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# Edmonds' Branching Theorem in Digraphs without Forward-infinite Paths

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# Edmonds' Branching Theorem in Digraphs without Forward-infinite Paths

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**Abstract**

By a well-known theorem of Edmonds if a finite digraph is $k$-edge-connected from vertex $r$ then it has $k$ edge-disjoint spanning arborescences rooted at $r$. As it was shown by R. Aharoni and C. Thomassen in [7] this does not remain true for infinite digraphs. Thomassen also proved that for the class of digraphs without backward-infinite paths the above theorem of Edmonds remains true. Our main result is that for digraphs without forward-infinite paths the theorem is also true, even in general form.

## 1 Notions

Digraphs $D = (V, A)$ considered here may have multiple edges and arbitrary size. Loops are also allowed but irrelevant to our subject. If $B \subseteq V$ then we write $D[B]$ for the subgraph of $D$ spanned by $B$. For $X \subseteq V$ let $\mathsf{in}_D(X)$ and $\mathsf{out}_D(X)$ be the sets of ingoing and outgoing edges of $X$ in $D$, and $\varrho_D(X)$, $\delta_D(X)$ their cardinalities, respectively. By a path we mean a directed, possibly infinite, simple path so that repetition of vertices is not allowed. $\mathsf{start}(P)$ and $\mathsf{end}(P)$ denotes the first and last point of the path $P$ if it exists. For $e = (x, y)$ let $\mathsf{start}(e) = x$ and $\mathsf{end}(e) = y$. For $X, Y \subseteq V$ let $\mathsf{e}_D(X, Y) = \{e \in A : \mathsf{start}(e) \in X,\ \mathsf{end}(e) \in Y\}$, for singletons we write $\mathsf{e}(x, y)$ instead of $\mathsf{e}(\{x\}, \{y\})$. We say that the path $P$ goes from $X$ to $Y$ if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$ ($\mathsf{start}(P) = \mathsf{end}(P)$ is allowed). For singletons we write here also $v$ instead of $\{v\}$. Call a digraph $\mathcal{A} = (V, A)$ an arborescence with root vertex $r$ if $\mathcal{A}$ is a directed tree such that all vertices are reachable from $r$. We call $\min\{\varrho_D(X) : \varnothing \neq X \subseteq V \setminus \{r\}\}$ the edge-connectivity of $D$ from $r$ and we say $D$ is $\kappa$-edge-connected from $r$ if this cardinal is at least $\kappa$. For an undirected graph $G$ denote by $V(G)$ and $E(G)$ its point-set and edge-set respectively. If $G$ is directed we write $A(G)$ instead of $E(G)$. If $\mathcal{A}$ is a subarborescence of $D$ and $e \in \mathsf{out}_D(V(\mathcal{A}))$ then sometimes we write $\mathcal{A} + e$ instead of $(V(\mathcal{A}) \cup \{\mathsf{end}(e)\}, A(\mathcal{A}) \cup \{e\})$.

## 2 Introduction

Edmonds proved in [1] his famous branching theorem which states that if a finite digraph is $k$-edge-connected from $r$ then it has $k$ edge-disjoint spanning arborescences rooted at $r$. Lovász gave a new elegant proof for this theorem of Edmonds in [2] and his techniques opened the door for further generalizations such as [3], [4], [5] and [6]. Infinite generalizations have been marred by a negative result by R. Aharoni and C. Thomassen [7]. They constructed, for any $k \in \mathbb{N}$, a countably-infinite, locally finite, simple graph $G$ such that $G$ has $k$-connected orientations and $G$ has vertices $u, v$ such that deleting the edges of an arbitrary $u - v$ path disconnects $G$.

Thomassen showed (unpublished) that if $D = (V, A)$ does not contain backward-infinite paths and it is $k$-edge-connected from $r$ then it has $k$ edge-disjoint spanning arborescences rooted at $r$. The main idea of his proof is the following. First one constructs a spanning subgraph $D' = (V, A')$ such that $D'$ is also $k$-edge-connected from $r$ and all points of $D'$ have finite in-degrees. After that one constructs the desired arborescences in $D'$ using the finite version of the theorem and compactness arguments.

Our main result is that disallowance of the forward-infinite paths instead of backward-infinite paths is also sufficient. The proof is longer and uses totally different techniques than the sketch of Thomassen's proof above.

# 3 Main result

In this section we state and prove our main result theorem 3.1.

**Theorem 3.1.** *Let $D = (V, A)$ be a digraph, $\kappa$ a (finite or infinite) cardinal and $\mathcal{A}_i = (V_i, A_i)$ $(i < \kappa)$ edge-disjoint subarborescences of $D$ such that for all $v \in V$ the set $H_v = \{i < \kappa : v \notin V_i\}$ is finite and let $D_f = (V, A \setminus \cup_{i<\kappa} A_i)$. Suppose that $D_f$ does not contain forward-infinite paths. Then these subarborescences can be extended to edge-disjoint spanning arborescences of $D$ without changing their roots if and only if*

$$\forall X \ (\varnothing \neq X \subseteq V \Longrightarrow \varrho_{D_f}(X) \geq |\{i : V_i \cap X = \varnothing\}|). \tag{1}$$

Let us formulate two important special case of the theorem above.

