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Spanning tree with lower bound on the degrees

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Abstract

We concentrate on some recent results of Egawa and Ozeki [1, 2], and He et al. [5]. We give shorter proofs and polynomial algorithms as well.

We present two new proofs for the sufficient condition for having a spanning tree with prescribed lower bounds on the degrees, achieved recently by Egawa and Ozeki [1]. The first one is a natural proof using induction, and the second one is a simple reduction to the theorem of Lovász [9]. Using an old algorithm of Frank [4] we show that the condition of the theorem can be checked in time $O(m\sqrt{n})$, and moreover, in the same running time – if the condition is satisfied – we can also generate the spanning tree required. This gives the first polynomial algorithm for this problem.

Next we show a nice application of this theorem for the simplest case of the Weak Nine Dragon Tree Conjecture, and for the game coloring number of planar graphs, first discovered by He et al. [5].

Finally we give a shorter proof and a polynomial algorithm for a good characterization of having a spanning tree with prescribed degree lower bounds, for the special case when $G[S]$ is a cograph, where S is the set of the vertices having degree lower bound prescription at least two. This theorem was proved by Egawa and Ozeki [2] in 2014, they did not give a polynomial algorithm.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph, $S \subseteq V$ and $f : S \to \{2, 3, 4, \dots\}$ be an integer-valued function on S. For disjoint sets of vertices X and Y, $d_G(X, Y)$ denotes the number of edges between X and Y, $d_G(X) = d_G(X, V-X)$ and $d_G(u) = d_G(\{u\})$. When the graph G is clear from the context, we omit it from the notation.

For a subset X of vertices let $f(X) = \sum_{x\in X} f(x)$. The open neighborhood is denoted by $\Gamma_G(X) = \{u \in V-X \mid \exists x \in X,\ ux \in E\}$, and the closed neighborhood is denoted by $\Gamma^*_G(X) = \Gamma_G(X) \cup X$. A subgraph induced by a vertex set $X \subseteq V$ is denoted by $G[X]$, the number of its edges by $i_G(X)$, and the number of its components by $c(G[X])$ or $c_G(X)$. We will use the convention that $\Gamma_G(\emptyset) = \emptyset$ and $c(G[\emptyset]) = 0$.

Egawa and Ozeki proved the following sufficient condition for having a forest (or spanning tree) with prescribed lower bounds on the degrees.

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Theorem 1.1 ([1]). *If for all nonempty subsets $X \subseteq S$ we have $|\Gamma^*_G(X)| > f(X)$ then there is a forest subgraph F of G, such that for all vertices $v \in S$ we have $d_F(v) \geq f(v)$.*

Corollary 1.2 ([1]). *If for all nonempty subsets $X \subseteq S$ we have $|\Gamma^*_G(X)| > f(X)$ and G is connected, then there is a spanning tree T of G, such that for all vertices $v \in S$ we have $d_T(v) \geq f(v)$.*

Special cases of this theorem appeared in the literature as follows. When G is bipartite and S is one of the classes, it was proved by Lovász in 1970 [9]. For general G, if S is a stable set, it was proved by Frank in 1976 [4] in a stronger form, as in this case the condition above is also necessary, giving a special case of Theorem 6.1 in Section 6.

Deciding whether for a triplet (G, S, f) there is a spanning tree T with degree lower bounds, i.e., $d_T(v) \geq f(v)$ for all $v \in S$, is NP-complete (let $S = V - \{u, v\}$ and $f(x) = 2$ for each $x \in S$; any appropriate spanning tree is a Hamiltonian path). However consider the following algorithmic problem. For given (G, S, f) check, whether the condition of Corollary 1.2 is satisfied, and if yes, then construct the appropriate spanning tree T. We show that this problem is polynomially solvable, namely in time $O(m\sqrt{n})$, where $n = |V|$ and $m = |E|$.

In the next section we give a simpler proof than that of Egawa and Ozeki, using induction. In Section 3 we give another proof, that is a simple reduction to the theorem of Lovász, yielding also a fast algorithm, detailed in Section 4. In Section 5 we show an application (as an example) of Theorem 1.1 for the game coloring number of some planar graphs. Finally, in Section 6 we show how we can use these ideas to prove a good characterization of Egawa and Ozeki [2] for a special case.

