Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2014-12. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

COMBINATORIAL RIGIDITY: GRAPHS AND MATROIDS IN THE THEORY OF RIGID FRAMEWORKS

Tibor Jordán[1]

September 23, 2014

[1]Department of Operations Research, Eötvös University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. email: jordan@cs.elte.hu

Contents

Chapter 1

Rigid and globally rigid frameworks

1.1 Preface

This paper is based on the material I presented at the Research Institute for Mathematical Sciences (RIMS), Kyoto University, in October 2012 in a series of lectures. Thus, on one hand, it serves as the lecture note of this minicourse *Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks*. On the other hand, this final, extended form is perhaps closer to a short monograph on combinatorial rigidity problems of two-dimensional frameworks. It contains the fundamental results of this area as well as a number of more recent results concerning extensions, variations and applications. I have also added several exercises and some new results[1].

In spite of the diversity of the results presented in this paper there is also a long list of interesting topics that had to be omitted. We shall consider finite bar-and-joint frameworks in generic position in two-dimensional Euclidean space and the associated matroid. Thus we shall not deal with infinite frameworks, other types of frameworks (body-bar, body-hinge, body-pin) or constraints (direction, angle, affine, etc.) or manifolds or metrics. We shall not consider symmetric or periodic frameworks or tensegrities. We shall not consider random graphs either or polymatroids and other count matroids.

After the first introductory chapter we shall focus on the two-dimensional results even though in many cases the proofs and results extend to higher dimensions.

Acknowledgements

I thank RIMS, especially Satoru Iwata and Shin-ichi Tanigawa for their hospitality during my stay in Kyoto. Quite a few results presented here can be found in joint papers with various co-authors of mine. I am especially grateful to Bill Jackson for the uncountably many enjoyable discussions on different graph and matroid problems.

[1]The results presented in the following subsections are new: 2.6.3, 3.6.1, 3.6.4, 3.9.3. In addition, some new observations appear as exercises.

1.2 Rigid frameworks

In this rest of this chapter we briefly summarize the fundamental geometric and algebraic definitions and facts about d-dimensional frameworks that lead us to the combinatorial problems investigated in this work. For a more detailed introduction the reader is referred to [21, 64, 63].

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$.

We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). A *flexing* of the framework (G, p) is a continuous function $\pi : (-1, 1) \times V \to \mathbb{R}^d$ such that $\pi_0 = p$, and such that the frameworks (G, p) and (G, π_t) are equivalent for all $t \in (-1, 1)$, where $\pi_t : V \to \mathbb{R}^d$ is defined by $\pi_t(v) = \pi(t, v)$ for all $v \in V$. The flexing π is *trivial* if the frameworks (G, p) and (G, π_t) are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial flexing. It is known [2, 19] that non-rigidity, flexibility and the existence of a non-trivial smooth flexing are all equivalent.

It is a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [55] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider *generic frameworks* i.e. frameworks in which there are no algebraic dependencies between the coordinates of the vertices.

The first-order version of a flexing of the framework (G, p) is called an *infinitesimal motion*. This is an assignment of *infinitesimal velocities* to the vertices of G, $\tilde{p} : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(\tilde{p}(u) - \tilde{p}(v)) = 0 \text{ for all pairs } u, v \text{ with } uv \in E. \tag{1.1}$$

If π is a smooth flexing of (G, p), then $\dot{\pi}_0 := \frac{d\pi}{dt}|_{t=0}$ is an infinitesimal motion of (G, p). A *trivial infinitesimal motion* of (G, p) has the form $\tilde{p}(v) = Ap(v)+b$, for all $v \in V$, for some $d \times d$ antisymmetric matrix A and some $b \in \mathbb{R}^d$. Equivalently, an infinitesimal motion is trivial if it belongs to the kernel of $R(K_{|V|}, p)$, where K_n denotes the complete graph on n vertices. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is *infinitesimally rigid*. Gluck [19] proved that if a framework (G, p) is infinitesimally rigid, then it is rigid. The converse of this is not true in general, but if we exclude certain 'degenerate' configurations, for example, when we consider generic frameworks, then rigidity and infinitesimal rigidity are equivalent (see Section 1.3 below).

The set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{d|V|}$, given by the system of $|E|$ linear equations (1.1). The matrix of this system of linear equations is the *rigidity matrix* $R(G, p)$ of (G, p) of size $|E| \times d|V|$, where, for each edge $e = v_iv_j \in E$, in the row corresponding to e, the entries in the two columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros.

Example. The rigidity matrix of the framework of Figure 1.1(a) is as follows. The rows correspond to edges ab, bc, ca, cd, in this order, and consecutive pairs of columns correspond to vertices a, b, c, d.

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Thus $\tilde{p}$ (viewed as a vector in $\mathbb{R}^{d|V|}$) is an infinitesimal motion if and only if $R(G, p)\tilde{p} = 0$. Each translation and rotation of $\mathbb{R}^d$ gives rise to a smooth motion of (G, p) and hence to an infinitesimal motion of (G, p). These rigid motions (or equivalently, the trivial infinitesimal motions) of $\mathbb{R}^d$ give rise to a subspace of dimension $\binom{d+1}{2}$ in the null-space of $R(G, p)$. Hence

Lemma 1.2.1. *[63, Lemma 11.1.3] Let (G, p) be a framework in $\mathbb{R}^d$. Then*

$$\text{rank}\, R(G, p) \leq S(n, d), \tag{1.2}$$

where $n = |V(G)|$ and

$$S(n, d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d + 2 \\ \binom{n}{2} & \text{if } n \leq d + 1. \end{cases}$$

Thus a framework (G, p) is infinitesimally rigid in $\mathbb{R}^d$ if the rank of its rigidity matrix $R(G, p)$ is maximum, i.e. if equality holds in (1.2). We say that (G, p) is *independent* if the rows of $R(G, p)$ are linearly independent. An independent and infinitesimally rigid framework is called *minimally infinitesimally rigid.*

Infinitesimal rigidity can also be characterized by equilibrium loads as follows. An *equilibrium load* on a configuration p of vertex set V is an assignment $L : V \to \mathbb{R}^d$ of vectors to the vertices "without net translational or rotational component". More precisely, an equilibrium load is a vector in $\mathbb{R}^{dn}$ orthogonal to the kernel of $R(K_{|V|}, p)$. In particular, (the d-tuples of) each row of the rigidity matrix $R(K_{|V|}, p)$ form an equilibrium load on p. Thus the row space of $R(G, p)$ is a subspace of the space of equilibrium loads. The equilibrium loads form a subspace of $\mathbb{R}^{d|V|}$ of dimension $S(|V|, d)$ (provided that the affine span of the points is $\mathbb{R}^d$, or they are affine independent).

A *resolution* of equilibrium load L by (G,p) is a *stress*, which is an assignment of scalars $\omega : E \to \mathbb{R}$ to the edges such that for each vertex $v_i \in V$:

$$L(v_i) + \sum_{j:v_iv_j \in E} \omega_{i,j}(p(v_i) - p(v_j)) = 0, \tag{1.3}$$

where we use $\omega_{i,j}$ to denote the stress on edge v_iv_j.
Let $R_{i,j}(p)$ denote the row of $R(G,p)$ corresponding to edge v_iv_j. With this notation we have that

$$L + \sum_{v_iv_j \in E} \omega_{i,j} R_{i,j}(p) = 0. \tag{1.4}$$

By definition, (G,p) is infinitesimally rigid if the dimension of the row space equals the dimension of the space of equilibrium loads. It follows that a d-dimensional framework (G,p) is infinitesimally rigid if and only if every equilibrium load L on p has a resolution in the bars of (G,p), see [63, Theorem 3.1.1].

A *self-stress* on framework (G,p) is an assignment $\omega : E \to \mathbb{R}$ such that, for each vertex $v_i \in V$:

$$\sum_{j:v_iv_j \in E} \omega_{i,j}(p(v_i) - p(v_j)) = 0. \tag{1.5}$$

Thus a self-stress is a resolution of the zero equilibrium load. The self-stresses are the row dependencies of the rigidity matrix $R(G,p)$. If the framework is independent then the resolution of an equilibrium load, if it exists, is unique. However, if the framework is dependent then we can add any multiple of a self-stress to a given resolution to get another resolution.

Let $\mathcal{S}(G,p)$ be the vector space of self-stresses of (G,p) and let $\mathcal{M}(G,p)$ be the vector space of infinitesimal motions of (G,p). The following equality is well-known: for a d-dimensional framework (G,p) we have

$$\operatorname{rank} R(G,p) = |E| - \dim(\mathcal{S}(G,p)) = d|V| - \dim(\mathcal{M}(G,p)) \tag{1.6}$$

1.2.1 Operations on frameworks

We shall frequently use the (two-dimensional versions of the) following simple operations. Given a graph $G = (V,E)$, the (d-dimensional) 0-*extension* operation, which is sometimes called *vertex d-addition* or a *Henneberg move of type I*, adds a new vertex v_0 and d new edges $v_0v_1, ..., v_0v_d$ for some $v_i \in V$, $1 \le i \le d$. The corresponding geometric operation on (G,p) adds a new vertex positioned at $p(v_0)$ and inserts d new bars from $p(v_0)$ to $p(v_i)$, $1 \le i \le d$.

Lemma 1.2.2. *[63, Lemma 11.1.1] Let (G,p) be a d-dimensional framework and let (G',p) be obtained from (G,p) by a 0-extension. If $p(v_0), p(v_1), ..., p(v_d)$ are in general position in d-space then $\operatorname{rank} R(G',p) = \operatorname{rank} R(G,p) + d$.*

Given a graph $G = (V,E)$ with a designated edge $e = v_iv_j$, and $d-1$ additional vertices $v_1, \dots, v_{d-1}$, the (d-dimensional) 1-*extension* operation on e, which is sometimes called *edge d-split* or a *Henneberg move of type II*, adds a new vertex v_0, removes e, and inserts $d+1$ new edges $v_0v_i, v_0v_j, v_0v_1, v_0v_2, ..., v_0v_{d-1}$. The corresponding geometric operation on (G,p) adds a new vertex positioned at $p(v_0)$, subdividing the bar of e, and inserts $d-1$ new bars from the new vertex to each $p(v_i)$, $1 \leq i \leq d-1$.

Lemma 1.2.3. *[63, Lemma 11.1.7.] Let (G,p) be a d-dimensional framework and let (G',p) be obtained from (G,p) by a 1-extension. If $p(v_i), p(v_j), p(v_1), ..., p(v_{d-1})$ are in general position in d-space then $\text{rank}\, R(G',p) = \text{rank}\, R(G,p) + d$.*

1.3 Rigid and globally rigid graphs

The analysis and characterization of rigid and globally rigid frameworks become more tractable if we consider generic frameworks: a framework (G,p) is *generic* if the set of coordinates of the points $p(v)$, $v \in V(G)$, is algebraically independent over the rationals[2].

It is known, see [63], that the rigidity and infinitesimal rigidity of a d-dimensional framework (G,p) are equivalent if (G,p) is generic. Thus the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G,p) depends only on the graph G and not the particular realization p, if (G,p) is generic. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1,2$ and is a major open problem for $d \geq 3$. A similar situation holds for global rigidity. Gortler, Healy and Thurston [20] proved that global rigidity of frameworks in $\mathbb{R}^d$ is a generic property for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. As for rigidity, the problem of characterizing when a generic framework is globally rigid in $\mathbb{R}^d$ has been solved for $d = 1,2$ and it is an important open problem to characterize globally rigid graphs when $d \geq 3$.

1.3.1 The rigidity matroid

The rigidity matrix of a d-dimensional framework (G,p) defines the *rigidity matroid* of (G,p) on the ground set E where a set of edges $F \subseteq E$ is independent if and only if the rows of the rigidity matrix indexed by F are linearly independent. (For more details on matroids and related combinatorial results the reader is referred to [15, 56, 53].) Since the entries of the rigidity matrix are polynomial functions with integer coefficients, any two generic d-dimensional frameworks (G,p) and (G,q) have the same rigidity matroid. We call this the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. It follows from the discussions above that a graph G on n vertices is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = S(n,d)$. We say that a graph $G = (V,E)$ is *M-independent* in $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. It is not difficult to see that $\mathcal{R}_1(G)$ is the circuit matroid of G. It remains an open problem to find good characterizations for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

[2]In fact, most results on rigid graphs and the rigidity matroid mentioned in this work remain valid with a much weaker version of genericity: it suffices to require that the rank of each edge-induced submatrix of $R(G,p)$ be maximum over all realizations of G.

Lemma 1.2.1 implies the following necessary condition for G to be M-independent. For a subset $X \subseteq V$ of vertices in graph $G = (V, E)$ we use $i(X)$ to denote the number of edges induced by X in G.

Lemma 1.3.1. *If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$.*

Note that, since G is simple, we automatically have $i(X) \leq S(|X|, d) = \binom{|X|}{2}$ when $|X| \leq d + 1$.

The converse of Lemma 1.3.1 also holds for $d = 1, 2$. The case $d = 1$ follows from the fact that the 1-dimensional rigidity matroid of G is the same as the circuit matroid of G. The case $d = 2$ is a result of Laman that we shall prove in the next chapter.

1.3.2 Globally rigid graphs

Hendrickson verified the following necessary conditions for a graph to be globally rigid in $\mathbb{R}^d$. We call a graph G *redundantly rigid* in $\mathbb{R}^d$ if G has at least two edges and $G - e$ is rigid in $\mathbb{R}^d$ for all $e \in E(G)$.

Theorem 1.3.2. *[22] If G is globally rigid in $\mathbb{R}^d$ then either G is a complete graph with at most $d + 1$ vertices, or G is $(d + 1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

The following sufficient condition was proved by Connelly, see [7, Proof of Corollary 1.7].

Theorem 1.3.3. *[7] Suppose that G can be obtained from K_{d+2} by a sequence of 1-extensions and edge additions. Then G is globally rigid in $\mathbb{R}^d$.*

This theorem will be a key step in proving that the necessary conditions for global rigidity given in Theorem 1.3.2 are also sufficient when $d = 2$.

1.3.3 Exercises

Exercise 1.3.4. *Let (G, p) be a framework in $\mathbb{R}^1$ for which $p(u) \neq p(v)$ for all edges $uv \in E(G)$. Prove that (G, p) is infinitesimally rigid if and only if G is connected.*

Exercise 1.3.5. *Verify that $\mathcal{R}_1(G)$ is isomorphic to the circuit matroid of G.*

Exercise 1.3.6. *Let G be a rigid graph in $\mathbb{R}^2$. Show that there is an infinitesimally rigid two-dimensional realization (G, p) in which all coordinates are integers between 1 and $|V|$.*

Exercise 1.3.7. *Let (G, p) be a d-dimensional framework and $v_h, v_k \in V(G)$. Prove that the following are equivalent:*
(i) $R_{h,k}(p)$ cannot be resolved,
(ii) every self-stress ω on $E \cup \{v_h v_k\}$ is zero on $v_h v_k$,
(iii) there is an infinitesimal motion u of (G, p), such that $(p(v_h) - p(v_k))(u(v_h) - u(v_k)) \neq 0$.

Exercise 1.3.8. *Develop a polynomial time algorithm for testing whether a graph G satisfies the sparsity condition of Lemma 1.3.1 (i) for $d = 2$, and (ii) for any fixed integer $d \geq 2$.*

1.4 Pinned frameworks

Let $G = (V, E)$ be graph and consider a d-dimensional realization (G, p) of G. We may fix (G, p) in $\mathbb{R}^d$ by restricting the infinitesimal motions of its vertices to given subspaces of $\mathbb{R}^d$. Suppose that for all vertices $v \in V$ we are given a subspace $U(v) \subseteq \mathbb{R}^d$, generated by a subset of the standard basis of $\mathbb{R}^d$. We call $U(v)$ the *track* of v and we say that (G, p) is *fixed* by the given set of tracks if the only infinitesimal motion $\tilde{p}$ of (G, p) satisfying $\tilde{p}(v) \in U(v)$ for all $v \in V$ is the zero vector $\tilde{p} = 0$. In most cases we shall be interested in the special case when each track is either zero- or d-dimensional. We say that $P \subseteq V$ is a *pinning set* if (G, p) is fixed by the tracks $U(v) = \{0\}$ if $v \in P$, $U(v) = \mathbb{R}^d$ if $v \notin P$. We also say that the vertices in P are *pinned down*, or that each vertex of P is a *pin*.

The following lemma establishes the connection between tracks (pins) that fix a framework and its rigidity matrix (see also [54, Statement 8.2.1]). Note that each track $U(v)$ of dimension k, $0 \leq k \leq d$, corresponds naturally to a subset of size k of the d columns of the rigidity matrix which belong to v.

Lemma 1.4.1. *Let (G, p) be a framework in $\mathbb{R}^d$, let $U = (U(v) : v \in V)$ be a family of tracks, and let R_U be the matrix consisting of all columns of $R(G, p)$ which correspond to the tracks $U(v)$, $v \in V$. Then*
(i) U fixes (G, p) if and only if the columns of R_U are linearly independent,
(ii) P is a pinning set if and only if the $d|V - P|$ columns of $R(G, p)$ indexed by $V - P$ are linearly independent.

One may ask for an *optimal family of tracks* that fixes a given framework by using the least possible total restriction, i.e. an assignment $U = (U(v), v \in V)$ for which U fixes (G, p) and

$$\sum_{v \in V} (d - \dim U(v))$$

is minimum. By Lemma 1.4.1(i) an optimal family of tracks is easy to find by using a greedy algorithm to identify a maximum size independent set of columns in $R(G, p)$. Furthermore, the optimum is unchanged if we restrict the matrix to a maximum size set of independent rows (or if we consider the corresponding subgraph of G). It is also clear that

$$\min\{\sum_{v \in V} (d - \dim U(v)) : U \text{ fixes } (G, p)\} = d|V| - \operatorname{rank} R(G, p).$$

We obtain a much more difficult problem if we impose restrictions on the dimension of the tracks. This is the case, for example, when we consider pinning sets. The *pinning number*, $pin_d(G, p)$, of (G, p) is defined to be the size of a smallest pinning set for (G, p). For $d = 2$ Lemma 1.4.1(ii) implies that the smallest pinning set problem can be formulated as a matroid matching problem in a linearly represented matroid and hence $pin_2(G, p)$ can be computed in polynomial time by using the algorithm of Lovász [48]. A combinatorial formula for $pin_2(G, p)$ was also given by Lovász [47]. Mansfield [52] proved that the problem of computing $pin_3(G, p)$ for a framework (G, p) is NP-hard.

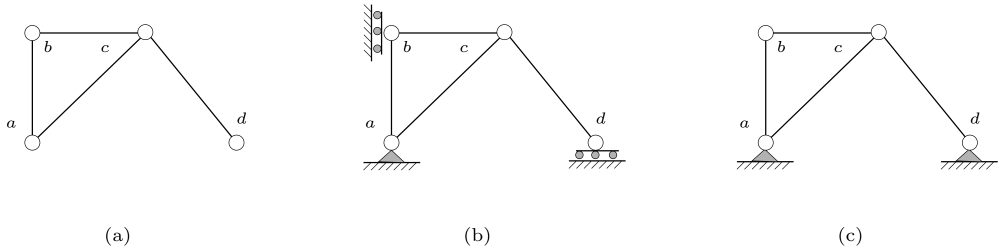

Figure 1.1: A framework in $\mathbb{R}^2$ on four vertices (left). The coordinates of the vertices are as follows: $p(a) = (0,0), p(b) = (0,1), p(c) = (1,1), p(d) = (2,0)$. Since $2|V| - \operatorname{rank} R(G,p) = 4$, to fix the framework one needs tracks of co-dimension four in total, which can be achieved by two one-dimensional tracks and a pin (middle) or two pins (right).

It is easy to see that any two generic d-dimensional frameworks on G have the same pinning number. Thus we may define the *pinning number of* G, $pin_d(G)$, as the pinning number of (G,p) of any generic framework (G,p) in $\mathbb{R}^d$. It is also easy to see that $pin_d(G) \le pin_d(G,p)$ for all frameworks (G,p). The next lemma implies that computing the pinning number of G is the same as finding a smallest complete graph whose addition to G makes it rigid [33]. For a set $P \subseteq V(G)$ let $G + K(P)$ denote the graph obtained from G by joining all pairs of non-adjacent vertices of P.

Lemma 1.4.2. *Let $G = (V,E)$ be a graph and $P \subseteq V$ with $|P| \ge d$. Let (G,p) be a generic realization of G in $\mathbb{R}^d$. Then P is a pinning set for (G,p) if and only if $G + K(P)$ is rigid in $\mathbb{R}^d$.*

Proof: Let $G' = G + K(P)$. First suppose that G' is rigid and consider the rigidity matrix $R(G',p)$. Since G' is rigid, the only solutions u to the equation $R(G',p)u = 0$ are from rigid congruences of $\mathbb{R}^d$. Thus, since (G',p) is generic, each non-zero solution leaves at most $(d-1)$ vertices fixed i.e. has at most $(d-1)$ zero entries. Suppose $R(G[V-P],p)$ has linearly dependent columns. Then we can find a non-zero solution u' to $R(G[V-P],p)u' = 0$. By extending u' to u by putting 0 in the components corresponding to P we obtain a non-zero solution to $R(G',p)u = 0$ with at least $|P| \ge d$ zeros, a contradiction. Thus P is a pinning set by Lemma 1.4.1(ii).

Now suppose that P is a pinning set and order the columns of $R = R(G',p)$ so that the columns of P come first and the rows of $E'' = E(G'[P])$ come first. (Then the upper right quarter is 0.) Hence $r(R) \ge r(R[P,E'']) + r(R[V-P, E-E'']) = d|P| - \binom{d+1}{2} + d|V-P| = d|V| - \binom{d+1}{2}$ (by using Lemma 1.4.1(ii) and that $G'[P]$ is rigid and $|P| \ge d$). Thus G' is rigid. •

Next we show that in the pinning problem we may assume that G is M-independent.

Lemma 1.4.3. *Let $F \subseteq E$ be a maximal edge set of $G = (V,E)$ for which $H = (V,F)$ is M-independent in $\mathbb{R}^d$. Then*
(i) each pinning set of G is a pinning set of H,
(ii) $pin_d(H) = pin_d(G)$.

Proof: To prove (i) suppose, for a contradiction, that there exists a pinning set P of G for which $H+K(P)$ is not rigid. Since $G+K(P)$ is rigid, we have $r_d(G+K(P)) > r_d(H+K(P))$, which implies that there is an edge $e \in E + E(K(P)) - (F + E(K(P))) = E - F$ for which $F + e$ is independent, contradicting the maximality of F. This proves (i), from which (ii) follows immediately. •

It follows from the observations above that the pinning problem in graphs (or in generic frameworks) can be attacked by purely combinatorial methods provided good characterizations for M-independent and rigid graphs are available. This is the case when $d = 2$. The solution of the 2-dimensional case will be discussed in Section 2.7.

1.5 Notation

Let $G = (V, E)$ be a graph. For $X, Y, Z \subset V$, let $G[X]$ be the induced subgraph of G on vertex set X and $E_G(X)$ be the set of edges of $G[X]$. We simply use $E(X)$ if the graph is clear from the context. Let $d(X, Y) = |E(X \cup Y) - (E(X) \cup E(Y))|$, and $d(X, Y, Z) = |E(X \cup Y \cup Z) - (E(X) \cup E(Y) \cup E(Z))|$. Thus $d(X, Y)$ is the number of edges between $X - Y$ and $Y - X$ and if X, Y are disjoint then $d(X, Y)$ denotes the number of edges from X to Y. We define the *degree* of X by $d(X) = d(X, V - X)$, that is, the number of edges with precisely one endvertex in X. The degree of a vertex v is simply denoted by $d(v)$. The minimum degree of a graph G is denoted by $\delta(G)$. For $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of X, that is, let $N(X) = \{v \in V - X : uv \in E \text{ for some } u \in X\})$.

A *k-separation* of a graph $H = (V, E)$ is a pair (H_1, H_2) of edge-disjoint subgraphs of G each with at least $k + 1$ vertices such that $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = k$. The graph H is said to be *k-connected* if it has at least $k + 1$ vertices and has no j-separation for all $0 \leq j \leq k - 1$. If (H_1, H_2) is a k-separation of H, then we say that $V(H_1) \cap V(H_2)$ is a *k-separator* of H.

We say that a graph $G = (V, E)$ is *k-edge-connected* if $d(X) \geq k$ for all proper subsets X of V. We call G *essentially k-edge-connected* if every $X \subset V$ with $d(X) \leq k - 1$ satisfies $|X| = 1$ or $|V - X| = 1$.

Chapter 2

Rigid graphs

The structure of the rigidity matrix easily implies that the one-dimensional rigidity matroid of a graph G is isomorphic to the circuit matroid of G. It also follows that G is M-independent in $\mathbb{R}^1$ if and only if G is a forest, that is, if $i(X) \leq |X| - 1$ holds for all non-empty subsets $X \subseteq V$. We shall characterize M-independence in $\mathbb{R}^2$ by a similar sparsity condition.

2.1 Sparse graphs

Let $G = (V, E)$ be a graph. We say that G is *sparse* if

$$i(X) \leq 2|X| - 3 \text{ for all } X \subseteq V \text{ with } |X| \geq 2. \tag{2.1}$$

We shall need the following equality, which is easy to check by counting the contribution of an edge to each of the two sides.

Lemma 2.1.1. *Let G be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y). \tag{2.2}$$

We call a set $X \subseteq V$ *critical* if $i(X) = 2|X| - 3$ holds.

Lemma 2.1.2. *Let $G = (V, E)$ be sparse and let $X, Y \subset V$ be critical sets in G with $|X \cap Y| \geq 2$. Then $X \cap Y$ and $X \cup Y$ are also critical, and $d(X, Y) = 0$.*

Proof: Since G is sparse, (2.1) holds. By (2.2) we have
$2|X| - 3 + 2|Y| - 3 = i(X) + i(Y) = i(X \cap Y) + i(X \cup Y) - d(X, Y) \leq$
$2|X \cap Y| - 3 + 2|X \cup Y| - 3 - d(X, Y) = 2|X| - 3 + 2|Y| - 3 - d(X, Y)$. Thus $d(X, Y) = 0$ and equality holds everywhere. Therefore $X \cap Y$ and $X \cup Y$ are also critical. •

Lemma 2.1.3. *Let $G = (V, E)$ be sparse and let $X, Y, Z \subset V$ be critical sets in G with $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Then $X \cup Y \cup Z$ is critical, and $d(X, Y, Z) = 0$.*

Proof: Since G is sparse and our sets are critical, we have $2|X|-3+2|Y|-3+2|Z|-3+d(X,Y,Z)=i(X)+i(Y)+i(Z)+d(X,Y,Z)\leq$
$i(X\cup Y\cup Z)\leq 2(|X\cup Y\cup Z|)-3=2(|X|+|Y|+|Z|-3)-3=2|X|-3+2|Y|-3+2|Z|-3$. Hence $d(X,Y,Z)=0$ and equality holds everywhere. Thus $X\cup Y\cup Z$ is critical. •

Let v be a vertex in a graph G with $d(v)=3$ and $N(v)=\{u,w,z\}$. The operation *splitting* means deleting v (and the edges incident to v) and adding a new edge, say uw, connecting two non-adjacent vertices of $N(v)$. The resulting graph is denoted by $G_v^{u,w}$ and we say that the splitting is made *on the pair* uv,wv. Note that v can be split in at most three different ways. Let $G=(V,E)$ be sparse and let v be a vertex with $d(v)=3$. Splitting v on the pair uv,wv is said to be *suitable* if $G_v^{u,w}$ is sparse. We call a vertex v *suitable* if there is a suitable splitting at v. We shall show that every vertex of degree three in a sparse graph is suitable.

Lemma 2.1.4. *Let v be a vertex in a sparse graph $G=(V,E)$.*
(a) If $d(v)=2$ then $G-v$ is sparse.
(b) If $d(v)=3$ then v is suitable.

Proof: Part (a) follows easily from (2.1) and from the definition of sparse graphs.

To prove (b) let $N(v)=\{u,w,z\}$. It is easy to see that splitting v on the pair uv,wv is not suitable if and only if there exists a critical set $X\subset V$ with $u,w\in X$ and $v,z\notin X$. Also observe that no critical set $Z\subseteq V-v$ can satisfy $d(v,Z)\geq 3$, since otherwise $E(G[Z\cup\{v\}])$ would violate (2.1). Thus if v is not suitable then there exist maximal critical sets $X_{uw},X_{uz},X_{wz}\subset V-v$ each containing precisely two neighbours ($\{u,w\},\{u,z\},\{w,z\}$, resp.) of v. By Lemma 2.1.2 and the maximality of these sets we must have $|X_{uw}\cap X_{uz}|=|X_{uw}\cap X_{wz}|=|X_{uz}\cap X_{wz}|=1$. Thus, by Lemma 2.1.3 the set $Y:=X_{uw}\cup X_{uz}\cup X_{wz}$ is also critical. Since $N(v)\subseteq Y$, we have $d(v,Y)\geq 3$. This is impossible by our previous observation. Therefore v is suitable. •

The sparse graph K_4-e shows that among the three possible splittings at a vertex of degree three there may be only one which is suitable.

Observe that the inverse operations of the vertex deletion and splitting operations used in Lemma 2.1.4 are the (two-dimensional) 0-extension and 1-extension operations, respectively, c.f. Lemmas 1.2.2 and 1.2.3. Recall that the 0-extension operation adds a new vertex v and two edges vu,vw with $u\neq w$. The 1-extension subdivides an edge uw by a new vertex v and adds a new edge vz for some $z\neq u,w$. An *extension* is either a 0-extension or a 1-extension. The next lemma follows easily from (2.1).

Lemma 2.1.5. *Let G be sparse and let G' be obtained from G by an extension. Then G' is sparse.*

2.2 Laman's theorem and the Henneberg construction

The following fundamental result, due to Laman, provides the characterization of independence in the two-dimensional rigidity matroid.

