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# Rigid graphs and an augmentation problem

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**Abstract**

A graph $G = (V, E)$ is called $(k, \ell)$-tight if $|E(X)| \leq k|X| - \ell$ for all $X \subseteq V$ with $|X| \geq 2$ and $|E| = k|V| - \ell$. A graph $G$ is called $(k, \ell)$-redundant if $G - e$ has a spanning $(k, \ell)$-tight subgraph for all $e \in E$. We consider the following augmentation problem. Given a graph $G = (V, E)$ that has a $(k, \ell)$-tight spanning subgraph, find a graph $H = (V, F)$ with minimum number of edges, such that $G + H$ is $(k, \ell)$-redundant.

In this paper, we give a polynomial algorithm for this augmentation problem for $k \geq \ell$. We also give a polynomial algorithm for the case where $G$ is a $(k, \ell)$-tight graph with $k < \ell \leq \frac{3}{2}k$. As $(k, \ell)$-tight graphs play an important role in rigidity theory, these algorithms can be used to make several types of bar-and-joint frameworks redundantly rigid by adding a smallest set of new bars.

## 1 Introduction

A graph $G = (V, E)$ is called $(k, \ell)$**-sparse** if $i_G(X) \leq k|X| - \ell$ for all $X \subseteq V$ with $|X| \geq 2$, where $i_G(X)$ denotes the number of edges of $G$ spanned by $X \subseteq V$. $(k, \ell)$-sparse graphs are usually defined for $\ell < 2k$, however here we will use a stronger assumption; we will assume that $\ell \leq \frac{3}{2}k$. A $(k, \ell)$-sparse graph is called $(k, \ell)$**-tight** if $|E| = k|V| - \ell$. A graph $G$ is called $(k, \ell)$**-rigid** if $G$ has a $(k, \ell)$-tight spanning subgraph. We will call an edge $e$ of $G$ a $(k, \ell)$**-redundant edge** if $G - e$ is $(k, \ell)$-rigid. A graph $G$ will be called a $(k, \ell)$**-redundant graph** if each edge of $G$ is $(k, \ell)$-redundant. Throughout this paper, $G = (V, E)$ is a loopless graph (that may contain parallel edges), $\ell$ is an integer and $k$ is a non-negative integer. We consider the following augmentation problem that we call here the **general (augmentation) problem**.

**Problem.** *Let $k$ and $\ell$ be integers with $k \geq 0$ and let $G = (V, E)$ be a loopless $(k, \ell)$-rigid graph. Find a graph $H = (V, F)$ on the same node set with minimum number of edges, such that $G + H = (V, E \cup F)$ is $(k, \ell)$-redundant.*

We call the special case of this problem, where the input graph $G$ is $(k, \ell)$-tight, the **reduced (augmentation) problem**.

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Sparsity properties are important in rigidity theory as they can be used in the characterization of many rigidity classes. For example, the generically rigid graphs in $\mathbb{R}^2$ are the $(2,3)$-rigid graphs by Laman's theorem [6] and the 'body-bar graph' induced by a given graph $G$ is generically rigid in $\mathbb{R}^d$ if and only if $G$ is a $\left(\binom{d+1}{2}, \binom{d+1}{2}\right)$-rigid graph by a theorem of Tay [8]. It is natural to ask how many new edges are needed to make our rigid graph redundantly rigid, that is, the augmented graph remains rigid if we omit an arbitrary edge of it. García and Tejel [3] showed that this is NP-hard for $(2,3)$-rigid graphs but can be solved polynomially for minimally rigid graphs, that is, when $G$ is $(2,3)$-tight. (The idea of this algorithm was also used later by Kohta et al. [5] to give a 2-approximate solution for the redundantly rigid 'Truss Topology Design' problem.) We also note that Frank and T. Király [2] gave a polynomial algorithm to augment a graph to a $(k,h)$-tree-connected graph using polyhedral techniques. (A graph $G=(V,E)$ is called **$(k,h)$-tree-connected** if $G-E'$ contains $k$ edge-disjoint spanning trees for every $E' \subseteq E$ with $|E'|=h$.) The famous result of Nash-Williams [7] states that the graphs that can be partitioned to $k$ edge-disjoint spanning trees are the $(k,k)$-tight graphs. Therefore, the algorithm of Frank and Király, with parameters $k \in \mathbb{Z}_+$ and $h=1$, can be used to give a polynomial algorithm for the general problem when $\ell = k$. This algorithm, together with Tay's result [8] shows that there is a polynomial algorithm that finds an edge set $F$ of minimum cardinality for a graph $G$ such that the body-bar graph induced by $G' = (V, E \cup F)$ is redundantly rigid. The algorithm, that will be presented here, will be a rather simple solution for these problems however it does not deal with the case of $h \geq 2$.

We will use the idea of García and Tejel [3] to give a polynomial algorithm that solves the reduced problem for $\ell \leq \frac{3}{2}k$. We will use this algorithm to give a polynomial algorithm that solves the general problem for $\ell \leq k$. On the other hand, [3] showed that the general problem is NP-hard for $k=2$ and $\ell=3$. This result will be extended for the case where $k$ is even and $\ell = \frac{3}{2}k$.

To obtain the solution for the general problem, we need more general concepts. Let $m : V \to \mathbb{Z}_+$ be a function where $\mathbb{Z}_+$ denotes the set of non-negative integers. For $X \subseteq V$, let $\widetilde{m}(X) := \sum_{v \in X} m(v)$. A graph $G=(V,E)$ is called **$(\mathbf{m},\ell)$-sparse** if $i_G(X) \leq \widetilde{m}(X) - \ell$ for all $X \subseteq V$ with $|X| \geq 2$. An $(\mathbf{m},\ell)$-sparse graph is called **$(\mathbf{m},\ell)$-tight** if $|E| = \widetilde{m}(V) - \ell$. A graph $G$ is called **$(\mathbf{m},\ell)$-rigid** if $G$ has an $(\mathbf{m},\ell)$-tight spanning subgraph. We will call an edge $e$ of $G$ an **$(\mathbf{m},\ell)$-redundant edge** if $G-e$ is $(\mathbf{m},\ell)$-rigid. A graph $G$ will be called an **$(\mathbf{m},\ell)$-redundant graph** if each edge of $G$ is $(\mathbf{m},\ell)$-redundant. Throughout this paper, we will assume that

$$m(v) \geq \ell \text{ for all } v \in V \text{ or } m \equiv k \text{ for a positive integer } k \text{ for which } k < \ell \leq \frac{3}{2}k. \tag{1}$$

Note that, when $m \equiv k$, an $(\mathbf{m},\ell)$-sparse/tight/rigid/redundant graph is $(k,\ell)$-sparse/tight/rigid/redundant, respectively.

We conclude the introduction by listing some notation used throughout this paper. $d_G(X)$ will denote the degree of a set $X \subseteq V$ and $d_G(X,Y)$ will denote the number of edges between $X-Y$ and $Y-X$. (When it is clear from the context, we omit the subscript $G$ from several notations.) If $G_1$ and $G_2$ are graphs, then $G_1 \subseteq G_2$ will denote that $G_1$ is a subgraph of $G_2$. For a graph $G$ and for a positive integer $c$, $cG$

will denote the graph that arises from $G$ by replacing each edge $e$ of $G$ by $c$ parallel copies of $e$.

# 2 Preliminaries

In this section, we list some well-known properties of $(\mathbf{m}, \ell)$-sparse graphs. We sketch their proofs for completeness. See [1, 9] for more details. It follows from the definition that an $(\mathbf{m}, \ell)$-tight subgraph of a $(\mathbf{m}, \ell)$-sparse graph is always an induced subgraph. Therefore, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ both are $(\mathbf{m}, \ell)$-tight subgraphs of an $(\mathbf{m}, \ell)$-sparse graph $G$, then $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is also a induced subgraph of $G$.

