition 2.3.

Remark 2.2. One can avoid using parallel edges (without losing the desired properties of the digraph) by dividing each of these edges with one-one new vertex and drawing between them k^2-many new directed edges, one-one for each ordered pair. One can also achieve k-connectivity instead of k-edge-connectivity by using some similarly easy modification.

Proposition 2.3. *For $\nu \in I^*$ let $V_\nu = \{r_{\nu\mu} : r \in \{s, t\},\ \mu \in I^*\}$. If $\nu \in I^*$, then $f_\nu : V \to V_\nu,\ f_\nu(r_\mu) \overset{def}{=} r_{\nu\mu}$ $(r \in \{s, t\})$ is an isomorphism between D and $D[V_\nu]$.*

Proof: It is a direct consequence of the definition of the edges, because the number of edges from r_μ to $r'_{\mu'}$ are the same as from $r_{\nu\mu}$ to $r'_{\nu\mu'}$ for all $r, r' \in \{s, t\},\ \nu, \mu, \mu' \in I^*$.
●

Proposition 2.4. *Denote by D_v the digraph that we get from D by contracting the set V_i to a vertex v_i $(i \in I)$. Then D_v is k-edge-connected.*

Proof: In the vertex-sequence $s, v_1, v_3, \ldots, v_{2k-1}$ there are k edges in both directions between the neighboring vertices such as in the sequence $v_0, v_2, \ldots, v_{2k-2}, t$. There are also at least k edges in both direction between the vertex-sets of the sequences above.
●

Recall, that $\lambda(u,v)$ denotes the local edge-connectivity from u to v in D (i.e. $\lambda(u,v) = \min\{|A'| : A' \subseteq A,$ there is no path from u to v in $(V, A \setminus A')\})$ and let $\lambda\{u,v\} \stackrel{\text{def}}{=} \min\{\lambda(u,v), \lambda(v,u)\}$.

Proposition 2.5. *D is connected.*

Proof: We will show that $\lambda\{s, r_\mu\} \geq 1$ for all $r \in \{s,t\}$, $\mu \in I^*$. We will use induction on length of μ (which is denoted by $|\mu|$) but first we handle the $|\mu| = 0, 1$ cases directly.

The path $s, t_0, s_1, t_2, s_3, \ldots, t_{2k-2}, s_{2k-1}, t$ shows that $\lambda(s,t) \geq 1$. Using the isomorphism f_i (see Proposition 2.3) we may fix an $s_i \to t_i$ path P_{s_i,t_i} in $D[V_i]$ for all $i \in I$. The path

$$t, P_{s_{2k-2},t_{2k-2}}, \ldots, P_{s_{2k-2j},t_{2k-2j}}, \ldots, P_{s_0,t_0}, P_{s_1,t_1}, s$$

justifies that $\lambda(t,s) \geq 1$ (thus $\lambda\{s,t\} \geq 1$). Then we may fix a $t_i \to s_i$ path P_{t_i,s_i} in $D[V_i]$ $(i \in I)$. The paths

$$s, P_{t_1,s_1}, P_{t_3,s_3}, \ldots, P_{t_{2j+1},s_{2j+1}}, \ldots, P_{t_{2k-1},s_{2k-1}}$$
$$P_{s_{2k-1},t_{2k-1}}, P_{s_{2k-3},t_{2k-3}}, \ldots, P_{s_{2k-1-2j},t_{2k-1-2j}}, \ldots, P_{s_1,t_1}, s$$

certify that $\lambda\{s, r_i\} \geq 1$ if $r \in \{s,t\}$, $i \in I_o$. The paths

$$t, P_{s_{2k-2},t_{2k-2}}, P_{s_{2k-4},t_{2k-4}}, \ldots, P_{s_{2k-2-2j},t_{2k-2-2j}} \ldots, P_{s_0,t_0}$$
$$P_{t_0,s_0}, P_{t_2,s_2}, \ldots, P_{t_{2j},s_{2j}}, \ldots, P_{t_{2k-2},s_{2k-2}}, t$$

certify that $\lambda\{t, r_i\} \geq 1$ if $r \in \{s,t\} \geq 1$, $i \in I_e$ and so (by $\lambda\{s,t\} \geq 1$ and by transitivity) $\lambda\{s, r_i\} \geq 1$ if $r \in \{s,t\}$, $i \in I_e$. Combining these, we have $\lambda\{s, r_i\} \geq 1$ if $r \in \{s,t\}, \mu \in I^*, |\mu| \leq 1$.