Theorem 2.2.1. *[43] Let $G = (V, E)$ be a graph. Then G is M-independent if and only if G is sparse.*

Proof: Necessity follows from Lemma 1.3.1. Sufficiency will follow if we can show that every sparse graph G has a realization (G, p) for which the rows of $R(G, p)$ are linearly independent. We prove this by induction on $|E|$. If G has only one edge uv then for any realization (G, p) in which $p(u) \neq p(v)$ we have $|E| = \operatorname{rank} R(G, p) = 1$, as required. Now suppose that G is a sparse graph with $|E| \geq 2$ and that the statement of the theorem holds up to $|E| - 1$ edges. We may suppose that $\delta(G) \geq 1$. By sparsity we have $|E| \leq 2|V| - 3$, which implies that the average degree of G is less than four and hence we have $\delta(G) \leq 3$. Let v be a vertex with $d(v) = \delta(G)$.

If $d(v) \leq 2$ then consider $G' = G - v$. Clearly, G' is sparse. By induction, there is an independent realization (G', p'). By applying Lemma 1.2.2 we may obtain a realization (G, p) for which $\operatorname{rank} R(G, p) = \operatorname{rank} R(G', p') + d(v) = |E(G')| + d(v) = |E|$ holds, as required.

If $d(v) = 3$ with $N(v) = \{u, w, z\}$ then consider a sparse graph G_v obtained from G by a suitable splitting (on the edge pair vu, vw, say). Such a graph exists by Lemma 2.1.4(b). By induction, there is an independent realization (G_v, p'). Since the set of configurations p in $\mathbb{R}^{2|V(G_v)|}$ for which $\operatorname{rank} R(G_v, p) = \operatorname{rank} R(G_v, p')$ is open, we may suppose that $p'(u), p'(w), p'(z)$ are not collinear. By applying Lemma 1.2.3 we may obtain a realization (G, p) for which $\operatorname{rank} R(G, p) = \operatorname{rank} R(G', p') + 2 = |E(G_v)| + 2 = |E|$. This completes the proof. •

We say that a rigid graph $G = (V, E)$ is *minimally rigid* if $G - e$ is not rigid for all $e \in E$. The edge sets of the minimally rigid graphs on vertex set V correspond to the bases of the rigidity matroid $\mathcal{R}(K_{|V|})$ and have the same size. The previous theorem implies the following characterization.

Theorem 2.2.2. *[43] A graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2|V| - 3$ and (2.1) holds.*

2.2.1 Exercises

Exercise 2.2.3. *Prove that G is minimally rigid if and only if G has three subtrees T_1, T_2, T_3 such that each vertex is incident with exactly two of the subtrees and there is no vertex set $X \subseteq V(G)$ of size at least two for which $T_i[X]$ is a tree for at least two subtrees. (Crapo [10].)*

Exercise 2.2.4. *Prove that G is minimally rigid if and only if the edge set of the augmented graph $G + uv$ can be partitioned into two spanning trees, for all $u, v \in V(G)$.*

2.2.2 Inductive constructions

Next we prove an inductive construction of minimally rigid graphs, which is sometimes called the Henneberg construction. We shall use the following simple connectivity properties of minimally rigid graphs.

Lemma 2.2.5. *Let $G = (V, E)$ be minimally rigid with $|V| \geq 3$. Then*
(a) G is 2-connected.
(b) For every $\emptyset \neq X \subset V$ we have $d(X) \geq 2$ and if $d(X) = 2$ holds then either $|X| = 1$ or $|V - X| = 1$.

Proof: Suppose that for some $v \in V$ the graph $G - v$ is disconnected and let $A \cup B$ be a partition of $V - v$ with $d(A, B) = 0$. Then (2.1) gives $|E| = 2|V| - 3 = i(A + v) + i(B + v) \leq 2(|A| + 1) - 3 + 2(|B| + 1) - 3 = 2(|A| + |B| + 1) - 4 = 2|V| - 4$, a contradiction. This proves (a).

Using (a), we have $d(X) \geq 2$ for every $\emptyset \neq X \subset V$. Suppose $|X|, |V - X| \geq 2$. By (2.1) we obtain $|E| = i(X) + i(V - X) + d(X) \leq 2|X| - 3 + 2|V - X| - 3 + d(X) = 2|V| - 6 + d(X) = |E| - 3 + d(X)$. This implies $d(X) \geq 3$ and proves (b). •

Theorem 2.2.6. *Let $G = (V, E)$ be minimally rigid and let $G' = (V', E')$ be a minimally rigid subgraph of G. Then G can be obtained from G' by a sequence of extensions.*

Proof: We shall prove that G' can be obtained from G by a sequence of splittings and deletions of vertices (of degree two). The theorem will then follow since these are the inverse operations of extensions.

The proof is by induction on $|V - V'|$. Since G' is rigid and G is minimally rigid, G' must be an induced subgraph of G. Thus the theorem holds trivially when $|V - V'| = 0$. Now suppose that $Y = V - V' \neq \emptyset$. Since G' and G are minimally rigid, it is easy to see that $|E - E'| = 2|Y|$ holds. Therefore, if $|Y| = 1$, then we must have $d(v) = 2$ for the unique vertex $v \in Y$. Hence G' can be obtained from G by deleting a vertex of degree two. Thus we may assume that $|Y| \geq 2$.

Claim 2.2.7. *If $|Y| \geq 2$ then $\sum_{v \in Y} d(v) \leq 4|Y| - 3$.*

Proof: Since $|V'| \geq 2$ and $|V - V'| \geq 2$, we can apply Lemma 2.2.5(b) to deduce that $d(Y) \geq 3$. Since $i(Y) + d(Y) = |E - E'| = 2|Y|$, we obtain

$$\sum_{v \in Y} d(v) = 2i(Y) + d(Y) = 4|Y| - d(Y) \leq 4|Y| - 3.$$

•

It follows from Claim 2.2.7 (and from the fact that the minimum degree in G is at least two) that there is a vertex $v \in Y$ with $2 \leq d(v) \leq 3$. Now Lemma 2.1.4 implies that either $H = G - v$ or $H = G_v^{u,w}$ is minimally rigid and is such that G' is a subgraph of H and $|V(H) - V(G')| < |V(G) - V(G')|$. The theorem now follows by induction. •

By choosing G' to be an arbitrary edge of G we obtain the following constructive characterization of minimally rigid graphs (called the Henneberg or Henneberg-Laman construction, c.f. [24, 43, 60]).

Theorem 2.2.8. *$G = (V, E)$ is minimally rigid if and only if G can be obtained from K_2 by a sequence of extensions.*

The next two lemmas about glueing (minimally) rigid graphs will also be useful.

Theorem 2.2.9. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two minimally rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid. Moreover, if $G_1 \cap G_2$ is minimally rigid then $G_1 \cup G_2$ is minimally rigid as well.*

Proof: Let F' be a maximal independent set in $\mathcal{R}(G_1 \cap G_2)$. Let K be the complete graph with vertex set $V(G_1 \cap G_2)$ and F be a base of $\mathcal{R}(K)$ containing F'. Let H be a minimally rigid spanning subgraph of $G_2 + (F - F')$ which contains F. Such an H exists, since G_2, and hence $G_2 + (F - F')$, is rigid. (To see that F and H exist we use the fact that any independent set in a matroid can be extended to a base.) Now Theorem 2.2.6 implies that H can be obtained by a sequence of extensions from $(V_1 \cap V_2, F)$. The same sequence of extensions, applied to G_1, yields a minimally rigid spanning subgraph of $G_1 \cup G_2$ by Lemma 2.1.5. This proves that $G_1 \cup G_2$ is rigid.

The second assertion follows from the fact that if $G_1 \cap G_2$ is minimally rigid then $F = F'$ and $H = G_2$. •

The following version is an immediate corollary.

Lemma 2.2.10. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid.*

2.2.3 Exercises

Exercise 2.2.11. *Given a graph G, the cone of G, denoted by G^*, is obtained from G by adding a new vertex v and making it adjacent to all vertices of G. Show that the cone graph of G is rigid if and only if G is connected.*

Exercise 2.2.12. *Develop a polynomial time algorithm for testing whether G can be obtained from K_2 by a sequence of 0-extensions.*

Exercise 2.2.13. *Develop a polynomial time algorithm for testing whether G has a spanning subgraph H that can be obtained from K_2 by a sequence of 0-extensions.*

Exercise 2.2.14. *Prove that there exists no 3-connected minimally rigid graph G in which for every edge $e \in E(G)$ there is a triangle in G containing e.*

Exercise 2.2.15. *We say that G is triangle reducible if it can be obtained from K_3 by a sequence of 1-extensions such that for each 1-extension operation, on edge uv and vertex w, say, the current graph contains a triangle on u, v, w. Develop a polynomial time algorithm for testing whether G is triangle-reducible.*

2.3 Rigid components and the rank function

In this section we determine the rank function of $\mathcal{R}(G)$ by using Laman's characterization of M-independence. We also introduce the rigid and the redundantly rigid components of a graph.

First we define covers of graphs, a concept that we shall frequently use later. Let $G = (V, E)$ be a graph. A *cover* of G is a collection $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V, each of size at least two, such that $\cup_{i=1}^{t} E(X_i) = E$. The cover is said to be *thin* if $|X_i \cap X_j| \leq 1$ for all $i \neq j$. The *value* $val(\mathcal{X})$ of the cover is $\sum_{i=1}^{t}(2|X_i| - 3)$.

Let $\mathcal{X}$ be a cover of G and let $F \subseteq E$ be a set of edges for which $H = (V, F)$ is M-independent. Then we have $|F \cap E_G(X_i)| \leq 2|X_i| - 3$ for all $1 \leq i \leq t$. Thus

$$|F| \leq val(\mathcal{X}). \tag{2.3}$$

We define a *rigid component* of a graph $G = (V, E)$ to be a maximal rigid subgraph of G. By Lemma 2.2.10 the vertex sets of the rigid components form a thin cover of G (and their edge sets form a partition of E).

Lemma 2.3.1. *Let $G = (V, E)$ be a graph, let $F \subseteq E$ be a maximal edge set in G for which $H = (V, F)$ is sparse. Then the family $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of maximal critical sets in H satisfies that*
(a) $\mathcal{X}$ is a thin cover of G with $|F| = val(\mathcal{X})$,
(b) $\mathcal{X}$ is equal to the family of vertex sets of the rigid components of G.

Proof: (a) The maximality of the critical sets and Lemma 2.1.2 implies that $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq t$. Since every single edge of F induces a critical set, it follows that $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ is a thin cover of H. Thus

$$|F| = \sum_{1}^{t} |E_H(X_i)| = \sum_{1}^{t}(2|X_i| - 3).$$

To complete the proof we show that $\mathcal{X}$ is a cover of G as well. Choose $uv \in E - F$. Since F is a maximal sparse subset of E, $F + uv$ is not sparse. Thus there exists a set $X \subseteq V$ such that $u, v \in X$ and $i_H(X) = 2|X| - 3$. Hence X is a critical set in H. This implies that $X \subseteq X_i$ and hence $uv \in E_G(X_i)$ for some $1 \leq i \leq t$.

(b) It follows from Theorem 2.2.2 that $G[X_i]$ is rigid for all $1 \leq i \leq t$. Thus we have to show that the vertex set of each rigid component of G is critical in H. Suppose, for a contradiction, that $H[C]$ is not critical, where C is the set of vertices of some rigid component of G. Then $|J| \leq 2|C| - 4$, where $J = E(H[C])$. Since $G[X_i]$ is rigid for all $1 \leq i \leq t$ and $\mathcal{X}$ is a thin cover of G, it follows from Lemma 2.2.10 that $\mathcal{X}' = \{X_i \in \mathcal{X} : |X_i \cap C| \geq 2\}$ is a thin cover of $G[C]$ with $|J| = val(\mathcal{X}')$. Thus we can use (2.3) to deduce that for any subset $F' \subseteq E(G[C])$ which induces an M-independent (and hence sparse) subgraph on vertex set C we have $|F'| \leq val(\mathcal{X}') = |J| \leq 2|C| - 4$. This contradicts the fact that $G[C]$ is rigid. •

Theorem 2.2.1, (2.3), and Lemma 2.3.1(a) show that the maximal edge sets of G that induce an M-independent subgraph have the same size. They also imply the following rank formula of the rigidity matroid, due to Lovász and Yemini.

Theorem 2.3.2. *[49] Let $G = (V, E)$ be a graph. Then*

$$r(G) = \min\{val(\mathcal{X}) : \mathcal{X} \text{ is a thin cover of } G\}.$$

We may simplify the min-max formula of Theorem 2.3.2 when the graph is obtained from an M-independent graph by 'pinning' a set of vertices. We shall use the next lemma later when we determine the pinning number of a graph. For a set $X \subseteq V$ let $e(X)$ denote the number of edges with at least one end-vertex in X.

Lemma 2.3.3. *Suppose that $G = (V, E)$ is an M-independent graph and let $P \subseteq V$ with $|P| \geq 2$. Let $G' = G + K(P)$. Then*

$$r(G') = \min_{P \subseteq Z}\{2|Z| - 3 + e(V - Z)\}.$$

Proof: Let $Z \subseteq V$ with $P \subseteq Z$ and consider the thin cover $\mathcal{Z} = \{Z \cup \{\{u, v\} : uv \in E - E(Z)\}\}$ of G'. Then $r(G') \leq val(\mathcal{Z}) = 2|Z| - 3 + e(V - Z)$.

To see that equality holds for some $Z \subseteq V$ choose a maximal edge set F in G' for which $H = (V, F)$ is M-independent and P is a critical set in H. Such an F can be constructed by extending the edge set F' of a minimally rigid subgraph of the complete graph $G'[P]$. Let $\mathcal{X}$ be the family of maximal critical sets of H. By Lemma 2.3.1(a) and Theorem 2.3.2 we have $r(G') = val(\mathcal{X})$. Since P is critical in H, there is a set $Z \in \mathcal{X}$ with $P \subseteq Z$. Thus, since G is M-independent and all edges of $K(P)$ are covered by Z, we have $i_G(X) = i_H(X) = 2|X| - 3$ for all $X \in \mathcal{X} - Z$. Hence $r(G') = val(\mathcal{X}) = 2|Z| - 3 + \sum_{X \in \mathcal{X} - Z} i_G(X) = 2|Z| - 3 + e(V - Z)$, which completes the proof. •

Next we prove a result of Gabow on the existence of an edge in a minimally rigid graph whose deletion leads to a graph in which all rigid components are small. Let $G = (V, E)$ be a minimally rigid graph and let $\mathcal{F} = \{A \subseteq E : |A| = 2|V(A)| - 3\}$ consist of the edge sets of the critical subsets of V. We call the members of $\mathcal{F}$ *tight.* First observe that

Lemma 2.3.4. *(a) Suppose that $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$. Then $A \cap B$ and $A \cup B$ are both tight.*
(b) Suppose that $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$. Then $A \cup B \notin \mathcal{F}$.

Proof: (a) follows from Lemma 2.1.2. (b) It follows from the definition that the cardinality of each member of $\mathcal{F}$ is odd. Thus for two disjoint sets $A, B \in \mathcal{F}$ we must have $A \cup B \notin \mathcal{F}$. •

Theorem 2.3.5. *[17] Let $G = (V, E)$ be a minimally rigid graph with $|E| = m$. Then there is an edge $e \in E$ for which each rigid component of $G - e$ has at most $\frac{m-1}{2}$ edges.*

Proof: Call a member A of $\mathcal{F}$ *big* if $|A| \geq \frac{m+1}{2}$. Note that m is odd. Since the edge set of a rigid component of $G-e$ is a member of $\mathcal{F}$, it suffices to prove that there is an edge $e \in E$ which belongs to all big tight sets.

Let A be a minimal tight set intersecting all big tight sets. Such a set exists, since E is tight. Let $e \in A$. We claim that $e \in E$ belongs to all big tight sets.

For a contradiction suppose that there is a big tight set M with $e \notin M$. By the choice of A we have $A \cap M \neq \emptyset$. Now $A \cap M$ is a proper subset of A and hence there is a big tight set B which is disjoint from $A \cap M$. We also have $A \cap B \neq \emptyset$. Clearly, $A \cap B$ and $A \cap M$ are disjoint. Since B and M are both big, we have $B \cap M \neq \emptyset$. By Lemma 2.3.4(a) the sets $A \cap B$, $B \cap M$ and $A \cup M$ are tight. Thus $(A \cup M) \cap B$ is also tight. But $(A \cup M) \cap B = (A \cap B) \cup (B \cap M)$, contradicting Lemma 2.3.4(b). •

A minimally rigid graph obtained from two disjoint minimally rigid graphs on $\frac{m-3}{2}$ edges each, connected by three edges forming a path shows that the bound is (almost) best possible. The graph $K_{2,n-2}$ plus an edge between the large degree vertices shows that there may be a unique edge e with this property.

We define a *redundantly rigid component* of a graph $G = (V, E)$ to be a maximal redundantly rigid subgraph of G (we call it a *non-trivial* redundantly rigid component) or a subgraph induced by an edge which belongs to no redundantly rigid subgraph of G (which is a *trivial* redundantly rigid component). It follows from Lemma 2.2.10 that two redundantly rigid components of G can have at most one vertex in common, and hence are edge-disjoint. Thus the redundantly rigid components of G partition E. Since each redundantly rigid component is rigid, this partition is a refinement of the partition of E given by the rigid components of G.

Let B be the set of edges of G that belong to no redundantly rigid subgraph of G. Then we have:

Lemma 2.3.6. *A subgraph H of G is a non-trivial redundantly rigid component of $G = (V, E)$ if and only if H is a rigid component of $G' = (V, E - B)$.*

2.3.1 Exercises

Exercise 2.3.7. *Show that a graph obtained from a minimally rigid graph by removing an edge has an even number of rigid components.*

Exercise 2.3.8. *Prove that a thin cover of a graph on n vertices has at most $\binom{n}{2}$ members.*

Exercise 2.3.9. *Consider the modified sparsity condition $i(X) \leq 2|X| - 2$ for all non-empty $X \subseteq V$. Show that this count also defines the independent sets of a matroid on the edge set of a graph. Determine its rank function.*

Exercise 2.3.10. *Prove that if G is redundantly rigid and G' is obtained from G by an edge addition or a 1-extension, then G' is redundantly rigid.*

Exercise 2.3.11. *Prove that if G is redundantly rigid and $\{u, v\}$ is a 2-separator in G then $d(u), d(v) \geq 4$.*

2.4 Highly connected graphs

A natural question is whether sufficiently high vertex-connectivity implies rigidity. While the answer is not known in higher dimensions, the two-dimensional case was settled by Lovász and Yemini.

Theorem 2.4.1. *[49] Every* 6*-connected graph is rigid.*

Proof: Let G be a counter-example with the smallest number of vertices and, with respect to this, the largest number of edges. Since G is not rigid, Theorem 2.3.2 implies that it has a thin cover $\mathcal{X} = \{X_1, ..., X_t\}$ with

$$\sum_1^t 2|X_i| - 3 < 2n - 3, \tag{2.4}$$

where n is the number of vertices of G. By the maximality of E, $G[X_i]$ is complete graph for all $1 \le i \le t$.

First we prove that each vertex v belongs to at least two X_i's. If this is not the case then consider the unique set, say X_1, with $v \in X_1$. Since $\mathcal{X}$ is a cover of G and the degree of v is at least 6, we must have $|X_1| \ge 7$. Let $G' = G - v$, $X_1' = X_1 - v$, and let $X_j' = X_j$ for all $2 \le j \le t$. Then $\mathcal{X}' = \{X_1', ..., X_t'\}$ is a cover of G' and, since $\sum_1^t 2|X_i'| - 3 < 2n' - 3$ holds, G' is not rigid. By the minimal choice of G it implies that G' is not 6-connected. Then either $n' = 6$ and $G = K_7$ (which is impossible, since K_7 is rigid), or there is a vertex separator T of size at most five in G'. Since T does not separate G, v is connected to each component of $G' - T$ in G. This contradicts the fact that the neighbour set of v induces a complete subgraph (as it is included in X_1).

Since the minimum degree of G is at least 6 and $\mathcal{X}$ is a cover of G, we have

$$\sum_{v \in X_i} (|X_i| - 1) \ge 6. \tag{2.5}$$

Next we show that each vertex $v \in V$ satisfies

$$\sum_{v \in X_i} (2 - \frac{3}{|X_i|}) \ge 2. \tag{2.6}$$

We may suppose that v is contained by the sets $X_1, ..., X_d$ and that $|X_1| \ge ... \ge |X_d|$ holds. By our first claim $d \ge 2$. Since each term in the sum is at least $\frac{1}{2}$, (2.6) is clear when $d \ge 4$. If $d = 3$ then (2.5) implies $|X_1| \ge 3$, and hence the left hand side is at least $1 + \frac{1}{2} + \frac{1}{2} = 2$. If $d = 2$ then (2.5) implies $|X_1| \ge 4$, and if $|X_1| = 4, 5$, or $|X_1| \ge 6$ holds then we have $|X_2| \ge 4, 3, 2$, respectively. Thus the sum is at least $\frac{5}{4} + \frac{5}{4}, \frac{7}{5} + 1, \frac{3}{2} + \frac{1}{2}$, which are not smaller than 2. Therefore (2.6) holds for all v.

By summing up these inequalities for all v we obtain

$$\sum_1^t |X_i|(2 - \frac{3}{|X_i|}) = \sum_1^t 2|X_i| - 3 \ge 2n, \tag{2.7}$$

contradicting (2.4). •

By rereading the proof and using the fact that the gap between the bounds of (2.4) and (2.7) is more than three, we can deduce the stronger statement that $G - F$ is rigid for any set F of at most three edges of G. This is sharp: consider two disjoint complete graphs on at least six vertices each and connect them by six disjoint edges. Generalizations and refinements of Theorem 2.4.1 can be found in [31, 34].

2.5 Algorithms

In this section we discuss the algorithmic aspects of the key results proven so far. We show, without providing detailed running time estimations, that the basic algorithmic questions can be solved in polynomial time.

To test whether $G = (V, E)$ is rigid, or more generally, to compute the rank of $\mathcal{R}(G)$, we need to find a base of $\mathcal{R}(G)$. This can be done greedily, by building up a maximal independent edge set by adding (or rejecting) edges one by one. The key of this procedure is the independence test: given an independent set I and an edge $e \in E - I$, check whether $I + e$ is independent or not. This problem can be formulated as a matching problem in a bipartite graph or as a network flow problem. Here we sketch an efficient method for this subroutine from [6], which is based on in-degree constrained orientations of G, see also [23, 44]. The following result and its algorithmic proof, due to Frank and Gyárfás, is our starting point.

Let $G = (V, E)$ be a graph. An *orientation* $D = (V, A)$ of G is a directed graph obtained from G by replacing each edge uv by a directed edge (directed from u to v or from v to u). If $D = (V, A)$ is a directed graph and $X \subseteq V$ then $\rho_D(X)$ denotes the number of directed edges entering X. This is the *in-degree* of X. The in-degree of a vertex v is denoted by $\rho_D(v)$. Let $g : V \to \mathbb{Z}_+$ assign non-negative integers to the vertices of G. For $X \subseteq V$ we use the notation $g(X) = \sum_{v \in X} g(v)$. We say that an orientation D of G is a *g-orientation* if $\rho_D(v) \leq g(v)$ holds for all $v \in V$.

Theorem 2.5.1. *[16] Let $G = (V, E)$ be a graph and $g : V \to Z_+$. Then G has a g-orientation if and only if*

$$i(X) \leq g(X) \quad \textit{for all} \quad X \subseteq V. \tag{2.8}$$

Proof: To see necessity suppose that D is a g-orientation of G and let $X \subseteq V$. Then $i(X) = \sum_{v \in X} \rho_D(v) - \rho_D(X) \leq g(X)$.

To prove sufficiency suppose that (2.8) holds and choose an orientation D' of G for which $h(D') = \sum_{v \in V} \max\{0, \rho(v) - g(v)\}$ is as small as possible. If $h(D') = 0$ then D' is a g-orientation. Otherwise there is a vertex s with $\rho_{D'}(s) > g(s)$. Let S denote the set of vertices from which there is a directed path to s in D'. Clearly, $\rho_{D'}(S) = 0$. If there is a vertex $t \in S$ with $\rho_{D'}(t) < g(t)$ then by reorienting the edges of a directed path from t to s we obtain an orientation D'' with $h(D'') = h(D') - 1$, contradicting the choice of D'. Thus we have $\rho_{D'}(v) \geq g(v)$ for each vertex $v \in S$, and hence, since $\rho_{D'}(s) > g(s)$, we obtain

$i(S) = \sum_{v \in S} \rho_{D'}(v) - \rho_{D'}(S) > \sum_{v \in S} g(v) = g(S)$, contradicting (2.8). This proves the theorem. •

This proof leads to an algorithm for finding a g-orientation, if it exists. It shows that if (2.8) holds then any orientation D' of G can be turned into a g-orientation by finding and reorienting directed paths $h(D')$ times. Such an elementary step (which decreases h by one) can be done in linear time.

Let $g_2 : V \to \mathbb{Z}_+$ be defined by $g_2(v) = 2$ for all $v \in V$. For two vertices $u, v \in V$ let $g_2^{uv} : V \to \mathbb{Z}_+$ be defined by $g_2^{uv}(u) = g_2^{uv}(v) = 0$, and $g_2^{uv}(w) = 2$ for all $w \in V - \{u, v\}$.

Lemma 2.5.2. *Let $G = (V, E)$ be a graph and suppose that $I \subset E$ is independent. Let $e = uv$ be an edge with $e \in E - I$. Then $I + e$ is independent if and only if (V, I) has a g_2^{uv}-orientation.*

Proof: Let $H = (V, I)$ and $H' = (V, I + e)$. First suppose that $I + e$ is not independent. Then there is a set $X \subseteq V$ with $i_{H'}(X) \geq 2|X| - 2$. Since I is independent, we must have $u, v \in X$ and $i_H(X) = 2|X| - 3$. Hence $i_H(X) = 2|X| - 3 > g_2^{uv}(X) = 2|X| - 4$, showing that H has no g_2^{uv}-orientation.

Conversely, suppose that $I + e$ is independent, but H has no g_2^{uv}-orientation. By Theorem 2.5.1 this implies that there is a set $X \subseteq V$ with $i_H(X) > g_2^{uv}(X)$. Since $i_H(X) \leq 2|X| - 3$ and $g_2^{uv}(X) = 2|X| - 2|X \cap \{u, v\}|$, this implies $u, v \in X$ and $i_H(X) = 2|X| - 3$. Then $i_{H'}(X) = 2|X| - 2$, contradicting the fact that $I + e$ is independent. •

A *weak g_2^{uv}-orientation* D of G satisfies $\rho_D(w) \leq 2$ for all $w \in V - \{u, v\}$ and has $\rho_D(u) + \rho_D(v) \leq 1$. It follows from the proof of Lemma 2.5.2 that a weak g_2^{uv}-orientation of (V, I) always exists.

If we start with a g_2-orientation of $H = (V, I)$ then the existence of a g_2^{uv}-orientation of H can be checked by at most four elementary steps (reachability search and reorientation) in linear time. Note also that H has $O(n)$ edges, since I is independent (where $n = |V|$).

This gives rise to a simple algorithm for computing the rank of E in $\mathcal{R}(G)$. By maintaining a g_2-orientation of the subgraph of the current independent set I, testing an edge needs only $O(n)$ time, and hence the total running time is $O(nm)$, where $m = |E|$. This can be improved to $O(n^2)$ by maintaining the list of the rigid components of (V, I) as follows. Let I be an independent set, let $e = uv$ be an edge with $e \in E - I$, and suppose that $I + e$ is independent. Let D be a g_2^{uv}-orientation of (V, I). Let $X \subseteq V$ be the maximal set with $u, v \in X, \rho_D(X) = 0$, and such that $\rho_D(x) = 2$ for all $x \in X - \{u, v\}$. Clearly, such a set exists, and it is unique. It can be found by identifying the set $V_1 = \{x \in V - \{u, v\} : \rho_D(x) \leq 1\}$, finding the set $\hat{V}_1$ of vertices reachable from V_1 in D, and then taking $X = V - \hat{V}_1$. The next lemma is easy to verify.

Lemma 2.5.3. *Let $H' = (V, I + e)$. Then $H'[X]$ is a rigid component of H'.*

Thus, when we add e to I, the set of rigid components is updated by adding $H'[X]$ and deleting each component whose edge set is contained by the edge set of $H'[X]$. Maintaining this list can be done in linear time. Furthermore, we can reduce the total running time to

$O(n^2)$ by performing the independence test for $I+e$ only if e is not spanned by any of the rigid components on the current list (and otherwise rejecting e, since $I+e$ is clearly dependent).

2.6 Special families of graphs

In this section we consider special families of graphs for which we can deduce simpler or different versions of some of the previous results concerning inductive constructions or the rank.

2.6.1 Minimally rigid plane graphs

Let $G = (V, E)$ be a plane graph, that is, a graph embedded in the plane without edge crossings. The *plane vertex splitting* operation at some vertex $x \in V$ picks an edge xy, partitions the edges incident to x (except xy) into two consecutive sets E_1, E_2 of edges (with respect to the natural cyclic ordering determined by the embedding), replaces x by two vertices x_1, x_2, replaces every edge wx with $wx \in E_i$ by an edge wx_i, $i = 1, 2$, and adds the edges yx_1, yx_2, x_1x_2. The embedding is modified only in the neighbourhood of x in such a way that it remains a planar embedding of the resulting graph. It is easy to see that plane vertex splitting, when applied to a minimally rigid plane graph, yields a minimally rigid plane graph. (Note that the standard version of vertex splitting, where the partition of the edges incident to x is arbitrary, preserves the property of being minimally rigid, but may destroy planarity. We shall discuss this operation later.)

We shall prove that every minimally rigid plane graph (that is, a minimally rigid graph with a planar embeddig) can be obtained from an edge by plane vertex splitting operations. To prove this we need to show that the inverse operation of plane vertex splitting can be performed on every minimally rigid plane graph with at least three vertices in such a way that the graph remains (plane and) minimally rigid. The inverse operation contracts an edge of a triangle face.