**Lemma 2.1.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-sparse graph, where* (1) *holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and let $G_i = (V_i, E_i)$ be $(\mathbf{m}, \ell)$-tight subgraphs of $G$ for $i = 1, 2, 3$.*

- *(a) If $\widetilde{m}(V_1 \cap V_2) \geq \ell$, then $G_1 \cup G_2$ and $G_1 \cap G_2$ are $(\mathbf{m}, \ell)$-tight graphs and there are no edges between $V_1 - V_2$ and $V_2 - V_1$. Otherwise, $G_1 \cup G_2$ is not $(\mathbf{m}, \ell)$-tight.*
- *(b) If $V_1 \cap V_2 = \{i_1\}$, $V_2 \cap V_3 = \{i_2\}$ and $V_3 \cap V_1 = \{i_3\}$ where $i_1, i_2, i_3$ are three different vertices of $V$, then $m \equiv k = \frac{2}{3}\ell$ and $G_1 \cup G_2 \cup G_3$ is $(k, \ell)$-tight.*

We note that the assumption of (a) holds always when $E_i \cap E_j \neq \emptyset$.

*Proof.* (a) The statements follow easily by the following. $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2| = \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell = \widetilde{m}(V_1 \cup V_2) - \ell + \widetilde{m}(V_1 \cap V_2) - \ell$.
(b) The statements follow easily by the following. $|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| = \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell + \widetilde{m}(V_3) - \ell = \widetilde{m}(V_1 \cup V_2 \cup V_3) + \widetilde{m}(\{i_1, i_2, i_3\}) - 3\ell$. Thus $3 \min_{v \in V} m(v) \leq \widetilde{m}(\{i_1, i_2, i_3\}) \leq 2\ell$. □

With the same idea as in Lemma 2.1(a), we get the following.

**Lemma 2.2.** *Let $\ell > k > 0$, let $G = (V, E)$ be a $(k, \ell)$-sparse graph and let $G_i = (V_i, E_i)$ be $(k, \ell)$-sparse subgraphs of $G$ for $i = 1, 2$. If $|V_1 \cap V_2| \leq 1$, then $G_1 \cup G_2$ cannot be a $(k, \ell)$-tight graph. Therefore, a $(k, \ell)$-tight graph is 2-connected for $\ell > k$.* □

**Lemma 2.3.** *If $\ell \geq 0$, then the minimum degree of a $(k, \ell)$-tight graph $G = (V, E)$ with $n \geq 3$ vertices is between $k$ and $2k - 1$.*

*Proof.* The upper bound follows from the fact that the average degree in $G$ is less than $2k$. To prove the lower bound, let $v$ be a node of minimum degree. Then $G - v$ has at most $k(n-1) - \ell$ edges thus the degree of $v$ is at least $k$ as $|E| = kn - \ell$. □

It is known that the edge sets of the $(\mathbf{m}, \ell)$-sparse subgraphs of a given graph form a matroid, called the **$(\mathbf{m}, \ell)$-sparsity matroid** or **count matroid**, when $m(u) + m(v) \geq \ell$ for each edge $uv$ (see [1, 9]). (However, this latter assumption follows by our assumptions on $m$ and $\ell$.) A circuit of this matroid is called an **$(\mathbf{m}, \ell)$-circuit**. Here, the notion of $(\mathbf{m}, \ell)$-circuit will be used for graphs and not only for edge sets. We summarize some properties of $(\mathbf{m}, \ell)$-circuits in the following lemma. The proof will be skipped as it directly follows by some matroid properties.

**Lemma 2.4.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph and $i, j \in V$.*

*(a) The graph $G + ij$ contains an unique $(\mathbf{m}, \ell)$-circuit, that will be denoted by $C^G_{(\mathbf{m},\ell)}(ij)$. Hence the subgraph of $G$ $T^G_{(\mathbf{m},\ell)}(ij) := C^G_{(\mathbf{m},\ell)}(ij) - ij$ is $(\mathbf{m}, \ell)$-tight.*

*(b) For every edge $e'$ of $C^G_{(\mathbf{m},\ell)}(ij)$, $G' = G+ij-e'$ is also $(\mathbf{m}, \ell)$-tight and the unique $(\mathbf{m}, \ell)$-circuit of $G'+e'$ is again $C^G_{(\mathbf{m},\ell)}(ij)$. Moreover, if $e'' \notin E(C^G_{(\mathbf{m},\ell)}(ij))$, then, $G' + ij - e''$ is not $(\mathbf{m}, \ell)$-tight.*

*(c) If $G' = (V', E')$ is an $(\mathbf{m}, \ell)$-tight subgraph of $G$ with $i, j \in V'$, then $T^G_{(\mathbf{m},\ell)}(ij) \subseteq G'$. Hence $T^G_{(\mathbf{m},\ell)}(ij) = \bigcap\{T_h : T_h$ an $(\mathbf{m}, \ell)$-tight subgraph of $G$ spanning both $i$ and $j\}$.* □

As it will be usually clear from the context, we will omit the subscript $(\mathbf{m}, \ell)$ and the superscript $G$ when we speak about $C^G_{(\mathbf{m},\ell)}(ij)$ or $T^G_{(\mathbf{m},\ell)}(ij)$. From now on, let $E(ij) := E(T(ij))$ and $V(ij) := V(T(ij))$. We will call an $(\mathbf{m}, \ell)$-tight subgraph $G'$ of $G$ **generated** if there are vertices $i, j$ such that $T(ij) = G'$; in this case, $i$ and $j$ will be called the **generators** of $G'$. We will use the appellation of **($(\mathbf{m}, \ell)$-)MGT subgraph** for the maximal generated $(\mathbf{m}, \ell)$-tight subgraphs of $G = (V, E)$ that are the generated subgraphs $T(ij)$ such that there is no $i', j' \in V$ with $T(ij) \subset T(i'j')$.

Let $R^G_{(\mathbf{m},\ell)}(i_1j_1, \dots, i_rj_r) = (V(i_1j_1, \dots, i_rj_r), E(i_1j_1, \dots, i_rj_r))$ denote the subgraph induced by the $(\mathbf{m}, \ell)$-redundant edges of $G = (V, E)$ in $G \cup \{i_1j_1, \dots, i_rj_r\}$ for $i_1, \dots, i_r, j_1, \dots, j_r \in V$. (Again we usually omit the subscript $(\mathbf{m}, \ell)$ and the superscript $G$ as it will be clear from the context.) Note that $R(ij) = T(ij)$ for any $i, j \in V$. The following lemma shows that if $G$ is $(\mathbf{m}, \ell)$-tight, then $R(i_1j_1, \dots, i_rj_r) = T(i_1j_1) \cup \dots \cup T(i_rj_r)$. Therefore, to make $G$ $(\mathbf{m}, \ell)$-redundant, we need the minimum number of generated $(\mathbf{m}, \ell)$-tight subgraphs of $G$ such that their edges cover all edges of $G$. We can assume that these generated $(\mathbf{m}, \ell)$-tight subgraphs are MGT subgraphs of $G$.