**Corollary 3.2.** *Let $D = (V, A)$ be a digraph, $k \in \mathbb{N}$ and $\mathcal{A}_i = (V_i, A_i)$ $(i < k)$ edge-disjoint subarborescences of $D$ and let $D_f = (V, A \setminus \cup_{i<k} A_i)$. Suppose $D_f$ does not contain forward-infinite paths. Then these subarborescences can be extended to edge-disjoint spanning arborescences of $D$ without changing their roots if and only if condition (1) holds.*

**Corollary 3.3.** *Let the digraph $D$ be $k$-edge-connected from the vertex $r$ for some $k \in \mathbb{N}$ and suppose that there are no forward-infinite paths in $D$. Then there are $k$ edge-disjoint spanning arborescences in $D$ rooted at $r$.*

**Remark 3.4.** Theorem 3.1 remains true if we change "forward-infinite" to "backward-infinite" because Thomassen's proof (see the sketch in the introduction) works also in this general situation.

*Proof of Theorem 3.1.* Without loss of generality we can assume that the roots of the $\{\mathcal{A}_i\}_{i<\kappa}$ arborescences are the same point by adding $rr_i$ edges to $D$, where $r$ is a new vertex and $r_i$ is the original root of $\mathcal{A}_i$ and extend $\mathcal{A}_i$ with $rr_i$ $(i<\kappa)$. The necessity of the condition is obvious, so we show only that it is sufficient. To do so, we need the following lemma.

**Lemma 3.5.** *For any $\zeta < \kappa$ and $v \notin V_\zeta$ there is a path $P$ in $D_f$ from $V_\zeta$ to $v$ such that condition (1) holds for $D$ and the arborescence-system $\{\mathcal{A}'_i\}_{i<\kappa}$ where* $\mathcal{A}'_i = \begin{cases} \mathcal{A}_i + P & \text{if } i = \zeta \\ \mathcal{A}_i & \text{otherwise} \end{cases}$.

Before the proof of the lemma 3.5 we need to build up some basic tools in spirit of Lovász's proof for the finite version of the theorem in [2].

Call a set $\varnothing \neq X \subseteq V$ accurate if $\varrho_{D_f}(X) = |\{i : V_i \cap X = \varnothing\}|$ and dangerous if it is accurate and $X \cap V_0 \neq \varnothing$. It is easy to see that if $e \in \mathsf{out}_{D_f}(V_0)$ then the extension $\mathcal{A}_0 \stackrel{\text{def}}{=} \mathcal{A}_0 + e$ violates condition (1) if and only if $e$ is an ingoing edge of some dangerous set.

**Proposition 3.6.** *If $X, Y$ are dangerous and $X \cap Y \neq \varnothing$ then $X \cap Y$ is dangerous.*

*Proof:* Let $s : \mathcal{P}(V) \to \mathbb{N}$, $s(X) = |\{i : V_i \cap X = \varnothing\}|$. Then $s$ is supermodular i.e. for $X, Y \subseteq V$ we have $s(X) + s(Y) \leq s(X \cup Y) + s(X \cap Y)$. Indeed, let $i < \kappa$ be arbitrary. If $V_i \cap X = \varnothing$ and $V_i \cap Y = \varnothing$ then $V_i \cap (X \cup Y) = \varnothing$ and $V_i \cap X \cap Y = \varnothing$, so the $V_i$'s contribution to both sides of the inequality is 2. If $V_i \cap X = \varnothing$ and $V_i \cap Y \neq \varnothing$ then the $V_i$'s contribution to both sides is 1.
Observe that equality holds if and only if there does not exist any $V_i$ such that $V_i \cap X \neq \varnothing$, $V_i \cap Y \neq \varnothing$ and $V_i \cap X \cap Y = \varnothing$. Let $p(X) = \varrho_{D_f}(X) - s(X)$. Then condition (1) is equivalent with $p(X) \geq 0$ for all $X \neq \varnothing$, and accurateness of $X$ means $p(X) = 0$. The function $\varrho_{D_f}$ is submodular, so $p$ is too i.e. $p(X) + p(Y) \geq p(X \cup Y) + p(X \cap Y)$ holds for all $X, Y \subseteq V$. Let $X, Y$ be dangerous and $X \cap Y \neq \varnothing$. Then by submodularity and condition (1), we get

$$0 + 0 = p(X) + p(Y) \geq p(X \cup Y) + p(X \cap Y) \geq 0 + 0,$$

so $X \cup Y$ and $X \cap Y$ are accurate therefore $s(X) + s(Y) = s(X \cup Y) + s(X \cap Y)$. By the observation about the function $s$ and by $X \cap V_0 \neq \varnothing$, $Y \cap V_0 \neq \varnothing$ it follows that $X \cap Y \cap V_0 \neq \varnothing$, so $X \cap Y$ is dangerous. ●