Let $e = uv$ be an edge of G. By the *contraction* of e we mean the operation which identifies the two end-vertices of e and deletes the resulting loop as well as one edge from each of the resulting pairs of parallel edges, if there exist any. The graph obtained from G by contracting e is denoted by G/e. We say that e is *contractible* in a minimally rigid graph G if G/e is also minimally rigid. Observe that by contracting an edge e the number of vertices is decreased by one, and the number of edges is decreased by the number of triangles that contain e plus one. Thus a contractible edge belongs to exactly one triangle of G.

Minimally rigid graphs in general do not necessarily contain triangles, see e.g. $K_{3,3}$. For minimally rigid plane graphs we can use Euler's formula to deduce the following.

Lemma 2.6.1. *Every minimally rigid plane graph $G = (V, E)$ with $|V| \geq 4$ contains at least two triangle faces (with distinct boundaries).*

It is easy to observe that an edge of a triangle face of a minimally rigid plane graph is not necessarily contractible. In addition, a triangle face may contain no contractible edges at all. See Figures 2.1 and 2.2 for examples. This is one reason why the proof of the inductive construction via vertex splitting is more difficult than that of Theorem 2.2.8, where the corresponding inverse operations of extensions can be performed at *every* vertex of degree two or three.

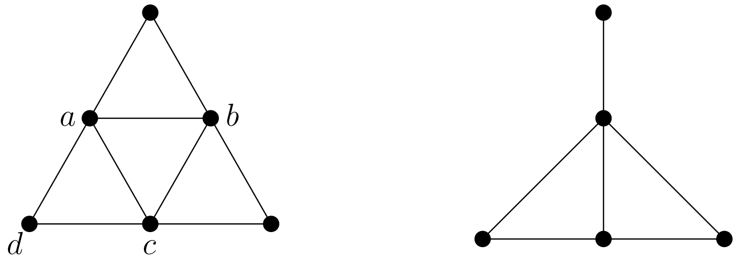

Figure 2.1: A minimally rigid graph G and a non-contractible edge ab on a triangle face abc. The graph obtained by contracting ab satisfies (2.1), but it has less edges than it should have. No edge on abc is contractible, but edges ad and cd are contractible in G.

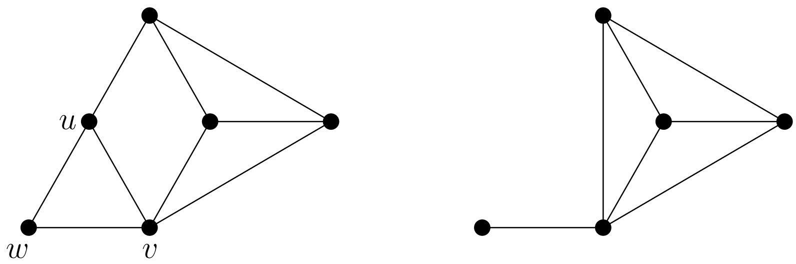

Figure 2.2: A minimally rigid graph G and a non-contractible edge uv on a triangle face uvw. The graph obtained by contracting uv has the right number of edges but it violates (2.1).

Lemma 2.6.2. *Let $G = (V, E)$ be a minimally rigid graph and let $X \subseteq V$ be a critical set. Let C be the union of some of the connected components of $G - X$. Then $X \cup C$ is critical.*

Proof: Let $C_1, C_2, ..., C_k$ be the connected components of $G - X$ and let $X_i = X \cup C_i$, for $1 \leq i \leq k$. We have $X_i \cap X_j = X$ and $d(X_i, X_j) = 0$ for all $1 \leq i < j \leq k$, and $\cup_{i=1}^k X_i = V$. Since G is minimally rigid and X is critical, we can count as follows: $2|V| - 3 = |E| = i(X_1 \cup X_2 \cup ... \cup X_k) = \sum_{i=1}^k i(X_i) - (k-1)i(X) \leq \sum_{i=1}^k (2|X_i| - 3) - (k-1)(2|X| - 3) = 2\sum_{i=1}^k |X_i| + 2(k-1)|X| - 3k + 3(k-1) = 2|V| - 3$. Thus equality must hold everywhere, and hence each X_i is critical.

Now Lemma 2.1.2 (and the fact that $|X| \geq 2$) implies that if C is the union of some of the components of $G - X$ then $X \cup C$ is critical. •

The next lemma characterises the contractible edges in a minimally rigid graph.

Lemma 2.6.3. *Let $G = (V, E)$ be a minimally rigid graph and let $e = uv \in E$. Then e is contractible if and only if there is a unique triangle uvw in G containing e and there exists no critical set X in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$.*

Proof: First suppose that e is contractible. Then G/e is minimally rigid, and, as we noted earlier, e must belong to a unique triangle uvw. For a contradiction suppose that X is a critical set with $u, v \in X$, $w \notin X$, and $|X| \geq 4$. Then e is an edge of $G[X]$ but it does not belong to any triangle in $G[X]$. Hence by contracting e we decrease the number of vertices and edges in $G[X]$ by one. This would make the vertex set of $G[X]/e$ violate (2.1) in G/e. Thus such a critical set cannot exist.

To see the 'if' direction suppose that there is a unique triangle uvw in G containing e and there exists no critical set X in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$. For a contradiction suppose that $G' := G/e$ is not minimally rigid. Let v' denote the vertex of G' obtained by contracting e. Since G is minimally rigid and e belongs to exactly one triangle in G, it follows that $|E(G')| = 2|V(G')| - 3$, so there is a set $Y \subset V(G')$ with $|Y| \geq 2$ and $i_{G'}(Y) \geq 2|Y| - 2$. Since G' is simple and uv belongs to a unique triangle in G, it follows that $V(G')$, all two-element subsets of $V(G')$, all subsets containing v' and w, as well as all subsets not containing v' satisfy (2.1) in G'. Thus we must have $|Y| \geq 3$, $v' \in Y$ and $w \notin Y$. Hence $X := (Y - v') \cup \{u, v\}$ is a critical set in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$, a contradiction. This completes the proof of the lemma. •

Thus two kinds of substructures can make an edge $e = uv$ of a triangle uvw non-contractible: a triangle uvw' with $w' \neq w$ and a critical set X with $u, v \in X$, $w \notin X$ and $|X| \geq 4$. Since a triangle is also critical, these substructures can be treated simultaneously. We say that a critical set $X \subset V$ is a *blocker* of edge $e = uv$ (with respect to the triangle uvw) if $u, v \in X$, $w \notin X$ and $|X| \geq 3$.

Lemma 2.6.4. *Let uvw be a triangle in a minimally rigid graph $G = (V, E)$ and suppose that $e = uv$ is non-contractible. Then there exists a unique maximal blocker X of e with respect to uvw. Furthermore, $G - X$ has precisely one connected component.*

Proof: There is a blocker of e with respect to uvw by Lemma 2.6.3. By Lemma 2.1.2 the union of two blockers of e, with respect to uvw, is also a blocker with respect to uvw. This proves the first assertion. The second one follows from Lemma 2.6.2: let C be the union of those components of $G - X$ that do not contain w, where X is the maximal blocker of e with respect to uvw. Since $X \cup C$ is critical, and does not contain w, it is also a blocker of e with respect to uvw. By the maximality of X we must have $C = \emptyset$. Thus $G - X$ has only one component (which contains w). •

Since a blocker X is a critical set in G, $G[X]$ is also minimally rigid.

Lemma 2.6.5. *Let $G = (V, E)$ be a minimally rigid graph, let uvw be a triangle, and let $f = uv$ be a non-contractible edge. Let X be the maximal blocker of f with respect to uvw. If $e \neq f$ is contractible in $G[X]$ then it is contractible in G.*

Proof: Let $e = rz$. Since e is contractible in $G[X]$, there exists a unique triangle rzy in $G[X]$ which contains e. For a contradiction suppose that e is not contractible in G. Then by Lemma 2.6.3 there exists a blocker of e with respect to rzy, that is, a critical set $Z \subset V$ with

$r, z \in Z$, $y \notin Z$, and $|Z| \geq 3$. Lemma 2.1.2 implies that $Z \cap X$ is critical. If $|Z \cap X| \geq 3$ then $Z \cap X$ is a blocker of e in $G[X]$, contradicting the fact that e is contractible in $G[X]$.

Thus $Z \cap X = \{r, z\}$. We claim that $w \notin Z$. To see this suppose that $w \in Z$ holds. Then $w \in Z - X$. Since $e \neq f$, and $|Z \cap X| = 2$, at least one of u, v is not in Z. But this would imply $d(X, Z) \geq 1$, contradicting Lemma 2.1.2. This proves $w \notin Z$.

Clearly, $Z - X \neq \emptyset$. Thus, since $Z \cup X$ is critical by Lemma 2.1.2, it follows that $Z \cup X$ is a blocker of f in G with respect to uvw, contradicting the maximality of X. This proves the lemma. •

Lemma 2.6.6. *Let $G = (V, E)$ be a minimally rigid plane graph, let uvw be a triangle face, and let $f = uv$ be a non-contractible edge. Let X be the maximal blocker of f with respect to uvw. Then all but one faces of $G[X]$ are faces of G.*

Proof: Consider the faces of $G[X]$ and the connected component C of $G - X$, which is unique by Lemma 2.6.4. Clearly, C is within one of the faces of $G[X]$. Thus all faces, except the one which has w in its interior, is a face of G, too. •

The exceptional face of $G[X]$ (which is not a face of G) is called the *special face* of $G[X]$. Since the special face has w in its interior, and uvw is a triangle face in G, it follows that the edge uv is on the boundary of the special face. If the special face of $G[X]$ is a triangle uvq, then the third vertex q of this face is called the *special vertex* of $G[X]$. If the special face of $G[X]$ is not a triangle, then X is a *nice blocker.* We say that an edge e is *face contractible* in a minimally rigid plane graph if e is contractible and the triangle containing e (which is unique by Lemma 2.6.3) is a face in the given embedding. The main result of this section will follow from the next result on the existence of a face contractible edge.

Theorem 2.6.7. *[13] Let $G = (V, E)$ be a minimally rigid plane graph with $|V| \geq 4$. Suppose that*
(i) if uvw is a triangle face, $f = uv$ is not contractible, and X is the maximal blocker of f with respect to uvw, then there is an edge in $G[X]$ which is face contractible in G,
(ii) for each vertex $r \in V$ there exist at least two face contractible edges which are not incident with r.

Proof: The proof is by induction on $|V|$. It is easy to check that the theorem holds if $|V| = 4$ (in this case G is unique and has essentially one possible planar embedding). So let us suppose that $|V| \geq 5$ and the theorem holds for graphs with less than $|V|$ vertices.

First we prove (i). Consider a triangle face uvw for which $f = uv$ is not contractible, and let X be the maximal blocker of f with respect to uvw. Since X is a critical set, the induced subgraph $G[X]$ is minimally rigid. Together with the embedding obtained by restricting the embedding of G to the vertices and edges of its subgraph induced by X, the graph $G[X]$ is a plane minimally rigid graph. Since $w \notin X$, $G[X]$ has less than $|V|$ vertices.

We call an edge e of $G[X]$ *proper* if $e \neq f$, e is face contractible in $G[X]$, and the triangle face of $G[X]$ containing e is a face of G as well. It follows from the definition and Lemma

2.6.5 that a proper edge e is face contractible in G as well. We shall prove (i) by showing that there is a proper edge in $G[X]$.

To this end first suppose that $|X| = 3$. Then $G[X]$ is a triangle, and each of its edges is contractible in $G[X]$. By Lemma 2.6.6 one of the two faces of $G[X]$ is a face of G as well. Thus each of the two edges of $G[X]$ which are different from f, is proper.

Next suppose that $|X| \geq 4$. By the induction hypothesis (ii) holds for $G[X]$ by choosing $r = u$. Thus there exist two face contractible edges e', e'' in $G[X]$ which are not incident with u (and hence e' and e'' must be different from f). If X is a nice blocker then the triangle face containing e' (or e'') in $G[X]$ is a face of G as well, by Lemma 2.6.6. Thus e' (or e'') is proper.

If X is not a nice blocker then it has a special triangle face uvq, which is not a face of G, and each of the other faces of $G[X]$ is a face of G by Lemma 2.6.6. Since e' and e'' are distinct edges which are not incident with u, at least one of them, say e', is not an edge of the triangle uvq. Hence the triangle face of $G[X]$ containing e' is a triangle face of G as well. Thus e' is proper. This completes the proof of (i).

It remains to prove (ii). To this end let us fix a vertex $r \in V$. We have two cases to consider.

Case 1. There exists a triangle face uvw in G with $r \notin \{u, v, w\}$.

If at least two edges on the triangle face uvw are face contractible then we are done. Otherwise we have blockers for two or three edges of uvw.

If none of the edges of the triangle uvw is contractible then there exist maximal blockers X, Y, Z for the edges vw, uw, and uv (with respect to u, v, and w), respectively. By Lemma 2.1.2 we must have $X \cap Y = \{w\}$, $X \cap Z = \{v\}$, and $Y \cap Z = \{u\}$ (since the sets are critical and $d(Y, Z), d(X, Y), d(X, Z) \geq 1$ by the existence of the edges of the triangle uvw). By our assumption r is not a vertex of the triangle uvw. Thus r is contained by at most one of the sets X, Y, Z. Without loss of generality, suppose that $r \notin X \cup Y$. By (i) each of the subgraphs $G[X], G[Y]$ contains an edge which is face contractible in G. These edges are distinct and avoid r. Thus G has two face contractible edges not containing r, as required.

Now suppose that uv is contractible but vw and uw are not contractible. Then we have maximal blockers X, Y for the edges vw, uw, respectively. As above, we must have $X \cap Y = \{w\}$ by Lemma 2.1.2. Since $r \neq w$, we may assume, without loss of generality, that $r \notin X$. Then it follows from (i) that there is an edge f in $G[X]$ which is face contractible in G. Thus we have two edges (uv and f), not incident with r, which are face contractible in G.

Case 2. Each of the triangle faces of G contains r.

Consider a triangle face ruv of G. Then uv is face contractible, for otherwise (i) would imply that there is a face contractible edge in $G[X]$, (in particular, there is a triangle face of G which does not contain r), a contradiction. Since G has at least two triangle faces by Lemma 2.6.1, it follows that G has at least two face contractible edges avoiding r. This proves the theorem. •

Theorem 2.6.7 implies that a minimally rigid plane graph on at least four vertices has a face contractible edge. A plane minimally rigid graph on three vertices is a triangle, and each of its edges is face contractible. Note that after the contraction of a face contractible edge the planar embedding of the resulting graph can be obtained by a simple local modification.

Since contracting an edge of a triangle face is the inverse operation of plane vertex splitting, the proof of the following theorem by induction is straightfoward.

Theorem 2.6.8. *[13] A plane graph is a minimally rigid plane graph if and only if it can be obtained from an edge by plane vertex splitting operations.*

It can be shown that any triangle face can be chosen as the starting configuration in Theorem 2.6.8.

A natural question is whether a 3-connected minimally rigid plane graph has a face contractible edge whose contraction preserves 3-connectivity as well. The answer is no: let G be a 3-connected minimally rigid plane graph and let G' be obtained from G by inserting a new triangle face $a'b'c'$ and adding the edges aa', bb', cc', for each of its triangle faces abc. Then G' is a 3-connected minimally rigid plane graph with no such edge. It is an open question whether there exist good local reduction steps which could lead to an inductive construction in the 3-connected case, such that all intermediate graphs are also 3-connected.

2.6.2 Line graphs

In this section we deduce a different formula for $r(G)$ when G is a line graph. We shall use this result in Section 4.2.1. The *line graph* $L(G)$ of a graph $G = (V, E)$ is the simple graph with vertex set $\{v_e : e \in E\}$, where two vertices v_e, v_f are adjacent if and only if e, f have a common end-vertex in G.

Let $G = (V, E)$ be a graph. For a family $\mathcal{F}$ of pairwise disjoint subsets of V let $E_G(\mathcal{F})$ denote the set, and $e_G(\mathcal{F})$ the number, of edges of G connecting distinct members of $\mathcal{F}$. For a partition $\mathcal{P}$ of V let

$$\mathrm{def}_G(\mathcal{P}) = 3(|\mathcal{P}| - 1) - 2e_G(\mathcal{P})$$

denote the *deficiency* of $\mathcal{P}$ in G and let

$$\mathrm{def}(G) = \max\{\mathrm{def}_G(\mathcal{P}) : \mathcal{P} \text{ is a partition of } V\}.$$

We say that a partition $\mathcal{P}$ of V is *tight* if $\mathrm{def}_G(\mathcal{P}) = \mathrm{def}(G)$ holds. Note that $\mathrm{def}(G) \geq 0$, since $\mathrm{def}_G(\{V\}) = 0$. The following rank formula shows that the 'degree of freedom' of $L(G)$ is equal to the deficiency of G.

Theorem 2.6.9. *[35] Let $G = (V, E)$ be a graph with minimum degree at least two. Then*

$$r(L(G)) = 2|E| - 3 - \mathit{def}(G). \tag{2.9}$$

Proof: First we prove that the right hand side is an upper bound on $r_2(L(G))$. Since $|V(L(G))| = |E|$, we have $r_2(L(G)) \leq 2|E| - 3$. Thus we may assume that $\mathrm{def}(G) \geq 1$. Let $\mathcal{Q} = \{Q_1, Q_2, ..., Q_t\}$ be a tight partition of V. Since $\mathrm{def}(G) \geq 1$, we must have $t \geq 2$.

For $v \in V$ let $B(v)$ denote the set of vertices in $L(G)$ corresponding to the edges incident with v in G. Since G has minimum degree at least two, we have $|B(v)| \geq 2$ for all $v \in V$. Let

$X_i = \cup_{v\in Q_i} B(v)$, for $1 \leq i \leq t$. Since each set $B(v)$ contains at least two vertices, we have $|X_i| \geq 2$ for $1 \leq i \leq t$. Furthermore, $|\{X_i : v_e \in X_i\}| \leq 2$ for each vertex v_e of $L(G)$ with equality if and only if $e \in E_G(\mathcal{Q})$. Thus $\sum_{i=1}^{t} |X_i| = |E| + e_G(\mathcal{Q})$. Since every edge of $L(G)$ is induced by some X_i and each set $X \subseteq V(L(G))$ with $|X| \geq 2$ induces at most $2|X| - 3$ independent edges in $\mathcal{R}_2(L(G))$, we can deduce that

$$\begin{aligned} r_2(L(G)) &\leq \sum_{i=1}^{t}(2|X_i| - 3) = 2|E| + 2e_G(\mathcal{Q}) - 3t \\ &= 2|E| - 3 - \mathrm{def}(G). \end{aligned}$$

To prove that equality holds consider the rigid components $H_1, H_2, ..., H_t$ of $L(G)$ and let $C_i = V(H_i)$ for $1 \leq i \leq t$. Since each set $B(v)$, $v \in V$, induces a complete (and hence rigid) subgraph in $L(G)$, we must have $B(v) \subseteq C_i$ for some $1 \leq i \leq t$. Furthermore, since $|B(v)| \geq 2$ for all $v \in V$, the maximality of the C_i's and the glueing lemma imply that each $B(v)$ is contained in exactly one set C_i. Let $Q_i = \{v \in V : B(v) \subseteq C_i\}$, $1 \leq i \leq t$. Observe that $Q_i \neq \emptyset$ for all $1 \leq i \leq t$, since each rigid component H_i has at least one edge, say $v_e v_f$. Hence there is a vertex $x \in V$ which is a common end-vertex of edges e, f in G. Thus $|B(x) \cap C_i| \geq 2$ and hence, by the glueing lemma, $B(x) \subseteq C_i$ and $x \in Q_i$ must hold. It follows that $\mathcal{Q} = \{Q_1, Q_2, ..., Q_t\}$ is a partition of V.

Claim 2.6.10. *$v_e \in C_i \cap C_j$ for some $v_e \in V(L(G))$ and $1 \leq i < j \leq t$ if and only if $e \in E_G(Q_i, Q_j)$.*

Proof: First suppose $v_e \in C_i \cap C_j$. Consider an edge $v_e v_f \in E(H_i)$. As above, we may deduce that there is a vertex $x \in V$, incident with e, f, with $x \in Q_i$. Similarly, by considering an edge $v_e v_h \in E(H_j)$ we obtain that there is a vertex $y \in V$, incident with e, h, with $y \in Q_j$. This implies that $e = xy$ and $e \in E_G(Q_i, Q_j)$.

Conversely, suppose that $e = xy \in E_G(Q_i, Q_j)$. Then $B(x) \subseteq C_i$, $B(y) \subseteq C_j$. Since $v_e \in (B(x) \cap B(y))$, we have $v_e \in C_i \cap C_j$, as required. •

By using Theorem 2.3.2 and Claim 2.6.10 we obtain

$$\begin{aligned} r_2(L(G)) &= \sum_{i=1}^{t}(2|C_i| - 3) = 2|E| + 2e_G(\mathcal{Q}) - 3t \\ &= 2|E| - 3 - \mathrm{def}(\mathcal{Q}) \geq 2|E| - 3 - \mathrm{def}(G), \end{aligned}$$

which completes the proof. •

The lower bound on the minimum degree of G in Theorem 2.6.9 cannot be weakened. This follows by observing that if G is a star then $L(G)$ is rigid but G is highly deficient.

Let $G = (V, P)$ be a multigraph. For $v \in V$ let $E_G(v)$ be the set of all edges of G incident to v. The *body-and-pin graph* of G is the graph G^* with $V(G^*) = V \cup P$ and

$$E(G^*) = \{vp \ : \ v \in V \text{ and } p \in E_G(v)\} \cup \{p_1 p_2 \ : \ v \in V \text{ and } p_1, p_2 \in E_G(v)\}.$$

By observing that G^* can be obtained from $L(G)$ by adding $|V|$ new vertices such that each new vertex is connected to a complete subgraph of size at least two, we can deduce the following.

Theorem 2.6.11. *Let $G = (V, P)$ be a multigraph with no isolated vertices. Then $r(G^*) = 2(|V| + |P|) - 3 - def(G)$.*

In the body-and-pin graph each pin is shared by exactly two bodies. We can obtain more general structures by relaxing this condition. Let $H = (V \cup P, I)$ be a bipartite graph without isolated vertices. The *identified body-and-pin graph* of H is the graph H^{BP} with $V(H^{BP}) = V \cup P$ and

$$E(H^{BP}) = \{vp \ : \ v \in V, p \in P, vp \in I\} \cup \{p_1 p_2 \ : \ v \in V \text{ and } p_1, p_2 \in E_H(v)\}.$$

(This definition extends the earlier definition for a graph G by taking H to be the bipartite graph obtained by subdividing each edge of G. We then have $G^* = H^{BP}$.) Let $\mathcal{F}$ be a partition of V. For each $p \in P$ let $w_{\mathcal{F}}(p)$ be the number of sets $F \in \mathcal{F}$ for which $N_H(p) \cap F \neq \emptyset$. Put $\mathrm{def}_H(\mathcal{F}) = 3(|\mathcal{F}| - 1) - 2(\sum_{p \in P}(w_{\mathcal{F}}(p) - 1))$ and let $\mathrm{def}(H) = \max_{\mathcal{F}}\{\mathrm{def}_H(\mathcal{F})\}$. By using the proof method of Theorem 2.6.9 it is not difficult to show that $r(H^{BP}) = 2(|V| + |P|) - 3 - \mathrm{def}(H)$. See also [27].

2.6.3 Regular graphs

In this section we give a short proof for (extensions of) two theorems due to S. Luo [50]. We call a graph *d-regular* if the degree of each vertex of G is equal to d.

Theorem 2.6.12. *Let $G = (V, E)$ be a connected d-regular graph. Then*
(i) [50] if $d = 4$ then $r(G) \geq \frac{8}{5}|V|$, and
(ii) [50] if $d = 5$ then $r(G) \geq \frac{5}{3}|V|$.
Furthermore,
(iii) if $d = 4$ and G is 3-edge-connected then we have $r(G) \geq \frac{7}{4}|V|$.

Proof: First we make some observations about thin covers of arbitrary graphs. Consider the thin cover $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ of G obtained from the rigid components G_i of G, $1 \leq i \leq m$. We have $G_i = G[X_i]$, $r(G_i) = 2|X_i| - 3$, $1 \leq i \leq m$, and $r(G) = val(\mathcal{X})$ by Lemma 2.3.1 and Theorem 2.3.2. A rigid component on two vertices is called a *trivial component*, otherwise it is a *non-trivial component.* Edges induced by trivial components are bridges, i.e. they belong to all bases of the rigidity matroid of G. So are the edges incident with a vertex v which is of degree two within some component of G. These observations follow from the fact that circuits of the rigidity matroid induce rigid subgraphs on at least four vertices, with minimum degree three (see Section 3.1).

If we have a vertex v of degree two in some component G_i, say with incident edges vx, vy, then we may delete v from X_i and add $\{v, x\}, \{v, y\}$ to $\mathcal{X}$ to obtain another thin cover of the same value. Clearly, $G[X_i - v]$ is also rigid. The subgraphs obtained by iteratively deleting the degree-two vertices from the non-trivial components will also be called components. To summarize, we may suppose that G has a thin cover $\mathcal{X}$ of smallest value for which each non-trivial component is rigid and has minimum degree at least three, and all redundant (i.e. non-bridge) edges of G are induced by some member of $\mathcal{X}$ of size at least four. The latter property implies that

$$r(G) = |E| - \sum_{X \in \mathcal{Y}} (|E(X)| - (2|X| - 3)), \tag{2.10}$$

where $\mathcal{Y} \subseteq \mathcal{X}$ is the family of non-trivial components.

For d-regular graphs of small degree we can say more. Since the cover is thin, for $d = 4, 5$ the properties described above imply that (*) *the non-trivial components of $\mathcal{X}$ are pairwise vertex-disjoint.*

Now focus on a non-trivial component $X \in \mathcal{Y}$. First suppose $d = 4$. Since G is connected, either $X = V$ (in which case G is rigid and the theorem follows) or at least two edges must leave X. Thus $|E(X)| \leq 2|X| - 1$ and hence $|E(X)| - (2|X| - 3) \leq 2$. Furthermore, if $|X| = 4$ then $|E(X)| - (2|X| - 3) = 1$. Thus we can find one redundant edge in each component of size four and at most two redundant edges in each component of size at least five. Hence, by using (*), we can deduce that the right hand side of (2.10) is minimized when each non-trivial component has five vertices and two outgoing edges, and these components cover V (assuming that $|V|$ is divisible by five, for simplicity). Therefore $r(G) \geq |E| - \frac{2}{5}|V| = \frac{8}{5}|V|$. This proves (i). If G is 3-edge-connected then we must have at least three edges leaving each non-trivial component X with $X \neq V$ and hence $|E(X)| - (2|X| - 3) \leq 1$ follows. In this case the right hand side of (2.10) is minimized when each non-trivial component has four vertices and four outgoing edges, and these components cover V. This gives $r(G) \geq |E| - \frac{1}{4}|V| = \frac{7}{4}|V|$, which proves (iii).

Next suppose $d = 5$. For simplicity suppose also that G is 2-edge-connected. Then, by using similar arguments, we can deduce that each non-trivial component X with $X \neq V$ has $|E(X)| \leq \frac{5}{2}|X| - 1$ and hence $|E(X)| - (2|X| - 3) \leq \frac{1}{2}|X| + 2$ with equality only if $|X| \geq 6$. Thus the right hand side of (2.10) is minimized when each non-trivial component has six vertices and these components cover V. In this case we have $r(G) \geq |E| - \frac{1}{2}|V| - \frac{2}{6}|V| = \frac{5}{3}|V|$. This proves (ii). •

The bounds given in Theorem 2.6.12 are sharp, see [50] for examples in case of (i) and (ii). Statement (iii) answers [50, Question 1.8]. We remark that if $d \geq 6$ then (*) may not hold, which makes the computations more difficult. However, a similar linear lower bound is easy to obtain and with more efforts the sharp bound is probably also not so difficult to reach.

2.7 Optimal pinning sets

Let $G = (V, E)$ be a graph. First consider the problem of finding an optimal family of tracks, $U = (U(v) : v \in V)$, which fixes (G, p) for a generic realization of G in $\mathbb{R}^2$. As we have observed earlier, we may assume that G is M-independent (or equivalently, that G is sparse). Thus $|E| = 2|V| - k$ for some integer $k \geq 3$. It is also clear that $\sum_{v \in V}(2 - \dim U(v)) = k$ for an optimal family of tracks. The following algorithm, due to Lee et al. [45], determines an optimal family of tracks in $O(n^2)$ time. It uses $k-2$ one-dimensional tracks (also called *sliders*) and one pin to fix (G, p). (For the remaining vertices the tracks are two-dimensional, that is, they induce no constraints.)

The algorithm works as follows. First identify the rigid components of G. Mark one of the components, say C, as the *base*. For some edge uv in C assign a pin to u and a slider to v. This fixes the base. Then repeat the following until one rigid component remains: pick an edge xy which leaves the base and assign a slider to y. Update G by adding a new edge yz, where xz is an edge in the base. Replace the base C by the rigid component of the updated graph containing uv.

The correctness of this algorithm follows from the fact that if C' is a rigid component that shares vertex x with C then the only motion of C' with respect to C is a rotation about x. Since the framework is generic, assigning a slider to y eliminates this motion and hence the distance between y and z becomes fixed. Thus every iteration increases the rank by one and therefore the algorithm will terminate with a rigid graph after adding at most $k-3$ sliders (in addition to the pin and the slider added to fix the original base). The algorithm, when applied to (a generic realization of) the graph of Figure 1.1(a), may give the family of tracks shown by Figure 1.1(b).

We remark that combinatorial characterizations for the generic rigidity of bar-and-slider frameworks (which are bar-and-joint frameworks equipped with one-dimensional tracks at given joints) have been given in [45], and also in [42], where the authors consider the version in which the directions of the slider lines are also given.

Next we consider the pinning number. Fekete [11] proved that $pin_2(G)$ can be computed in polynomial time. The key observation is as follows.