**Lemma 2.5.** *If $G$ is $(\mathbf{m}, \ell)$-tight, then $R(i_1j_1, \dots, i_rj_r) = T(i_1j_1) \cup \dots \cup T(i_rj_r)$.*

*Proof.* $R(i_1j_1) = T(i_1j_1)$ by definition hence $T(i_1j_1) \cup \dots \cup T(i_rj_r) \subseteq R(i_1j_1, \dots, i_rj_r)$. For the other direction, let $e \in E(i_1j_1, \dots, i_rj_r)$ be an arbitrary edge. Now, $G - e$ is $(\mathbf{m}, \ell)$-sparse and $|E - e| = \widetilde{m}(V) - \ell - 1$. $G \cup \{i_1j_1, \dots, i_rj_r\} - e$ is $(\mathbf{m}, \ell)$-rigid hence $E \cup \{i_1j_1, \dots, i_rj_r\} - e$ has a rank of $\widetilde{m}(V) - \ell$ in the $(\mathbf{m}, \ell)$-sparsity matroid. Thus there is an edge $f$ in $\{i_1j_1, \dots, i_rj_r\}$ for which $E - e + f$ is a basis of the $(\mathbf{m}, \ell)$-sparsity matroid. Since $E - e + f$ is independent in the $(\mathbf{m}, \ell)$-sparsity matroid, we must have $e \in T(f)$. □

The following lemma shows that every $(k, \ell)$-MGT subgraph induces at least 3 nodes. (A similar statement holds for the general $(\mathbf{m}, \ell)$ case, but we will not use it.)

**Lemma 2.6.** *Assume that $G = (V, E)$ is a $(k, \ell)$-tight graph with $k < \ell \leq \frac{3}{2}k$. If $T(ij)$ is $(k, \ell)$-MGT and $n \geq 4$, then $|V(ij)| \geq 3$.*

*Proof.* $(2k-\ell)K_n$ is always $(k,\ell)$-redundant for $n \geq \frac{k+1}{2k-\ell}+1$ (that holds for $n \geq 4$ by $\ell \leq \frac{3}{2}k$). Hence $R(E((2k-\ell)K_n)-E)=G$. For any $ij \in E((2k-\ell)K_n)-E$, $|V(ij)| \geq 3$ since $|V(ij)|=2$ if and only if there are $2k-\ell$ parallel edges between $i$ and $j$ in $G$. By Lemma 2.5, $G = R(E((2k-\ell)K_n)-E) = \bigcup_{ij \in E((2k-\ell)K_n)-E} T(ij)$. Hence the edges of $G$ can be covered by generated $(\mathbf{m},\ell)$-tight subgraphs with at least 3 nodes. Therefore, each MGT subgraph of $G$ must induce at least 3 nodes. □

# 3 Augmenting an $(\mathbf{m},\ell)$-tight graph to an $(\mathbf{m},\ell)$-redundant graph

Before we present the algorithm, we prove the following lemmata.

**Lemma 3.1.** *Let $G=(V,E)$ be an $(\mathbf{m},\ell)$-tight graph (where* (1) *holds for $m: V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$) with at least* 4 *vertices and let $T(ij)$ be an MGT subgraph of $G$. Then $\widetilde{m}(V(ij) \cap V(ij')) \geq \ell$ and $T(ij) \cup T(ij')$ is $(\mathbf{m},\ell)$-tight for any $j' \in V-i$.*

*Proof.* The lemma follows by Lemma 2.1(a) for the case where $m \geq \ell$. Assume now that $m \equiv k < \ell \leq \frac{3}{2}k$. Then the statement holds for $j' \in V(ij)$ because then $T(ij') \subseteq T(ij)$ by Lemma 2.4(c).

Suppose that $j' \notin V(ij)$ and suppose for a contradiction, that $|V(ij) \cap V(ij')| < 2$.

*Case 1:* If $|V(ij') \cap V(jj')| \geq 2$, then $T(ij') \cup T(jj')$ is a $(k,\ell)$-tight graph by Lemma 2.1(a), and as it contains $i$ and $j$, we have $T(ij) \subseteq T(ij') \cup T(jj')$ by Lemma 2.4(c). But as $E(ij) \cap E(ij') = \emptyset$, we have $T(ij) \subset T(jj')$ contradicting the maximality of $T(ij)$.

*Case 2:* If $|V(ij) \cap V(jj')| \geq 2$, then $T(ij) \cup T(jj')$ is a $(k,\ell)$-tight graph by Lemma 2.1(a), and as it contains $i$ and $j'$, we have $T(ij') \subseteq T(ij) \cup T(jj')$ by Lemma 2.4(c). But as $E(ij) \cap E(ij') = \emptyset$, we have $T(ij') \subset T(jj')$ and we are again at *Case 1.*

*Case 3:* Finally if we are neither in *Case 1* nor in *Case 2*, it is because $G_1 = T(ij)$, $G_2 = T(ij')$ and $G_3 = T(jj')$ satisfy the conditions of Lemma 2.1(b), hence $\ell = \frac{3}{2}k$ and $G_1 \cup G_2 \cup G_3$ is $(k,\ell)$-tight. Moreover, by the maximality of $G_1$ and $|V| \geq 4$, there is at least one additional vertex $i' \notin \{i,j\}$ in $G_1$ by Lemma 2.6. Now, $T(i'j') \subseteq G_1 \cup G_2 \cup G_3$ by Lemma 2.4(c). Since any path between $i'$ and $j'$ in $G_1 \cup G_2 \cup G_3$ must contain $i$ or $j$; and since a $(k,\ell)$-tight graph is 2-connected for $\ell > k$ by Lemma 2.2, $i$ and $j$ must be in $V(i'j')$. Hence $T(ij) \subset T(i'j')$ by Lemma 2.4(c), contradicting again the maximality of $T(ij)$.

As $E(ij) \cap E(ij') \neq \emptyset$, $k|V(ij) \cap V(ij')| \geq \ell$ and hence $T(ij) \cup T(ij')$ is $(k,\ell)$-tight by Lemma 2.1(a). □

**Lemma 3.2.** *Let $k < \ell \leq \frac{3}{2}k$ and let $G$ be a $(k,\ell)$-tight graph with at least* 4 *vertices and let $i$ be a node of degree between $k$ and $2k-1$. Then $E(ij) \cap E(ij') \neq \emptyset$ and $T(ij) \cup T(ij')$ is $(k,\ell)$-tight if $j,j' \in V-i$, $j \neq j'$ and* $\min(|V(ij)|,|V(ij')|) \geq 3$.

*Proof.* As the minimum degree in a $(k,\ell)$-tight graph with more then 3 vertices is at least $k$ by Lemma 2.3, $d_{T(ij)}(i) \geq k$ and $d_{T(ij')}(i) \geq k$. Thus there is at least one edge incident to $i$ that is in both $T(ij)$ and $T(ij')$ since $d_G(i) \leq 2k-1$.

As $E(ij) \cap E(ij') \neq \emptyset$, $T(ij) \cup T(ij')$ is $(k,\ell)$-tight by Lemma 2.1(a). □

## 3.1 Algorithm to find a small covering rooted at a single node

Now we show that the subroutine of the algorithm of García and Tejel works similarly for $(\mathbf{m},\ell)$-tight graphs as it worked for Laman graphs. This subroutine outputs a covering of the node set of $G$ with minimum number of sets of the form $V(i_1j)$ for a given node $i_1$ that we call the root of the covering.

**Algorithm 3.3.** INPUT: *$m: V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ holding* (1) *and an $(\mathbf{m},\ell)$-tight graph $G=(V,E)$ with a given node $i_1 \in V$ and a set $L \subseteq V$.*
OUTPUT: *A list $V'(i_1,L)=\{j_2,\dots,j_r\}$ of vertices such that $V=\bigcup_{s=2}^{r} V(i_1j_s)$. [If $L$ is $\emptyset$ then we will refer to this set as $V'(i_1)$.]*

1. *Initialize $V'(i_1,L)=\emptyset$. All vertices are unmarked. Mark $i_1$.*
2. *Explore all vertices $j \in L$ and after this all other vertices $j \in V-L$:*
   *If $j$ is unmarked do*
   *Calculate $T(i_1j)$;*
   *Mark all unmarked vertices in $T(i_1j)$;*
   *$V'(i_1,L) := [V'(i_1,L)-V(i_1j)]+j$.*

With the algorithm of [1, Section 13.5.4] $T(i_1j)$ can be calculated in polynomial time thus the following claim holds.