Let be $l \geq 1$ and suppose $\lambda\{s, r_\mu\} \geq 1$ if $r \in \{s,t\}$, $\mu \in I^*$, $|\mu| \leq l$. Let $\nu = \mu i$, where $i \in I$ and $|\mu| = l$. By the induction hypothesis we have $\lambda\{s, s_\mu\} \geq 1$. By the induction hypothesis for $l = 1$ we have $\lambda\{s, r_i\} \geq 1$ and so $\lambda\{s_\mu, r_{\mu i}\} \geq 1$ by the isomorphism f_μ. Combining these, we get $\lambda\{s, r_{\mu i}\} \geq 1$. ●

Lemma 2.6. *D is k-edge-connected.*

Proof: Let $k > l \geq 1$.

Proposition 2.7. *For all $\mu \in I^*$ if we delete at most l edges of the digraph $D[V_\mu]$ in such a way that its subgraphs $D[V_{\mu i}]$ $(i \in I)$ remain connected after the deletion, then $D[V_\mu]$ also remains connected after the deletion.*

Proof: Because the isomorphisms f_μ $(\mu \in I^*)$ it is enough to deal with the case $\mu = \varnothing$. Denote by D' the digraph that we have after the deletion. Let D'_v be the digraph that we get from D' by contracting the set V_i to a vertex v_i $(i \in I)$. The digraphs $D'[V_i]$ $(i \in I)$ are connected by assumption, thus D' is connected iff D'_v is connected. The digraph D'_v arises by deleting at most l edges of the k-edge-connected digraph D_v (see Proposition 2.4) so it is connected. ●

We will prove that if D is l-edge-connected then it is also $l+1$ edge-connected. This is enough because we have already proved 1-connectivity of D in Proposition 2.5. Assume that D is l-edge-connected. Let $C \subset A$, $|C| = l$ arbitrary and $D' \stackrel{\text{def}}{=} (V, A \backslash C)$. By the definition of $l+1$-edge connectivity we need to show that D' is connected. Suppose for contradiction that it is not. The connectivity of the subgraphs $D'[V_i]$ ($i \in I$) implies the connectivity of D' (by Proposition 2.7) hence there is an $i_0 \in I$ such that $D'[V_{i_0}]$ is not connected. The connectivity of the subgraphs $D'[V_{i_0 i}]$ ($i \in I$) implies the connectivity of $D'[V_{i_0}]$ hence there is an $i_1 \in I$ such that $D'[V_{i_0 i_1}]$ is not connected... By recursion we have an infinite sequence $(i_n)_{n \in \mathbb{N}}$ such that the digraphs $D'[V_{i_0 \ldots i_n}]$ ($n \in \mathbb{N}$) are all disconnected. Note that the digraphs $D[V_{i_0 \ldots i_n}]$ ($n \in \mathbb{N}$) are l-connected because D is l-connected by assumption and they are isomorphic to it, so necessarily $C \subset \mathsf{span}(V_{i_0 \ldots i_n})$ for all $n \in \mathbb{N}$. But then

$$C \subset \bigcap_{n=0}^{\infty} \mathsf{span}(V_{i_0 \ldots i_n}) = \mathsf{span}\left(\bigcap_{n=0}^{\infty} V_{i_0 \ldots i_n}\right) = \mathsf{span}(\varnothing) = \varnothing$$

which is a contradiction because $|C| = l \geq 1$. ■

Lemma 2.8. *Does not exist edge-disjoint back and forth paths between s and t in D.*