**Proposition 3.7.** *Let $B$ be a dangerous set. Then for any $w \in B$ there is a path $R$ from $V_0 \cap B$ to $w$ in $D_f[B]$.*

*Proof:* Let $B'$ be the set of vertices which are reachable from $V_0 \cap B$ in $D_f[B]$. Suppose, seeking a contradiction, that $B' \neq B$. Then $B \setminus B'$ violates condition (1) which is a contradiction. ●

**Proposition 3.8.** *For all $w \in V$ there is a system of edge-disjoint paths $\{P_i\}_{i \in H_w}$ in $D_f$ such that $P_i$ goes from $V_i$ to $w$.*

*Proof:* We extend $D_f$ to $D'_f$ by adding new vertices and edges (see figure 1). Let $V(D'_f) = V \cup \{s\} \cup \{v_i\}_{i \in H_w}$, $|\mathsf{e}_{D'_f}(s, v_i)| = 1$ $(i \in H_w)$ and $|\mathsf{e}_{D'_f}(v_i, u)| = \aleph_0$ $(i \in H_w, u \in V_i)$. If there are $|H_w|$-many edge-disjoint paths from $s$ to $w$ in $D'_f$ then we are done. Suppose, seeking contradiction, that there aren't. By Menger's theorem there is a $w \in X \subseteq V(D'_f) \setminus \{s\}$ with $\varrho_{D'_f}(X) < |H_w|$. Let $l = \left|\{v_i\}_{i \in H_w} \setminus X\right|$. Note that $0 < l$ otherwise $sv_i \in \mathsf{in}_{D'_f}(X)$ $(i \in H_w)$ and hence $|H_w| \leq \varrho_{D'_f}(X)$ would follow. By the infinitely many parallel edges $X \cap V$ is disjoint from at least $l$ arborescences. Otherwise $\varrho_{D'_f}(X) = \varrho_{D_f}(X \cap V) + |H_w| - l$ so $\varrho_{D_f}(X \cap V) = l + \varrho_{D'_f}(X) - |H_w| < l$ but then $X \cap V$ violates condition (1) in $D$ which is a contradiction. ●

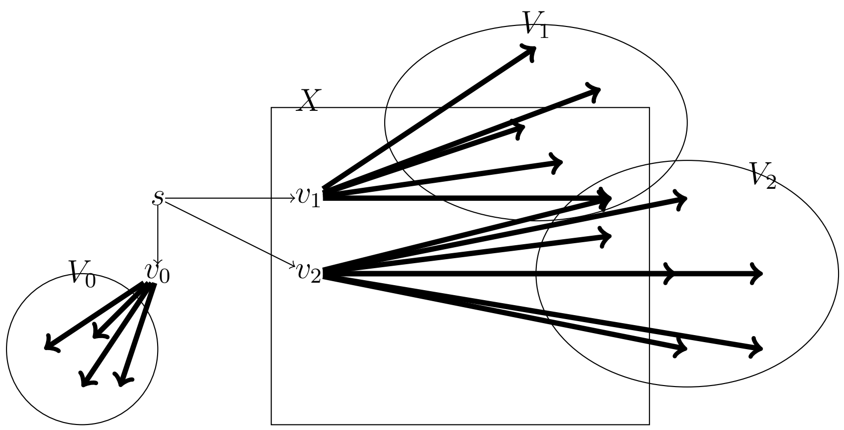


Figure 1: The construction of $D'_f$ and the cut $X$ from the proof above in the case $H_w = \{0, 1, 2\}$, $l = 1$. (Thick arrows stand for countably infinite parallel edges).

**Proposition 3.9.** *Let $B_2 \subseteq B_1$ be dangerous sets. Denote the endpoints of edges in $\mathsf{in}_{D_f}(B_1)$ with multiplicity by $s_1, \ldots, s_m$ (where the multiplicity of a vertex $z$ is $\left|\mathsf{in}_{D_f}(B_1) \cap \mathsf{in}_{D_f}(z)\right|$). Then there is a system of edge-disjoint paths $\{P_j\}_{j=1}^m$ in $D_f[B_1]$ such that $P_j$ goes from $s_j$ to $B_2$. If $\{P_j\}_{j=1}^m$ is a path-system like above and $\varrho_{D_f}(B_2) = \varrho_{D_f}(B_1)$ then $\mathsf{e}_{D_f}(B_1 \setminus B_2, B_2) \subseteq \cup_{j=1}^m A(P_i)$.*

*Proof:* Extend $D_f[B_1]$ with a new vertex $s$ and new edges $ss_1, \ldots, ss_m$. For the first part of the proposition it is enough to show that the resulting graph $H$ is $m$-edge-connected from $s$. Suppose, seeking contradiction, that it is not. Then by Menger's theorem there is a $\varnothing \neq X \subseteq B_1$ such that $\varrho_H(X) < m$. Using dangerousness of $B_1$ we obtain $m = s(B_1) \leq s(X)$ so $\varrho_{D_f}(X) = \varrho_H(X) < m \leq s(X)$ thus $X$ violates condition (1) which is a contradiction.