**Claim 3.4.** *The running time of Algorithm 3.3 is polynomial.* □

Next, we show how to use Algorithm 3.3 to cover its edge set with MGT subgraphs. The proofs will slightly differ for the case of $m \equiv k < \ell \le \frac{3}{2}k$ and for the case of $m \ge \ell$.

First we show that the subgraphs $T(i_1j_s)$ $(s=2,\dots,r)$ cover the edges of $G$ if $i_1$ is a node of minimum degree.

**Lemma 3.5.** *Let $G=(V,E)$ be an $(\mathbf{m},\ell)$-tight graph with at least* 4 *nodes (where* (1) *holds for $m: V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that $i_1$ has minimum degree in $G$. Then $E=\bigcup_{j_s \in V'(i_1)} E(i_1j_s)$ and, if $m \equiv k < \ell$, $|V(i_1j_s)| \ge 3$ for every $j_s \in V'(i_1)$.*

*Proof.* If $r=|V(i_1)|+1=2$, then $V(i_1j_2)=V$ thus $T(i_1j_2)=G$ that proves the lemma. From now on assume that $r \ge 3$.

Assume first that $m \ge \ell$. Then $\bigcup_{j_s \in V'(i_1)} T(i_1j_s)$ is $(\mathbf{m},\ell)$-tight by Lemma 2.1(a) as $i_1 \in V(i_1j_s) \cap V(i_1j_{s'})$ for any $j_s, j_{s'} \in V'(i_1)$. As $G$ is $(\mathbf{m},\ell)$-tight and $\bigcup_{j_s \in V'(i_1)} T(i_1j_s)$ is an induced $(\mathbf{m},\ell)$-tight subgraph of $G$, these two graphs must coincide.

Next assume that $m \equiv k < \ell$. As the minimum degree of $G$ is between $k$ and $2k-1$ by Lemma 2.3, $E(i_1j_s) \cap E(i_1j_{s'}) \ne \emptyset$ and $T(i_1j_s) \cup T(i_1j_{s'})$ is $(k,\ell)$-tight if $|V(i_1j_s)| \ge 3$ and $|V(i_1j_{s'})| \ge 3$ for $j_s, j_{s'} \in V'(i_1)$ by Lemma 3.2. As $d(i_1) \le 2k-1$ and $\ell \le \frac{3}{2}k$, $i_1$ has at most 3 neighbors $v'$ with $d(i_1,v')=2k-\ell$. Thus there is at least one node $j \in V-i_1$ such that $d_G(i_1,j) < 2k-\ell$. (Note that in the case of

$n = 4$, if $d_G(i_1, v') = 2k - \ell$ for every $v' \in V - i_1$, then by the minimality of the degree of $i_1$ and $\ell \le \frac{3}{2}k$, $|E| \ge \frac{3(2k-\ell)4}{2} \ge \frac{9}{2}k - \ell > 4k - \ell$ thus $G$ is not $(k, \ell)$-sparse.) Therefore, there is at least one $j_s \in V'(i_1)$ such that $T(i_1 j_s)$ has at least 3 vertices. Let $T := \bigcup\{T(i_1 j_s) : j_s \in V'(i_1), |V(i_1 j_s)| \ge 3\}$. This is a $(k, \ell)$-tight graph as we have seen before.

Since $T$ is $(k, \ell)$-tight and $|V(T)| \ge 3$, $d_T(v) \ge k$ by Lemma 2.3. As $d_G(i_1, v') = 2k - \ell$ for $v' \in V - V(T)$, $d_G(i_1) \ge k + |V - V(T)|(2k - \ell) \ge k + |V - V(T)|\frac{k}{2}$ by $\ell \le \frac{3}{2}k$. Therefore, $|V - V(T)| \le 1$ by Lemma 2.3 as the degree of $i_1$ is minimum. Moreover, if $V - V(T) = \{v'\}$ then $d_G(v') \ge d_G(i_1) \ge 3k - \ell$. Thus if $|V - V(T)| = 1$ then $kn - \ell = |E| \ge (k + 2k - \ell) + |E(T)| = (2k - \ell) + k|V(T)| - \ell = kn - \ell + (k - \ell) > kn - \ell + 1$, a contradiction. Hence $|V - V(T)| = 0$ and $T$ is a $(k, \ell)$-tight subgraph of $G$ covering $V$. Therefore, $G = T = \bigcup_{j_s \in V'(i_1)} T(i_1 j_s)$, as we claimed. □

From the proof of the previous lemma we also get the following.

**Corollary 3.6.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that $i_1$ has minimum degree in $G$. Then $T(i_1 j) \cup T(i_1 j')$ is $(\mathbf{m}, \ell)$-tight for every $j, j' \in V(i_1)$.* □

We say that a set $J \subseteq V$ is an **MGT-generator of** $G$ if $T(jj')$ is MGT for every two distinct elements $j, j' \in J$; $j'' \notin T(jj')$ for every three distinct elements $j, j', j'' \in J$; and for every MGT subgraph $T$ of $G$ there is a pair $j, j' \in J$ for which $T = T(jj')$. The following lemma gives a constructive proof for the existence of MGT-generators.

**Lemma 3.7.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that $i_1$ has minimum degree in $G$. Then $V'(i_1)$ or $V'(i_1) \cup \{i_1\}$ is an MGT-generator of $G$.*

*Proof.* If $r = 2$, then $G = T(i_1 j_2)$ is MGT hence $\{i_1, j_2\}$ is an MGT-generator of $G$. Assume that $r \ge 3$.

By the definition of $V'(i_1)$, $j_s \notin V(i_1 j_{s'})$ for $s, s' \in \{2, \ldots, r\}$, $s \ne s'$. As by Lemma 3.5 $E = \bigcup_{s=2}^{r} E(i_1 j_s)$, all the edges incident to $j_s$ are in $T(i_1 j_s)$.

Let $T(vv')$ be an MGT subgraph of $G$. Let $j \in V'(i_1)$ such that $v \in V(i_1 j)$. If $v' \in V(i_1 j)$, then $T(vv') \subseteq T(i_1 j)$ hence $T(vv') = T(i_1 j)$ by maximality. Thus we can assume that $v' \in V(i_1 j')$ for $j' \in V'(i_1)$, $j' \ne j$. $E(i_1 j) \cap E(jj') \ne \emptyset$ since $T(i_1 j)$ spans all edges incident to $j$. Thus $T(i_1 j) \cup T(jj')$ is $(\mathbf{m}, \ell)$-tight by Lemma 2.1(a). As $i_1, j' \in V(i_1 j) \cup V(jj')$, $T(i_1 j') \subset T(i_1 j) \cup T(jj')$. Since $v' \in V(i_1 j')$ but $v' \notin V(i_1 j)$, $v' \in V(jj')$. Similarly, $v \in V(jj')$ and hence $T(vv') \subseteq T(jj')$. Thus $T(vv') = T(jj')$ by the maximality.

$T(i_1 j) \cup T(i_1 j')$ is $(\mathbf{m}, \ell)$-tight by Corollary 3.6. Hence $T(jj') \subseteq T(i_1 j) \cup T(i_1 j')$. Thus $j'' \notin V(vv')$ for $j'' \in V'(i_1)$, $j'' \ne j, j'$.

Therefore, each MGT subgraph of $G$, that covers at least 2 elements of $V'(i_1)$, covers exactly 2 of them and these 2 elements are its generators. As any two node of $G$ is covered by an MGT subgraph of $G$, we get that $T(jj')$ is MGT for every $j, j' \in V'(i_1)$. Moreover, we have seen that every MGT subgraph of $G$ has the form of $T(jj')$ for some $j, j' \in V'(i_1) + i_1$.