Proof: Suppose, seeking a contradiction, that there are. Let $P_{s,t}$ be an $s \to t$ path and $P_{t,s}$ be a $t \to s$ path such that they are edge-disjoint and have a minimal sum of lengths among these path-pairs. For $u, v \in V$ call a set $U \subseteq V$ an uv-cut iff $u \in U$ and $v \notin U$. The set $\{t\} \cup \bigcup\{V_i : i \in I_e\}$ is a ts-cut and its outgoing edges are $\{(t_i, s_{i+1})\}_{i \in I_e}$. Let $i_0 \in I_e$ be the maximal index such that $P_{t,s}$ uses the edge $(t_{i_0} s_{i_0+1})$. Then an initial segment of $P_{t,s}$ is necessarily of the form $t, P_{s_{2k-2},t_{2k-2}}, P_{s_{2k-4},t_{2k-4}}, \ldots, P_{s_{i_0},t_{i_0}}, s_{i_0+1}$ where P_{s_i,t_i} is an $s_i \to t_i$ path in $D[V_i]$. The set $T \stackrel{\text{def}}{=} \{t\} \cup \bigcup\{V_i : i_0 \leq i \in I\}$ is also a ts-cut and all the tails of its outgoing edges are in $\{t_{i_0}, t_{i_0+1}\}$. $P_{t,s}$ has already used the edge (t_{i_0}, s_{i_0+1}) so it may not use another edge with tail t_{i_0} hence $P_{t,s}$ leave T using an edge with tail t_{i_0+1}. But then $P_{t,s}$ contains an $s_{i_0+1} \to t_{i_0+1}$ subpath $P_{s_{i_0+1},t_{i_0+1}}$ in $D[V_{i_0+1}]$.
$S \stackrel{\text{def}}{=} \{s\} \cup \bigcup\{V_i : i_0 + 1 \geq i \in I\}$ is an st-cut and all the tails of its outgoing edges are in $\{s_{i_0}, s_{i_0+1}\}$. Therefore $P_{s,t}$ has an initial segment in $D[S]$ that ends in this set. We know that $P_{s,t}$ does not use the edge (t_{i_0}, s_{i_0+1}) because $P_{t,s}$ has already used it, so there is an $m \in \{i_0, i_0+1\}$ such that $P_{s,t}$ has a $t_m \to s_m$ subpath P_{t_m,s_m} in $D[V_m]$. But then the paths P_{t_m,s_m} and P_{s_m,t_m} are proper subpaths of $P_{s,t}$ and $P_{t,s}$ respectively. By Proposition 2.3 f_m is an isomorphism between D and $D[V_m]$ and thus the inverse-images of the paths P_{t_m,s_m} and P_{s_m,t_m} are edge-disjoint back and forth paths between s and t with strictly less sum of lengths than the added length of paths $P_{s,t}$ and $P_{t,s}$, which contradicts to the choice of $P_{s,t}$ and $P_{t,s}$.■

□

2.2 Undirected case

For infinite, undirected graphs the analog finite Theorem 1.2 remains true (contrary to the directed case).

Theorem 2.9. *Let $G = (V, E)$ be a possibly infinite, $(2n-1)$-edge-connected, undirected graph for some $n > 0$. Let $x_1, y_1, \ldots, x_n, y_n \in V$ be not necessarily distinct vertices. Then there exists a system of pairwise edge-disjoint paths P_j $(j \leq n)$ in G such that P_j links x_j and y_j $(j \leq n)$.*

Proof. We use induction on n. The case $n = 1$ is trivial. Assume we know the theorem for some n, then we have to show that it is true for $n+1$. Let $G = (V, E)$ be a $(2n+1)$-edge-connected graph and $x_1, y_1, \ldots, x_{n+1}, y_{n+1} \in V$. Fix a system P_j $(j \leq 2n+1)$ of pairwise edge-disjoint paths between x_{n+1} and y_{n+1}. By the induction hypothesis we can also fix pairwise edge-disjoint paths between x_k and y_k for $k = 1, \ldots, n$. We color the edges of these paths with pairwise distinct colors.

Case 1 There is a $j_0 \leq 2n+1$ such that P_{j_0} does not have colored edges.
Then P_{j_0} and the colored paths form a desired system.