Let $\{P_j\}_{j=1}^m$ be a desired path system and $l \stackrel{\text{def}}{=} |\{j : s_j \in B_2\}|$. Then there are exactly $l$ of the pahts above that have 0 length, the other $m - l$ paths use one-one edges of $\mathsf{e}_{D_f}(B_1 \setminus B_2, B_2)$. Otherwise $\mathsf{e}_{D_f}(B_1 \setminus B_2, B_2) = \varrho_{D_f}(B_2) - l = m - l$. ●

**Corollary 3.10.** *Let $B$ be a dangerous set and $e \in \mathsf{in}_{D_f}(B)$. Then for any $w \in B$ there is a path $R$ from $\mathsf{end}(e)$ to $w$ in $D_f[B]$.*

*Proof of Lemma 3.5.* By symmetry it is sufficient to prove the lemma for $\zeta = 0$. Let us introduce the following recursive process (informal first). Here we use the notions $\mathcal{A}_0, D_f$ as variables.

Fix a path $P_1$ from $V_0$ to $v$ in $D_f$ (it exists by proposition 3.7). Extend $\mathcal{A}_0$ with the edges of $P_1$ and their endpoints one by one in sequence until it is doable without the violation of condition (1). If $e_1$ is the first edge that we can not add to $\mathcal{A}_0$ then there is a dangerous set $B_1$ such that $e_1 \in \mathsf{in}_{D_f}(B_1)$. Fix a path $P_2$ that goes from $V_0 \cap B_1$ to $\mathsf{end}(e_1)$ in $D_f[B_1]$ (it exists by proposition 3.7). Extend $\mathcal{A}_0$ with the edges of $P_2$ and their endpoints one by one in a sequence until it does not violate condition (1). If they never violate condition (1) we reach the vertex $\mathsf{end}(e_1)$, and then we step back to path $P_1$ and continue the process starting with the last vertex of $P_1$ which is in $V_0$. Otherwise let $e_2 \in A(P_2)$ be the first violating edge. Then $e_2 \in \mathsf{in}_{D_f}(B_2)$ for some dangerous set $B_2$ and we can assume, by proposition 3.6, that $B_2 \subseteq B_1$. We take a $P_3$ path from $V_0 \cap B_2$ to $\mathsf{end}(e_2)$ in $D_f[B_2]$ and continue...

Surprisingly, with this naive process we reach the vertex $v$ after finitely many steps. To show this we make precise the informal description above. We define by recursion a sequence $(\mathcal{A}_0^n, T_n, x_n)$ where $\mathcal{A}_0^n = (V_0^n, A_0^n)$ is the extension of the arborescence $\mathcal{A}_0$ after the $n$-th step, $T_n$ is a rooted tree with root vertex $s$ and $x_n \in V(T_n)$. Let $\mathcal{A}_i^n = (V_i^n, A_i^n) \overset{\text{def}}{=} \mathcal{A}_i$ if $0 < i$ and $D_f^n \overset{\text{def}}{=} (V, A \setminus \bigcup_{i<\kappa} A_i^n)$. Call $B \subseteq V$ $n$-dangerous if it is dangerous with respect to the arborescence-system $\{\mathcal{A}_i^n\}_{i<\kappa}$. Let $T = (\bigcup_{n=0}^{\infty} V(T_n), \bigcup_{n=0}^{\infty} E(T_n))$. Denote by $b(t)$ the smallest $n$ such that $t \in V(T_n)$. The vertices of the trees $T_n$ are in the form $t = (B_t, P_t, e_t)$ (except the root which is $s = (B_s, P_s)$) where $B_t \subseteq V$ is an $b(t)$-dangerous set, $P_t$ is path from $V_0^{b(t)}$ to $\mathsf{end}(e_t)$ in $D_f^{b(t)}[B_t]$ and $e_t \in \mathsf{in}_{D_f^{b(t)}}(B_t)$. Denote by $\mathsf{parent}_T(t)$, $\mathsf{child}_T(t)$ the parent of the vertex $t$ and the set of children of vertex $t$ in the tree $T$ respectively.