2 First proof – by induction

We prove Theorem 1.1 by induction on the number of edges. If G is a forest or $S = \emptyset$ then the theorem is obviously true.

We call a set $X \subseteq S$ tight, if it satisfies the condition $|\Gamma^*_G(X)| \geq f(X) + 1$ with equality.

If uv is an edge and $G - uv$ satisfies the condition then we are done by induction. So we may assume that for every edge uv the graph $G - uv$ has a set $X \subseteq S$ violating the condition. This implies that there are no edges outside S, and also that for each edge uv either u or v is contained in a tight set X, where the other one is connected to X by exactly one edge. If u is contained in tight set X with $d(v, X) = 1$ then we orient edge uv from v to u, otherwise from u to v. (If both u and v is contained in such a tight set, we choose arbitrarily.) This oriented graph $\vec{G}$ has the property that no arc leaves S. The in-degree of a vertex u (set X) is denoted by $\varrho(u)$ (or $\varrho(X)$ resp.).

Claim 2.1. *For each $u \in S$ we have $f(u) \geq \varrho(u)$.*

Proof of the Claim. If $\varrho(u) > 0$ then u is contained in a tight set. As $|\Gamma^*_G(X)|$ is a submodular set function, the intersection and the union of two intersecting tight sets are both tight. Thus the intersection $I(u)$ of all tight sets containing u is also a tight set, and every arc vu enters $I(u)$.

If $|I(u)| = 1$ then, by the tightness, we have $f(u) = d_G(u) \geq \varrho(u)$. Otherwise $I(u) - u$ is not a violating set, and for each arc vu, the vertex v has no neighbors in $I(u) - u$, thus we have $f(I(u)) - f(u) + 1 \leq |\Gamma^*_G(I(u) - u)| \leq |\Gamma^*_G(I(u))| - \varrho(u) = f(I(u)) + 1 - \varrho(u)$, giving the claim. □

To finish the proof of the theorem it is enough to prove that G is a forest. Let C be any cycle in G, and $X = V(C) \cap S$. Let $\overline{X}$ be the closure of X relative to S: $\overline{X} = \{v \in S \mid \exists x \in X$, such that v and x are in the same component of $G[S]\}$. Clearly $c(G[\overline{X}]) \leq c_G(X)$ and, by the observation made above, no arc leaves $\overline{X}$.

If $V(C) \subseteq S$ then, using Claim 2.1 and the fact that $G[\overline{X}]$ is now connected and contains a cycle, $f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| + \varrho(\overline{X}) \geq |\Gamma^*_G(\overline{X})|$, this is a contradiction.

Otherwise $G[S]$ is a forest, and $|V(C) - S| \geq c(G[\overline{X}])$. Now, by Claim 2.1, $f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| - c(G[\overline{X}]) + \varrho(\overline{X})$. As at least two arcs go from any vertex of $V(C) - S$ to $\overline{X}$, these vertices (at least $c_G(X)$) count twice in $\varrho(\overline{X})$, so $|\Gamma^*_G(\overline{X})| \leq |\overline{X}| + \varrho(\overline{X}) - c(G[\overline{X}]) \leq f(\overline{X})$, a contradiction again. □ □

3 Second proof – reduction to Lovász' theorem

In this section we prove Theorem 1.1 using a theorem of Lovász [9]. We quote this old theorem reformulated for fitting the notions used in this paper. We denote by f^+ the function $f + 1$, i.e., $f^+(x') = f(x') + 1$ for $x' \in S'$.

Theorem 3.1 ([9]). *Let $B = (S' \cup V, E')$ be a bipartite graph and $f : S' \to \{1, 2, 3, \dots\}$ be a function. B has a forest subgraph F_0 with the property $d_{F_0}(x') = f^+(x')$ for every $x' \in S'$, if and only if for all nonempty $X' \subseteq S'$ we have $|\Gamma_B(X')| > f(X')$.*

Proof of Theorem 1.1. We have (G, S, f) given, and let S' be a set disjoint from V with elements $S' = \{u' \mid u \in S\}$, and extend f to S' in the obvious way: $f(u') := f(u)$ for each $u \in S$.