Now, we turn to show that if $T(i_1 j)$ is MGT for a $j \in V'(i_1)$, then $T(i_1 j')$ is MGT for every $j' \in V'(i_1)$. Assume for a contradiction that $T(i_1 j')$ is not MGT for a $j' \in V'(i_1)$. Then $T(i_1 j') \subseteq T(j' j'')$ for some $j'' \in V'(i_1)$ since all (possible) MGT subgraph of $G$ covering $j'$ has the form of $T(j' j'')$ with $j'' \in V'(i_1) + i_1$. By Lemma 3.1, we have $T(j' j'') \subseteq T(jj') \cup T(jj'')$ since these latter two graphs are MGT or $j = j''$.

Now $i_1 \in V(j' j'') \subseteq V(jj') \cup V(jj'')$ hence $i_1 \in V(jj^*)$ where $j^* = j'$ or $j''$ (but $j^* \neq j$). Therefore, $T(i_1 j) \subseteq T(jj^*)$. But by the construction of $V'(i_1)$, $j^* \notin V(i_1 j)$ that contradicts the maximality of $T(i_1 j)$. Moreover, this argument show that if $i_1 \in V(jj^*)$ for $j, j^* \in V'(i_1)$, then $T(i_1 j)$ is not an MGT subgraph of $G$. □

Now we have the following two corollaries.

**Corollary 3.8.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that $i_1$ has minimum degree in $G$ and $d_G(i_1) = m(i_1)$. Then $V'(i_1) + i_1$ is an MGT-generator of $G$.*

*Proof.* As $d_G(i_1) = m(i_1)$, if $T$ is an $(\mathbf{m}, \ell)$-tight subgraph of $G$ with $i_1 \in V(T)$ and $|V(T)| \geq 3$, $T - i_1$ is also $(\mathbf{m}, \ell)$-tight. Thus $i_1$ is the generator of all MGT subgraphs containing $i_1$ and we are done by Lemma 3.7. □

If Corollary 3.8 cannot be applied, then we will determine whether $i_1$ is needed to MGT subgraphs with another run of Algorithm 3.3.

**Corollary 3.9.** *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that $i_1$ has minimum degree in $G$. Let $j_2 \in V'(i_1)$ and let $V'' = V'(j_2, V'(i_1) - j_2 + i_1)$. Then $V'' + j_2$ is an MGT-generator of $G$. Moreover, $G = \bigcup_{j \in V''} T(j_2 j)$.*

*Proof.* By Lemma 3.7 and by the construction of $V''$, $V'' = V'(i_1) - j_2$ or $V'' = (V'(i_1) - j_2) + i_1$ and we have the first part of the corollary. But then we have $G = \bigcup_{j \in V''} T(j_2 j)$ by Lemma 3.1 and by the construction of $V''$. □

## 3.2 Algorithm to find the minimum covering of an $(\mathbf{m}, \ell)$-tight graph

The following two properties of MGT subgraphs will be useful to reduce the number of MGT subgraphs covering $G$.

**Lemma 3.10.** *Let $G_1 := T(i_1 j_1) = (V_1, E_1)$ and $G_2 := T(i_2 j_2) = (V_2, E_2)$ be two MGT subgraphs of an $(\mathbf{m}, \ell)$-tight graph $G$.*

*(a) Suppose that $m \equiv k < \ell$ and $E_1 \cap E_2 = \emptyset$. Then $E(i_1 i_2) \cap E(j_1 j_2) \neq \emptyset$ and similarly $E(i_1 j_2) \cap E(j_1 i_2) \neq \emptyset$.*

*(b) Suppose that $m \geq \ell > 0$ and $V_1 \cap V_2 = \emptyset$. Then $V(i_1 i_2) \cap V(j_1 j_2) \neq \emptyset$ and similarly $V(i_1 j_2) \cap V(j_1 i_2) \neq \emptyset$.*

*Moreover in both cases,* $G_1 \cup G_2 \subset T(i_1i_2) \cup T(j_1j_2)$ *and* $T(i_1i_2) \cup T(j_1j_2) = T(i_1j_2) \cup T(j_1i_2)$.

*Proof. Case (a):* As $E_1 \cap E_2 = \emptyset$ and $G_1, G_2$ are both MGT, $i_1, i_2, j_1, j_2$ are 4 different vertices by Lemma 3.1.

Let us choose one of the $(k,\ell)$-tight graphs $T(i_1i_2), T(j_1j_2), T(i_1j_2), T(j_1i_2)$, say $T(i_1i_2)$. By Lemma 3.1, $E' := E(i_1i_2) \cap E_1 \neq \emptyset$ and $E'' := E(i_1i_2) \cap E_2 \neq \emptyset$. Since $E_1 \cap E_2 = \emptyset$, $E' \cap E'' = \emptyset$. Thus $E' \cup E''$ cannot be the edge set of a $(k,\ell)$-tight graph by Lemma 2.1(a) and hence $F(i_1i_2) := E(i_1i_2) - (E' \cup E'') \neq \emptyset$. Similarly, we can define the non-empty edge sets $F(j_1j_2), F(i_1j_2), F(j_1i_2)$.

By Lemma 3.1, $T := G_1 \cup T(i_1i_2) \cup G_2$ is a $(k,\ell)$-tight graph with the edge set $E_1 \cup E_2 \cup F(i_1i_2)$. As $j_1, j_2 \in V(T)$, $T(j_1j_2) \subseteq T$ and hence $F(j_1j_2) \subseteq F(i_1i_2)$. With telling the same reasoning for the other pairs, we get $F(i_1i_2) = F(j_1j_2) = F(i_1j_2) = F(j_1i_2) =: F$.

Therefore, $E(i_1i_2) \cap E(j_1j_2) \neq \emptyset$ and hence $T(i_1i_2) \cup T(j_1j_2)$ is $(k,\ell)$-tight containing all the 4 vertices $i_1, i_2, j_1, j_2$. Thus $G_1, G_2 \subseteq T(i_1i_2) \cup T(j_1j_2)$. Moreover, $(E(i_1i_2) \cup E(j_1j_2)) - (E_1 \cup E_2) = F \neq \emptyset$. The same properties can be proved for $T(i_1j_2) \cup T(j_1i_2)$.

*Case (b):* As $V_1 \cap V_2 = \emptyset$ and $G_1$ and $G_2$ are MGT graphs, $i_1, i_2, j_1, j_2$ are 4 different vertices.

Let us choose one of the $(\mathbf{m},\ell)$-tight graphs $T(i_1i_2), T(j_1j_2), T(i_1j_2), T(j_1i_2)$, say $T(i_1i_2)$. Now, $i_1 \in V' := V(i_1i_2) \cap V_1 \neq \emptyset$ and $i_2 \in V'' := V(i_1i_2) \cap V_2 \neq \emptyset$. Moreover, both $V'$ and $V''$ span an $(\mathbf{m},\ell)$-tight subgraph of $G$. Since $V_1 \cap V_2 = \emptyset$ , $V' \cap V'' = \emptyset$. Thus $V' \cup V''$ cannot span an $(\mathbf{m},\ell)$-tight graph by Lemma 2.1(a) (as $\ell > 0$) and hence $W(i_1i_2) := V(i_1i_2) - (V' \cup V'') \neq \emptyset$. Similarly, we can define the non-empty node sets $W(j_1j_2), W(i_1j_2), W(j_1i_2)$.

By Lemma 2.1(a), $T := G_1 \cup T(i_1i_2) \cup G_2$ is an $(\mathbf{m},\ell)$-tight graph with the node set $V_1 \cup V_2 \cup W(i_1i_2)$. As $j_1, j_2 \in V(T)$, $T(j_1j_2) \subseteq T$ and hence $W(j_1j_2) \subseteq W(i_1i_2)$. With telling the same reasoning for the other pairs, we get $W(i_1i_2) = W(j_1j_2) = W(i_1j_2) = W(j_1i_2) =: W$.