In the beginning let $\mathcal{A}_0^0 = \mathcal{A}_0$, $T_0$ has only its root $s = (B_s, P_s)$ where $B_s = V$ and $P_s$ is an arbitrary path from $V_0$ to $v$ in $D_f$ (it exists by proposition 3.7) and $x_0 = s$. The recursion is as follows

**Case 1.** $v \in V_0^n$

$$\mathcal{A}_0^{n+1} \overset{\text{def}}{=} \mathcal{A}_0^n,\ T_{n+1} \overset{\text{def}}{=} T_n,\ x_{n+1} \overset{\text{def}}{=} x_n$$

**Case 2.** $v \notin V_0^n$ and $\mathsf{end}(P_{x_n}) \in V_0^n$

$$x_{n+1} \overset{\text{def}}{=} \mathsf{parent}_{T_n}(x_n),\ \mathcal{A}_0^{n+1} \overset{\text{def}}{=} \mathcal{A}_0^n,\ T_{n+1} \overset{\text{def}}{=} T_n$$

**Case 3.** $v$, $\mathsf{end}(P_{x_n}) \notin V_0^n$ and the extension $\mathcal{A}_0^n + e$ does not violate condition (1) where $e \in A(P_{x_n})$ is the outgoing edge of the last vertex of $P_{x_n}$ which is in $V_0^n$,

$$\mathcal{A}_0^{n+1} \overset{\text{def}}{=} \mathcal{A}_0^n + e,\ T_{n+1} \overset{\text{def}}{=} T_n,\ x_{n+1} \overset{\text{def}}{=} x_n$$

**Case 4.** $v$, $\mathsf{end}(P_{x_n}) \notin V_0^n$ and the extension $\mathcal{A}_0^n + e$ violates condition (1) where $e \in A(P_{x_n})$ is the outgoing edge of the last vertex of $P_{x_n}$ which is in $V_0^n$.

Then $e \in \mathsf{in}_{D_f^n}(B)$ for some $n$-dangerous $B$. Let $T_{n+1}$ be the extension of $T_n$ with a new vertex $t$ such that $t$ is a child of $x_n$. Let $B_t \overset{\text{def}}{=} B \cap B_{x_n}$ (it is $n$-dangerous by proposition 3.6), $P_t$ a path from $B_t \cap V_0^n$ to $\mathsf{end}(e)$ in $D_f^n[B_t]$ (it exists by proposition 3.7), $e_t \stackrel{\text{def}}{=} e$ and $\mathcal{A}_0^{n+1} \stackrel{\text{def}}{=} \mathcal{A}_0^n,\ x_{n+1} \stackrel{\text{def}}{=} t$.

It is routine to check the following properties of the $(\mathcal{A}_0^n, T_n, x_n)_{n\in\mathbb{N}}$ sequence.

1. If $t_2 \in \mathsf{child}_T(t_1)$ then $B_{t_2} \subset B_{t_1}$ $(t_1, t_2 \in V(T))$.
2. $e_t \in A(P_{\mathsf{parent}_T(t)}) \cap \mathsf{e}(B_{\mathsf{parent}_T(t)} \setminus B_t, B_t)$ $(t \in V(T) \setminus \{s\})$.
3. $e_t \in A(D_f^n)$ $(t \in V(T) \setminus \{s\}, n \in \mathbb{N})$
4. If $x_n \neq x_{n+1}$ then $\{x_n, x_{n+1}\} \in E(T)$ $(n \in \mathbb{N})$.
5. $x_{b(t)} = t$ and if $t \neq s$ then $x_{b(t)-1} = \mathsf{parent}_T(t)$.

**Proposition 3.11.** *If $t_1, t_2 \in \mathsf{child}_T(t)$ and $b(t_1) < b(t_2)$, then $e_{t_1}$ precedes $e_{t_2}$ in $P_t$.*

*Proof:* It is enough to show that $\mathsf{end}(e_{t_1}) \in V_0^{b(t_2)-1}$ (see case 4). Let $m = \min\{n \in \mathbb{N} : b(t_1) \leq n,\ x_n = t\}$. Then $m \leq b(t_2) - 1$ because $x_{b(t_2)-1} = t$ (see property 5) and $b(t_1) \leq b(t_2) - 1$. By $x_{b(t_1)} = t_1$ and property 4 at the $m-1$-th step of the recursion arise case 2 and $x_{m-1} = t_1$. So by the definition of case 2 $\mathsf{end}(e_{t_1}) \in V_0^{m-1}$ and so $\mathsf{end}(e_{t_1}) \in V_0^{b_{t_2}-1}$. ●

**Corollary 3.12.** *For all $t \in V(T)$ $|\mathsf{child}_T(t)| \leq |A(P_t)|$.*

We need to show that case 1 occurs in the recursion. Suppose, seeking a contradiction, that it does not i.e. $v \notin V_0^n$ for all $n$.

We show that $T$ is an infinite tree. Suppose, seeking a contradiction, that it is not. Then $T = T_m$ for some $m$. In all case 3-type step we put an edge of $P_t$ to the 0-th arborescence for some $t \in V(T_m)$, hence after the $m$-th step there are at most $\sum_{t\in V(T_m)} |A(P_t)|$ case 3-type steps and at most $\mathsf{height}(T_m)$ case 2-type steps, so there must be a case 4-type step after the $m$-th step, thus $T_m \neq T$ which is a contradiction.