Construct a bipartite graph $B = (V \cup S', E')$ as follows. For each ordered vertex pair $(u \in S,\ v \in V)$ we put edge $u'v$ into E', if $uv \in E$, and we also put the 'vertical' edges $u'u$ for each $u \in S$. Observe that for each $X \subseteq S$ (if X' denotes the corresponding subset of S') we have $\Gamma_B(X') = \Gamma^*_G(X)$. Therefore the condition of Theorem 3.1 is satisfied and thus we have a forest in B with $d_{F_0}(x') = f^+(x')$ for every $x' \in S'$.

First we claim that we can modify F_0 to get another forest F_1, so that $d_{F_1}(x') \geq f^+(x')$ for every $x' \in S'$, but F_1 contains every vertical edge. Suppose $u'u \notin E'(F_0)$. If u' and u are in different components of F_0 then we add the edge $u'u$, otherwise there is unique path $u'vu'_1 \dots u'_t u$ in F_0, in this case we delete edge $u'v$ and add edge $u'u$ still resulting in a forest (with the same degrees inside S).

Finally we construct the desired forest F by contracting each vertical edge (we contract u' to vertex u). It is easy to see, that in this way F becomes a forest subgraph of the graph G, and $d_F(x) \geq d_{F_1}(x') - 1 \geq f(x)$ for every $x \in S$. □

4 A polynomial algorithm for checking the condition and constructing the tree

In this section we first describe the algorithmic proof of Frank [4] for Theorem 3.1. After cloning each vertex $u' \in S'$ into $f(u')$ copies and running e.g., the Hopcroft-Karp algorithm for maximum bipartite matching, we either get a forest F' with degrees $d_{F'}(u') = f(u')$ for every $u' \in S'$ (and $d_{F'}(v) \leq 1$ for each $v \in V$), or we get a subset $X' \subseteq S'$ that violates the Hall condition, namely $|\Gamma_B(X')| < f(X')$. In this latter case the corresponding $X \subseteq S$ clearly violates the condition of the theorem as well.

We make an auxiliary digraph $D = (U, A)$, where $U = V \cup S' \cup \{r\}$. We orient edges of F' from S' to V, other edges of B from V to S', and finally add arcs rv for each $v \in V$ uncovered by F'. We run a BFS from vertex r in digraph D, it gives an arborescence T rooted at r which spans all vertices reachable from r. If every vertex in S' is reachable from r, then for each arc of T leading from V to S' we add the corresponding edge to F' resulting in the desired forest F_0 in B. Otherwise, if X' denotes the set of vertices of S' that are not reachable from r, then we claim that X' violates the condition of Theorem 3.1. If not, then there exist $u \in V$ and $x' \in X'$ such that $ux' \in A$ but either u is uncovered by F' or its F'-neighbor y' is in $S' - X'$. In both cases u is reachable from r (in the first case ru is an arc, in the second case y' is reachable from r and $y'u$ is an arc); consequently x' is also reachable from r, a contradiction.

Making F_1 from F_0 is algorithmically easy, as well as contracting the vertical edges.

Running time. Observe that for running the algorithm of Hopcroft and Karp, we do not need to make the cloning in reality, it is enough to do it imaginarily. Doing so keeps the running time $O(m\sqrt{n})$. All the other parts (together with constructing a spanning tree from the forest) can be done in $O(m)$.

5 A nice application: WNDT Conjecture and game coloring number of planar graphs

We show a nice application of Theorem 1.1 as an example. For a subset X of vertices, if $|X| > 1$, then we define $\lambda_G(X) = \frac{i(X)}{|X|-1}$ and $\mathrm{Arb}(G) = \max\{\lambda_G(X) \mid X \subseteq V,\ |X| > 1\}$. The Weak Nine Dragon Tree (WNDT for short) Conjecture is the following: if for integers k and d we have $\mathrm{Arb}(G) \leq k + \frac{d}{d+k+1}$, then there are k forests $F_1, \ldots, F_k$, such that the maximum degree in $G - F_1 - \ldots - F_k$ is at most d. It was proved by Kim et al. [7] for the case of $d > k$.

Here we show that if we further restrict ourselves to the special case of $k = 1$, then this result is a simple consequence of Theorem 1.1.