Therefore, $V(i_1i_2) \cap V(j_1j_2) \neq \emptyset$ and hence $T(i_1i_2) \cup T(j_1j_2)$ is $(\mathbf{m},\ell)$-tight containing all the 4 vertices $i_1, i_2, j_1, j_2$. Thus $G_1, G_2 \subseteq T(i_1i_2) \cup T(j_1j_2)$. Moreover, $(V(i_1i_2) \cup V(j_1j_2)) - (V_1 \cup V_2) = W \neq \emptyset$. The same properties can be proved for $T(i_1j_2) \cup T(j_1i_2)$. □

This general lemma will be applied in the following way.

**Corollary 3.11.** *Let* $T(ij_1)$, $T(ij_2)$ *and* $T(ij_3)$ *be three different MGT subgraphs of the* $(\mathbf{m},\ell)$*-tight graph* $G$ *(where* (1) *holds for* $m : V \to \mathbb{Z}_+$ *and* $\ell \in \mathbb{Z}$*) and let* $T$ *be the* $(\mathbf{m},\ell)$*-tight graph* $T(ij_1) \cup T(ij_2) \cup T(ij_3)$. *Then at least two of the three graphs* $T(ij_1) \cup T(j_2j_3)$, $T(ij_2) \cup T(j_1j_3)$ *and* $T(ij_3) \cup T(j_1j_2)$ *coincide with* $T$.

Note that $T$ is tight by Lemma 3.1 in the case of $m \equiv k < \ell \leq \frac{3}{2}k$ and by Lemma 2.1(a) in the case of $m \geq \ell$.

*Proof.* First note that if $\ell \leq 0$, then all the three graphs coincide with $T$ by Lemmata 2.1(a) and 2.4(c). Let us assume now $\ell > 0$.

Note that if $E(ij_{s_1}) \cap E(j_{s_2}j_{s_3}) \neq \emptyset$ and $m \equiv k < \ell$ or if $V(ij_{s_1}) \cap V(j_{s_2}j_{s_3}) \neq \emptyset$ and $m \geq \ell > 0$, then $T(ij_{s_1}) \cup T(j_{s_2}j_{s_3}) \subseteq T$ is $(\mathbf{m}, \ell)$-tight for every $s_1, s_2, s_3$ with $\{s_1, s_2, s_3\} = \{1, 2, 3\}$ by Lemma 2.1(a). Hence $T = T(ij_1) \cup T(ij_2) \cup T(ij_3) \subseteq T(ij_{s_1}) \cup T(j_{s_2}j_{s_3})$ by Lemma 2.4(c) as $i, j_1, j_2, j_3 \in V(ij_{s_1}) \cup V(j_{s_2}j_{s_3})$. Moreover, if for a triple $\{s_1, s_2, s_3\} = \{1, 2, 3\}$, $E(ij_{s_1}) \cap E(j_{s_2}j_{s_3}) = \emptyset$ and $m \equiv k < \ell$ or $V(ij_{s_1}) \cap V(j_{s_2}j_{s_3}) = \emptyset$ and $m \geq \ell$, then we can apply Lemma 3.10 and we get that the other two intersections are non-empty. □

Now we are ready to prove that the following algorithm (that extends the algorithm of García and Tejel [3] for $(\mathbf{m}, \ell)$-tight graphs) gives the minimum covering of an $(\mathbf{m}, \ell)$-tight graph with its MGT subgraphs.

**Algorithm 3.12.** Input: *$m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ with* (1) *and an $(\mathbf{m}, \ell)$-tight graph $G = (V, E)$ with at least 4 vertices.* Output: *A list of edges $F$ for which $G + F$ is $(\mathbf{m}, \ell)$-redundant.*

***1.*** *Search for a node with minimum degree in $G$. Let $v$ be a vertex of minimum degree.*

*If $d_G(v) = m(v)$ then let $i_1 := v$, $V' := \emptyset$ and go to step 3.*

*Otherwise, let $i_1 := v$.*

***2.*** *Run Algorithm 3.3 with inputs $m, \ell, G, i_1$ and $L := \emptyset$. Let the output $V'(i_1) = \{j_2, \ldots, j_r\}$.*

*If $r = 2$, then RETURN $F := \{i_1 j_2\}$.*

*Else let $L := V'(i_1) - j_2 + i_1$ and let $i_1 := j_2$.*

***3.*** *Run Algorithm 3.3 with inputs $m, \ell, G, i_1$ and $L$. Let the output be $V'(i_1, L) = \{i_2, \ldots, i_h\}$.*

*If $h = 2$ then RETURN $F := \{i_1 i_2\}$.*

*Else, let $V' := V'(i_1, L) + i_1 = \{i_1, \ldots, i_h\}$ and let $F = \emptyset$.*

***4.*** *While $h \geq 4$ do:*

*Calculate $T(i_1 i_{h-2})$ and $T(i_{h-1} i_h)$.*

*If $\widetilde{m}(V(i_1 i_{h-2}) \cap V(i_{h-1} i_h)) \geq \ell$, then*

*$F := F + i_{h-1} i_h$.*

*Else*

*$F := F + i_{h-2} i_h$,*

*$i_{h-2} := i_{h-1}$.*

*$h := h - 2$.*

***5.*** *Final step.*

*If $h = 2$, then $F := F + i_1 i_2$.*

*If $h = 3$, then $F := F \cup \{i_1 i_2, i_1 i_3\}$.*

*RETURN $F$.*

**Theorem 3.13.** *There is a polynomial time algorithm to obtain a set of edges $F = \{e_1, \dots e_t\}$ of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ with* (1) *and of an $(\mathbf{m},\ell)$-tight graph $G$, such that $G = T(e_1) \cup \dots \cup T(e_t)$.*

*Proof.* We will prove that Algorithm 3.12 calculates an optimal covering of $G$ and it runs in polynomial time. If the algorithm finishes in step 2 or 3, then it is because $G = T(e_1)$ thus we have found the minimum covering.

Otherwise, Corollaries 3.8 and 3.9 show that any MGT subgraph of $G$ covering two different $i, i' \in V'' = \{i_1, \dots, i_h\}$ coincides with $T(ii')$ and does not contain any other $i'' \in V''$. Therefore, at least $\lceil \frac{h}{2} \rceil$ MGT subgraphs are needed to cover the node set $V''$.

By Lemma 3.5 and Corollary 3.9 $G = \bigcup_{s=2}^{h} T(i_1 i_s)$. The construction of $E'$, Lemma 2.1(a) and Corollary 3.11 provides that after each iteration in step 4 the set of edges covered by $\bigcup_{s=2}^{h} T(i_1 i_s) \cup \bigcup_{e \in F} T(e)$ does not reduce. Therefore, $G = \bigcup_{e \in F} T(e)$ at the end of Algorithm 3.12. (Note that for $\ell \leq 0$ the algorithm can be simplified in this point by taking an arbitrary perfect matching on $L$ (or a matching that covers all but one node $v$ and an additional edge covering $v$) that follows from the proof of Corollary 3.11.)

We have seen that Algorithm 3.3 is polynomial. Thus the running time of steps 1-3 is polynomial. The running time of step 4 is also polynomial since it has $O(n)$ iterations and in each iteration we need to calculate 2 generated $(\mathbf{m},\ell)$-tight graphs. Finally, the running time of step 5 is $O(1)$. □

# 4 Augmenting an $(\mathbf{m},\ell)$-rigid graph to an $(\mathbf{m},\ell)$-redundant graph for $m \geq \ell$

Throughout this section, $R = (V, \bar{E})$ will denote an $(\mathbf{m},\ell)$-rigidgraph and $G = (V, E)$ will denote an $(\mathbf{m},\ell)$-tight spanning subgraph of $R$. Obviously, every edge in $\bar{E} - E$ is $(\mathbf{m},\ell)$-redundant in $R$. By Lemma 2.5, the $(\mathbf{m},\ell)$-redundant edges of $G$ in $R$ are the edges of $R^G(\bar{E} - E) = \bigcup_{uv \in \bar{E} - E} T^G(uv)$. As we have solved the augmentation problem for $(\mathbf{m},\ell)$-tight graphs, we assume that $\bar{E} - E \neq \emptyset$.