So $T$ is a countably-infinite tree with finite degrees (see corollary 3.12). Then by König's lemma there is an infinite path $(t_n)_{n\in\mathbb{N}}$ in $T$ such that $t_{n+1} \in \mathsf{child}_T(t_n)$ $(n \in \mathbb{N})$. We may assume that $t_0 \neq s$.

Let's look at the monotone decreasing sequence $(B_{t_n})_{n\in\mathbb{N}}$ (see property 1).

**Proposition 3.13.** $\sup_{n\in\mathbb{N}} s(B_{t_n}) < \infty$.

*Proof:* It is enough to show that $C \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} B_{t_n} \neq \varnothing$, because if $u \in C$, then by assumption $H_u = \{i < \kappa : v \notin V_i\}$ is a finite set and $s(B_{t_n}) \leq |H_u|$ $(n \in \mathbb{N})$. We construct a path $R$ by recursion such that $R$ enters into all of the sets $\{B_{t_n}\}_{n\in\mathbb{N}}$ and does not enter into any of the sets $\{V \setminus B_{t_n}\}_{n\in\mathbb{N}}$. $R$ can not be forward-infinite by assumption so if we succeed, then $\mathsf{end}(R) \in C$ and the proof is completed. Let $R_0$ be the length 1 path $e_{t_0}$. By using corollary 3.10 with the arborescences $\{\mathcal{A}_i^{b(t_{n+1})}\}_{i<\kappa}$ there is a path $R_{n+1}$ in $D_f^{b(t_{n+1})}[B_{t_n}]$ from $\mathsf{end}(R_n)$ to $B_{n+1}$. By connecting the paths $\{R_n\}_{n\in\mathbb{N}}$ we get the desired path $R$. ●

The monotone increasing sequence $s(B_{t_n})$ is bounded (see proposition 3.13). By deleting few initial elements of the sequence $(t_n)_{n\in\mathbb{N}}$ we may assume that the sequence $s(B_{t_n})$ is constant, i.e. the sets $B_{t_n}$ are disjoint from the same $l$-many $V_i$'s. Let $s_n(X) = |\{i : V_i^n \cap X = \varnothing\}|$. Then the sequence $s_{b(t_n)}(B_{t_n})$ is also constant, namely $l-1$ or $l$ depending on whether the sets $B_{t_n}$ are disjoint from $V_0$ or not. Denote this number by $m$. Note that $m > 0$ because $e_{t_n} \in \mathsf{in}_{D_f^{b(t_n)}}(B_{t_n})$ and $B_{t_n}$ is $b(t_n)$-dangerous. Apply proposition 3.9 to the sets $B_{t_n}$ and $B_{t_{n+1}}$ with the arborescence-system $\{\mathcal{A}_i^{b(t_{n+1})}\}_{i<\kappa}$ $(n = 0, 1, \dots)$. By connecting the results we get a system of edge-disjoint paths $\{P_i\}_{j=1}^m$ entering all of the sets $\{B_{t_n}\}_{n\in\mathbb{N}}$. The edges in $\{e_{t_n}\}_{n\in\mathbb{N}}$ are pairwise distinct because $\mathsf{start}(e_{t_{n+1}}) \in B_{t_n} \setminus B_{t_{n+1}}$ (see property 2) and by the second part of proposition 3.9 and by properties 2, 3 $\{e_{t_n}\}_{n\in\mathbb{N}} \subseteq \bigcup_{j=1}^m A(P_i)$. So at least one of the paths are infinite which contradicts to our assumption. ●●

Now we continue the proof of theorem 3.1. If $v \in V$ then by lemma 3.5 and finiteness of the sets $H_v$ we can extend the arborescences $\{\mathcal{A}_i\}_{i\in H_v}$ without the violation of condition (1) with finitely many new points and edges such that all of these extensions contain $v$. In the countable case we can construct the desired spanning arborescenes by an easy recursion. In the $n$-th step do the extensions above with the arborecences after the previous step and the following vertex $v_n$ where $V = \{v_n\}_{n=0}^{\infty}$. In the uncountable case we have to be more careful because we need to assure that we do not violate condition (1) in limit steps. The main idea is that if we extend one of the arborescences with some vertex $v$, then at the successor of these steps we put $v$ into all arboresences which missed it.

Let us make precise the idea above. Let $V \overset{\text{def}}{=} \{v_\alpha\}_{\alpha<\lambda}$, where $\lambda = |V|$. We extend the arborecences by transfinite recursion on $\lambda$. Let $\mathcal{A}_i^\alpha = (V_i^\alpha, A_i^\alpha)$ be the arborescence which we get from $\mathcal{A}_i$ after the $\alpha$-th step $(i < \kappa,\ \alpha \le \lambda)$.