Lemma 2.7.1. *Let $G = (V, E)$ be an M-independent graph and let $P \subseteq V$ with $|P| \geq 2$. Then P is a pinning set for G if and only if $2|X| \leq e(X)$ for all $X \subseteq V - P$.*

Proof: Suppose, for a contradiction, that P is a pinning set and $2|X| > e(X)$ for some $X \subseteq V - P$ and let $Z = V - X$. Then $\mathcal{X} = \{Z \cup \{\{u, v\} : uv \in E - E(Z)\}\}$ is a thin cover of $G + K(P)$ with $val(\mathcal{X}) \leq 2|Z| - 3 + e(X) < 2|V| - 3$. Thus, by Theorem 2.3.2, $G + K(P)$ is not rigid. Hence P is not a pinning set by Lemma 1.4.2, a contradiction.

Now suppose $2|X| \leq e(X)$ for all $X \subseteq V - P$. It follows from Lemma 2.3.3 that there is a thin cover $\mathcal{X}$ of $G + K(P)$ with $P \subseteq Z$ for some $Z \in \mathcal{X}$ and $r(G + K(P)) = val(\mathcal{X}) = 2|Z| - 3 + e(V - Z)$. Since $e(V - Z) \geq 2|V - Z|$ this gives $val(\mathcal{X}) = 2|V| - 3$. Hence $G + K(P)$ is rigid and, by Lemma 1.4.2, P is a pinning set. •

Thus finding a smallest pinning set is equivalent to finding a largest set $Y \subseteq V$ for which $e(X) \geq 2|X|$ for all $X \subseteq Y$. This can be formulated as a matching problem in an auxiliary graph and can be solved in $O(n^2)$ time, see [11]. The reduction is as follows.

Consider the bipartite graph $B(G) = \{E, V^*; E^*\}$ obtained from G by assigning one vertex to every edge of G and two vertices v_1, v_2 to every vertex of G, forming the two sides E and V^* of the bipartition, and connecting a pair $e \in E, v_i \in V^*$ by an edge $ev_i \in E^*$ if and only if the edge e is incident with v in G. Let

$$\nu_p(B(G)) = \max\{|U| : U \subseteq V, \text{ there is a matching in } B(G) \text{ covering the vertices of } U^*\},$$

where U^* is the union of all pairs v_1, v_2 with $v \in U$. Observe that for some $Y \subseteq V$ we have $2|X| \leq e(X)$ for all $X \subseteq Y$ in G if and only if Y^* satisfies the condition of Hall's theorem on matchings in bipartite graphs in $B(G)$. Thus we have:

Lemma 2.7.2. $\max\{|Y| : 2|X| \leq e(X)$ *for all* $X \subseteq Y\} = \nu_p(B(G))$.

Now extend $B(G)$ by adding the edges v_1v_2 for all $v \in V$. Let $B^*(G)$ denote the resulting graph and let $\nu(B^*(G))$ denote its matching number (i.e. the size of a maximum matching in $B^*(G)$). The final observation is the following.

Lemma 2.7.3. $\nu_p(B(G)) = \nu(B^*(G)) - |V|$.

Proof: Let $Y \subseteq V$ be a maximum size subset for which Y^* can be covered by a matching in $B(G)$ and let M be the corresponding matching. We can extend M to a larger matching in $B^*(G)$ by adding the edge u_1u_2 for all pairs u_1, u_2 with $u \in V - Y$. Hence $\nu(B^*(G)) \geq |V| + \nu_p(B(G))$.

Now consider a maximum size matching N in $B^*(G)$. If N covers only one of the two vertices u_1, u_2 for some $u \in V$ then by replacing this edge of N by the edge u_1u_2 we obtain another maximum size matching in $B^*(G)$. Thus we may suppose that for all pairs u_1, u_2 with $u \in V$, u_1 and u_2 are both covered by N. This implies that $\nu_p(B(G)) \geq \nu(B^*(G)) - |V|$. So we must have equality and the lemma follows. •

We can now conclude that the problem of finding a smallest pinning set in G can be reduced to the problem of finding a maximum matching in $B^*(G)$, and hence it can be solved in polynomial time.

Fekete [11] also provides a min-max formula for $pin_2(G)$. Makai and Szabó [51] deduce this formula by using polymatroidal methods. We also remark that Servatius, Shai, and Whiteley [58] consider a different version of the pinning problem and provide a characterization and a decomposition result for the so-called pinned isostatic graphs.

Chapter 3

The rigidity matroid

In this chapter we prove a number of structural properties of the rigidity matroid and use them to solve various rigidity problems. We start with the circuits, that is, the minimal M-dependent sets.

3.1 M-circuits

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}(G)$. For example, K_4, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all M-circuits. Using Laman's characterization of M-independence, i.e. the sparsity count (2.1), we may deduce:

Lemma 3.1.1. *Let $G = (V, E)$ be a graph. The following statements are equivalent.*
(a) G is an M-circuit.
(b) $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$.
(c) $|E| = 2|V| - 2$ and

$$i(X) \leq 2|X| - 3 \quad \textit{for all } X \subseteq V \textit{ with } 2 \leq |X| \leq |V| - 1. \tag{3.1}$$

Note that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{R}(G)$ i.e. an M-circuit of G. The proof of the following elementary property of M-circuits is similar to that of Lemma 2.2.5.

Lemma 3.1.2. *Let $H = (V, E)$ be an M-circuit and let $\emptyset \neq X \subset V$. Then $d(X) \geq 3$ and if $d(X) = 3$ holds then either $|X| = 1$ or $|V - X| = 1$.*

Let $H = (V, E)$ be a 2-connected graph and suppose that (H_1, H_2) is a 2-separation of G with $V(H_1) \cap V(H_2) = \{a, b\}$. For $1 \leq i \leq 2$, let $H_i' = H_i + ab$ if $ab \notin E(H_i)$ and otherwise put $H_i' = H_i$. We say that H_1, H_2 are the *cleavage graphs* obtained by *cleaving* G *along* $\{a, b\}$. Given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and two designated edges $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$, the 2-*sum* of H_1 and H_2 (along the edge pair u_1v_1, u_2v_2), denoted by $H_1 \oplus_2 H_2$, is the graph obtained from $H_1 - u_1v_1$ and $H_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2.

We shall use the following results on 2-sums and cleaving.

Lemma 3.1.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be M-circuits and let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$. Then the 2-sum $G_1 \oplus_2 G_2$ along the edge pair u_1v_1, u_2v_2 is an M-circuit.*

Lemma 3.1.4. *Let $G = (V, E)$ be an M-circuit and $\{a, b\}$ be a 2-separator of G. Then $ab \notin E$. Furthermore, if G' and G'' are the graphs obtained from G by cleaving G along $\{a, b\}$ then G' and G'' are both M-circuits.*

Let $G = (V, E)$ be a graph. Let $V_3 = \{v \in V : d(v) = 3\}$ denote the set of vertices of degree three in G. For convenience, vertices of degree three will also be called *nodes* in the next few sections. We call $G[V_3]$ the *subgraph of nodes of* G. A node of G with degree at most one (exactly two, exactly three) in the subgraph of nodes of G is called a *leaf node* (*series node*, *branching node*, respectively). A *wheel* $W_n = (V, E)$ is a graph on $n \geq 4$ vertices which has a vertex z which is adjacent to all the other vertices and for which $W_n[V - z]$ is a cycle. Thus the subgraph of nodes of a wheel W_n with $n \geq 5$ is a cycle.

Lemma 3.1.5. *If $G = (V, E)$ is an M-circuit then either G is a wheel or $G[V_3]$ is a forest.*

Proof: Suppose that the subgraph of nodes of G contains a cycle and choose a shortest (diagonal free) cycle C of $G[V_3]$. Since G is not a cycle, $V - V(C) \neq \emptyset$. Let $\bar{C} = V - V(C)$. Since each vertex of C is a node (so it has degree three) and C has no diagonals, $|\bar{C}| = 1$ implies that G is a wheel. Hence we may assume that $|\bar{C}| \geq 2$. In this case $i(\bar{C}) = 2|V| - 2 - i(C) - d(C, \bar{C}) = 2|V| - 2 - |C| - |C| = 2(|V| - |C|) - 2 = 2|\bar{C}| - 2$, contradicting (3.1). •

Lemma 3.1.6. *Let $G = (V, E)$ be an M-circuit and let $X \subset V$ be a critical set with $|V - X| \geq 2$. Then $V - X$ contains at least two nodes.*

Proof: Let X be a critical set with $|V - X| \geq 2$ and let $Y = V - X$ denote its complement. By definition, $|X| \geq 2$. Since we also have $|Y| \geq 2$, Lemma 3.1.2 implies $d(Y) \geq 4$. Clearly, $\sum_{v \in Y} d(v) = 2i(Y) + d(Y)$. Thus, using the fact that G is an M-circuit and X is critical, we can count as follows:

$$\begin{aligned}\sum_{v \in Y} d(v) &= 2i(Y) + d(Y) = 2(|E| - i(X) - d(Y)) + d(Y) \\ &= 2(2|V| - 2 - (2|X| - 3) - d(Y)) + d(Y) \\ &= 4|Y| + 2 - d(Y) \leq 4|Y| - 2 \end{aligned} \tag{3.2}$$

Since the degree of each vertex is at least three in G, (3.2) implies that Y contains at least two nodes, as required. •

3.1.1 Exercises

Exercise 3.1.7. *Find all non-isomorphic M-circuits on at most six vertices. Find one which is (a) not 3-connected, (b) has an independent set of size three.*

Exercise 3.1.8. *Show that the planar dual of a planar (3-connected) M-circuit is a planar (3-connected) M-circuit.*

Exercise 3.1.9. *Consider the family of those simple graphs that can be obtained from a minimally rigid graph by adding an edge. Find an inductive construction for this family that uses extensions.*

Exercise 3.1.10. *Let T be a tree on vertex set V. Show that there is an M-circuit on vertex set V which contains T.*

Exercise 3.1.11. *Prove that $\mathcal{R}(G)$ is a uniform matroid if and only if G is M-independent or G is an M-circuit.*

3.2 Inductive construction of M-circuits

Let $G = (V, E)$ be a graph and let $uw \in E$. Recall that an *1-extension* of G along uw is obtained from G by subdividing the edge uw by a new vertex v (i.e. replacing the edge uw by a path uvw) and adding a new edge vz for some $z \in V - \{u, w\}$. The next lemma is easy to prove.

Lemma 3.2.1. *Let G' be obtained from G by an 1-extension. If G is an M-circuit then so is G'. If G is 3-connected then so is G'.*

In this section we shall prove an inductive construction of M-circuits using 1-extensions and 2-sums (or 1-extension alone, provided the M-circuit is 3-connected). The latter version (and its generalization, Theorem 3.4.6) will be the key combinatorial result that leads to the characterization of globally rigid graphs.

Let v be a node in a graph H with $N(v) = \{u, w, z\}$. Recall that the splitting operation at v deletes one of the edges incident to v, say zv, and replaces the remaining two edges uv, wv by a new edge uw (and then deletes v as well). The resulting graph is denoted by H_v^{uw}, or simply H_v, when the split pair of edges is clear.

Let G be an M-circuit and let v be a node in G. A pair of edges uv, wv incident to v (and the corresponding splitting) is called *admissible* if G_v^{uw} is also an M-circuit. We say that node v is *admissible* if there is an admissible splitting at v (among the three possible splittings). Otherwise v is *non-admissible*. First we show that every 3-connected M-circuit on at least five vertices has an admissible node. For example, consider a wheel W_n with $n \geq 5$. In this graph there is a (unique) admissible splitting at each node v (for which the splitting results in a smaller wheel).

Lemma 3.2.2. *Let $G = (V, E)$ be an M-circuit and let v be a node in G with $N(v) = \{u, w, z\}$. Then splitting off v on the pair uv, wv is not admissible if and only if there is a critical set $X \subset V$ with $u, w \in X$ and $v, z \notin X$.*

Proof: Let $G_v = G_v^{uw}$. First suppose that X is a critical set in G with $u, w \in X$ and $v, z \notin X$. Then splitting off the pair uv, wv (and hence adding a new edge uw) increases $i(X)$ by one. Since $z \notin X$, X violates (3.1) in G_v. Thus splitting off v on the pair uv, wv is not admissible.

Conversely, suppose that $X \subset V(G_v) = V - v$ violates (3.1) in G_v. Then $i_{G_v}(X) \geq 2|X| - 2$. Since $i_{G_v}(X) \leq i(X) + 1$, it follows that X is critical in G and $u, w \in X$. If $z \in X$ then $i(X + v) = i(X) + 3 = 2|X| - 3 + 3 = 2|X + v| - 2$, contradicting (3.1). Thus $z \notin X$ holds, too, as required. •

If v is a node with $N(v) = \{u, w, z\}$ and X is a critical set with $u, w \in X$ and $v, z \notin X$ then we call X a *v-critical set on u and w*, or simply a *v-critical set.* If $d(z) = 3$ then it is obvious that splitting off v on uv, wv is non-admissible, since such a split would make $d_{G_v}(z) = 2$. (This observation also shows that all branching nodes are non-admissible.) In this case $V - \{v, z\}$ is a "trivial" v-critical set on u and w. "Non-trivial" critical sets will be of particular interest: if X is a v-critical set on u and w for some node v with $N(v) = \{u, w, z\}$, and $d(z) \geq 4$, then X is called *node-critical.*

Lemma 3.2.3. *Let $G = (V, E)$ be a 3-connected M-circuit with $|V| \geq 5$ and suppose that v is a non-admissible leaf node of G. Then there exist two v-critical sets X, Y such that $|X \cap Y| \geq 2$ and $X \cup Y = V - v$. Moreover, if v is adjacent to a node z, then X and Y can be chosen to satisfy $z \in X \cap Y$ as well.*

Proof: Let $N(v) = \{x, y, z\}$. Since v is non-admissible, Lemma 3.2.2 implies that there exist three v-critical sets X, Y, Z on y and z, x and z, x and y, respectively. Suppose that no two of these sets intersect each other in at least two vertices. Let m denote the number of those edges in $G[X \cup Y \cup Z]$ which do not belong to the edge set of $G[X]$, $G[Y]$, or $G[Z]$. Then

$$2|X \cup Y \cup Z| - 3 \geq i(X \cup Y \cup Z) = i(X) + i(Y) + i(Z) + m$$

$$= 2|X| - 3 + 2|Y| - 3 + 2|Z| - 3 + m$$

$$= 2(|X| + |Y| + |Z| - 3) - 3 + m = 2|X \cup Y \cup Z| - 3 + m$$

Thus equality holds everywhere, and hence $X \cup Y \cup Z$ is critical and $m = 0$. Since $d(v, X \cup Y \cup Z) = 3$, this implies that $X \cup Y \cup Z = V - v$ (otherwise $(X \cup Y \cup Z) + v$ violates (3.1)). Hence, since $|V| \geq 5$, at least one of the three critical sets X, Y, Z (say, X) satisfies $|X| \geq 3$. But we have $m = 0$, and hence y, z is a cutpair in G, contradicting the fact that G is 3-connected. This contradiction shows that we have two sets (say, X and Y) with $|X \cap Y| \geq 2$. Hence $X \cup Y$ is also critical by Lemma 3.9.12 and so $X \cup Y = V - v$ follows, since $d(v, X \cup Y) = 3$.

To see the second part of the statement of the lemma suppose that z is a node. If the edges xz and yz are both present in G, then x, y is a cutpair, contradicting the fact that G is 3-connected. Thus we may assume, without loss of generality, that $yz \notin E$. Then for the v-critical set X on y and z we must have $|X| \geq 3$. By Lemma 2.2.5(b) $G[X]$ is 2-connected and hence z has two neighbours in X. If z has no neighbours in Y then $xz \notin Y$, $|Y| \geq 3$, and z is an isolated vertex in $G[Y]$. This would contradict Lemma 2.2.5(b). Hence z has a neighbour in Y and this implies that $|X \cap Y| \geq 2$. By Lemma 3.9.12 this gives that $X \cup Y$ is also critical, and since $d(v, X \cup Y) \geq 3$, we must have $X \cup Y = V - v$, as required. •

The next lemma is crucial in the proof of the main result of this section.

Lemma 3.2.4. *Let $G = (V, E)$ be a 3-connected M-circuit and suppose that G is not a wheel. Let $v \in V$ be a node with $N(v) = \{x, y, z\}$ and $d(z) \geq 4$, and let X be a v-critical set on x, y. Furthermore, suppose that either*
(a) there is a non-admissible series node $u \in V - X - v$ with precisely one neighbour w in X, and w is a node, or
(b) there is a non-admissible leaf node $t \in V - X - v$.
Then there is a node-critical set X' in G that properly contains X.

Proof: (a) Let $u \in V - X - v$ be a non-admissible series node with $N(u) = \{w, p, q\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since u is a series node, we can assume that $d(p) = 3$ and $d(q) \geq 4$. Since u is non-admissible, there exists a u-critical set Y on w and p by Lemma 3.2.2. Now G is not a wheel, and hence $G[V_3]$ contains no cycles by Lemma 3.1.5. Thus $pw \notin E$ and hence $|Y| \geq 3$. This implies, by Lemma 2.2.5(b), that $G[Y]$ is 2-connected, and hence Y contains two neighbours of w. If $|X| = 2$ then X induces an edge. If $|X| \geq 3$ then Lemma 2.2.5(b) implies that $G[X]$ is 2-connected. In any case, we conclude that $G[X]$ is connected, and hence at least one of the neighbours of w in Y must be in X. Thus $|X \cap Y| \geq 2$. Furthermore, $X' := X \cup Y \subseteq V - u - q$. By Lemma 2.1.2 it follows that X' is a u-critical set on w and p. Thus, since $d(q) \geq 4$ and $p \notin X$, the set X' is a node-critical set that properly contains X, as required.

(b) Since t is a non-admissible leaf node, Lemma 3.2.3 implies that there exist two t-critical sets Y_1 and Y_2 with $Y_1 \cup Y_2 = V - t$, $|Y_1 \cap Y_2| \geq 2$, and so that if t has a neighbour r which is a node then we can also assume $r \in Y_1 \cap Y_2$. Note that Y_1 and Y_2 are node-critical and $|Y_1|, |Y_2| \geq 3$. If $|X| = 2$ then X induces the edge xy. Since $x, y \in Y_1 \cup Y_2$ and, by Lemma 2.1.2, we have $d(Y_1 - Y_2, Y_2 - Y_1) = 0$, it follows that xy is induced by Y_1 or Y_2. Thus $|X| = 2$ implies that $X \subset Y_1$ or $X \subset Y_2$, which proves part (b) of the lemma by choosing $X' = Y_1$ or $X' = Y_2$. Hence we may assume $|X| \geq 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geq 3$, we have $|X \cap Y_1| \geq 2$ or $|X \cap Y_2| \geq 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geq 2$ holds.

We must have $d(t, X) \leq 2$, since $d(t, X) = 3$ would imply that $X + t$ violates (3.1). If $d(t, X) = 2$ then $X + t$ is also critical and by choosing $X' = X + t$ the lemma follows. Thus we may assume that $d(t, X) \leq 1$ (and hence $|N(t) \cap X| \leq 1$).

It follows by Lemma 2.1.2 that $X \cup Y_1$ is a critical set. If $N(t) \cap X \subseteq Y_1$ then the lemma follows by choosing $X' = X \cup Y_1$, since this set is node critical and properly contains X. Thus we may assume that $N(t) \cap X = \{s\}$ and $s \notin Y_1$. Hence $d(s) \geq 4$ holds, for otherwise $d(s) = 3$ would imply $s \in Y_1 \cap Y_2$. We have $s \in Y_2$, since $Y_1 \cup Y_2 = V - t$. So if $|X \cap Y_2| \geq 2$ then we are done, as above, by choosing the node-critical set $X' = X \cup Y_2$. We may now assume that $|X \cap Y_2| = 1$. Since $d(t, X \cup Y_1) = 3$ and $X \cup Y_1$ is critical, we have $X \cup Y_1 = V - t$. We also have $(X - s) \subseteq Y_1$, since $Y_1 \cup Y_2 = V - t$. It follows that $V - Y_1 = \{s, t\}$. But we have $d(s) \geq 4$, which contradicts Lemma 3.1.6. This completes the proof. •

We are now ready to prove the result on admissible nodes.

Theorem 3.2.5. *[5] Let $G = (V, E)$ be a 3-connected M-circuit with $|V| \geq 5$. Then G has an admissible node.*

Proof: The theorem trivially holds if G is a wheel, so we may assume that G is not a wheel. Hence the subgraph of nodes of G is a forest (on at least four nodes) by Lemma 3.1.5. Let $\mathcal{X} = \{X \subset V : X \text{ is a node-critical set in } G\}$. If $\mathcal{X} = \emptyset$ then we are done since either G is a wheel, or $G[V_3]$ is a forest, in which every leaf or series is admissible if no node-critical sets exist. Otherwise let X be a maximal member of $\mathcal{X}$. Since X is node-critical, there exists a node v and $t \in N(v)$ such that X is a v-critical set, $d(t) \geq 4$, and $t \notin X$. Clearly, $X + v$ is also critical and $|V - X - v| \geq 2$. By applying Lemma 3.1.6 to $X + v$ we obtain that $V - X - v$ contains at least two nodes. Let $W := V_3 \cap (V - X - v)$. Consider $G[W]$, the subforest of $G[V_3]$ on vertex set W. Since $|W| \geq 2$, $G[W]$ has at least two leaves u, w. Note that these nodes are not necessarily leaves in $G[V_3]$. Observe that each vertex z in $V - X - v - t$ (and hence each node in W) has at most one neighbour in X, since otherwise either $X + z$ would also be a node-critical set, contradicting the maximality of X, or $X + z$ would contradict (3.1). Therefore u and w cannot be branching nodes in G.

If u is a leaf node in G then Lemma 3.2.4(b) and the maximality of X imply that u is an admissible node. If u is a series node in G then, since u has at most one neighbour in X and since u is a leaf in $G[W]$, it follows that it has precisely one neighbour y in X and y is a node. Thus Lemma 3.2.4(a) and the maximality of X imply that u is an admissible node. •

A more sophisticated argument gives the following stronger statement.

Theorem 3.2.6. *[5] Let $G = (V, E)$ be a 3-connected M-circuit with $|V| \geq 5$. Then either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes.*

Theorem 3.2.6 is best possible in the sense that there exist 3-connected M-circuits containing precisely four admissible nodes but no three pairwise non-adjacent admissible nodes (a wheel on five vertices) and there exist 3-connected M-circuits containing precisely three non-adjacent admissible nodes but no four admissible nodes ($K_{3,3} + e$). Furthermore, there exist M-circuits with no admissible nodes (the 2-sum of two K_4's), showing that 3-connectivity is essential.

The following theorem shows that every M-circuit can be obtained from disjoint K_4's by 2-sums and 1-extensions. Note that the 2-sum operation is always performed on two distinct connected components, and the 1-extension operation is performed within one connected component.

Theorem 3.2.7. *[5] $G = (V, E)$ is an M-circuit if and only if G is a connected graph obtained from disjoint copies of K_4's by taking 2-sums and applying 1-extensions.*

Proof: By Lemma 3.1.3 and Lemma 3.2.1 it follows that a connected graph built up from disjoint copies of K_4's by 2-sums and 1-extensions is an M-circuit. To prove the other direction by induction on the number of vertices, we need to show that if G is an M-circuit on at least five vertices then the inverse operation of either the 2-sum or the 1-extension can be applied to G in such a way that the resulting graphs are also M-circuits. The inverse operation of 2-sum is 2-separation. If G has a cutpair then Lemma 2.2.5(c) and Lemma 3.1.4 imply that a 2-separation can be applied to G in such a way that the resulting graphs are

M-circuits, as required. Hence we may assume that G is 3-connected. Now it follows from Theorem 3.2.5 that G has an admissible node v. By performing an admissible splitting at v we obtain a smaller M-circuit. Since splitting off is the inverse of 1-extension, this completes the proof of the theorem. •

If G is a 3-connected M-circuit then there is a more powerful inductive construction that uses 1-extensions only. This strengthening is based on the following theorem. We call a node v of a 3-connected M-circuit G *feasible* if there is an admissible splitting at v for which the resulting graph G_v is 3-connected.

Theorem 3.2.8. *[5] Let $G = (V, E)$ be a 3-connected M-circuit with $|V| \geq 5$. Then G has two non-adjacent feasible nodes.*

As a corollary of Theorem 3.2.8 we obtain the following constructive characterization of 3-connected M-circuits, which was conjectured by Connelly.

Theorem 3.2.9. *[5] $G = (V, E)$ is a 3-connected M-circuit if and only if G can be built up from K_4 by a sequence of 1-extensions.*

3.2.1 Exercises

Exercise 3.2.10. *Prove that the square of a graph G is 3-connected and redundantly rigid if and only if G is essentially 2-edge-connected.*

Exercise 3.2.11. *Let G be 4-regular. Show that $G-F$ is rigid for all $F \subset E(G)$ with $|F| \leq 2$ if and only if G is essentially 6-edge-connected.*

Exercise 3.2.12. *Let G be 4-regular. Show that if G is essentially 6-edge-connected then $G - v$ is an M-circuit for all $v \in V$.*

Exercise 3.2.13. *Show that every 4-regular graph G for which $G-F$ is rigid for all $F \subset E(G)$ with $|F| \leq 2$ can be obtained from K_5 by 2-extensions.*

Exercise 3.2.14. *Prove the following sharpening of Theorem 3.2.7: $G = (V, E)$ is an M-circuit if and only if G can be obtained from K_4 by taking 2-sums with K_4's and applying 1-extensions.*

Exercise 3.2.15. *Show that it is possible to find a construction sequence for a 3-connected M-circuit so that the edges of a designated triangle of G belong to the starting K_4 (and so they are never involved in the 1-extensions).*

3.3 M-connected graphs

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected.* If $\mathcal{M}$ has components $E_1, E_2, \ldots, E_t$ and $\mathcal{M}_i$ is the matroid restriction of $\mathcal{M}$ onto E_i then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \ldots \oplus \mathcal{M}_t$, where $\oplus$ denotes the direct sum of matroids, see [53].

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}(G)$ is connected. For example, $K_{3,m}$ is M-connected for all $m \geq 4$.

Lemma 3.3.1. *Suppose that G is M-connected. Then G is redundantly rigid.*

Proof: Let H be a maximal redundantly rigid subgraph of G. Since G is M-connected, H exists. Since H is a vertex-induced subgraph, the lemma follows if $V(H) = V(G)$ holds. If not, let e be an edge in $E(G) - E(H)$ and let $f \in E(H)$. The M-connectivity of G implies that G has an M-circuit C with $e, f \in E(C)$. Now Lemma 2.2.10 implies that $H \cup C$ is rigid. Since each edge of $H \cup C$ is in some M-circuit, $H \cup C$ is also redundantly rigid. This contradicts the maximality of H. •

The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. We say that an M-component is *trivial* if it has only one edge, that is, if it is induced by an edge which belongs to no M-circuit. Such an edge is also called an *M-bridge.* Note that the trivial M-components and the trivial redundantly rigid components are the same. Since the non-trivial M-components of G are redundantly rigid by Lemma 3.3.1, the partition of $E(G)$ given by the M-components is a refinement of the partition given by the redundantly rigid components and hence a further refinement of the partition given by the rigid components, see Figure 3.1. It also follows that the M-components are pairwise edge-disjoint vertex-induced subgraphs of G.

Furthermore, $\mathcal{R}(G)$ can be expressed as the direct sum of the rigidity matroids of the rigid components of G, the redundantly rigid components of G, or the M-components of G.

In the rest of this section and in the next section we characterize M-connected graphs in three different ways. We shall need the following result. We say that a graph G is *nearly 3-connected* if G can be made 3-connected by adding at most one new edge.

Theorem 3.3.2. *[25] Suppose that G is nearly 3-connected and every edge of G is in some M-circuit. Then G is M-connected.*

Proof: For a contradiction suppose that G is not M-connected and let H_1, H_2,...,H_q be the M-components of G. Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices belonging to no other M-component than H_i, and let $Y_i = V(H_i) - X_i$ for $1 \leq i \leq q$. Let $n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$. Clearly, $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^q x_i + |\cup_{i=1}^q Y_i|$. Moreover, we have $\sum_{i=1}^q y_i \geq 2|\cup_{i=1}^q Y_i|$. Since every edge of G is in some M-circuit, and every M-circuit has at least four vertices, we have that $n_i \geq 4$ for $1 \leq i \leq q$. Furthermore, since G is nearly 3-connected, $y_i \geq 2$ for all $1 \leq i \leq q$, and $y_i \geq 3$ for all but at most two M-components.

Let us choose a base B_i in each rigidity matroid $\mathcal{R}(H_i)$. Using the above inequalities we

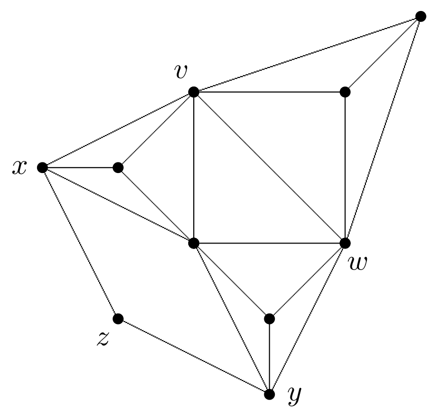

Figure 3.1: This graph is rigid so has exactly one rigid component. There are three redundantly rigid components, consisting of the union of the three copies of K_4, and the remaining two copies of K_2. There are five M-connected components: each of the three copies of K_4, and the remaining two copies of K_2.

have

$$\begin{aligned} |\cup_{i=1}^{q} B_i| &= \sum_{i=1}^{q} |B_i| = \sum_{i=1}^{q}(2n_i - 3) = 2\sum_{i=1}^{q} n_i - 3q \geq \\ & (2\sum_{i=1}^{q} x_i + \sum_{i=1}^{q} y_i) + \sum_{i=1}^{q} y_i - 3q \geq 2|V| + 3q - 2 - 3q = 2|V| - 2. \end{aligned}$$

Since $\mathcal{R}(G)$ has rank at most $2|V|-3$, this implies that $\cup_{i=1}^{q} B_i$ contains a circuit, contradicting the fact that the B_i's are bases for the $\mathcal{R}(H_i)$'s and $\mathcal{R}(G) = \oplus_{i=1}^{q}\mathcal{R}(H_i)$. •

As a corollary we obtain:

Theorem 3.3.3. *[25] Suppose that G is 3-connected and redundantly rigid. Then G is M-connected.*

We also need the following four lemmas to complete our first characterization of M-connected graphs. The first two lemmas follow from Lemmas 3.1.3 and 3.1.4, respectively.