The idea of our proof comes from Jackson and Jordán [4] where the authors proved that the $(k,k)$-redundant edges of a $(k,k)$-rigid (that is, a $k$-tree-connected) graph $R$ form induced subgraphs of $R$ with disjoint node sets.

We divide the problem into two cases depending on whether $\ell < 0$ or $\ell \geq 0$.

**The case of $\ell \geq 0$**

First, let us consider the case where $m \geq \ell \geq 0$. In this case, the $(\mathbf{m},\ell)$-redundant edges of $G$ form some vertex disjoint $(\mathbf{m},\ell)$-tight induced subgraphs of $G$ by Lemma 2.1(a). By shrinking each of these subgraphs $T_i$'s to a single node and by defining $m'$ to be $\ell$ on the shrunken nodes and to be $m(v)$ on each non-shrunken node $v$, we get the shrunken graph $G' = (V', E')$. We claim the following.

**Proposition 4.1.** *$G'$ is $(\mathbf{m}',\ell)$-tight. Moreover, the pre-image of any $(\mathbf{m}',\ell)$-tight subgraph of $G'$ is $(\mathbf{m},\ell)$-tight and the shrunken image of an $(\mathbf{m},\ell)$-tight subgraph of $G$ is $(\mathbf{m}',\ell)$-tight.*

*Proof.* $G'$ is $(\mathbf{m}',\ell)$-tight as the number of $G'$-edges in $X' \subseteq V'$ equals the number of $G$-edges in its pre-image $X$ (that is, at most $\widetilde{m}(X) - \ell$) minus the number of edges in the shrunken components $T_i$'s of $X$ (that is, $\sum(\widetilde{m}(V(T_i)) - \ell)$).

Now let $T'$ be an $(\mathbf{m}',\ell)$-tight subgraph of $G'$, that is, $widetildem'(V(T')) = \widetilde{m}(V(T') - S) + \ell|V(T') \cap S| - \ell = |E(T')|$ where $S$ denotes the set of shrunken nodes in $V'$. Let $T$ be the pre-image of $T'$. Then $|E(T)| = |E(T')| + \sum^* |E(T_i)|$ where the sum is on the shrunken components with image in $V(T')$. Therefore, $|E(T)| = \widetilde{m}(V(T') - S) + \ell|V(T') \cap S| - \ell + \sum^*(\widetilde{m}(V(T_i)) - \ell) = \widetilde{m}(V(T)) - \ell$, that is, $T$ is $(\mathbf{m},\ell)$-tightas the $(\mathbf{m},\ell)$-sparsity follows by the $(\mathbf{m},\ell)$-sparsity of $G$.

For the last statement let $T$ be an $(\mathbf{m},\ell)$-tight subgraph of $G$. As if we take the union of $T$ with the shrunken $T_i$ components whose node set is intersected by $V(T)$, we get another $(\mathbf{m},\ell)$-tight subgraph of $G$ by Lemma 2.1(a) and hence its shrunken image, that coincides with the shrunken image of $T$, is $(\mathbf{m}',\ell)$-tightsimilarly as $G'$. □

Therefore, a covering of $G$ with $(\mathbf{m},\ell)$-tight subgraphs gives a covering of $G'$ with $(\mathbf{m}',\ell)$-tight subgraphs. Hence the minimum number of edges that we need to make $G$ $(\mathbf{m},\ell)$-redundant is at least the minimum number of edges that we need to make $G'$ $(\mathbf{m}',\ell)$-redundant. The following statement shows that these two values are equal.

**Proposition 4.2.** *Let $F'$ denote an edge set of minimum cardinality on $V'$ for which $G' \cup F'$ is $(\mathbf{m}',\ell)$-redundant. Let $F$ be an arbitrary pre-image of $F'$ with the same cardinality, that is, we get $F'$ form $F$ by shrinking the $(\mathbf{m},\ell)$-tight subgraphs of redundant edges of $G$. Then $R \cup F$ is $(\mathbf{m},\ell)$-redundant.*

*Proof.* The shrunken image of $T^G_{(\mathbf{m},\ell)}(uv)$ is an $(\mathbf{m}',\ell)$-tight subgraph of $G'$ that spans the image $u'$ and $v'$ of $u$ and $v$ both by Proposition 4.1. Thus it is a subgraph of $T^{G'}_{(\mathbf{m}',\ell)}(u'v')$ by Lemma 2.4(c). Since the image of each non-$(\mathbf{m},\ell)$-redundant edge of $R$ is in $G'$ and the subgraphs $\{T^{G'}_{(\mathbf{m}',\ell)}(u'v') : u'v' \in F'\}$ cover the edge set of $G'$, the subgraphs $\{T^G_{(\mathbf{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\mathbf{m},\ell)$-redundant edge of $R$. Hence $R \cup F$ is $(\mathbf{m},\ell)$-redundant. □

We have reduced the problem of augmenting an $(\mathbf{m},\ell)$-rigid graph to an $(\mathbf{m},\ell)$-redundant graph to the problem of augmenting an $(\mathbf{m}',\ell)$-tight graph to an $(\mathbf{m}',\ell)$-redundant graph that we can solve with our previous algorithm as $m' \geq \ell$ holds obviously. We have got the following theorem.

**Theorem 4.3.** *There is a polynomial time algorithm to obtain a set of edges $F$ of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}_+$ with $m \geq \ell$ and of an $(\mathbf{m},\ell)$-rigid graph $R$, such that $R \cup F$ is $(\mathbf{m},\ell)$-redundant.* □

### The case of $\ell < 0$

Now, we consider the case where $\ell < 0$. In this case, the $(\mathbf{m},\ell)$-redundant edges of $G$ form one $(\mathbf{m},\ell)$-tight induced subgraph $T$ of $G$ by Lemma 2.1(a). By shrinking $T$ to a single node and by defining $m'$ to be 0 on the shrunken node and to be $m(v)$ on each non-shrunken node $v$, we get the shrunken graph $G' = (V', E')$. If $X \subseteq V$ is disjoint from $V(T)$, then $i_G(X) \leq \widetilde{m}(X)$ since $i_G(X) + \widetilde{m}(V(T)) - \ell = i_G(X) + i_G(V(T)) \leq i_G(X \cup V(T)) \leq \widetilde{m}(X) + \widetilde{m}(V(T)) - \ell$. Hence if $X \subseteq V'$ does not contain the shrunken node, then $i_{G'}(X) \leq \widetilde{m'}(X)$. Moreover, the number of $G'$-edges in $X' \subseteq V'$ containing the shrunken node equals to the number of $G$-edges in its pre-image $X$ (that is at most $\widetilde{m}(X) - \ell$) minus the number of edges in $T$ (that is $\widetilde{m}(V(T)) - \ell$). Therefore, $G'$ is $(\mathbf{m}', 0)$-tight. Like in Proposition 4.1, the image of an $(\mathbf{m},\ell)$-tight subgraph of $G$ is $(\mathbf{m}', 0)$-tight(as if we take its union with the $T$, we get another $(\mathbf{m},\ell)$-tight subgraph of $G$ by Lemma 2.1(a)). Moreover, by a similar argument, the union of the pre-image of any $(\mathbf{m}', 0)$-tight subgraph of $G'$ and $T$ is $(\mathbf{m},\ell)$-tight. (Note that an $(\mathbf{m},\ell)$-tight subgraph of $G$ must intersect $T$ in this case.) Therefore, a covering of $G$ with $(\mathbf{m},\ell)$-tight subgraphs gives a covering of $G'$ with $(\mathbf{m}', 0)$-tight subgraphs. Hence the minimum number of edges that we need to make $G$ $(\mathbf{m},\ell)$-redundant is at least the minimum number of edges that we need to make $G'$ $(\mathbf{m}', 0)$-redundant. The following statement shows that these two numbers are equal.