Theorem 5.1. *[7] If $d \geq 2$ is an integer and $\mathrm{Arb}(G) \leq 1 + \frac{d}{d+2}$, then there is a forest subgraph F of G, such that for every vertex v we have $d_G(v) - d_F(v) \leq d$.*

Actually we prove a stronger form (also proved in [7]).

Theorem 5.2. *[7] If $d \geq 2$ is an integer and for each nonempty subset X of the vertices we have $2(d+1) \cdot |X| > (d+2) \cdot i(X)$, then there is a forest subgraph F of G, such that for every vertex v we have $d_G(v) - d_F(v) \leq d$.*

Proof. Let $S = \{v \in V \mid d(v) \geq d+2\}$ and let f be defined on S by $f(v) = d(v) - d$. For a subset $X \subseteq S$ let $\Gamma_j(X) = \{v \in V - X \mid d(v, X) = j\}$, and let $\tilde{X} = X \cup \bigcup_{j=2}^{n} \Gamma_j(X)$. By the condition of the theorem $(d+1) \cdot |\tilde{X}| > \frac{d+2}{2} \cdot i(\tilde{X})$, i.e., $(d+1) \cdot |X| + (d+1) \cdot \sum_{j=2}^{n} |\Gamma_j(X)| > \frac{d+2}{2} \cdot \big(i(X) + \sum_{j=2}^{n} j \cdot |\Gamma_j(X)|\big)$. Realigned we get

$$(d+1) \cdot |X| > \frac{d+2}{2} \cdot i(X) + \sum_{j=2}^{n} \Big[\Big(j \cdot \frac{d+2}{2} - (d+1)\Big) \cdot |\Gamma_j(X)|\Big] \geq$$

$$\geq 2 \cdot i(X) + \sum_{j=2}^{n} \Big[\Big((j-2) \cdot \frac{d}{2} + j - 1\Big) \cdot |\Gamma_j(X)|\Big] \geq 2 \cdot i(X) + \sum_{j=2}^{n} \Big[(j-1) \cdot |\Gamma_j(X)|\Big],$$

as $d \geq 2$. We have $f(X) = 2 \cdot i(X) + \sum_{j=1}^{n} j \cdot |\Gamma_j(X)| - d \cdot |X| = 2 \cdot i(X) + \sum_{j=2}^{n} \Big[(j-1) \cdot |\Gamma_j(X)|\Big] + \big(|\Gamma(X)| - d \cdot |X|\big) < (d+1) \cdot |X| + \big(|\Gamma(X)| - d \cdot |X|\big) = |\Gamma^*(X)|$, thus the condition of Theorem 1.1 is satisfied and the forest F produced fulfills the statement of our theorem. □

Of course, we can apply the algorithm described in the previous section and efficiently make this decomposition.

Let G be a simple connected planar graph with girth $g \geq 5$. We know by Euler's formula that $i(X) < \frac{g}{g-2} \cdot |X|$ for every subset X of the vertices, and $\frac{g}{g-2} \leq \frac{2d+2}{d+2}$ if $d \geq \frac{4}{g-4}$. We get the following Corollary (which is a strengthening of a theorem proved first by He et al. in [5], the improvement was reported to be proved in [8]).

Corollary 5.3. *[5, 8] If G is a simple planar graph with girth at least g (where $g = 5$ or $g = 6$), then there is a forest subgraph F of G, such that for every vertex v we have $d_G(v) - d_F(v) \leq \lceil \frac{4}{g-4} \rceil$.*

The game coloring number was defined by Zhu [12] via a two-person game (for upper bounding the so-called "game chromatic number"). Alice and Bob remove vertices of G by turns, the backdegree of the vertex is the number of its previously removed neighbors. The game coloring number $\mathrm{col}_g(G)$ is the smallest $k+1$, where Alice can achieve that every vertex has backdegree at most k. An easy observation of Zhu [12] states that if the edges of G can be partitioned into graphs G' and H, then $\mathrm{col}_g(G) \leq \mathrm{col}_g(G') + \Delta(H)$, where Δ denotes the maximum degree. Faigle et al. [3] proved that the game coloring number of a tree is at most 4. Consequently we get the following result, that is also a strengthening of a theorem proved by He et al. in [5]). We also note, that by our algorithmic results we also provide a simple polynomial algorithm for Alice for winning the game.