Lemma 3.3.4. *Suppose G_1 and G_2 are M-connected. Then $G_1 \oplus_2 G_2$ is M-connected.*

Lemma 3.3.5. *Suppose G_1 and G_2 are obtained from G by cleaving G along a 2-separator. If G is M-connected then G_1 and G_2 are also M-connected.*

Let $G = (V, E)$ be a 2-connected graph, $c \geq 3$ be an integer, and let $(X_1, X_2, ..., X_c)$ be cyclically ordered subsets of V satisfying (by taking $X_{c+1} = X_1$):

(i) $|X_i \cap X_j| = 1$, for $|i-j| = 1$, and $X_i \cap X_j = \emptyset$ for $|i-j| \geq 2$, and
(ii) $\{E(X_1), E(X_2), ..., E(X_c)\}$ is a partition of E.

Then we say that $(X_1, X_2, ..., X_c)$ is a *polygon (of size c)* in G. (The graph in Figure 3.1 is a polygon of size 3, where the sets X_1, X_2, X_3 are given by the vertex sets of its redundantly rigid components.) It is easy to see that if u and v are distinct vertices with $\{u\} = X_{i-1} \cap X_i$ and $\{v\} = X_j \cap X_{j+1}$, for some $1 \leq i, j \leq c$, then either $\{u, v\}$ is a 2-separator in G or $i = j$ and $X_i = \{u, v\}$.

Lemma 3.3.6. *Suppose that $G = (V, E)$ has a polygon of size c. Then*
(a) G is not M-connected.
(b) If $c \geq 4$ then G is not rigid.

Proof: Let $X_1, X_2, ..., X_c$ be a polygon and let $E_i = E(X_i)$ for $1 \leq i \leq c$. Note that $E_1, E_2, ..., E_c$ is a partition of E. Using the polygon structure we obtain

$$r(E) \leq \sum_{i=1}^{c} r(E_i) \leq \sum_{i=1}^{c} (2|X_i| - 3) = 2|V| + 2c - 3c = 2|V| - c. \tag{3.3}$$

Thus for $c \geq 4$ we have $r(E) \leq 2|V| - 4$, and hence G is not rigid. This proves (b). To prove (a) suppose that G is M-connected. Then G is rigid and $r(E) = 2|V| - 3$. By (b) this yields $c = 3$. Moreover, equality must hold everywhere in (3.3). Thus $r(E) = \sum_{i=1}^{c} r(E_i)$. It follows that no two edges in different sets E_i belong to an M-circuit, see [53, Proposition 4.2.1]. This contradicts the fact that $\mathcal{R}(G)$ is a connected matroid. •

We say that a 2-separator $\{x_1, x_2\}$ *crosses* another 2-separator $\{y_1, y_2\}$ in a 2-connected graph G, if x_1 and x_2 are in different components of $G - \{y_1, y_2\}$. It is easy to see that if $\{x_1, x_2\}$ crosses $\{y_1, y_2\}$ then $\{y_1, y_2\}$ crosses $\{x_1, x_2\}$. Thus, we can say that these 2-separators are *crossing*. It is also easy to see that crossing 2-separators induce a polygon of size four in G. Thus Lemma 3.3.6(b) has the following corollary:

Lemma 3.3.7. *Suppose that G is rigid (and hence 2-connected). Then there are no crossing 2-separators in G.*

Let $G = (V, E)$ be a 2-connected graph with no crossing 2-separators. The *cleavage units* of G are the graphs obtained by recursively cleaving G along each of its 2-separators. Since G has no crossing 2-separators this sequence of operations is uniquely defined and results in a unique set of graphs each of which have no 2-separators. Thus each cleavage unit of G is either 3-connected or else a complete graph on three vertices. (The graph G in Figure 3.1 has three cleavage units, obtained by cleaving G along the 2-separators $\{v, w\}$ and $\{x, y\}$.) The stronger hypothesis that G has no polygons will imply that each cleavage unit of G is a 3-connected graph. In this case, an equivalent definition for the cleavage units is to first construct the *augmented graph* $\hat{G}$ from G by adding all edges uv for which $\{u, v\}$ is a 2-separator of G and $uv \notin E$, and then take the cleavage units to be the maximal 3-connected subgraphs of $\hat{G}$. (These definitions are a special case of a general decomposition theory for 2-connected graphs due to Tutte [61].)

Theorem 3.3.8. *[25] A graph G is M-connected if and only if it is 2-connected, has no polygon, and each of its cleavage units is redundantly rigid.*

Proof: If G is M-connected, then G is rigid and hence 2-connected by Lemma 2.2.5(a), G has no polygons by Lemma 3.3.6(a), each cleavage unit of G is M-connected by Lemma 3.3.5, and hence each cleavage unit is redundantly rigid by Lemma 3.3.1. On the other hand, if G is 2-connected, has no polygons and each cleavage unit is redundantly rigid, then each cleavage unit is M-connected by Theorem 3.3.2, and G is M-connected by Lemma 3.3.4. •

The weaker hypothesis that G is 2-connected, has no polygons, and is redundantly rigid is not sufficient to imply that G is M-connected. This can be seen by considering the graph G obtained from the triangular prism H by replacing each edge v_iv_j of H by a complete graph with vertex set $\{v_i, v_j, v_i', v_j'\}$, where $v_i', v_j' \notin V(H)$. The graph G is redundantly rigid since it is rigid and every edge belongs to an M-circuit (a complete graph on four vertices). To see that G is not M-connected we first note that H is minimally rigid and hence it is not redundantly rigid. We may now deduce that G is not M-connected since H is a cleavage unit of G, and every cleavage unit of an M-connected graph is M-connected by Lemma 3.3.5.

We can also obtain a characterization of M-connected graphs in terms of covers. The following lemma is easy to prove by standard matroid techniques.

Lemma 3.3.9. *Let $\mathcal{M} = (E, r)$ be a matroid on ground set E with rank function r and let $E_1, E_2, \ldots, E_t$ be the components of $\mathcal{M}$. Then*
(i) $r(E) = \sum_1^t r(E_i)$, and
(ii) if $r(E) = \sum_1^q r(F_i)$ for some partition $F_1, F_2, \ldots, F_q$ of E and E_i is a component of $\mathcal{M}$ for some $1 \leq i \leq t$, then $E_i \subseteq F_j$ for some $1 \leq j \leq q$.

The next lemma shows how this general result can be formulated in terms of subgraphs and covers in the special case when the matroid is the rigidity matroid of a graph. We say that a cover is *non-trivial* if it contains at least two sets.

Theorem 3.3.10. *[12] $G = (V, E)$ is M-connected if and only if $val(\mathcal{X}) \geq 2|V| - 2$ for all non-trivial covers $\mathcal{X}$ of G.*

Proof: First suppose that G is M-connected. Then G is rigid, and hence $val(\mathcal{X}) \geq 2|V| - 3$ for all covers $\mathcal{X}$ of G by (the easy direction of) Theorem 2.3.2. Suppose that $val(\mathcal{X}) = 2|V| - 3$ for some non-trivial cover $\mathcal{X} = \{X_1, X_2, ..., X_q\}$ of G. Let $F_i = E(G[X_i])$, $1 \leq i \leq q$. We have $r(F_i) = 2|X_i| - 3$ for all $1 \leq i \leq q$, as $\mathcal{X}$ is a cover of G which minimizes $val(\mathcal{X})$. Thus $r(E) = val(\mathcal{X}) = \sum_1^q r(F_i)$, which contradicts Lemma 3.3.9(ii).

To prove the other direction suppose that $val(\mathcal{X}) \geq 2|V| - 2$ for all non-trivial covers $\mathcal{X}$ of G, but G is not M-connected. Let $H_1, H_2, ..., H_t$ be the M-components of G. Lemma 3.3.9(i) now implies that $2|V| - 3 \geq r(E) = \sum_1^t r(E(H_i)) = \sum_1^t (2|V(H_i)| - 3)$. Thus, since each edge of G belongs to some M-component and $t \geq 2$, $\mathcal{X} = \{V(H_1), V(H_2), ..., V(H_t)\}$ is a non-trivial cover of G with $val(\mathcal{X}) \leq 2|V| - 3$. This contradicts our assumption. •

3.3.1 Exercises

Exercise 3.3.11. *Prove the following statement. Let $G = (V, E)$ be a 2-connected graph and $\{u, v\}$ be a 2-separator of G such that $uv \in E$. Then G is M-connected if and only if $G - uv$ is M-connected.*

Exercise 3.3.12. *Prove the following statement, which is the M-connected version of the glueing lemma. If G_1, G_2 are M-connected graphs with $|V(G_1) \cap V(G_2)| \geq 2$ then $G_1 \cup G_2$ is M-connected.*

Exercise 3.3.13. *Suppose that the sparsity condition $i(X) \leq 2|X| - 2$ holds for all $X \subseteq V$ in graph $G = (V, E)$. Show that the M-circuits of G are pairwise edge-disjoint.*

Exercise 3.3.14. *Suppose that the sparsity condition $i(X) \leq 2|X| - 1$ holds for all $X \subseteq V$ in an M-connected graph $G = (V, E)$. Show that the complements of the edge-sets of the M-circuits of G are pairwise disjoint.*

Exercise 3.3.15. *Suppose that the sparsity condition $i(X) \leq 2|X|$ holds for all $X \subseteq V$ in an M-connected graph $G = (V, E)$. Prove that for every edge $e \in E$ the graph $H = G - e$ satisfies $i_H(X) \leq 2|X| - 1$ for all $X \subseteq V$.*

3.4 The ear-decomposition of the rigidity matroid

A well-known simple result of graph theory states that a graph G is 2-connected if and only if it has an ear-decomposition. This statement can also be obtained from a more general result concerning connected matroids, stated below, by applying it to the circuit matroid of graph G. Next we apply this general result to the two-dimensional rigidity matroid of G. As we shall see, it will lead us to an inductive construction of (3-connected and) M-connected graphs.

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $C_1, C_2, ..., C_t$ be a non-empty sequence of circuits of $\mathcal{M}$. Let $D_j = C_1 \cup C_2 \cup ... \cup C_j$ for $1 \leq j \leq t$. We say that $C_1, C_2, ..., C_t$ is a *partial ear decomposition* of $\mathcal{M}$ if for all $2 \leq i \leq t$ the following properties hold:

(E1) $C_i \cap D_{i-1} \neq \emptyset$,

(E2) $C_i - D_{i-1} \neq \emptyset$,

(E3) no circuit C'_i satisfying (E1) and (E2) has $C'_i - D_{i-1}$ properly contained in $C_i - D_{i-1}$.

The set $C_i - D_{i-1}$ is called the *lobe* of circuit C_i, and is denoted by $\tilde{C}_i$. An *ear decomposition* of $\mathcal{M}$ is a partial ear decomposition with $D_t = E$. As an example, we construct an ear-decomposition C_1, C_2, C_3 of the rigidity matroid of the graph obtained from $K_{3,5}$ by adding an edge, see Figure 3.2.

We need the following facts about ear decompositions. The proof of (a) and (b) in the next lemma can be found in [9]. The proof of (c) is easy and is omitted.

Lemma 3.4.1. *Let $\mathcal{M}$ be a matroid with rank function r. Then*
(a) $\mathcal{M}$ is connected if and only if $\mathcal{M}$ has an ear decomposition.

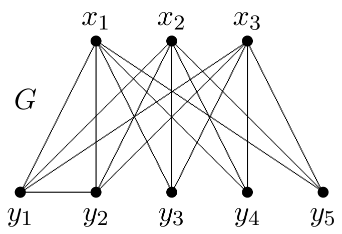

Figure 3.2: If $C_1 = E(G - y_1)$, $C_2 = E(G - y_2)$ and $C_3 = E(G - \{y_4, y_5\})$, then C_1, C_2, C_3 is an ear decomposition of the rigidity matroid of G. We have $\tilde{C}_2 = \{x_1y_1, x_2y_1, x_3y_1\}$ and $\tilde{C}_3 = \{y_1y_2\}$.

(b) If $\mathcal{M}$ is connected then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$.
(c) If $C_1, C_2, ..., C_t$ is an ear decomposition of $\mathcal{M}$ then

$$r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1 \quad \textit{for} \quad 2 \leq i \leq t. \tag{3.4}$$

Next we apply Lemma 3.4.1 to the rigidity matroid of a graph G, which is defined on the edge set of G, and deduce various results about certain subgraphs and vertex sets of G.

Lemma 3.4.2. *Let $G = (V, E)$ be an M-connected graph and $H_1, H_2, \ldots, H_t$ be the M-circuits of G induced by an ear decomposition $C_1, C_2, ..., C_t$ of $\mathcal{R}(G)$ with $t \geq 2$. Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$, and let $X = V(H_t) - Y$. Then:*
(a) Either $Y = \emptyset$ and $|\tilde{C}_t| = 1$, or $Y \neq \emptyset$ and every edge $e \in \tilde{C}_t$ is incident to Y.
(b) $|\tilde{C}_t| = 2|Y| + 1$.
(c) If $Y \neq \emptyset$ then X is critical in H_t.
(d) $G[Y]$ is connected.
(e) If G is 3-connected then $|X| \geq 3$.

Proof: Since M-connected graphs are rigid, it follows that G, $\cup_{i=1}^{t-1} H_i$, and H_t are all rigid. Thus (E3) implies that (a) holds. Furthermore, $r(E) = 2|V| - 3$ and $r(\cup_{i=1}^{t-1} C_i) = 2|V - Y| - 3$. By Lemma 3.4.1(c) this implies that $2|Y| = |\tilde{C}_t| - 1$. This gives (b).

Since H_t is an M-circuit, we have $|E(H_t)| = 2|V(H_t)| - 2$. Hence, since $|X| \geq 2$, (b) implies that X is critical in H_t and hence (c) holds.

To prove (d) suppose that Y can be partitioned into two non-empty sets Y_1, Y_2 with $d(Y_1, Y_2) = 0$. Since X is critical and H_t is an M-circuit, we must have $i(Y_j) + d(Y_j, X) \leq 2|Y_j|$ for $j = 1, 2$. This gives $|\tilde{C}_t| = \sum_{j=1}^{2} i(Y_j) + d(Y_j, X) \leq 2(|Y_1| + |Y_2|) \leq 2|Y|$, contradicting (b). Property (e) follows from the fact that either $Y \neq \emptyset$ and X is a separator in G (using (c)), or $Y = \emptyset$ and $|X| = |V(H_t)| \geq 4$ (since H_t is an M-circuit). •

We say that a graph $G = (V, E)$ is *minimally M-connected* if G is M-connected but $G - e$ is not M-connected for all $e \in E$. To illustrate the typical application of ear decompositions, we show that the minimum degree of a minimally M-connected graph is equal to three.

Theorem 3.4.3. *Let $G = (V, E)$ be an M-connected graph and suppose that $G - e$ is not M-connected for all $e \in E$. Then there is a vertex $v \in V$ with $d(v) = 3$.*

Proof: If G is an M-circuit then the theorem follows from Lemma 3.1.2 and the fact that the average degree in an M-circuit is less than four. Now suppose that G is not an M-circuit. Take an ear-decomposition $C_1, C_2, ..., C_t$ of $\mathcal{R}(G)$ and let $H_1, H_2, \ldots, H_t$ be the M-circuits of G induced by the ears. Since G is not an M-circuit, we have $t \geq 2$. Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$, and let $X = V(H_t) - Y$.

By Lemma 3.4.2(a) and the minimality of G we have $Y \neq \emptyset$. Since H_t is 3-edge-connected, we can use Lemma 3.4.2(b) to deduce that there is a vertex $v \in Y$ with $d(v) = 3$, as required. •

Our inductive construction uses edge additions and 1-extensions.

Lemma 3.4.4. *If G is M-connected and G' is obtained from G by an edge addition or a 1-extension, then G' is M-connected.*

Proof: First suppose that G' is obtained from G by adding an edge e. Since G is M-connected, it is rigid by Lemma 3.3.1. Thus there is an M-circuit H in G' with $e \in E(H)$. Now the M-connectivity of G' follows from transitivity.

Next consider the case when G' is obtained from G by a 1-extension which subdivides an edge uw of G by a new vertex v and adds a new edge vz for some $z \notin \{u, w\}$. Let $f \in E(G)$ be an edge which is incident with z. Since $f \neq uw$, we also have $f \in E(G')$. We shall prove that for all edges $g \in E(G') - f$ there exists an M-circuit H in G' with $f, g \in E(H)$. This will imply that G' is M-connected by transitivity.

If $g \in E(G)$ then there is an M-circuit H' in G with $f, g \in E(H')$. If $uw \notin E(H')$ then we are done by choosing $H = H'$. Otherwise we let H be the 1-extension of H' (on the edge uw and vertex z), which is a subgraph of G', and is also an M-circuit by Lemma 3.2.1. Finally, if $g \notin E(G)$, that is, if $g \in \{vu, vw, vz\}$, then we take an M-circuit H'' of G with $uw, f \in E(H'')$ and let H be the 1-extension of H'' (on the edge uw and vertex z). As above, H is an M-circuit of G' with $f, g \in E(H)$. •

By further analysing the structure of M-connected graphs and their ear-decompositions it possible to strengthen Theorem 3.4.3 and show that if G is 3-connected and minimally M-connected then the last ear contains a degree three vertex which can be split off so that the resulting graph is also M-connected. This statement (which is also an extension of Theorem 3.2.5) can be strengthened even further as follows.

Theorem 3.4.5. *[25] Let $G = (V, E)$ be 3-connected and M-connected and suppose that $G - e$ is not 3-connected or not M-connected for all $e \in E$. Then either $G = K_4$ or G has a vertex v of degree three for which G_v is 3-connected and M-connected for some splitting G_v of G at v.*

Theorem 3.4.5 and Lemma 3.4.4 imply the following inductive construction of 3-connected and M-connected graphs, which will be used in the characterization of globally rigid graphs in Chapter 4.

Theorem 3.4.6. *[25] $G = (V, E)$ is 3-connected and M-connected if and only if G can be obtained from K_4 by 1-extensions and edge additions.*

By using the above results one may obtain an inductive construction for M-connected graphs similar to that of M-circuits, based on 1-extensions, edge-additions and 2-sums.

3.5 Algorithms

We can use the algorithmic methods developed in Section 2.5 to identify the M-connected components of a graph. Let $G = (V, E)$ be a graph. Suppose that $I \subset E$ is independent but $I + e$ is dependent. Then there is a unique circuit in the restriction of $\mathcal{R}(G)$ to $I + e$, called the fundamental circuit of e with respect to I. Similarly, the *fundamental M-circuit* of e with respect to I is the (unique) M-circuit contained in $(V, I + e)$. The main idea is that when we run the algorithm of Section 2.5 to find a base of $\mathcal{R}(G)$ we also identify the set of fundamental M-circuits with respect to the output base I for all edges $E - I$. To find the fundamental M-circuit of $e = uv$ with respect to I we proceed as follows. Let D be a weak g_2^{uv}-orientation of (V, I) (with $\rho_D(v) = 1$, say). As we noted earlier, such an orientation exists. Let $Y \subseteq V$ be the (unique) minimal set with $u, v \in Y, \rho_D(Y) = 0$, and such that $\rho_D(x) = 2$ for all $x \in Y - \{u, v\}$. This set exists, since $I + e$ is dependent. Y is easy to find: it is the set of vertices that can reach v in D.

Lemma 3.5.1. *The vertex set of the fundamental M-circuit of e with respect to I in $(V, I+e)$ is equal to Y.*

Thus if $I + e$ is dependent, we can find the fundamental M-circuit of e in linear time. Our algorithm will maintain a list of M-components and compute the fundamental M-circuit of $e = uv$ only if u and v are not in the same M-component of (V, I). Otherwise e is added to the (unique) M-component that contains u and v. When a new fundamental M-circuit is found, its subgraph will be merged into one new M-component with all the current M-components whose edge set intersects it. It can be seen that the final list of M-components will be equal to the set of M-components of G, and the edges not induced by any of these components will form the set of M-bridges (i.e. trivial M-components) of G. It can also be shown that the algorithm computes $O(n)$ fundamental circuits, so the total running time is still $O(n^2)$. By using similar techniques we can also determine an ear-decomposition of $\mathcal{R}(G)$ for an M-connected graph G.

With the set of M-bridges in hand (which are also the trivial redundantly rigid components) we can find all redundantly rigid components of G by using Lemma 2.3.6.

3.6 Redundantly rigid graphs

In this section we use the structural results on M-connected graphs to verify various results about redundantly rigid graphs.

3.6.1 Minimally redundantly rigid graphs

It is clear that a redundantly rigid graph $G = (V, E)$ has $|E| \geq 2|V| - 2$, where we have equality if and only if G is an M-circuit. In this section we prove a tight upper bound on the number of edges of a redundantly rigid graph G for which $G - e$ is no longer redundantly rigid for all edges e of G.

We say that $G = (V, E)$ is *minimally redundantly rigid* if G is redundantly rigid but $G - e$ is not redundantly rigid for all $e \in E$. First observe that:

Proposition 3.6.1. *(a) If G is minimally redundantly rigid then each M-component of G is minimally redundantly rigid, and*
(b) if G is M-connected and minimally redundantly rigid then G is minimally M-connected.

In the next lemma we consider the special case when G is M-connected.

Lemma 3.6.2. *Let $G = (V, E)$ be a minimally M-connected graph. Then $|E| \leq 3|V| - 6$.*

Proof: Since G is M-connected, $\mathcal{R}(G)$ has an ear-decomposition $C_1, C_2, ..., C_t$. We prove the lemma by induction on t. If $t = 1$ then G is an M-circuit and hence we have $|V| \geq 4$ and

$$|E| = 2|V| - 2 \leq 3|V| - 6. \tag{3.5}$$

Now suppose that $t \geq 2$ and let $H_1, H_2, \ldots, H_t$ be the M-circuits of G induced by the circuits of the ear decomposition.

Claim 3.6.3. *Let $G' = \cup_{i=1}^{s} H_i$ be the subgraph of G induced by the first s ears of the ear-decomposition, where $1 \leq s \leq t$. Then G' is minimally M-connected.*

Proof: We may suppose that $s = t - 1$. It is clear that G' is M-connected. For a contradiction suppose that $G' - e$ is M-connected for some $e \in E(G')$. Let H' be an M-circuit of G' containing e and let $f \in \tilde{C}_t$. By the strong circuit axiom there is an M-circuit H in G with $E(H) \subseteq (E(H') \cup E(H_t)) - \{e\}$ and $f \in E(H)$. We must have $E(H) \cap E(H') \neq \emptyset$. Fix an ear-decomposition $P_1, P_2, ..., P_l$ of $G' - e$. (We must have $l = t - 2$ but we shall not use this fact.) We claim that $P_1, P_2, ..., P_l, E(H)$ is an ear-decomposition of $G - e$. By the choice of H we have (E1) and (E2) for $i = t - 1$. (E3) follows from the facts that $(C_1 \cup C_2 \cup ... \cup C_{t-1}) - e = P_1 \cup P_2 \cup ... \cup P_l$ and that C_t satisfies (E3) with respect to $C_1, C_2, ..., C_{t-1}$. It also follows that the union of the ears is equal to $E(G - e)$. This leads to the desired contradiction since $G - e$ is not M-connected and hence it cannot have an ear-decomposition. •

Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$. Observe that $Y \neq \emptyset$, for otherwise, by Lemma 3.4.2(a), $|\tilde{C}_t| = \{e\}$ holds for some $e \in E$ and hence $G - e$ is M-connected, contradicting the assumption that G is minimally M-connected. By Claim 3.6.3 $G' = \cup_{i=1}^{t-1} H_i$, that is, the subgraph of G induced by the first $t-1$ ears is minimally M-connected.

By induction, we have $|E'| \leq 3|V'| - 6$, and hence by Lemma 3.4.2(b)

$$|E| \leq |E'| + |\tilde{C}_t| \leq 3|V - Y| - 6 + 2|Y| + 1 \leq 3|V| - 6 \tag{3.6}$$

follows, as required. •

Theorem 3.6.4. *Let $G = (V, E)$ be a minimally redundantly rigid graph. Then $|E| \leq 3|V| - 6$.*

Proof: Suppose that G has q M-components $G_i = (V_i, E_i)$, $1 \leq i \leq q$. By multiplying the rank of G by $\frac{3}{2}$ and using Lemma 3.3.9(i), Proposition 3.6.1, and Lemma 3.6.2 we can deduce that

$$3|V| - \frac{9}{2} = \sum_{i=1}^{q}(3|V_i| - \frac{9}{2}) \geq \sum_{i=1}^{q} |E(G_i)| + q\frac{3}{2} = |E| + q\frac{3}{2}, \tag{3.7}$$

and hence

$$|E| \leq 3|V| - \frac{9}{2} - q\frac{3}{2} \leq 3|V| - 6. \tag{3.8}$$

•

The complete bipartite graphs $K_{3,m}$ for $m \geq 4$ show that we have arbitrarily large minimally redundantly rigid graphs with $|E| = 3|V| - 9$. In fact, with a careful analysis of the first two M-circuits of an ear-decomposition, we can show that we have $|E| \leq 3|V| - 9$ for all minimally redundantly rigid graphs $G = (V, E)$ with $|V| \geq 7$.

The argument is as follows. First observe that we may suppose that G is M-connected (this follows from (3.8) and the fact that the number q of M-components of a rigid graph is odd).

Next observe that we are done if H_1 has at least seven vertices (this follows from (3.5)). In fact, (3.5) and (3.6) imply that we are done unless $|V(H_1)| = 4$ and $|Y_2| \leq 3$, $|V(H_1)| = 5$ and $|Y_2| \leq 2$, or $|V(H_1)| = 6$ and $|Y_2| = 1$, where $Y_2 = V(H_2) - V(H_1)$. However, in each of these exceptional cases we can show that $H_1 \cup H_2$ is not minimally M-connected, a contradiction by Claim 3.6.3.

To verify the latter claim we first check that if $|V(H_1)| = 4$ then $H_1 = K_4$, if $|V(H_1)| = 5$ then $H_1 = W_5$ (a wheel on five vertices) and if $|V(H_1)| = 6$ then H_1 is W_6 or the 2-sum of two K_4's, or $K_{3,3} + e$ or the prism plus an edge. Note that all M-circuits on at most six vertices contain at most two pairwise non-adjacent vertices, except $K_{3,3} + e$, which has a unique independent vertex set of size three.

The simplest case is when $|V(H_1)| = 6$ and $|Y_2| = 1$: then either the neighbour set of Y_2 in G contains a pair of adjacent vertices or $H_1 = K_{3,3} + e$ and the neighbour set of Y_2 is the independent vertex set of size three in H_1. In the former case $H_1 \cup H_2$ can be obtained from H_1 by a 1-extension and then adding an edge, so it is not minimally M-connected (c.f. Lemma 3.4.4). In the latter case $H_1 \cup H_2$ contains $K_{3,4}$ as a spanning subgraph, which leads to a similar contradiction.

If $|V(H_1)| \le 5$ and $|Y_2| \le 2$ then either the neighbour set of Y_2 has size at least three (in which case a similar argument works, possibly with a sequence of two 1-extensions), or Y_2 has exactly two neighbours. In the latter case $H_1 \cup H_2$ contains the 2-sum of H_1 and H_2 (which is an M-circuit) as a spanning subgraph, a contradiction.

The last case is when $|V(H_1)| = 4$ and $|V(Y_2)| = 3$. Then $|H_2| \ge 5$ and hence we are in one of the cases above by interchanging H_1 and H_2, that is, by starting the ear-decomposition of $H_1 \cup H_2$ with H_2.

3.6.2 Exercises

Exercise 3.6.5. *Show that by deleting any set of at most $n-3$ edges from K_n we obtain a rigid graph.*

Call a graph $G = (V, E)$ *birigid* if G is rigid and $G - v$ is rigid for all $v \in V$. A birigid graph G is *minimally birigid* if $G - e$ is not birigid for all $e \in E$.

Exercise 3.6.6. *Prove that every birigid graph is 3-connected and redundantly rigid.*

Exercise 3.6.7. *Show that a birigid graph $G = (V, E)$ satisfies $|E| \ge 2|V| - 1$ (see [57]).*

Exercise 3.6.8. *Give a linear (in $|V|$) upper bound for the number of edges in a minimally birigid graph $G = (V, E)$. (The tight bound is given in [41].)*

Exercise 3.6.9. *Develop a polynomial-time algorithm that can test whether a given minimally rigid graph $G = (V, E)$ on n vertices satisfies that for all $1 \le i \le n-1$ there is a subset $X \subset V$ with $|X| = i$ for which both subgraphs $G[X]$ and $G[V - X]$ are rigid.*

3.6.3 An inductive construction

Theorem 2.2.8 provided an inductive construction for minimally rigid graphs. A similar construction for the family of redundantly rigid graphs, in which all intermediate graphs are redundantly rigid, appears to be hard to find. For a somewhat larger family, however, such a construction exists. We say that G is *redundant* if it has at least one edge and each edge of G is in an M-circuit. It follows that a graph G is redundantly rigid if and only if G is rigid and redundant.

Suppose that $G = (V, E)$ is a 2-connected graph and let (G_1, G_2) be a 2-separation of G with $V(G_1) \cap V(G_2) = \{u, v\}$. For $1 \le i \le 2$, let $G_i' = G_i + uv$ if $uv \notin E(G_i)$ and otherwise put $G_i' = G_i$. We say that G_1', G_2' are the *cleavage graphs* obtained by *cleaving G along $\{u, v\}$*.

Lemma 3.6.10. *Suppose that G is a 2-connected redundant graph. Let $\{u, v\}$ be a 2-separator of G and let $\tilde{H}_1$ and $\tilde{H}_2$ be the cleavage graphs obtained by cleaving G along $\{u, v\}$. Then at least one of the following holds:*
(i) $\tilde{H}_i$ is redundant for $i = 1, 2$;
(ii) there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u, v\}$ for which H_i is redundant for $i = 1, 2$.