**Proposition 4.4.** *Let $F'$ denote an edge set of minimum cardinality on $V'$ for which $G' \cup F'$ is $(\mathbf{m}', 0)$-redundant. Let $F$ be an arbitrary pre-image of $F'$ with the same cardinality, that is, we get $F'$ form $F$ by shrinking $T$. Then $R \cup F$ is $(\mathbf{m},\ell)$-redundant.*

*Proof.* The shrunken image of $T^G_{(\mathbf{m},\ell)}(uv)$ is an $(\mathbf{m}', 0)$-tight subgraph of $G'$ that spans the image $u'$ and $v'$ of $u$ and $v$ both. Thus it is a subgraph of $T^{G'}_{(\mathbf{m}',0)}(u'v')$ by Lemma 2.4(c). Since the image of each non-$(\mathbf{m},\ell)$-redundant edge of $R$ is in $G'$ and the subgraphs $\{T^{G'}_{(\mathbf{m}',0)}(u'v') : u'v' \in F'\}$ cover the edge set of $G'$, the subgraphs $\{T^G_{(\mathbf{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\mathbf{m},\ell)$-redundant edge of $R$. Hence $R \cup F$ is $(\mathbf{m},\ell)$-redundant. □

We have reduced the problem of augmenting an $(\mathbf{m},\ell)$-rigid graph to an $(\mathbf{m},\ell)$-redundant graph to the problem of augmenting an $(\mathbf{m}', 0)$-tight graph to an $(\mathbf{m}', 0)$-redundant graph that we can solve with our previous algorithm as $m' \geq 0$ holds obviously. We have got the following theorem.

**Theorem 4.5.** *There is a polynomial time algorithm to obtain a set of edges $F$ of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}_-$ and of an $(\mathbf{m},\ell)$-rigid graph $R$, such that $R \cup F$ is $(\mathbf{m},\ell)$-redundant.* □

# 5 Concluding remarks

## Further extensions

Let $G$ be a graph such that $cG$ is $(\mathbf{m},\ell)$-tight (for some $m$ and $\ell$ which satisfy (1)). $R(i_1j_1,\dots,i_cj_c) = T(i_1j_1)\cup\dots\cup T(i_cj_c)$ by Lemma 2.5 hence $R(ij,\dots,ij) = T(ij)\cup\dots\cup T(ij) = T(ij)$. Thus if we get $G'$ by adding some edges to $G$ we get the same $(\mathbf{m},\ell)$-redundant edges in $cG'$ as if we add just one (and not $c$) copy of these edges to $cG$. Hence our previous algorithms can be used to prove the following corollaries.

**Corollary 5.1.** *There is a polynomial time algorithm to obtain a set of edges $F$ of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with (1) and of a graph $G$ for which $cG$ is $(\mathbf{m},\ell)$-tight, such that $cG \cup F$ is $(\mathbf{m},\ell)$-redundant.* □

**Corollary 5.2.** *There is a polynomial time algorithm to obtain a set of edges $F$ of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with $m \geq \ell$ and of a graph $G$ for which $cG$ is $(\mathbf{m},\ell)$-rigid, such that $cG \cup F$ is $(\mathbf{m},\ell)$-redundant.* □

## An NP-hardness result

Finally, we show that the general problem is NP-hard in the case where $k$ is even and $\ell = \frac{3}{2}k$.

**Theorem 5.3.** *Let $k$ be a fixed positive even number. Then the general problem for $(k, \frac{3}{2}k)$-rigid graphs cannot be solved with a polynomial algorithm unless P=NP.*

*Proof.* García and Tejel [3] proved the statement of this theorem for $k = 2$. Now let $k \geq 4$. For a contradiction, let us assume that we have a polynomial algorithm $\mathcal{A}$ for the general problem for $(k, \frac{3}{2}k)$-rigid graphs. We will show that $\mathcal{A}$ can be used to solve the general problem for $(2,3)$-rigid graphs, contradicting [3]. To this end, let $G = (V,E)$ be a $(2,3)$-rigid graph. It is easy to check that $\frac{k}{2}G$ is $(k, \frac{3}{2}k)$-rigid. Let $F$ be a set of new edges that $\mathcal{A}$ returns on the input graph $\frac{k}{2}G$. By using Lemma 2.5 for an arbitrary $(k, \frac{3}{2}k)$-tight subgraph of $\frac{k}{2}G$, we get that $F$ does not include parallel edges. We will show that $G^* = G + F$ is $(2,3)$-redundant and there is no smaller set of new edges with this property.

Let $G' = (V,E')$ be a $(2,3)$-tight spanning subgraph of $G$. Then it is easy to see that $\frac{k}{2}G'$ is $(k, \frac{3}{2}k)$-tight. Let $e \in E'$ be an arbitrary edge. As $\frac{k}{2}G + F$ is $(k, \frac{3}{2}k)$-redundant, there is an edge $f \in \frac{k}{2}(E-E')\cup F$ such that the graph $\frac{k}{2}G' - e + f$ is $(k, \frac{3}{2}k)$-tight. We show that $\frac{k}{2}(G'-e+f)$ is also $(k, \frac{3}{2}k)$-tight and hence $G'-e+f$ is $(2,3)$-tight. To prove the $(k, \frac{3}{2}k)$-sparsity, it is enough to prove it on every $X \subseteq V$ for which $f$ is induced by $X$ as $i_{\frac{k}{2}(G'-e+f)}(X) = i_{\frac{k}{2}(G'-e)}(X) \leq k|X| - \frac{3}{2}k$ whenever $X$ does not induce $f$ and $|X| \geq 2$. Since the graph $\frac{k}{2}G' - e + f$ is $(k, \frac{3}{2}k)$-sparse, $0 < k|X| - \frac{3}{2}k - i_{\frac{k}{2}G'-e}(X) \leq k|X| - \frac{3}{2}k - i_{\frac{k}{2}(G'-e)}(X)$. By the definition of $\frac{k}{2}(G'-e)$, $\frac{k}{2}\Big| i_{\frac{k}{2}(G'-e)}(X)$ and thus $\frac{k}{2}\Big| k|X| - \frac{3}{2}k - i_{\frac{k}{2}(G'-e)}(X)$ for every $X \subseteq V$. As $\frac{k}{2}\Big| k|X| - \frac{3}{2}k - i_{\frac{k}{2}(G'-e)}(X) > 0$, $k|X| - \frac{3}{2}k - i_{\frac{k}{2}(G'-e)}(X) \geq \frac{k}{2}$ and hence $i_{\frac{k}{2}(G'-e)}(X) + \frac{k}{2} \leq k|X| - \frac{3}{2}k$ for every $X \subseteq V$

for which $f$ is induced in $X$. Therefore, $\frac{k}{2}(G' - e + f)$ is also $(k, \frac{3}{2}k)$-sparse. Thus, by counting the edges, we get that $\frac{k}{2}(G' - e + f)$ is $(k, \frac{3}{2}k)$-tight and hence $G' - e + f$ is $(2, 3)$-tight. Therefore, $G^*$ is $(2, 3)$-redundant as $e$ was arbitrarily chosen.

Finally, assume that $G^{**} = G + F'$ is $(2, 3)$-redundant for a set of new edges $F'$. Then it follows easily that $\frac{k}{2}G^{**}$ is $(k, \frac{3}{2}k)$-redundant. By Lemma 2.5, we can conclude that $\frac{k}{2}G + F'$ is also $(k, \frac{3}{2}k)$-redundant, and we get $|F| \leq |F'|$ by the minimality of $|F|$. □


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