Corollary 5.4. *[5, 8] If G is a simple planar graph with girth at least* 5*, then* $\mathrm{col}_g(G) \leq 8$*. If G is a simple planar graph with girth at least* 6*, then* $\mathrm{col}_g(G) \leq 6$*.*

6 Good characterization for a special case

In [2] Egawa and Ozeki proved the following theorem stating a good characterization, if $G[S]$ is a cograph, i.e., it does not contain an induced P_4. By the definition, an induced subgraph of a cograph is a cograph, and for two different vertices of the same connected component of a cograph, they are either connected or have a common neighbor.

Egawa and Ozeki also showed by a simple example, that this characterization does not remain true if $G[S] = P_4$: let the vertices of the P_4 be v_1, v_2, v_3, v_4 and let G have two more vertices, a and b, such that a is connected to v_1 and v_4 while b connected to v_2 and v_3; and let $f(v_i) = 2$.

Theorem 6.1 ([2])**.** *If $G[S]$ is a cograph, then G has a forest subgraph with degree lower bounds f on S if and only if for all nonempty subsets $X \subseteq S$ we have*

$$|\Gamma_G(X)| + 2|X| - c_G(X) > f(X).$$

Proof. We follow the outline of the proof in [2] but we make some simplifications and also give a polynomial algorithm. It is not hard to see that the condition above is necessary. Let F be a forest with $d_F(u) \geq f(u)$ for each $u \in S$, and let $X \subseteq S$. Now $i_G(F[X]) = |X| - c(F[X])$ and $f(X) \leq 2i_G(F[X]) + d_F(X) = 2|X| - 2c(F[X]) + d_F(X) \leq (d_F(X) - c(F[X])) + 2|X| - c_G(X)$ and $d_F(X) - c(F[X]) \leq |\Gamma_G(X)| - 1$, because F is a forest.

Let $\Gamma_Z(X) = \Gamma(X) \cap Z$. We denote the cograph $G[S]$ by H and $V - S$ by W.

Claim 6.2. *If $A, B \subseteq S$, then*

$$|\Gamma_{A\cup B}(A \cup B)| - c_H(A \cup B) + |\Gamma_{A\cup B}(A \cap B)| - c_H(A \cap B) \leq$$
$$\leq |\Gamma_{A\cup B}(A)| - c_H(A) + |\Gamma_{A\cup B}(B)| - c_H(B).$$

Proof. We first prove the claim for the case when $G[A \cup B]$ is a connected cograph and $A \cap B \neq \emptyset$. We use the well-known observation of Erdős and Rado stating that a graph or its complement is connected. As for any $x, y \in A \cap B$ they are either in a same component of $G[A]$ or in same component of $G[B]$, (if they are not connected, then they have a common neighbor in $A \cup B$), we may assume that $G[A \cap B]$ is inside a component K of $G[A]$. Let $K_1, \dots, K_a$ denote the other components of $G[A]$, and $L_1, \dots, L_b, I_1, \dots, I_c$ denote the components of $G[B]$, where I_j are the components intersecting $A \cap B$. As $c_H(A \cap B) \geq c$, it is enough to prove

$$0 + |\Gamma_{A\cup B}(A \cap B)| - 1 - c \leq |\Gamma_{A\cup B}(A)| + |\Gamma_{A\cup B}(B)| - (a + 1) - (b + c),$$

i.e., $|\Gamma_{A\cup B}(A)| + |\Gamma_{A\cup B}(B)| - |\Gamma_{A\cup B}(A \cap B)| \geq a + b$. This can be easily seen, as $\Gamma_{A\cup B}(A \cap B) \subseteq \Gamma_{A\cup B}(A) \cup \Gamma_{A\cup B}(B)$ and each K_i and L_j contains a vertex not connected to $A \cap B$ but connected to $B - A$ (or $A - B$, resp.). If $A \cap B = \emptyset$, then the situation is very similar, we need to prove $|\Gamma_{A\cup B}(A)| + |\Gamma_{A\cup B}(B)| \geq a + b - 1$, however, as we assumed $G[A \cup B]$ to be connected, this is always satisfied with a strict inequality.