Proof. First we prove that each edge $f \in E(\tilde{H}_1) - uv$ belongs to an M-circuit in $\tilde{H}_1$. Since G is redundant, there is an M-circuit C in G which contains f. If C is a subgraph of $\tilde{H}_1$ then we are done. If not, then $\{u, v\}$ is a 2-separator of C. In this case it follows from Lemma 3.1.4 that the cleavage graphs C_1 and C_2 obtained by cleaving C along $\{u, v\}$ are both M-circuits. Hence C_1 is an M-circuit in $\tilde{H}_1$ which contains f. By symmetry we also have that each edge $f' \in E(\tilde{H}_2) - uv$ belongs to an M-circuit in $\tilde{H}_2$.

Thus, if uv belongs to an M-circuit in both cleavage graphs then (i) holds. Now suppose that, say, uv is in no M-circuit in $\tilde{H}_1$. As above, this implies that if $uv \in E(G)$ then all M-circuits of G containing uv must be in $\tilde{H}_2$ and if $uv \notin E(G)$ then all M-circuits of G containing some edge of $E(\tilde{H}_1) - uv$ must be in $\tilde{H}_1 - uv$.

By moving the edge uv from one side of the 2-separation to the other, if necessary, we may assume that there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u, v\}$ and $uv \notin E(H_1)$. The arguments above now imply that H_1 and H_2 are both redundant. Thus (ii) holds. $\square$

We have seen that 3-connected redundant graphs are M-connected. Thus they can be built up from K_4 by 1-extensions and edge-additions. If a redundant graph has a separator of size two then we can use the previous lemma to cut it into smaller redundant graphs. Separators of size at most one can be handled in a similar way. By putting these facts together and applying induction we obtain the following result.

Theorem 3.6.11. *[39] $G = (V, E)$ is redundant if and only if G can be obtained from disjoint copies of K_4's by recursively applying 1-extensions, edge-additions within some connected component, 2-sums to two connected components, and merging components along at most two vertices.*

The inductive construction of Theorem 3.6.11 can be obtained in polynomial time. This leads to an efficient method for replacing the edges (bars) of a redundantly rigid graph by cables and struts so that the resulting tensegrity graph has an infinitesimally rigid realization, see [39].

3.6.4 Merging redundantly rigid graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with exactly k vertices and at least one edge in common, each with at least $k + 1$ vertices. Let $H = (V, E)$ be the graph obtained by identifying (or superimposing) their common vertices and edges, and then deleting a common edge. We say that H is obtained by a *k-merge* operation from G_1, G_2.

Connelly [8] asked whether the 3-merge operation applied to two redundantly rigid graphs yields a redundantly rigid graph (and proved the corresponding result where redundantly rigid is replaced by globally rigid). We give an affirmative answer. The result of this subsection is from an unpublished joint work with Bill Jackson.

Let $k = 3$ and let G_1, G_2, H be two redundantly rigid graphs and their 3-merged graph, respectively, as defined above. Let $e = uv$ be the deleted edge and let $\{u, v, x\}$ be the set of vertices in common.

Lemma 3.6.12. *H is rigid.*

Proof: Since G_i is redundantly rigid, $G_i - e$ is rigid for $i = 1, 2$. Thus H is obtained from two rigid graphs by glueing them together along three vertices. Hence Lemma 2.2.10 implies that H is rigid. •

We say that a pair of vertices x, y in G is *linked* in G, or that xy is an *implied edge* of G if $r(G + xy) = r(G)$. It is easy to see that x, y is a linked pair if and only if there is a rigid component of G containing x and y.

Lemma 3.6.13. *Let $f \in E(G_1) - E(G_2)$. Then $H - f$ is rigid.*

Proof: Since G_2 is redundantly rigid, $G_2 - e$ is rigid. This implies that each pair of vertices of V_2 is linked in $H - f$. In particular, x, y is an implied edge. By using the fact that $G_1 - f$ is rigid, it follows that each pair of vertices of V_1 is linked in $H - f$. Since $k = 3$, Lemma 2.2.10 implies that $H - f$ is rigid. •

It remains to prove that the edges induced by $V_1 \cap V_2$ are redundant in H. First we prove this in the special case when G_1, G_2 are both M-circuits.

Lemma 3.6.14. *Suppose that G_1, G_2 are M-circuits and let $f = xv$ be an edge induced by $V_1 \cap V_2$ in H. Then $H - f$ is rigid.*

Proof: We have two cases to consider, depending on the number of edges induced by $V_1 \cap V_2$ in H. Let $n_i = |V_i|$, $i = 1, 2$.

Case 1. $V_1 \cap V_2$ induces exactly one edge in H.

We have $|E(H)| = 2n_1 - 2 + 2n_2 - 2 - 3 = 2(n_1 + n_2 - 3) - 1$.

Since G_i is redundantly rigid, each vertex $w \in V_i$ has degree at least three, for $i = 1, 2$, and at most two of them are induced by $V_1 \cap V_2$. Thus there exist edges xa and vb in H with $a \in V(G_1) - V(G_2)$ and $b \in V(G_2) - V(G_1)$. By Lemma 3.6.13 there exist M-circuits C_a, C_b in H with $xa \in C_a$ and $vb \in C_b$. If $v \in V(C_a)$ or $x \in V(C_b)$ then f is redundant in H, since there is a redundantly rigid subgraph (namely, an M-circuit) of H containing both of its end-vertices.

If C_a and C_b have at least two vertices in common, then (by Lemma 2.2.10) their union is a redundantly rigid subgraph of H containing the end-vertices of f, so as above, it follows that f is redundant.

Thus we may suppose that C_a and C_b are edge-disjoint, $\delta_H(x) \subseteq E(C_a)$ and $\delta_H(v) \subseteq E(C_b)$, and $u \in V(C_a) \cap V(C_b)$. (Here $\delta_H(w)$ denotes the set of edges incident with w in H.) Since $|E(H)| = 2|V(H)| - 1$ and H has two edge-disjoint M-circuits, there are no other M-circuits in H. Now Lemma 3.6.13 implies that all edges but f belong to either $E(C_a)$ or $E(C_b)$ and hence u is a cut-vertex in $H - f$ (and also in $H + e - f$) separating the end-vertices of f.

This contradicts the fact that f belongs to a circuit in G_1, and hence also in $H + e$.

Case 2. $V_1 \cap V_2$ induces two edges in H.

We have $|E(H)| = 2n_1 - 2 + 2n_2 - 2 - 4 = 2(n_1 + n_2 - 3) - 2$.

Thus H has a unique M-circuit C. By Lemma 3.6.13 all edges in $E(G_1) - E(G_2)$ and $E(G_2) - E(G_1)$ are in $E(C)$. Thus $x, v \in V(C)$, which implies that f is redundant. •

Theorem 3.6.15. *Let H be obtained from the redundantly rigid graphs G_1, G_2 by a 3-merge operation. Then H is redundantly rigid.*

Proof: By Lemma 3.6.12 H is rigid. Thus we need to show that every edge of H is redundant. This follows from Lemma 3.6.13 for all edges in $(E(G_1)-E(G_2)) \cup (E(G_2)-E(G_1))$. Consider an edge $f = xv$ induced by $V_1 \cap V_2$ in H.

Let C_1 and C_2 be M-circuits of G_1 and G_2, respectively, containing f. We may assume that e belongs to C_1 and C_2, otherwise f is clearly redundant in H.

Thus $V(C_1) \cap V(C_2) = \{u, v, x\}$ holds and H contains the 3-merge of C_1 and C_2 (with edge e deleted) as a subgraph H'. By Lemma 3.6.14 f is redundant in H' and hence also in H. This completes the proof. •

Theorem 3.6.15 holds for 2-merge, too, but not for 4-merge. (To see this suppose that G_1 and G_2 share a vertex v of degree three plus all the neighbours of v. Then deleting an edge incident with v from their merged graph H creates a vertex of degree two.)

3.6.5 Redundantly rigid components

The next lemma shows that if two non-adjacent vertices u, v are not contained in the same redundantly component of G then there exists an edge e for which u and v are not linked in $G - e$. This fact is useful in the analysis of the so-called globally linked pairs of vertices [29], see Section 4.3.

Lemma 3.6.16. *Let $G = (V, E)$ be a rigid graph and $u, v \in V$ with $uv \notin E$. Then $\{u, v\}$ is contained in a redundantly rigid component of G if and only if $\{u, v\}$ is contained in a rigid component of $G - e$ for all $e \in E$.*

Proof: We first prove necessity. Suppose u, v is contained in a redundantly rigid component H of G. Then $H \neq K_2$ and so $H - e$ is a rigid subgraph of G for all $e \in E$. Hence u, v is contained in a rigid component of $G - e$ for all $e \in E$.

We next prove sufficiency. Suppose u, v is not contained in a redundantly rigid component of G. Then G is not redundantly rigid so at least one edge of G is an M-bridge, that is, it does not belong to any M-circuit in G. Let $F = \{e_1, e_2, ..., e_m\}$ be the set of M-bridges of G. By Lemma 2.3.6 the rigid components of $G - F$ are exactly the non-trivial redundantly rigid components of G. Thus u, v is not contained in a rigid component of $G - F$. Let H' be a maximal M-independent subgraph of $G - F$. Note that the vertex sets of the rigid components of $G - F$ and H' are the same and $H' + F$ is an M-independent (and rigid) spanning subgraph of G.

Let F' be a maximal proper subset of F for which u, v is not contained in a rigid component of $H' + F'$. If $F - F' = \{f\}$ then we are done by choosing $e = f$. This follows from the fact that u, v is not contained in a rigid component of $H' - f$ and hence is not contained in a rigid component of $G - f$ as well. So we may suppose that we have two distinct edges $f_1, f_2 \in F - F'$. By the maximality of F' there is a rigid subgraph $G_i = (V_i, E_i)$ of $H' + F' + f_i$ which contains u and v, for $i = 1, 2$. Since $H' + F$ is M-independent, these subgraphs are induced subgraphs of $H' + F$ and we must have $f_1, f_2 \notin G_1 \cap G_2$. Then $G_1 \cap G_2$ is a rigid subgraph of $H' + F'$ which contains u and v. This contradicts the choice of F'. •

3.7 Rigidity matroids of highly connected graphs

Different graphs may have isomorphic rigidity matroids. For example, the rigidity matroid of every M-independent graph with k edges is isomorphic to the free matroid on k elements. We shall prove that if G is sufficiently highly connected then its rigidity matroid uniquely determines G. Note that a celebrated result of Whitney implies a similar result in $\mathbb{R}^1$: the circuit matroid of a 3-connected graph G uniquely determines G. The result of this section can be found [37], along with further results on highly connected rigidity matroids.

The proof method of the next theorem is motivated by a proof for (a special case of) Whitney's theorem, due to J. Edmonds (see [53]). Let $J \subseteq E$ be a set of elements in matroid $\mathcal{M}$. We say that J is a 2-*hyperplane* of $\mathcal{M}$ if $r(J) = r(E) - 2$ and J is closed, that is, for all $e \in E - J$ we have $r(J + e) = r(E) - 1$.

Theorem 3.7.1. *[37] Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If G is 7-connected then G is isomorphic to H.*

Proof: We say that a 2-hyperplane J of $\mathcal{R}(G)$ is M-connected if the matroid restriction of $\mathcal{R}(G)$ to J is M-connected. Since G is 7-connected, Theorems 2.4.1 and 3.3.2 imply that G is rigid and $E(G - v)$ (i.e. the edge set E minus the vertex bond of v) is an M-connected 2-hyperplane of $\mathcal{R}(G)$ for all $v \in V(G)$.

Now consider an arbitrary M-connected 2-hyperplane J of $\mathcal{R}(G)$. By Lemma 3.3.1 the subgraph $L = (V(J), J)$ of G on the set of end vertices of J is rigid. Thus $r(J) = 2|V(J)| - 3$ and, since 2-hyperplanes are closed sets, it follows that L is an induced subgraph of G. By using the fact that G is rigid, we obtain $|V(G)| = |V(J)| + 1$. Thus the complement of J corresponds to a vertex bond of G.

It follows that there is a bijection between $V(G)$ and the M-connected 2-hyperplanes of $\mathcal{R}(G)$ and that $\mathcal{R}(G)$ uniquely determines the vertex-edge incidencies in G.

By the assumption of the theorem $\mathcal{R}(G)$ and $\mathcal{R}(H)$ are isomorphic. It follows from Theorems 2.4.1 and 3.3.2 that $\mathcal{R}(G)$ is M-connected. Thus $\mathcal{R}(H)$ is also M-connected and hence H is rigid by Lemma 3.3.1. This implies that $2|V(G)| - 3 = r(G) = r(H) = 2|V(H)| - 3$ and hence $|V(G)| = |V(H)|$. Thus $\mathcal{R}(H)$ has $|V(H)|$ M-connected 2-hyperplanes. So G and H are isomorphic, as claimed. •

The bound on the connectivity of G in Theorem 3.7.1 could perhaps be improved to 6, but it cannot be replaced by 5, see [37] for an example.

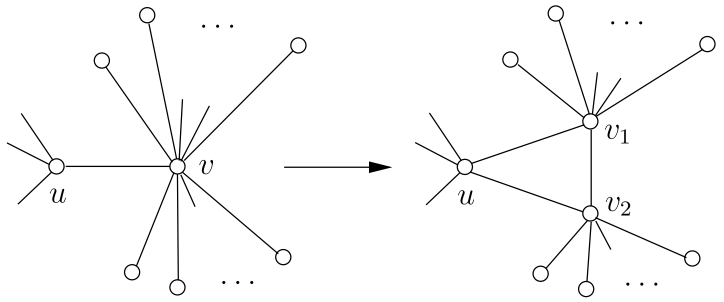

Figure 3.3: The vertex splitting operation on edge uv and vertex v.

3.8 Vertex splitting in redundantly rigid graphs

Another familiar operation in combinatorial rigidity is vertex splitting. Given a graph $G = (V, E)$, an edge $uv \in E$, and a bipartition F_1, F_2 of the edges incident to v (except uv), the (2-dimensional) *vertex splitting* operation *on edge uv at vertex v* replaces vertex v by two new vertices v_1 and v_2, replaces the edge uv by three new edges uv_1, uv_2, v_1v_2, and replaces each edge $wv \in F_i$ by an edge wv_i, $i = 1, 2$, see Figure 3.3. The vertex splitting operation is said to be *non-trivial* if F_1, F_2 are both non-empty, or equivalently, if each of the split vertices v_1, v_2 has degree at least three. In this section we verify that non-trivial vertex splitting preserves the property of being 3-connected and redundantly rigid.

It is known that vertex splitting preserves rigidity [62, 63]. For completeness we sketch a different proof of this fact, which relies on the inductive construction of Theorem 2.2.8.

Lemma 3.8.1. *Let G be a rigid graph and let G' be obtained from G by a vertex splitting operation. Then G' is rigid.*

Proof. Let G' be obtained from G by a vertex splitting on edge uv at vertex v, with bipartition F_1, F_2. Let H be a minimally rigid spanning subgraph of G which contains the edge uv and consider a sequence of graphs $H_1, H_2, \ldots, H_m$ with $H_1 = uv$ and $H_m = H$, for which H_i is obtained from H_{i-1} by an extension, for $2 \leq i \leq m$. Such a sequence exists by Theorem 2.2.6. Let us define a bipartition F_1^j, F_2^j of the edges incident to v (except uv) in each H_j, starting with H_m, as follows. Let $F_i^m = F_i$ for $i = 1, 2$. Now suppose that $j < m$ and let wv be an edge different from uv in H_j. If $wv \in E(H_j) \cap E(H_{j+1})$ then let wv belong to the same class of the bipartition as in H_{j+1}. Otherwise, if $wv \in E(H_j) - E(H_{j+1})$, then H_{j+1} is obtained from H_j by a 1-extension which replaces wv by two edges yv, yw, where y is a new vertex. In this case let the bipartition class of wv be defined to be the same as that of yv in H_{j+1}.

To see that G' is rigid apply the 'same' sequence of extensions defined above to build a graph but start with a triangle on vertices u, v_1, v_2 instead of the edge uv and so that whenever a new edge incident to v is to be added in H_j, connect the corresponding edge to either v_1 or v_2 according to the bipartition F_1^j, F_2^j. The graph H_m' obtained this way is a minimally rigid spanning subgraph of G'. Thus G' is rigid. $\square$

Next we show that a non-trivial vertex splitting operation takes an M-circuit to an M-circuit.

Lemma 3.8.2. *Let C be an M-circuit and let C' be obtained from G by a non-trivial vertex splitting. Then C' is an M-circuit.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. We shall use the characterization of M-circuits to show that C' is indeed an M-circuit. Since C is an M-circuit, it is clear that $|E(C')| = 2|V(C')| - 2$. Consider a proper subset X' of $V(C')$ and let X denote the corresponding subset of $V(C)$ obtained by identifying vertices v_1 and v_2. Clearly, if $|X' \cap \{u, v_1, v_2\}| \in \{0, 1, 3\}$ or $X' \cap \{u, v_1, v_2\} = \{v_1, v_2\}$ then X is a proper subset of $V(C)$ and we have $2|X'| - i_{C'}(X') \geq 2|X| - i_C(X) \geq 3$. If $X' \cap \{u, v_1, v_2\} = \{u, v_i\}$ for some $i = 1, 2$ then either X is a proper subset of $V(C)$ and we have $2|X'| - i_{C'}(X') \geq 2|X| - i_C(X) \geq 3$, or $V(C') - X' = \{v_j\}$, $j \neq i$. In the latter case we can use the fact that $|E(C')| = 2|V(C')| - 2$ and $F_j \neq \emptyset$ to deduce that $2|X'| - i_{C'}(X') \geq 3$. This completes the proof. □

Applying a vertex splitting operation to an arbitrary redundantly rigid graph G may destroy redundant rigidity, even if the operation is non-trivial, see Figure 3.4. We shall prove that this cannot happen when G is 3-connected. First we need the following observation.

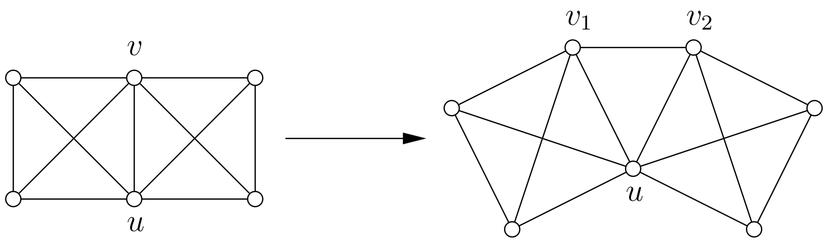

Figure 3.4: Non-trivial vertex split may destroy redundant rigidity.

Lemma 3.8.3. *Let G be a 3-connected graph and let G' be obtained from G by a non-trivial vertex splitting operation. Then G' is 3-connected.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. For a contradiction suppose that G' is not 3-connected. Then there is a small separator, i.e. a set $S \subset V(G')$ with $|S| \leq 2$ for which $G' - S$ is disconnected. Since each vertex has degree at least three in G', it follows that each connected component of $G' - S$ contains at least two vertices. Furthermore, since u, v_1, v_2 induce a triangle in G', there is exactly one component of $G' - S$ which intersects $\{u, v_1, v_2\}$. This implies that G, which can be obtained from G' by contracting the edge $v_1 v_2$ (i.e. by performing the inverse of the vertex splitting operation) also has a separator of size at most two, a contradiction. □

We are now ready to prove the main result of this section.

Theorem 3.8.4. *[40] Let G be a 3-connected and redundantly rigid graph and let G' be obtained from G by a non-trivial vertex splitting operation. Then G' is also 3-connected and redundantly rigid.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. It follows from Lemmas 3.8.1 and 3.8.3 that G' is 3-connected and rigid. It remains to prove that $G' - xy$ is rigid for all edges $xy \in E(G')$. If $xy \notin \{uv_1, uv_2, v_1v_2\}$ then this follows from Lemma 3.8.1 and the hypothesis that G is redundantly rigid, since, for such an edge, $G' - xy$ can be obtained from $G - xy$ by a vertex splitting operation on edge uv at v.

To deal with the remaining edges let us choose and edge $va \in F_1$ and consider an M-circuit C in G with $\{uv, va\} \subset E(C)$. Such an M-circuit exists, since G is M-connected by Theorem 3.3.2. First suppose that $E(C) \cap F_2 \neq \emptyset$. In this case G' contains, as a subgraph, a graph C' obtained from C by a non-trivial vertex splitting on edge uv at v with bipartition F_1', F_2', where $F_1' = F_1 \cap E(C)$ and $F_2' = F_2 \cap E(C)$. Since $\{uv_1, uv_2, v_1v_2\} \subset E(C')$, it follows from Lemma 3.8.2 that uv_1, uv_2, and v_1v_2 are also redundant edges in G' and hence we are done.

Next suppose that $E(C) \cap F_2 = \emptyset$. In this case G' contains a subgraph which is isomorphic to C and contains the edge uv_1 (i.e. the subgraph obtained from C by replacing uv by uv_1, and replacing each edge $wv \in E(C) \cap F_1$ by wv_1). Thus uv_1 is a redundant edge in G'. By symmetry we obtain that uv_2 is redundant as well. Hence we are done if v_1v_2 is also redundant in G'.

Otherwise, when $G' - v_1v_2$ is not rigid, and hence v_1v_2 is in no M-circuit in G', the above arguments imply that each edge of $G' - v_1v_2$ belongs to an M-circuit. Since G' is 3-connected, $G' - v_1v_2$ is nearly 3-connected. Theorem 3.3.2 now implies that $G' - v_1v_2$ is M-connected, and hence rigid, a contradiction. This completes the proof of the theorem. □

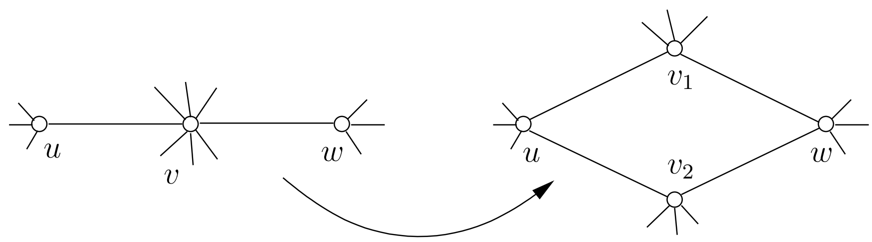

Figure 3.5: The diamond split operation.

We remark that there is a second form of vertex splitting in two dimensions. Let uv, vw be two adjacent edges and let F_1, F_2 be a bipartition of the edges incident to v (except uv, vw). The operation *diamond split* replaces vertex v by two new vertices v_1, v_2, replaces the edges uv, vw by a four-cycle uv_1, uv_2, wv_1, wv_2, and then replaces each edge $zv \in F_i$ by an edge zv_i, for $i = 1, 2$. See Figure 3.5. It is known that diamond split preserves rigidity [4] and it is not difficult to show that a diamond split operation takes an M-circuit to an M-circuit. In general, however, it may destroy redundant rigidity as well as 3-connectivity. Further useful operations as well as results about the effect of the above split operations to planar circuits and their duals can be found in [4].

3.9 Geometric sensitivity

Let (G,p) be a minimally infinitesimally rigid bar-and-joint framework and let L be an equilibrium load on p (defined in Section 1.2). Let $S \subseteq V(G)$ be a designated set of joints and suppose that each joint with a non-zero load belongs to S. As mentioned earlier, L can be resolved by a (unique) stress, that is, an assignment of scalars to the bars of the framework. The *active zone* of S (with respect to p and L) is the set of those bars in which the stress, which resolves L, is non-zero.

In this section we show that if the framework is generic then for almost all loads the active zone depends only on the graph G of the framework and can be found by efficient combinatorial algorithms. These results can be extended to arbitrary infinitesimally rigid generic frameworks and (apart from the algorithmic results) to all dimensions, see [36].

Let (G,p) be a minimally rigid generic framework and let $S \subseteq V(G)$ be a designated vertex set with $|S| \geq 2$. We denote the set of those equilibrium loads on p which satisfy $(V-S) \subseteq \{v \in V(G) : L(v) = 0\}$ by $\mathcal{L}_S(p)$. We use K_S to denote the set of all edges with both end-vertices in S. The graph obtained from G by adding the edges of K_S is denoted by $G + K_S$. Here we do not add new copies of those edges that are already present in G, so $G + K_S$ is also a simple graph, in which S induces a complete subgraph.

Theorem 3.9.1. *[36] Let (G,p) be a minimally rigid generic framework and let $S \subseteq V(G)$ be a designated vertex set with $|S| \geq 2$. Then for almost all (that is, for an open and dense subset of $\mathcal{L}_S(p)$) loads $L \in \mathcal{L}_S(p)$ the active zone $A_S(G,L,p)$ is equal to the union of the edge sets of those M-components of $G + K_S$ that intersect K_S, restricted to $E(G)$.*

Proof: Let $A = \cup_{L \in \mathcal{L}_S(p)} A_S(G,L,p)$. Note that $R_e(p) \in \mathcal{L}_S(p)$ for each $e \in K_S$, where $R_e(p)$ is the row vector of the rigidity matrix of $(G + K_S, p)$ associated with e. Since (G,p) is generic, for each edge $e \in K_S - E_G(S)$ the active zone $A_S(G, R_e(p), p)$ is equal to $E(C_e) - \{e\}$, where C_e is the unique M-circuit (i.e. the fundamental circuit of e) in $G + e$. Clearly, for $e \in E_G(S)$ we have $A_S(G, R_e(p), p) = \{e\}$. Hence A contains $E_G(S)$ as well as the edge set of C_e for every $e \in K_S - E_G(S)$. Furthermore,

$$A = E_G(S) \cup \bigcup_{e \in K_S - E_G(S)} (E(C_e) - \{e\}) \tag{3.9}$$

holds since $\mathcal{L}_S(p)$ is the vector space spanned by $\{R_e(p) : e \in K_S\}$.

We claim that for almost all $L \in \mathcal{L}_S(p)$ we have $A_S(G,L,p) = A$. To see this define, for each $e = v_i v_j \in A$, the vector space $\mathcal{L}_S^e(p)$ to be the space of those equilibrium loads $L \in \mathcal{L}_S(p)$ for which the unique resolution ω of L by (G,p) satisfies $\omega_{i,j} = 0$. By the definition of A the space $\mathcal{L}_S^e(p)$ is a proper linear subspace of $\mathcal{L}_S(p)$ for all $e \in A$. Hence $\mathcal{L}_S(p) - \bigcup_{e \in A} \mathcal{L}_S^e(p)$ is an open and dense subspace of $\mathcal{L}_S(p)$, which verifies the claim.

Let B be the union of the edge sets of those M-components of $G + K_S$ that intersect K_S, restricted to $E(G)$. We complete the proof by showing that $A = B$. It is clear from (3.9) that each edge of A is included in some M-component of $G + K_S$ intersecting K_S. Thus $A \subseteq B$. For all edges $f \in B \cap K_S$ we have $f \in A$ by (3.9). Next consider an edge $f \in B - K_S$ and let C be an M-circuit in $G + K_S$ with $f \in E(C)$. Since G is M-independent, we have

$E(C)\cap(K_S-E_G(S)) \neq \emptyset$. Suppose that C is chosen so that $|E(C)\cap(K_S-E_G(S))|$ is as small as possible. We claim that $|E(C)\cap(K_S-E_G(S))| = 1$ must hold. Let $e \in E(C)\cap(K_S-E_G(S))$ and let C_e be the unique M-circuit in $G+e$. The claim follows if we have $f \in E(C_e)$, hence we may suppose that $f \in E(C)-E(C_e)$ holds. By applying the strong circuit axiom[1] to the edge sets of the M-circuits C, C_e, we obtain that there is an M-circuit C' with $E(C') \subseteq E(C) \cup E(C_e)$, $f \in E(C')$ and $e \notin E(C')$. Since this contradicts the choice of C, the claim follows. Thus $f \in E(C_e)$ for some $e \in K_S - E_G(S)$, which implies $B \subseteq A$. This gives $A = B$, as required. •

Thus, when the framework is generic then for almost all loads the active zone depends only on the graph and we may simply denote it by $A_S(G)$ and call it the *active zone of S in G*.

Theorem 3.9.1 can be extended to arbitrary rigid graphs (rigid generic frameworks (G,p)). Let $S \subseteq V(G)$ with $|S| \geq 2$ be a designated vertex-set and let $L \in \mathcal{L}_S(p)$. The *active zone of S*, with respect to p and L, is the set of those edges of G in which the stress is non-zero for some resolution of L. It is not hard to see that if (G,p) is generic then for almost all loads an edge $e \in E(G)$ belongs to the active zone of S if and only if there is a minimally rigid spanning subgraph H of G for which e is in the active zone of S in H. This provides a purely combinatorial definition of the *active zone of S in G* for (generic realizations of) rigid graphs, for almost all loads. The extended characterization is as follows.

Theorem 3.9.2. *[36] Let $G = (V,E)$ be a rigid graph and let $S \subseteq V$ with $|S| \geq 2$. Then an edge $f \in E$ belongs to the active zone of S in G if and only if the M-component containing f in $G + K_S$ intersects K_S.*

Next we simplify the previous characterization of active zones (which is in fact valid in all dimensions) which will lead to an efficient algorithm for identifying them.

Let $G = (V,E)$ be a minimally rigid graph. For a given $S \subseteq V$ with $|S| \geq 2$ let C_S be a minimal minimally rigid subgraph of G with $S \subseteq V(C_S)$. It follows from Lemma 2.1.2 that C_S is unique. The unique minimal minimally rigid subgraph C_S of G with $S \subseteq V(C_S)$ is called the *rigid core* (or simply the *core*) of vertex set S in G and is denoted by $C_S(G)$. When $S = \{a,b\}$ for some $a, b \in V$ then we may also use the notation $C_{a,b}(G)$.