Let $\mathcal{A}_i^0 = \mathcal{A}_i$ $(i < \kappa)$.
If $\alpha < \lambda$ is a limit ordinal, then $\mathcal{A}_i^\alpha \overset{\text{def}}{=} (\bigcup_{\beta<\alpha} V_i^\beta, \bigcup_{\beta<\alpha} A_i^\beta)$.
If $\alpha = \beta + 1$, where $\beta < \lambda$ is a limit ordinal and the arborescences $\{\mathcal{A}_i^\beta\}_{i<\kappa}$ satisfy condition (1) then add $v_\beta$ to all of the arborescences $\{\mathcal{A}_i^\beta\}_{i<\kappa}$ which missed it by using lemma 3.5 repeatedly. Denote the resulting arborescences by $\mathcal{A}_i^\alpha$ $(i < \kappa)$.
If $\alpha = \beta + 2$ where $\beta < \lambda$ is an arbitrary ordinal and the arborescences $\{\mathcal{A}_i^{\beta+1}\}_{i<\kappa}$ satisfy condition (1), and the set $N^\beta \overset{\text{def}}{=} \{v_{\beta+1}\} \cup \bigcup_{i<\kappa} V_i^{\beta+1} \setminus V_i^\beta$ is finite, then add the points of $N^\beta$ one by one to all of the arborescences $\{\mathcal{A}_i^{\beta+1}\}_{i<\kappa}$ that missed it, by using lemma 3.5 repeatedly. Denote the resulting arborescences by $\mathcal{A}_i^\alpha$ $(i < \kappa)$.

**Proposition 3.14.** *The transfinite recursion above does not stop before the $\lambda$-th step.*

*Proof:* Suppose, seeking a contradiction, that it does. The limit steps are always doable. At successor steps we do not violate condition (1) and we extended the arborescences with only finitely many new points and edges, so if a successor step is doable then the successor of it is also. Hence the first step where the recursion can not be continued is necessarily a successor of a limit ordinal $\beta$. But then $\beta$ is the first ordinal such that the arborescences $\{\mathcal{A}_i^\beta\}_{i<\kappa}$ violate condition (1). Let

$s_\beta(X) = |\{i : V_i^\beta \cap X = \varnothing\}|$, $\varnothing \neq Y \subseteq V$ arbitrary and $D_f^\beta = (V, A \setminus \bigcup_{i<\kappa} A_i^\beta)$. If $\mathsf{in}_{D_f}(Y) = \mathsf{in}_{D_f^\beta}(Y)$, then $\varrho_{D_f^\beta}(Y) = \varrho_{D_f}(Y) \geq s(Y) \geq s_\beta(Y)$. If $e \in \mathsf{in}_{D_f}(Y) \setminus \mathsf{in}_{D_f^\beta}(Y)$ then there is an $i_0$ such that $e \in A_{i_0}^\beta$. Let $\gamma < \beta$ be the smallest ordinal such that $e \in A_{i_0}^\gamma$. Then $\gamma$ is a successor ordinal so by the recursion we get $\mathsf{end}(e) \in \mathcal{A}_i^{\gamma+1}$ $(i < \kappa)$, thus $\mathsf{end}(e) \in \mathcal{A}_i^\beta$ $(i < \kappa)$, hence $\varrho_{D_f^\beta}(Y) \geq 0 = s_\beta(Y)$. So there is no violating $Y$ for the arborescenses $\{\mathcal{A}_i^\beta\}_{i<\kappa}$ which is a contradiction. ●

Finally the arborescences $\{\mathcal{A}_i^\lambda\}_{i<\kappa}$ are the desired spanning-arborecences. □

# 4 Conjectures

## 4.1 A condition about paths

We show by a counterexample that the condition of finiteness of the sets $H_v = \{i < \kappa : v \notin V_i\}$ is necessary in theorem 3.1 and formulate a conjecture with a condition about paths (which is a strengthening of the condition (1)) motivated by this counterexample.

We construct a digraph $D$ (see figure 2) and a system of subarborescences of it such that $D_f$ does not contain even 5-length paths and the system satisfies condition (1) but the desired extensions of the arborescences do not exist. Let $D = (V, A)$ where $V = \{r_n\}_{n\in\mathbb{N}} \cup \{v\}$, $|\mathsf{e}_D(r_0, r_n)| = \aleph_0$ $(n \in \mathbb{N}^+)$, $|\mathsf{e}_D(r_n, v)| = 1$ $(n \in \mathbb{N}^+)$, $|\mathsf{e}_D(v, r_0)| = \aleph_0$ and $\mathcal{A}_n = (\{r_n\}, \varnothing)$ $(n \in \mathbb{N})$. Observe that if $P$ is a path of length 3 in $D$ then $r_0$ and $v$ must belong to, hence $D_f (= D)$ does not contain a path of length 3. We show that the system above does not violate condition (1). Let $\varnothing \neq X \subseteq V$ be arbitrary. Assume first $r_0 \notin X$. If $r_n \in X$ for some $n \in \mathbb{N}^+$ then $\varrho_{D_f}(X) = \aleph_0$, if not then $X = \{v\}$ and $\varrho_{D_f}(X) = \aleph_0$ again. Assume $r_0 \in X$. If $v \notin X$ then $\varrho_{D_f}(X) = \aleph_0$. Else if each elements of $\{r_n\}_{n\in\mathbb{N}^+} \setminus X$ has one-one outgoing edge to $X$ so there is equality in condition (1).
Otherwise, obviously there is no system of edge-disjoint paths $\{P_n\}_{n\in\mathbb{N}}$ such that $P_n$ goes from $r_n$ to $v$ thus we can not extend the $\mathcal{A}_n$-s to edge-disjoint spanning arborescences.