Case 2 Each P_j has at least one colored edge.
Let e_j be the first colored edge in P_j with respect to the $x_{n+1} \to y_{n+1}$ direction. These $2n+1$ edges are colored with at most n colors thus there is a color, say red, which appears at lest three times. Let $\mathcal{R}$ be the set of those P_j paths whose first colored edge is red. Let U be the set of those vertices which can be reached in some $P_j \in \mathcal{R}$ from x_{n+1} using only uncolored and red edges. Denote by u, v the first and the last intersection of the red path (R from now on) with U respectively in respect to a fix direction of R (see figure 2 bellow).

Replace the segment of R between u and v by the $[u, x_{2n+1}]$ and $[x_{2n+1}, v]$ segments of the appropriate P_j paths or the $[u, v]$ segment of the appropriate P_j path if u and v are in the same P_j. Observe that the modified R is still a path with the original endpoints and it is still edge-disjoint from the other colored paths.

Then jump to the beginning of the case distinction and repeat the process with the modified set of colored paths.

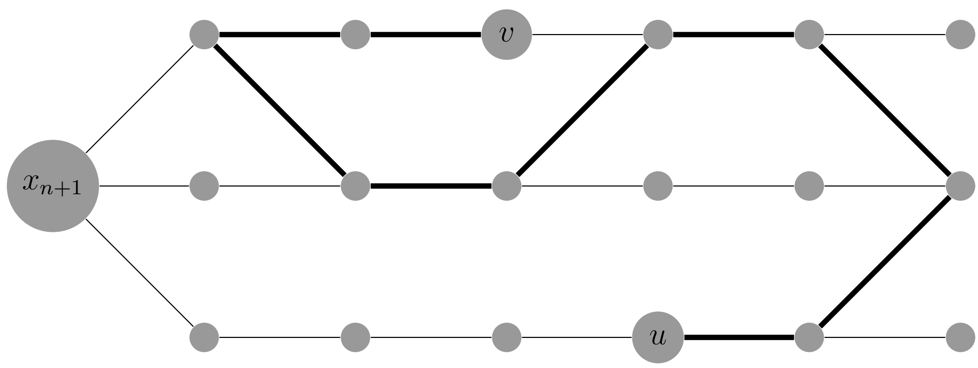

Figure 2: A subgraph of G with point-set U in case $|\mathcal{R}| = 3$, thin edges stand for uncolored and thick edges stand for red edges.

Lemma 2.10. *Case 2 can occur at most $|\cup_j E(P_j)|$-times thus the process above terminates after finitely many steps.*

Proof: We define an equivalence relation on the colored edges of the P_j paths. Let $e \sim f$ iff e and f are on the same P_j and the edges of this P_j between e and f (including e and f) have the same color. Note that for every path in $\mathcal{R}$ its first equivalence class is red and these equivalence classes are subsets of the edges of R between u and v. We will show that after the replacement in case 2 the number of equivalence classes is reduced by at least one. When we remove the segment of R between u and v in case 2 if we remove an edge then we also remove its equivalence class because of the choice of u or v. For at most two paths from $\mathcal{R}$ (paths that contain u and v), their first equivalence class may still be red after the replacement in R, but for every other path $P \in \mathcal{R}$ ($|\mathcal{R}| \geq 3$ so at least one) its first equivalence class will vanish, since its edges were on the old R but not on the new one. ■

□

3 An open problem

First we remark an easy consequence of our main result in the directed case. Let $D = (V, A)$ be a k-edge-connected digraph without back and forth edge-disjoint paths between $s, t \in V$ for some fixed $k \in \mathbb{N}$. Such a digraph exists for all $k \in \mathbb{N}$ by Theorem 2.1. Take three disjoint copies of this digraph (denoted by D, D', D''), identify the vertices t, t', t'' and denote the resulting digraph by D^*. Let $T = \{\langle s, s' \rangle, \langle s', s'' \rangle, \langle s'', s \rangle\}$ then it is a routine to check that for all two-element subsets T' of T the problem EPP(D^*, T') is unsatisfiable. This motivates the following conjecture.

Conjecture 3.1. *For all $k \in \mathbb{N}$ there exists a k-edge-connected digraph $D = (V, A)$ and a k-element set $T \subset V^2$ such that for all two-element subsets T' of T EPP(D, T') is unsatisfiable.*

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