Lemma 3.9.3. *Let G be a minimally rigid graph and let $S \subseteq V$ with $|S| \geq 2$. Then $C_S(G) = E_G(S) \cup_{ab\in K_S-E_G(S)} C_{a,b}(G)$.*

Proof: Let $C = E_G(S) \cup_{ab\in K_S-E_G(S)} C_{a,b}(G)$. Observe that, by definition, $C_{a,b}(G) \subseteq C_S(G)$ and hence $C \subseteq C_S(G)$. Thus, since $S \subseteq C$, it remains to prove that C is minimally rigid. To see this note that C can be obtained from the complete (and hence rigid) graph on vertex set S by attaching the subgraphs $C_{a,b}(G)$, for all $ab \in K_S - E_G(S)$ (which preserves rigidity by Lemma 2.2.10 and makes all edges of $K_S - E_G(S)$ redundant) and then deleting the edges of $K_S - E_G(S)$. •

[1] Let C, C' be two circuits of some matroid with $e \in C \cap C'$ and $f \in C - C'$. Then there is a circuit $C'' \subseteq C \cup C'$ with $e \notin C''$ and $f \in C''$.

Since the M-circuits are rigid, the core $C_{a,b}$ for a given vertex pair a, b of G is equal to the unique M-circuit of $G + ab$, minus the edge ab. This fact, together with equality (3.9) obtained in the proof of Theorem 3.9.1, and Lemma 3.9.3 imply that the active zone of S is equal to the edge set of its core. Furthermore, since there exist efficient algorithms for finding the core of a pair of vertices (see Section 3.5), it follows that the active zone can be found in polynomial time.

Theorem 3.9.4. *[36] Let G be a minimally rigid graph and let $S \subseteq V$ with $|S| \geq 2$. Then $A_S(G) = E(C_S(G))$.*

3.9.1 The influenced zone and the joint sensitivity index

We can use the characterization of active zones in the following related problem. Let (G, p) be a minimally infinitesimally rigid framework and let $v \in V(G)$ be a designated vertex. Let (G, p') be another minimally infinitesimally rigid realization of G in which $p'(v) \neq p(v)$ but $p'(u) = p(u)$ for all $u \in V(G)$ with $u \neq v$. Let $\mathcal{L}_v(p)$ denote the space of equilibrium loads $L : V \to \mathbb{R}^2$ on p with $L(v) = 0$. Consider an equilibrium load $L \in \mathcal{L}_v(p)$. It is easy to see that we also have $L \in \mathcal{L}_v(p')$. Let ω and ω' be the stresses in the frameworks (G, p) and (G, p'), respectively, that resolve L.

If v is incident with exactly two edges then $\omega(vu) = \omega'(vu) = 0$ must hold for all $vu \in \delta(v)$ and we have $\omega = \omega'$. However, the resolving stress may be different in some of the bars when the degree of v is larger. Suppose that $d(v) \geq 3$. In this case the *influenced zone* of v, denoted by $I_v(G, L, p, p')$, is defined to be the union of $\delta(v)$ and the set of those edges $v_iv_j \in E(G) - \delta(v)$ for which $\omega_{ij} \neq \omega'_{ij}$.

It turns out that for generic frameworks the influenced zone depends only on G and we may simply denote it by $I_v(G)$ and call it the *influenced zone of v in G*, see [36]. (If v is of degree two, its influenced zone is defined to be empty.) It was also shown in [36] that for minimally rigid graphs G and vertices of degree at least three we have $I_v(G) = A_{N(v)}(G)$. Thus we can deduce that the influenced zone of v is the edge set of the core of $N(v)$.

Theorem 3.9.5. *[36] Let G be a minimally rigid graph and let $v \in V(G)$ with $d(v) \geq 3$. Then $I_v(G) = E(C_{N(v)}(G))$.*

This result leads us to the following combinatorial problem. One way to measure the sensitivity of a generic framework (or graph G) with respect to local changes is the *joint sensitivity index* $s(G)$ defined by

$$s(G) = \frac{\sum_{v \in V} |V(I_v(G))|}{|V|^2}.$$

We can use this parameter to design families of graphs with extremely high (or low) sensitivity.

We call a minimally rigid graph G *special* if every proper rigid subgraph H of G is complete (and hence is a complete graph on two or three vertices). The graphs $K_{3,3}$ and the prism (that is, the complement graph of the six-cycle) are both special. It is known that the family of special graphs is infinite [28]. Moreover, each special graph on at least six vertices has minimum degree three. Therefore, by Theorem 3.9.5, the influenced zone $I_v(G)$ is the edge set of a minimally rigid subgraph of G on at least four vertices for all $v \in V(G)$. Hence $V(I_v(G)) = V(G)$ must hold. This implies that the joint sensitivity index of special graphs is the highest possible.

Lemma 3.9.6. *Let G be a special graph on at least six vertices. Then $s(G) = 1$.*

There exist non-special graphs, too, whose joint sensitivity index attains the extremal value. For example, the minimally rigid graph G' obtained by connecting the minimally rigid graph $K_4 - e$ and a four-cycle C_4 by four disjoint edges has $s(G') = 1$, but G' is not special.

To find families of low sensitivity first observe that in a minimally rigid graph $G = (V, E)$ with $|V| \geq 4$ we must have $|V_3| \geq 2$, where V_3 is the set of vertices of degree at least three. Furthermore, for each $v \in V_3$ we have $|V(I_v(G))| \geq d_G(v) + 1$. Hence

$$\sum_{v \in V} |V(I_v(G))| \geq \sum_{v \in V} (d_G(v) - 2) + 3|V_3| \geq 2|E| - 2|V| + 6 = 4|V| - 6 - 2|V| + 6 = 2|V|.$$

This bound is attained for the graph obtained from the complete bipartite graph $K_{2,n-2}$ by adding an edge to the smaller colour class.

For graphs with minimum degree at least three we have

$$\sum_{v \in V} |V(I_v(G))| \geq \sum_{v \in V} (d_G(v) + 1) = 5|V| - 6.$$

To get close to this bound we need graphs in which for all vertices v (except possibly a few vertices with a small influenced zone) $G[N(v) + v]$ is rigid. Such graphs can be constructed from a triangle by repeated applications of the operation which adds a new vertex v and two edges va, vb, where ab is an edge incident with a degree two vertex, and by performing two edge splitting operations.

3.9.2 Exercises

Let H be a graph and $v \in V(H)$. The graph H^* is obtained from H by adding a new vertex v' and new edges $v'u$ for all $u \in N_H(v)$. The set of edges incident with v is denoted by $\delta(v)$.

Exercise 3.9.7. *Let G be a minimally rigid graph and let $v \in V(G)$. Suppose that $d_G(v) \geq 3$. Then there is a unique non-trivial M-component J in G^* and we have $\delta(v) \cup \delta(v') \subseteq E(J)$.*

Exercise 3.9.8. *Verify that the graphs $K_{3,3}$ and the prism are both special, as well as all graphs which can be obtained from $K_{3,3}$ by repeated applications of the following operation: replace two incident edges ab, bc by six edges $aa', a'b, bc', c'c, ac', a'c$, where a', c' are new vertices.*

Exercise 3.9.9. *Show that every special graph is 3-connected.*

Exercise 3.9.10. *Prove that (G, p) is redundantly rigid if and only if it is rigid and it has a stress which is non-zero on all edges.*

3.9.3 Optimal generation of stresses

An optimization problem related to active zones is the following: given a generic minimally infinitesimally rigid framework (or a minimally rigid graph) find a smallest set of joints for which a generic load, acting on this set of joints, generates a non-zero stress in all bars.

Let $G = (V, E)$ be a minimally rigid graph. Observe that for all subsets $X \subset V$ with $|X| \leq |V| - 2$ we have $e(X) \geq 2|X|$, with equality if and only if $G - X$ is minimally rigid. (Recall that $e(X)$ denotes the number of edges incident with X.) For $|X| = |V| - 1$ we have $e(X) = 2|X| - 1$.

We call a non-empty subset $X \subset V$ with $|X| \leq |V| - 2$ *co-rigid* if $e(X) = 2|X|$ holds. We say that $S \subseteq V$ is a *co-rigid cover* of G if $S \cap X \neq \emptyset$ for all co-rigid sets X of G. By the previous results of this section S is a co-rigid cover if and only if by applying a generic load to the joints corresponding to S in a generic bar-and-joint realization of G we generate a non-zero stress in all bars. Thus we want to find a smallest co-rigid cover in G.

The next lemma is easy to verify by observing that each edge contributes to the two sides by the same amount.

Lemma 3.9.11. *Let $G = (V, E)$ be a graph and let $X, Y \subseteq V$. Then*

$$e(X) + e(Y) = e(X \cap Y) + e(X \cup Y) + d(X - Y, Y - X).$$

Lemma 3.9.12. *Let X be a minimal co-rigid set, let Y be a co-rigid set, and suppose that $X - Y$ and $X \cap Y$ are both non-empty. Then one of the following holds:*
(i) $|X \cup Y| = |V| - 1$ and $d(X - Y, Y - X) = 0$,
(ii) $X \cup Y = V$.

Proof: First suppose $|X \cup Y| \leq |V| - 2$. Since X and Y are co-rigid, we can use Lemma 3.9.11 and the minimality of X to obtain

$$2|X| + 2|Y| = e(X) + e(Y) = e(X \cap Y) + e(X \cup Y) + d(X - Y, Y - X) \geq$$

$$\geq 2|X \cap Y| + 1 + 2|X \cup Y| = 2|X| + 2|Y| + 1,$$

a contradiction. If $|X \cup Y| = |V| - 1$ then $e(X \cup Y) = 2|X \cup Y| - 1$, which implies, by using the same chain of inequalities, that $d(X - Y, Y - X) = 0$. •

Theorem 3.9.13. *Let $\mathcal{C}$ be the family of minimal co-rigid sets of G. Then the sets in $\mathcal{C}$ are pairwise disjoint or there is a pair $v, w \in V$ for which $\{v, w\} \cap X \neq \emptyset$ for all $X \in \mathcal{C}$.*

Proof: We may assume that there exist intersecting pairs in $\mathcal{C}$. First suppose that there is a pair $X, Y \in \mathcal{C}$ with $X \cap Y \neq \emptyset$ and $|X \cup Y| = |V| - 1$. Let $V - (X \cup Y) = \{v\}$ and let $\mathcal{C}_v = \{X \in \mathcal{C}, v \notin X\}$. If $\mathcal{C}_v$ contains only the pair X, Y then we are done by choosing $w \in X \cap Y$. Now consider a set $Z \in \mathcal{C}_v$ different from X, Y. By the minimality of X and Y we must have $Z \cap X \neq \emptyset \neq Z \cap Y$, and hence Lemma 3.9.12 gives $X - Y \subset Z$ and $Y - X \subset Z$. We can apply the same argument to each set in $\mathcal{C}_v$ to deduce that $\mathcal{P} = \{V - X - v : X \in \mathcal{C}_v\}$ is a subpartition of $V - v$, i.e. it consists of pairwise disjoint subsets of V. If $\cup_{P \in \mathcal{P}} P$ is a proper subset of $V - v$ then the theorem follows by choosing a vertex $w \notin \cup_{P \in \mathcal{P}} P$, different from v. If $\cup_{P \in \mathcal{P}} P = V - v$ then, since we have $d(X - Y, Y - X) = 0$ for all $X, Y \in \mathcal{C}_v$ by Lemma 3.9.12, it follows that v is a cutvertex of G. This contradicts the fact that G is rigid (and has at least three vertices).

Next suppose that for all pairs $X, Y \in \mathcal{C}$ with $X \cap Y \neq \emptyset$ we have $X \cup Y = V$. Fix a vertex $v \in Y - X$. Suppose that there is a set $Z \in \mathcal{C}$ different from X, Y. The minimality of Y implies that Z intersects X and hence Lemma 3.9.12 (and the assumption that the union of intersecting pairs in $\mathcal{C}$ is V) gives $Y - X \subseteq Z$ and $v \in Z$. Thus v belongs to all sets of $\mathcal{C}$ but X. The theorem now follows by choosing a vertex $w \in X$. •

Clearly, S is a co-rigid cover if and only if S intersects all minimal co-rigid sets. Let $c(G)$ denote the size of a smallest co-rigid cover of G. Theorem 3.9.13 implies that $c(G) \leq 2$ or $c(G) \geq 3$ and the minimal co-rigid sets are pairwise disjoint. In the latter case $c(G)$ is equal to the number of minimal co-rigid sets and any minimal co-rigid cover is a smallest co-rigid cover.

To test whether a given subset S forms a co-rigid cover of G we have to check whether the core of S is equal to G or not. Thus the smallest co-rigid cover of G can be found in polynomial time: first check whether there is a pair of vertices forming a (smallest) co-rigid cover. If not, then form a minimal co-rigid cover by starting with the co-rigid cover V and greedily deleting vertices as long as possible.

Note that a single vertex v can never be a co-rigid cover: it does not cover the complement of an edge incident with v. In a special graph any pair of non-adjacent vertices forms a (smallest) co-rigid cover.

3.10 Collinear realizations

In this section we consider the existence of infinitesimally rigid realizations of a graph that satisfy some additional geometric properties. Since the proofs of certain generic properties often rely on non-generic realizations (e.g. the proof of the fact that 1-extension preserves independence or rigidity), these questions are fairly natural. We focus on realizations in which three given vertices are collinear.

So we ask the following question: given three vertices x, y, z of G, when do we have an infinitesimally rigid realization (G, p) with $p(x), p(y), p(z)$ collinear? We shall give a necessary and sufficient condition which leads to an efficient algorithm and verifies that the existence of such a realization is a generic property.

First we suppose that $G = (V, E)$ is M-independent, and $x, y, z \in V$ are three distinct vertices. An *obstacle (for the triple* (x, y, z)*)* is a triple of critical sets (X, Y, Z) for which $X \cap Y = \{z\}$, $X \cap Z = \{y\}$, and $Y \cap Z = \{x\}$. It follows from Lemmas 2.1.1 and 2.1.3 and the fact that G is M-independent that if (X, Y, Z) is an obstacle then $X \cup Y \cup Z$ is critical and $d(X, Y, Z) = 0$ holds. It is not hard to check that if G has an obstacle for the given set of vertices then G has no infinitesimally rigid realization (G, p) with $p(x), p(y), p(z)$ collinear.

The following lemma is about the cores of the vertex pairs of the set $\{x, y, z\}$.

Lemma 3.10.1. *Let $x, y, z \in V$ be three distinct vertices in a minimally rigid graph $G = (V, E)$. Then there exists an obstacle for (x, y, z) if and only if $|C_{x,y} \cap C_{x,z}| = |C_{x,y} \cap C_{y,z}| = |C_{x,z} \cap C_{y,z}| = 1$.*

For the proof of (an extension of) the next theorem we refer the reader to [26].

Theorem 3.10.2. *[26] Let $G = (V, E)$ be a minimally rigid graph and let $x, y, z \in V$ be distinct vertices. Then G has an infinitesimally rigid realization (G, p) in which $p(x), p(y), p(z)$ are collinear if and only if G contains no obstacle for the triple (x, y, z).*

Corollary 3.10.3. *Let $G = (V, E)$ be minimally rigid with $|V| \geq 4$ and let $x, y \in V$ be non-adjacent vertices. Then there is a vertex $z \in V - \{x, y\}$ such that G has an infinitesimally rigid realization (G, p) in which $p(x), p(y), p(z)$ are collinear.*

Proof: Consider $C_{x,y}$ (the unique minimal critical set containing the pair x, y in G). Since $G[C_{x,y}]$ is not complete, there is a vertex $z \in C_{x,y} - \{x, y\}$. Now it follows from Lemma 3.10.1 that there is no obstacle for the triple (x, y, z) in G, and hence G has the required realization by Theorem 3.10.2. •

The characterization of collinearity for an arbitrary rigid graph $G = (V, E)$ is in terms of the circuits of $\mathcal{R}(G+T)$, where $G+T$ is the graph obtained by adding a set $T = \{xy, yz, zx\}$ of three new edges to G. In Theorem 3.10.5 below, we show that G has an infinitesimally rigid realization (G, p) with $p(x), p(y), p(z)$ collinear if and only if some circuit of $\mathcal{R}(G + T)$ contains at least two edges of T.

In what follows we shall consider a rigid graph $G = (V, E)$ with three designated vertices $x, y, z \in V$. Let $T = \{xy, yz, zx\}$ be a set of three new edges. For a subgraph H with $x, y, z \in V(H)$ we use $H + T$ to denote the graph obtained by adding the edges of T to H. Note that, if there exists an edge in H between x, y, z, then $H + T$ will contain a pair of parallel edges which will induce an M-circuit in $H + T$.

Lemma 3.10.4. *Let $G = (V, E)$ be rigid and $x, y, z \in V$. Let $T = \{xy, yz, zx\}$ be a set of three new edges. Then each minimally rigid spanning subgraph H of G has an (x, y, z)-obstacle if and only if the edges xy, yz, zx belong to three different M-components in $G + T$.*

Proof: First we prove the theorem in the special case when G is minimally rigid. Suppose that G has an (x, y, z)-obstacle. Then we have three edge-disjoint M-circuits C_1, C_2, C_3 in $G + T$ with $xy \in E(C_1), yz \in E(C_2)$ and $zx \in E(C_3)$ by Lemma 3.10.1. For a contradiction suppose that there is an M-circuit C in $G + T$ with $|E(C) \cap \{xy, yz, zx\}| \geq 2$.

Suppose $\{xy, yz, zx\} \subset E(C)$. Then the circuit axiom, applied to C and C_3, gives an M-circuit C' with $E(C') \subseteq E(C) \cup E(C_3)$ and $zx \notin E(C')$. Since C_1 (resp. C_2) is the unique M-circuit in $G+xy$ ($G+yz$, resp.), C_1, C_2, C_3 are edge-disjoint, and $E(C_i)$ cannot be a subset of $E(C)$ for $i = 1, 2$, we must have $xy, yz \in E(C')$.

Thus we may assume, by symmetry, that $E(C)$ contains xy and yz but not zx. As above, the circuit axiom applied to C and C_2 implies that there is an M-circuit C'' with $E(C'') \subset E(C) \cup E(C_2)$ and $yz \notin E(C'')$. Hence C'' is an M-circuit in $G + xy$, $C'' = C_1$, and $E(C_1) \subseteq E(C)$ follows, a contradiction.

Now suppose that G has no (x,y,z)-obstacle. By Lemma 3.10.1 we may assume that there exist two M-circuits C_1, C_2 in $G+T$ with $xy \in E(C_1), yz \in E(C_2)$ and $|V(C_1) \cap V(C_2)| \geq 2$. Then xy and yz belong to the same M-component of $G+T$ by Exercise 3.3.12. This completes the proof when G is minimally rigid.

In the rest of the proof we consider an arbitrary rigid graph G. Suppose that there is an M-circuit C in $G+T$ with $xy, yz \in E(C)$. Let H be a minimally rigid spanning subgraph of G obtained by extending $E(C) - \{xy, yz, zx\}$ to a basis of $\mathcal{R}(G)$. Since xy and yz belong to the same M-component of $H+T$, it follows from the first part of the proof that H has no (x,y,z)-obstacle.

Conversely, suppose that the edges xy, yz, zx belong to three different M-components in $G+T$. Then they belong to different M-components in $H+T$ for each minimally rigid spanning subgraph H of G. It now follows from the first part of the proof that there is an (x,y,z)-obstacle in each minimally rigid spanning subgraph H of G. •

The result for rigid graphs follows from Theorem 3.10.2 and Lemma 3.10.4.

Theorem 3.10.5. *[26] Let $G=(V,E)$ be a rigid graph and let $x,y,z \in V$ be distinct vertices. Let $T = \{xy, yz, zx\}$ be a set of three new edges. Then G has an infinitesimally rigid realization (G,p), in which $(p(x), p(y), p(z))$ are collinear if and only if there is an M-component C of $G+T$ with $|E(C) \cap \{xy, yz, zx\}| \geq 2$.*

3.10.1 Exercises

Exercise 3.10.6. *Prove Lemma 3.10.1.*

The existence of infinitesimally rigid realizations in which a given pair of vertices is coincident has been characterized in [14]. The necessary conditions of the next exercise turn out to be sufficient.

Exercise 3.10.7. *Let $G=(V,E)$ be a rigid graph and let $u,v \in V$ be a designated vertex pair. Suppose that G has an infinitesimally rigid realization (G,p) with $p(u)=p(v)$. Prove that (a) if $uv \in E$ then $G-uv$ is rigid , and (b) $G/\{u,v\}$ is rigid.*

Chapter 4

Globally rigid graphs

The characterization of globally rigid graphs, which is the other main target of ours, can be deduced from some of the structural results presented in the previous chapter.

4.1 Globally rigid graphs

Theorems 3.3.3 and 3.4.6 imply the following inductive construction.

Theorem 4.1.1. *[25] Let G be a 3-connected and redundantly rigid graph. Then G can be obtained from K_4 by a sequence of 1-extensions and edge additions.*

Together with the two-dimensional versions of Theorems 1.3.2 and 1.3.3 this implies the following characterization of globally rigid graphs.

Theorem 4.1.2. *[25] Let G be a graph. Then G is globally rigid if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

Since there exist efficient algorithms for testing 3-connectivity and redundant rigidity, global rigidity can also be tested in polynomial time. We note that a different proof of Theorem 4.1.2 can be found in [59].

We also obtain a useful sufficient condition in terms of the vertex-connectivity number of the graph. It can be used e.g. in the analysis of globally rigid random graphs, see [30, 32]. Recall that every 6-connected graph is redundantly rigid (c.f. Theorem 2.4.1). By using Theorem 4.1.2 the next result is an immediate corollary.

Theorem 4.1.3. *[25] Every 6-connected graph G is globally rigid.*

4.1.1 Exercises

Exercise 4.1.4. *Prove that the 3-merge operation, defined in Section 3.6.4, applied to two globally rigid graphs on at least four vertices, yields a globally rigid graph.*

4.2 Global rigidity of special families of graphs

In this section we consider global rigidity properties of two special families of graphs and give a simpler characterization of global rigidity and a sufficient condition in terms of the minimum degree, respectively.

4.2.1 Zeolites

A d-dimensional *body-and-hinge framework* is a different structural model consisting of full dimensional rigid bodies and hinges. Each hinge is a $(d-2)$-dimensional affine subspace that joins some pair of bodies. The bodies are free to move continuously in $\mathbb{R}^d$ subject to the constraint that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. The framework is rigid if every such motion preserves the distances between all pairs of points belonging to different rigid bodies, i.e. the motion extends to an isometry of $\mathbb{R}^d$. In the underlying graph of the framework the vertices correspond to the bodies and two vertices are adjacent if and only if the corresponding bodies are joined by a hinge. We can obtain an equivalent bar-and-joint framework by replacing each body by a bar-and-joint realization of a large enough complete graph in such a way that two bodies joined by a hinge share $d-1$ joints.

The special case when $d = 2$ and each body is incident with 3 hinges (pins) gives rise to the (2-dimensional) *combinatorial zeolites*. These are bar-and-joint frameworks whose graph is the line graph of a 3-regular graph (the underlying graph of the framework). The investigation of these structures is motivated in part by the existence (and flexibility properties) of real zeolites, which are molecules formed by corner-sharing tetrahedra. Planar plate frameworks (which contain planar combinatorial zeolites as a special case), in which the bodies are pairwise congruent regular polygons, have also been studied in the rigidity literature. The global rigidity of a generic combinatorial zeolite depends only on the edge-connectivity of its underlying graph.

Theorem 4.2.1. *[35] Let $G = (V, E)$ be a 3-regular graph. Then $L(G)$ is globally rigid if and only if G is 3-edge-connected.*

Proof: First suppose that $G - F$ has two connected components D_1, D_2 for some $F \subseteq E$ with $|F| \leq 2$. Since G is 3-regular, there must be an edge in D_i for $i = 1, 2$. This implies that the vertex set in $L(G)$ corresponding to F is a separating vertex set in $L(G)$. Thus $L(G)$ is not 3-vertex-connected. This proves the 'only if' direction.

To see the 'if' part, suppose that G is 3-edge-connected. This implies that $L(G)$ is 3-vertex-connected, since each separating vertex set in $L(G)$ gives rise to a separating edge set of G of the same size.

Next we show that $L(G)$ is redundantly rigid. We need the following claim.

Claim 4.2.2. *Let H be a graph with minimum degree at least two and suppose that H can be made 3-edge-connected by adding at most one edge. Then $L(H)$ is rigid.*

Proof: By Theorem 2.6.9 it suffices to show that $\mathrm{def}(H) = 0$. Consider a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of $V(H)$ with $t \geq 2$. Since H can be made 3-edge-connected by adding at most one edge, all but at most two members X_i of $\mathcal{P}$ satisfy $e_H(X_i, V(H) - X_i) \geq 3$, and all members satisfy $e_H(X_i, V(H) - X_i) \geq 2$. Hence

$$2e_H(\mathcal{P}) \geq 3t - 2 > 3(|\mathcal{P}| - 1).$$

Thus $\operatorname{def}(H) = 0$. •

Now consider and edge $p = v_e v_f$ of $L(G)$. This edge corresponds to a pair of edges $e = xy, f = xz$ in G with a common end-vertex. Since G is 3-edge-connected, we can apply Claim 4.2.2 to $H = G - e$ to deduce that $L(H)$ is rigid.

It is easy to check that $L(G) - p$ can be obtained from $L(H)$ by adding a new vertex and connecting it to three distinct vertices of $L(H)$. This operation preserves rigidity (in fact, connecting the new vertex to two vertices of $L(H)$, which is a 0-extension, would already preserve rigidity). Thus $L(G) - p$ is rigid. This proves that $L(G)$ is redundantly rigid, as required. The theorem follows by applying Theorem 4.1.2. •

A much more general result, characterizing the globally rigid generic body-hinge frameworks (in all dimensions), has been obtained in [38].

4.2.2 Graphs of large minimum degree

We may obtain a sufficient condition for global rigidity in terms of the minimum degree of G. The lower bound in the next theorem is best possible, as shown by two complete graphs of equal size with two vertices in common.

Theorem 4.2.3. *[30] Let $G = (V, E)$ be a graph on $n \geq 4$ vertices with $\delta(G) \geq \frac{n+1}{2}$, where $\delta(G)$ denotes the minimum degree in G. Then G is globally rigid.*

Proof. By Theorem 4.1.2 it suffices to show that G is 3-connected and redundantly rigid. If $n \leq 4$ then G is complete, so we may suppose that $n \geq 5$. The hypothesis that $\delta(G) \geq (n+1)/2$ implies that G cannot have a vertex cut of size less than three and hence G is 3-connected. For a contradiction suppose that $H = G - e$ is not rigid for some $e \in E$. Let C be a rigid component of H with as few vertices as possible. Put $D = H - V(C)$. The facts that distinct rigid components of H can share at most one vertex and $\delta(G) \geq \frac{n+1}{2}$, imply that $|V(D)| \geq 4$ and $|V(C)| \leq \frac{n-1}{2}$. Since C is a rigid component of H, each vertex of D is adjacent to at most one vertex of C in H by Lemma 2.1.5. Since $\delta(G) \geq \frac{n+1}{2}$, this implies that $\delta(D) \geq \frac{n-3}{2}$ and all but at most two non-adjacent vertices of D have degree at least $\frac{n-1}{2}$. Hence we may construct a graph $\bar{D}$ with $\delta(\bar{D}) \geq \frac{n-1}{2}$ by adding at most one edge to D. Since $|V(C)| \geq 2$, we have $|V(D)| \leq n - 2$. We may now use induction on n to deduce that $\bar{D}$ is globally rigid. Since $|V(D)| \geq 4$, $\bar{D}$ is redundantly rigid and hence D is rigid. Since $|V(C)| \leq \frac{n-1}{2}$ and $\delta(G) \geq \frac{n+1}{2}$, each vertex of C is adjacent to at least one vertex of D in H, and all but at most two non-adjacent vertices of C are adjacent to at least two vertices of D. We may now use Lemma 2.1.5 to deduce that H is rigid, a contradiction. □

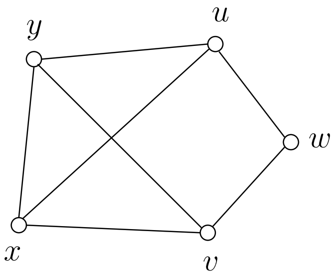

Figure 4.1: A realization (G, p) of a rigid graph G. The pair $\{u, v\}$ is globally linked in (G, p).

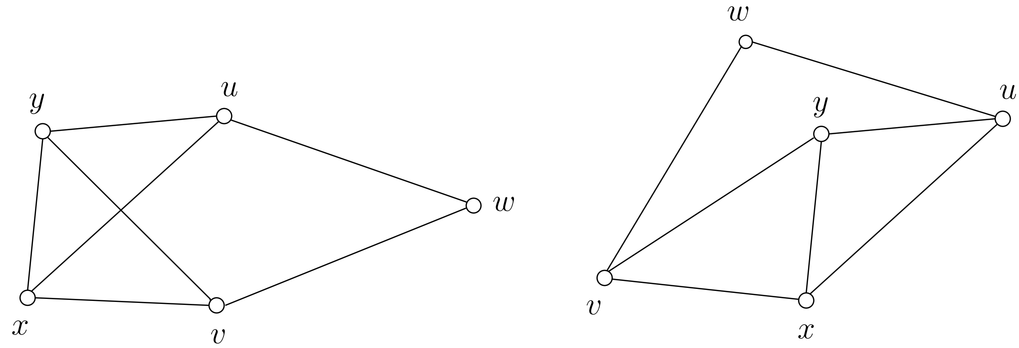

Figure 4.2: Two equivalent realizations of the rigid graph G of Figure 4.1, which show that the pair $\{u, v\}$ is not globally linked in G. Note that the existence of this pair of realizations is due to the fact that the edges uw and vw are sufficiently long.

4.3 Globally linked pairs

Next we consider properties of generic frameworks which are weaker than global rigidity. A pair of vertices $\{u, v\}$ in a framework (G, p) is *globally linked* in (G, p) if, in all equivalent frameworks (G, q), we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. The pair $\{u, v\}$ is *globally linked* in G if it is globally linked in all generic frameworks (G, p). Thus G is globally rigid if and only if all pairs of vertices of G are globally linked. Unlike global rigidity, however, 'global linkedness' is not a generic property in $\mathbb{R}^2$. Figures 4.1 and 4.2 give an example of a pair of vertices in a rigid graph G which is globally linked in one generic realization, but not in another. Note that if $d = 1$ then global linkedness is a generic property: $\{u, v\}$ is globally linked in G if and only if G has two openly disjoint uv-paths.