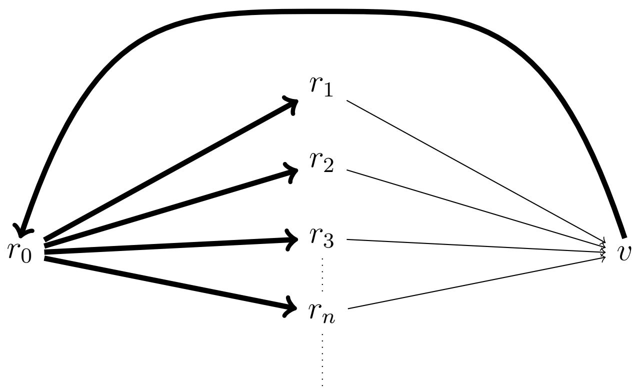


Figure 2: The counterexample. (Thick arrows stand for countably infinite parallel edges).

**Conjecture 4.1.** *Let $D = (V, A)$ a digraph, $\mathcal{A}_i = (V_i, A_i)$ $(i < \kappa)$ edge-disjoint subarborescences of $D$ and $D_f = (V, A \backslash \cup_{i<\kappa} A_i)$. Suppose $D_f$ does not contain forward-infinite paths and for all $v \in V$ there is a system of edge-disjoint paths $\{P_i\}_{i<\kappa}$ in $D_f$ such that $P_i$ goes from $V_i$ to $v$. Then there are extensions of the $\mathcal{A}_i$-s to edge-disjoint spanning arborescences of $D$ without changing their roots.*

## 4.2 An arborescence-analog of the Erdős-Menger theorem

The edge-version of the directed Menger's theorem states that if $D = (V, A)$ is a finite digraph with distinct vertices $s, t$ then the maximum number of the edge-disjoint $s - t$ paths is equal to $\lambda_D(s, t) \stackrel{\text{def}}{=} \min\{\varrho_D(X) : t \in X \subseteq V \setminus \{s\}\}$. If $D$ is infinite but $\lambda_D(s, t)$ is still finite then Menger's theorem remains true, actually it is derivable from the finite version. Let $D$ be an infinite digraph and $\kappa = \lambda_D(s, t) \geq \aleph_0$. Erdős observed that the theorem above can be extended to this case easily in the form that there is a system $\{P_i\}_{i<\kappa}$ of edge-disjoint $s - t$ paths. He also realized that his generalization is too "rough" and conjectured that there is also a system $\{P_i\}_{i<\kappa}$ of edge-disjoint $s - t$ paths and a set $t \in X \subseteq V \setminus \{s\}$ such that $\mathsf{in}_D(X) \subseteq \bigcup_{i<\kappa} A_i$ and $|A_i \cap \mathsf{in}_D(X)| = 1$ for all $i < \kappa$. It was called the Erdős-Menger conjecture and after a sequence of partial results was decided affirmatively by R. Aharoni and E. Berger in [8].

An analog of Erdős' generalization above is true in the context of arborescences. Namely let $D$ be $\kappa$-edge-connected from the vertex $r$, where $\kappa \geq \aleph_0$. Then $D$ has $\kappa$ edge-disjoint spanning arborescences rooted at $r$. The proof is not too hard, the main idea is that one can build the desired spanning arborescences by an "almost greedy" tranfinite recursion taking care only of the property that after certain steps the pointset of the initial arborescences should be the same. In the spirit of the Erdős-Menger theorem we formulate a strengthening of the fact above.

**Conjecture 4.2.** *Let $D = (V, A)$ be a digraph and $\kappa \geq \aleph_0$ its edge-connectivity from $r$, then there is a system of edge-disjoint spanning arborescences $\mathcal{A}_i = (V, A_i)$ $(i < \kappa)$ of $D$ rooted at $r$ and $\varnothing \neq X \subseteq V \backslash \{r\}$ such that $\mathsf{in}_D(X) \subseteq \bigcup_{i<\kappa} A_i$ and $|A_i \cap \mathsf{in}_D(X)| = 1$ holds for all $i < \kappa$.*

Note that infiniteness of the cardinal $\kappa$ can not be omitted by [7].

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