The complete characterization of globally linked pairs of graphs is not known. We present partial results and conjectures.

The 1-extension operation preserves the property that a pair of vertices is globally linked as long as the split edge is redundant.

Theorem 4.3.1. *[28] Let G, H be graphs such that G is obtained from H by a 1-extension on edge xy and vertex w. Suppose that $H - xy$ is rigid and that $\{u, v\}$ is globally linked in*

H. Then $\{u, v\}$ is globally linked in G.

Let $H = (V, E)$ be a graph and $x, y \in V$. We use $\kappa_H(x, y)$ to denote the maximum number of pairwise openly disjoint xy-paths in H. Note that if $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$.

Lemma 4.3.2. *[28] Let (G, p) be a generic framework, $x, y \in V(G)$, $xy \notin E(G)$, and suppose that $\kappa_G(x, y) \leq 2$. Then $\{x, y\}$ is not globally linked in (G, p).*

We used Theorem 4.3.1 and Lemma 4.3.2 to characterize globally linked pairs for the family of M-connected graphs.

Theorem 4.3.3. *[28] Let $G = (V, E)$ be an M-connected graph and $x, y \in V$. Then $\{x, y\}$ is globally linked in G if and only if $\kappa_G(x, y) \geq 3$.*

Theorem 4.3.3 has the following immediate corollary.

Corollary 4.3.4. *[28] Let $G = (V, E)$ be a graph and $x, y \in V$. If either $xy \in E$, or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$, then $\{x, y\}$ is globally linked in G.*

It is easy to show that the 0-extension operation preserves the property that a pair of vertices is not globally linked.

Lemma 4.3.5. *[28] If $\{u, v\}$ is not globally linked in H and G is a 0-extension of H then $\{u, v\}$ is not globally linked in G.*

A counterpart of Theorem 4.3.1 is as follows.

Theorem 4.3.6. *[29] Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

This result was used to deduce the next result.

Theorem 4.3.7. *[29] Let $G = (V, E)$ be a rigid graph, $u, v \in V$, and $R = (U, F)$ be a redundantly rigid component of G. Suppose that $G - e$ is not rigid for all $e \in E - F$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$ or $\{u, v\}$ is globally linked in R.*

The special case of Theorem 4.3.7 when G has no non-trivial redundantly rigid components characterises globally linked pairs in minimally rigid graphs.

Theorem 4.3.8. *[29] Let $G = (V, E)$ be a minimally rigid graph and $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$.*

The truth of the following conjecture would imply a complete characterization.

Conjectures 4.3.9. *[28] The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ if and only if either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$.*

It may be useful to consider two related conjectures.

Conjectures 4.3.10. *[28] Suppose that* $\{x, y\}$ *is a globally linked pair in a graph* G*. Then there is a redundantly rigid component* R *of* G *with* $\{x, y\} \subseteq V(R)$*.*

Conjectures 4.3.11. *[28] Let* G *be a graph. Suppose that there is a redundantly rigid component* R *of* G *with* $\{x, y\} \subseteq V(R)$ *and* $\{x, y\}$ *is globally linked in* G*. Then* $\{x, y\}$ *is globally linked in* R*.*

It follows from Theorem 3.3.2 that Conjecture 4.3.9 implies both Conjectures 4.3.10 and 4.3.11. The 'if' direction of Conjecture 4.3.9 follows from Corollary 4.3.4. We shall prove that the 'only if' direction follows from Conjectures 4.3.10 and 4.3.11.

Proof: (of the 'only if' part of Conjecture 4.3.9 by assuming Conjectures 4.3.10 and 4.3.11 are true.) Suppose that $\{x, y\}$ is globally linked in $G = (V, E)$. We use induction on $|V|$ to show that either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$. Since the statement is trivially true if $|V| \leq 3$, we may assume that $|V| \geq 4$ and that $xy \notin E$. It follows from the truth of Conjectures 4.3.10 and 4.3.11 that there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$ and such that $\{x, y\}$ is globally linked in R. This implies that $\kappa_R(x, y) \geq 3$ by Lemma 4.3.2. If R is 3-connected then R is M-connected by Theorem 3.3.3, and we are done by choosing $H = R$.

Now suppose that there is a 2-separator $\{u, v\}$ of R and let R_1, R_2 be the cleavage graphs obtained by cleaving R along $\{u, v\}$. Since $\kappa_R(x, y) \geq 3$, we may assume, without loss of generality, that $x, y \in V(R_1)$. Let us also suppose that the 2-separator has been chosen so that R_2 is inclusionwise minimal. This implies that R_2 is 3-connected. (Note that $|V(R_2)| \geq 4$, since R is redundantly rigid.)

Claim 4.3.12. *There is an* M*-circuit* C *in* R_2 *with* $uv \in E(C)$*.*

Proof: Since R is redundantly rigid, every edge $e \in E(R)$ belongs to an M-circuit C_e. Each M-circuit C' is a 2-connected subgraph of R. This fact and Lemma 3.3.5 imply that, if $C_e \nsubseteq R_2$ for some $e \in E(R_2) - uv$, then the claim will follow by choosing $C = (C_e \cap R_2) + uv$. Thus we may suppose that $C_e \subset R_2 - uv$ for all $e \in E(R_2) - uv$. Since R_2 is 3-connected, Theorem 3.3.3 implies that $R_2 - uv$ is M-connected, and hence rigid. Thus there is an M-circuit C in R_2 with $uv \in E(C)$. •

Since $\{x, y\}$ is globally linked in R, $\{u, v\}$ is a 2-separator of R and $uv \in E(R_1)$, it follows that $\{x, y\}$ is globally linked in R_1. By induction, there is an M-connected subgraph H' of R_1 with $x, y \in V(H')$ and $\kappa_{H'}(x, y) \geq 3$. If $uv \notin E(H')$ then let H be an M-component of G containing H'. Thus we may suppose that $uv \in E(H')$. By Lemma 3.3.4, $H'' = H' \oplus_2 C$ is an M-connected subgraph of G containing x, y with $\kappa_{H''}(x, y) \geq 3$. The conjecture now follows by choosing an M-component H of G containing H''. •

4.4 Globally loose pairs

We say that a pair of vertices $\{u, v\}$ is *globally loose* in a graph G if $\{u, v\}$ is not globally linked in all generic realizations of G. It follows from Lemma 4.3.2 and Theorem 4.3.3 that if G is M-connected then each pair $\{u, v\}$ is either globally linked or globally loose in G, and that $\{u, v\}$ is globally loose if and only if $\kappa_G(u, v) = 2$. On the other hand, the pair $\{u, v\}$ in the rigid graph given in Figure 4.1 is neither globally linked nor globally loose.

We shall obtain a sufficient condition for a pair $\{u, v\}$ to be globally loose in a graph G. An edge e of a globally rigid graph H is *critical* if $H - e$ is not globally rigid.

Theorem 4.4.1. *[28] Let $G = (V, E)$ be a graph and $u, v \in V$. Suppose that $uv \notin E$, and that G has a globally rigid supergraph H in which uv is a critical edge. Then $\{u, v\}$ is globally loose in G.*

Proof: Let (G, p) be a generic framework and let H be a globally rigid supergraph of G in which uv is critical. Since uv is critical in H, it follows that $(H - uv, p)$ is not globally rigid. Thus there is an equivalent, but not congruent realization $(H - uv, q)$. Clearly, $||p(u) - p(v)|| \neq ||q(u) - q(v)||$ must hold. Now G is a subgraph of $H - uv$, and hence the framework (G, q) verifies that $\{u, v\}$ is globally loose in G. •

It follows from the definition that if G is special and $uv \notin E(G)$ then $G + uv$ is a 3-connected M-circuit. Thus $G + uv$ is globally rigid by Lemma 3.3.1 and Theorem 4.1.2, and uv is critical in $G + uv$. Hence Theorem 4.4.1 implies that each pair of vertices in a special graph is either globally linked or globally loose:

Theorem 4.4.2. *[28] Let G be special and suppose that $u, v \in V$. Then $\{u, v\}$ is globally loose in G if and only if $uv \notin E$.*

The following stronger result was proved in [29]: if G is minimally rigid and $G + uv$ is an M-circuit for two non-adjacent vertices u, v of G, then $\{u, v\}$ is globally loose. The special case when $G + uv$ is a 3-connected M-circuit follows from Theorem 4.4.1. The example in Figure 4.1 shows that the stronger conclusion, that $\{u, v\}$ is not globally linked in all generic realisations of G, may not hold when $G + uv$ is not an M-circuit.

4.5 Globally rigid graphs with pinned vertices

Let $G = (V, E)$ be a graph. As we saw earlier, a smallest set $P \subseteq V$ for which $G + K(P)$ is rigid can be found in polynomial time. We used this fact in the solution of the optimal pinning set problem. Motivated by a natural question in the network localization problem (see [30]) we are also interested in the optimization problem in which the goal is to find a smallest set $P \subseteq V$ for which $G + K(P)$ is globally rigid. The complexity of this problem is still open. In this section we present partial results from [12] which are based on structural results of the previous chapter and which give rise to an efficient algorithm for finding near optimal solutions.

In the *M-connected pinning problem* the goal is to find a smallest set $P \subseteq V$ for which $G + K(P)$ is M-connected. The following lemma establishes the connection between the feasible solutions of the M-connected pinning problem and the M-components of G.

Lemma 4.5.1. *Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$2|V| - 2 \leq 2|Z| - 3 + \sum_{H_i \in \mathcal{H}_Z} (2|V(H_i)| - 3) \tag{4.1}$$

holds for all $Z \subset V$ with $P \subseteq Z$, $Z \neq V$, where $\mathcal{H}_Z = \{H_i \in \mathcal{H} : V(H_i) \cap (V - Z) \neq \emptyset\}$.

Proof: First suppose that $G + K(P)$ is M-connected. Since every edge of G belongs to an M-component of G and $P \subseteq Z$, it follows that $\{Z\} \cup \{V(H_i) : H_i \in \mathcal{H}, V(H_i) \cap (V - Z) \neq \emptyset\}$ is a cover of $G + K(P)$. This cover is non-trivial, since $Z \neq V$. Thus (4.1) follows from Lemma 3.3.10.

To prove the other direction suppose, for a contradiction, that (4.1) holds but $G' = G + K(P)$ is not M-connected. Let $\mathcal{H}' = \{H'_1, H'_2, ..., H'_q\}$ denote the M-components of G'. Since complete graphs on at least four vertices are M-connected, and $|P| \geq 4$, it follows that $G'[P]$ is M-connected. Thus there is an M-component of G', say H'_1, for which $P \subseteq V(H'_1)$. Let $Z' = V(H'_1)$ and $\mathcal{H}_{Z'} = \{H_i \in \mathcal{H} : V(H_i) \cap (V - Z') \neq \emptyset\}$. Note that $Z' \neq V$.

Claim 4.5.2. *Let $X \subseteq V$ be a set of vertices. Then $X = V(H'_j)$ for some M-component H'_j of G' with $2 \leq j \leq q$ if and only if $X = V(H)$ for some $H \in \mathcal{H}_{Z'}$.*

Proof: First consider an M-component $H'_j \in \mathcal{H}'$ with $j \geq 2$ and let $X = V(H'_j)$. Since $P \subseteq Z'$ and H'_1 is an induced subgraph of G' which has no edge in common with H'_j, it follows that $G[X]$ is M-connected and $X \cap (V - Z') \neq \emptyset$. Thus $X = V(H)$ for some $H \in \mathcal{H}_{Z'}$.

Next consider an M-component $H_i \in \mathcal{H}_{Z'}$ of G and put $X = V(H_i)$. $G'[X]$ is clearly M-connected. For a contradiction suppose that there is an M-component H'_j of G' with $V(H'_j) = Y \subseteq V$ for which X is a proper subset of Y. Then $|Y \cap Z'| \geq |Y \cap P| \geq 2$ must hold. Since $X \cap (V - Z') \neq \emptyset$, we have $j \geq 2$. This contradicts the fact that the M-components of G' are pairwise edge-disjoint. Thus $G'[X]$ is an M-component of G', which completes the proof. •

By using Claim 4.5.2 and Lemma 3.3.9(i), and by applying (4.1) with $Z = Z'$, we obtain

$$2|V| - 3 \geq r(G') = 2|V(H'_1)| - 3 + \sum_{H_i \in \mathcal{H}_{Z'}} (2|V(H_i)| - 3) \geq 2|V| - 2,$$

a contradiction. •

Let $G = (V, E)$ be a graph and let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G. Let $H(G) = (V, \mathcal{E})$ be the hypergraph which contains $2|V(H_i)| - 3$ copies of the hyperedge $V(H_i)$ for each $H_i \in \mathcal{H}$, $1 \leq i \leq t$. Note that since the M-components are rigid it follows from Lemma 3.3.9(i) that $|\mathcal{E}| = r(G) \leq 2|V| - 3$. By letting $Y = V - Z$ in Lemma 4.5.1 and using the above definitions we obtain:

Lemma 4.5.3. *Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$2|Y| + 1 \leq e_{H(G)}(Y) \tag{4.2}$$

holds for all non-empty subsets $Y \subseteq V - P$, where $e_{H(G)}(Y)$ denotes the number of hyperedges $e \in \mathcal{E}$ with $e \cap Y \neq \emptyset$.

A hypergraph $F = (V, \mathcal{F})$ satisfying $| \cup \mathcal{F}'| \geq |\mathcal{F}'| + 1$ for all $\emptyset \neq \mathcal{F}' \subseteq \mathcal{F}$ is called a *hyperforest*. Inequality (4.2) can be reformulated in terms of hyperforests as follows. Let $L(G) = (W, \mathcal{U})$ be the hypergraph obtained from the dual hypergraph of $H(G)$ by duplicating every hyperedge. For a set $X \subseteq V$ let $\mathcal{U}(X)$ denote the set of hyperedges corresponding to X in $L(G)$. Thus $|\mathcal{U}(X)| = 2|X|$.

Lemma 4.5.4. *Let $G = (V, E)$ be a graph and let $P \subseteq V$ with $|P| \geq 4$. Then P satisfies (4.2) if and only if $\mathcal{U}(V - P)$ is a hyperforest.*

Lorea [46] proved that the edge sets of the subhypergraphs of a hypergraph H' which are hyperforests form the family of independent sets of a matroid. A matroid arising this way is called the *circuit matroid* of the hypergraph H' and will be denoted by $\mathcal{M}_{H'}$. We call a matroid which is the circuit matroid of a hypergraph a *hypergraphic matroid*.

It follows from Lemma 4.5.4 that the problem of finding a smallest set P for which $G + K(P)$ is M-connected can be formulated as finding a largest matroid matching[1] in the hypergraphic matroid $\mathcal{M}_{L(G)}$, in which the doubled hyperedges form the pairs. Hypergraphic matroids are known to be linear, but it is not known how to find a suitable linear representation. The complexity status of the matroid matching problem in hypergraphic matroids is still open. Nevertheless, this formulation can be used to design a constant factor approximation algorithm (which works for the minimum cost version as well).

To describe the approximation algorithm we need the following concepts. A *2-polymatroid* is a pair (S, f), where S is a finite ground set and f is a non-negative, monotone increasing, integer-valued, and submodular function on the subsets of S, for which $f(s) \leq 2$ for all $s \in S$. A set $X \subseteq S$ is *spanning* if $f(X) = f(S)$.

Let $G = (V, E)$ be a graph and $X \subseteq V$. Let us define $b : 2^V \to \mathbb{Z}_+$ by letting

$$b(X) = r^*(\mathcal{U}(X)), \tag{4.3}$$

where r^* is the rank function of the matroid dual of the hypergraphic matroid $\mathcal{M}_{L(G)}$. Then (V, b) is a 2-polymatroid.

For a spanning set $X \subseteq V$ we have $r^*(\mathcal{U}(X)) = b(X) = b(V) = r^*(\mathcal{U}(V))$. Thus X is spanning if and only if the set corresponding to $\mathcal{U}(V - X)$ is independent in $\mathcal{M}_{L(G)}$. Together with Lemmas 4.5.3 and 4.5.4 this implies:

[1]Let $\mathcal{M}$ be a matroid on ground-set S and suppose that S is partitioned into a set A of pairs. A subset $M \subseteq A$ is a *matroid matching* if the union of the pairs in M is independent in $\mathcal{M}$. In the *matroid matching problem* the goal is to find a largest matroid matching, see [56, Chapter 43]. Lovász [47] proved that this problem may require exponential time in general but can be solved polynomially if the matroid is represented by a set of vectors in some linear space.

Lemma 4.5.5. *Let $G = (V, E)$ be a graph and $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if P is a spanning set of the 2-polymatroid (V, b).*

Given a 2-polymatroid (S, f) and a cost function $c : S \to \mathbb{R}$, the *minimum cost spanning set problem* is to find a spanning set X of the 2-polymatroid that minimizes $c(X) = \sum_{s \in X} c(s)$. Baudis et al. [3] verified that the GSS (Greedy Spanning Set) algorithm is a constant factor approximation algorithm for this problem. Algorithm GSS starts with $X = \emptyset$ and, as long as $f(X) < f(S)$ holds, adds a new element s to X for which $\frac{c(s)}{f(X+s)-f(X)}$ is minimum.

Theorem 4.5.6. *[3] Let (S, f) be a 2-polymatroid, let $c : S \to \mathbb{R}$ be a cost function, and let X_{opt} be a spanning set of minimum cost. Then*

$$c(X) \leq \frac{3}{2} c(X_{opt}),$$

where X is the spanning set output by algorithm GSS.

Lemma 4.5.5 and Theorem 4.5.6 give rise to a $\frac{3}{2}$-approximation algorithm for the M-connected pinning problem (as well as for its minimum cost version). To see this it remains to note that by using bipartite matching algorithms it is easy to test independence in $\mathcal{M}_{L(G)}$ and evaluate $b(X)$ for some $X \subseteq V$ in polynomial time.

We can also use this approximation algorithm as a subroutine in an approximation algorithm for finding a smallest (or minimum cost) subset P for which $G + K(P)$ is globally rigid. In this algorithm we also need a subroutine for finding a smallest (or minimum cost) subset P for which $G + K(P)$ is 3-connected. The following lemma implies that the latter problem is easy: an optimal solution can be found, in a greedy manner, in linear time.

Let $H = (V, E)$ be a 2-connected graph. We say that $X \subset V$ is *tight* if $|N(X)| = 2$ and $X \cup N(X) \neq V$.

Lemma 4.5.7. *Let $H = (V, E)$ be 2-connected and let $P' \subseteq V$. Then $H + K(P')$ is 3-connected if and only if $P' \cap X \neq \emptyset$ for all minimal tight sets X of H. Furthermore, the minimal tight sets of H are pairwise disjoint and can be found in linear time.*

Recall that redundant rigidity and M-connectivity are the same for 3-connected graphs by Lemma 3.3.1 and Theorem 3.3.3. Thus, by combining the approximation algorithm for the (minimum cost) M-connected pinning problem and the algorithm for the (minimum cost) 3-connected pinning problem we obtain a constant factor approximation algorithm for the (minimum cost) globally rigid pinning problem. This can be seen as follows. Let $G = (V, E)$ and $c : V \to \mathbb{R}$ be the input graph together with a cost function on its vertex set. Let c^* denote the cost of an optimal solution. By checking all feasible solutions $P \subseteq V$ with $|P| = 3$ we may suppose that the optimal solution has at least four vertices. First we compute a close-to-optimal solution P for the minimum cost M-connected pinning problem with $c(P) \leq \frac{3}{2} c^*$. Since $G' = G + K(P)$ is M-connected, it is 2-connected. Next we compute an optimal solution P' for the minimum cost 3-connected pinning problem on G'. Clearly, $c(P') \leq c^*$. It is also clear that $G + K(P \cup P')$ is 3-connected and M-connected. Furthermore, $c(P \cup P') \leq c(P) + c(P') \leq \frac{5}{2} c^*$ holds.

We remark that the above methods can be used to design a constant factor approximation algorithm for the corresponding *augmentation problem* as well, in which the goal is to add a smallest set F of new edges to G such that $G+F$ is globally rigid. See also [18] for the solution of a related augmentation problem.

Bibliography

[1] T. G. Abbot, Generalizations of Kempe's universality theorem, MSc thesis, MIT (2008). `http://web.mit.edu/tabbott/www/papers/mthesis.pdf`

[2] L. Asimow and B. Roth, The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245: 279–289, 1978.

[3] G. Baudis, C. Gröpl, S. Hougardy, T. Nierhoff, and H.J. Prömel, Approximating minimum spanning sets in hypergraphs and polymatroids, Tech. Report, Humboldt-Universität zu Berlin, 2000.

[4] L. Berenstein, L. Chavez, and W. Whiteley, Inductive techniques for plane rigidity using tree coverings, manuscript, 2002.

[5] A. Berg and T. Jordán, A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid, *J. Combinatorial Theory, Ser. B.*, Vol. 88, 77-97, 2003.

[6] A. Berg and T. Jordán, Algorithms for graph rigidity and scene analysis, Proc. 11th Annual European Symposium on Algorithms (ESA) 2003, (G. Di Battista, U. Zwick, eds) Springer Lecture Notes in Computer Science 2832, pp. 78-89, 2003.

[7] R. Connelly, Generic global rigidity, *Discrete Comput. Geom.* 33 (2005), no. 4, 549–563.

[8] R. Connelly, Combining globally rigid frameworks, *Proc. of the Steklov Institute of Mathematics*, 2011, Vol. 275, pp. 191-198.

[9] C.R. Coullard and L. Hellerstein, Independence and port oracles for matroids, with an application to computational learning theory, *Combinatorica* 16 (1996), no. 2, 189–208.

[10] H. Crapo, On the generic rigidity of plane frameworks, INRIA Tech. report, 1990.

[11] Z. Fekete, Source location with rigidity and tree packing requirements, *Operations Research Letters* 34, Issue 6, pp. 607-612, 2006.

[12] Z. Fekete and T. Jordán, Uniquely localizable networks with few anchors, Proc. Algosensors 2006, (S. Nikoletseas and J.D.P. Rolim, eds) Springer Lecture Notes in Computer Science 4240, pp. 176-183, 2006.

[13] Z. Fekete, T. Jordán, and W. Whiteley, An inductive construction for plane Laman graphs via vertex splitting, Proc. 12th Annual European Symposium on Algorithms (ESA) 2004, (S. Albers and T. Radzik, eds) Springer Lecture Notes in Computer Science 3221, pp. 299-310, 2004.

[14] Z. Fekete, T. Jordán, and V.E. Kaszanitzky, Rigid realizations of graphs with two coincident points, *Graphs and Combinatorics*, January 2014.

[15] A. Frank, *Connections in Combinatorial Optimization.* Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2011.

[16] A. Frank and A. Gyárfás, How to orient the edges of a graph, Combinatorics, (Keszthely), Coll. Math. Soc. J. Bolyai 18, 353-364, North-Holland, 1976.

[17] H.N. Gabow, Centroids, representations, and submodular flows, *Journal of Algorithms*, Vol. 18, Issue 3, May 1995, Pages 586-628.

[18] A. Garcia and J. Tejel, Augmenting the rigidity of a graph in R^2, *Algorithmica*, February 2011, Volume 59, Issue 2, pp 145-168.

[19] H. Gluck, Almost all simply connected closed surfaces are rigid, *Geometric topology (Proc. Conf., Park City, Utah, 1974)*, pp. 225–239. Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975.

[20] S. Gortler, A. Healy, and D. Thurston, Characterizing generic global rigidity, *American Journal of Mathematics*, Volume 132, Number 4, August 2010, pp. 897-939.

[21] J. Graver, B. Servatius, and H. Servatius, Combinatorial Rigidity, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[22] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.* **21** (1992), no. 1, 65-84.

[23] B. Hendrickson and D. Jacobs, An algorithm for two-dimensional rigidity percolation: the pebble game, *J. Computational Physics* 137, 346-365 (1997).

[24] L. Henneberg, Die graphische Statik der starren Systeme, Leipzig 1911.

[25] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs, *J. Combin. Theory Ser. B*, Vol. 94, 1-29, 2005.

[26] B. Jackson and T. Jordán, Rigid two-dimensional frameworks with three collinear points, *Graphs and Combinatorics*, Vol. 21, No. 4, 427-444, 2005.

[27] B. Jackson and T. Jordán, Pin-collinear body-and-pin frameworks and the molecular conjecture, *Discrete and Computational Geometry* 40: 258-278, 2008.

[28] B. Jackson, T. Jordán, and Z. Szabadka, Globally linked pairs of vertices in equivalent realizations of graphs, *Discrete and Computational Geometry*, Vol. 35, 493-512, 2006.

[29] B. Jackson, T. Jordán, and Z. Szabadka, Globally linked pairs of vertices in rigid frameworks, in: Rigidity and Symmetry (R. Connelly, W. Whiteley and A. Ivic Weiss, eds.), Fields Institute Communications, 70, pp. 177-203 (2014).

[30] B. Jackson and T. Jordán, Graph theoretic techniques in the analysis of uniquely localizable sensor networks, in: G. Mao, B. Fidan (eds), Localization algorithms and strategies for wireless sensor networks, IGI Global, 2009, pp. 146-173.

[31] B. Jackson and T. Jordán, A sufficient connectivity condition for generic rigidity in the plane, *Discrete Applied Mathematics*, 157 (2009) 1965-1968.

[32] B. Jackson, B. Servatius, and H. Servatius, The 2-dimensional rigidity of certain families of graphs, *J. Graph Theory*, Vol. 54, Issue 2, pages 154-166, 2007.

[33] T. Jordán, Rigid and globally rigid graphs with pinned vertices, in: Bolyai Society Mathematical Studies, 20, G.O.H. Katona, A. Schrijver, T. Szőnyi, eds., Fete of Combinatorics and Computer Science, 2010, Springer, pp. 151-172.

[34] T. Jordán, On the existence of k edge-disjoint 2-connected spanning subgraphs, *J. Combinatorial Theory*, Ser. B., Vol. 95, 257-262, 2005.

[35] T. Jordán, Generically globally rigid zeolites in the plane, *Information Proc. Letters*, Vol. 110, Issues 18-19, 2010, Pages 841-844.

[36] T. Jordán, G. Domokos, and K. Tóth, Geometric sensitivity of rigid graphs, *SIAM J. Disc. Math.*, vol. 27, No. 4, pp. 1710-1726, 2013.

[37] T. Jordán and V.E. Kaszanitzky, Highly connected rigidity matroids have unique underlying graphs, *European J. Combinatorics*, Volume 34, Issue 2, February 2013, pp. 240–247.

[38] T. Jordán, C. Király, and S. Tanigawa, Generic global rigidity of body-hinge frameworks, Egerváry Research Group, Budapest, TR-2014-06.

[39] T. Jordán, A. Recski, and Z. Szabadka, Rigid tensegrity labelings of graphs, *European J. Combinatorics* 30 (2009) 1887-1895.

[40] T. Jordán and Z. Szabadka, Operations preserving the global rigidity of graphs and frameworks in the plane, *Computational Geometry*, 42 (2009) 511-521.

[41] V.E. Kaszanitzky and Cs. Király, On minimally highly vertex-redundantly rigid graphs, Egerváry Research Group, Budapest, TR-2014-08.

[42] N. Katoh and S. Tanigawa, On the infinitesimal rigidity of bar-and-slider frameworks, Proc. ISAAC (Algorithms and Computation), 2009, Springer.

[43] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[44] A. Lee and I. Streinu, Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 308(8):1425–1437, 2008.

[45] A. Lee, I. Streinu, and L. Theran, The slider-pinning problem, Proceedings of the 19th Canadian Conference on Computational Geometry, Ottawa, Ontario, August 2007.

[46] M. Lorea, Hypergraphes et matroides, *Cahiers Centre Etud. Rech. Oper.* 17 (1975) pp. 289-291.

[47] L. Lovász, The matroid matching problem, Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), pp. 495-517, Colloq. Math. Soc. János Bolyai, 25, North-Holland, Amsterdam-New York, 1981.

[48] L. Lovász, Matroid matching and some applications, *J. Combin. Theory Ser. B* 28 (1980), no. 2, 208-236.

[49] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* 3 (1982), no. 1, 91–98.

[50] S. Luo, Lower bound for the rank of 2-dimensional generic rigidity matroid for regular graphs of degree four and five, arXiv:1205.4382, July 2012.

[51] M. Makai and J. Szabó, The parity problem of polymatroids without double circuits, *Combinatorica* 28 (6) pp. 679-692, 2008.

[52] A. Mansfield, On the computational complexity of a rigidity problem, *IMA J. Appl. Math.* 27 (1981), no. 4, 423-429.

[53] J.G. Oxley, *Matroid theory*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992. xii+532 pp.

[54] A. Recski, *Matroid Theory and its Applications in Electric Network Theory and in Statics*, Akadémiai Kiadó, Budapest, 1989.

[55] J.B. Saxe, Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[56] A. Schrijver, *Combinatorial optimization - polyhedra and efficiency*, Springer-Verlag, Berlin, 2003.

[57] B. Servatius, Birigidity in the plane, *SIAM J. Discrete Math.* 2 (1989), no. 4, 582–589.

[58] B. Servatius, O. Shai, and W. Whiteley, Combinatorial characterization of the Assur graphs from engineering, *European J. Combin.* 2010, pp. 1091-1104.

[59] S. Tanigawa, Sufficient conditions for the global rigidity of graphs, arXiv:1403.3742, March 2014.

[60] T-S. Tay and W. Whiteley, Generating isostatic frameworks, *Structural Topology* 11, 1985, pp. 21-69.

[61] W.T. Tutte, *Connectivity in Graphs*, University of Toronto Press (1966).

[62] W. Whiteley, Vertex splitting in isostatic frameworks, *Structural Topology* No. 16 (1990), 23-30.

[63] W. Whiteley, Some matroids from discrete applied geometry, *in:* Matroid theory (J.E. Bonin, J.G. Oxley and B. Servatius eds., Seattle, WA, 1995), Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.

[64] W. Whiteley, Rigidity and scene analysis, Handbook of Discrete and Computational Geometry, 2nd edition, CRC Press, 1327-1354, 2004.