If $G[A \cup B]$ has several components, then it is enough to prove the claim for every component separately, thus the same proof works. We remark that for the case $A \cap B = \emptyset$ we still have strict inequality if there is an edge between A and B. □

Let $b_0(X)$ and $b(X)$ be set-functions on the subsets of S defined by $b_0(X) = |\Gamma(X)| - c_G(X)$, and $b(X) = |\Gamma(X)| - c_G(X) + 2|X| - f(X)$. As $|\Gamma_Z(X)|$ is submodular for each Z, and for $A, B \subseteq S$ we have $b_0(A) = |\Gamma_{A\cup B}(A)| + |\Gamma_{V-(A\cup B)}(A)| - c_H(A)$, clearly b_0 is submodular, moreover, if $U(A, B)$ denotes the set of vertices in $V-(A \cup B)$ connected to both $A-B$ and $B-A$ but not to $A \cap B$ (i.e., $U(A, B) = \Gamma(A) \cap \Gamma(B) - \Gamma(A \cap B)$), then $b_0(A \cup B) + b_0(A \cap B) + |U(A, B)| \leq b_0(A) + b_0(B)$. As $b(X)$ is the sum of $b_0(X)$ and the modular function $2|X| - f(X)$, the same statement holds for b as well. We call a nonempty subset $X \subseteq S$ *tight* if $b(X) = 1$.

Corollary 6.3. *Suppose the condition of the theorem, i.e., $b(X) \geq 1$ for all $\emptyset \neq X \subseteq S$ holds. The intersection and union of two intersecting tight sets A and B is tight, and $|U(A, B)| = 0$. If A and B are disjoint tight sets and $U(A, B) \neq \emptyset$ or there is an edge connecting A and B, then $A \cup B$ is tight, moreover either $|U(A, B)| = 1$ and no edge connects A to B; or $U(A, B) = \emptyset$.*

We may assume that every vertex $v \in S$ is contained in a tight set, otherwise we can increase $f(v)$ without violating the condition. By the corollary, $I(v)$, the *intersection of all tight sets containing* v is also a tight set. We also suppose that for every edge wv (where $w \in W = V-S$) the graph $G - wv$ would violate the condition, i.e., $v \in S$ and $d(w, I(v)) = 1$. (If this would be not the case, then we simply delete the edge wv.) Unfortunately we may not assume a similar condition about edges induced by S because deleting such an edge can introduce an induced P_4.

We denote the components of $G[S]$ by $Z_1 \ldots, Z_t$. Call a vertex $u \in Z_i$ *proper* if $I(u) \subseteq Z_i$. Our goal is to prove that every vertex in S is proper. We need some preliminary observations.

Claim 6.4. *Suppose tight sets A and B intersect the same component Z_i of $G[S]$. Then $A \cup B$ is tight. Consequently if $u, v \in Z_i$, then $I(u) \cup I(v)$ is tight, moreover, a vertex $w \in W$ cannot be connected to both u and v.*

Proof. Either the sets A and B are intersecting, or connected by an edge, or otherwise – using that $G[Z_i]$ is a cograph – they have a common neighbor $x \in Z_i$ which is not in $A \cup B$, so $x \in U(A, B)$. Thus the first statement is a consequence of Corollary 6.3.

Suppose wu, wv are edges. As wu does not enter $I(v)$ and wv does not enter $I(u)$, we have $w \in U(I(u), I(v))$, so using Corollary 6.3 again we get a contradiction. □

We make an auxiliary bipartite graph G' by contracting each component of $G[S]$ (we delete the loops arising). By Claim 6.4 no parallel edges can arise. Recall that the components of $G[S]$ are $Z_1, Z_2, \ldots, Z_t$, the corresponding contracted vertices of G' are called $z_1, z_2 \ldots, z_t$. Actually in the next lemma we use a modification of this concept and we will use this auxiliary graph later, however we defined it here to make our goal traceable.

Lemma 6.5. *Every vertex $u \in S$ is proper.*

Proof. Suppose this is not the case and let $u \in S$ be an unproper vertex for which $I(u)$ is minimal. Suppose $u \in Z_1$ and $I = I(u)$ intersects also $Z_2, \ldots, Z_r$. For a vertex $v \in I$, either $I(v) = I$ or $I(v) \subset I$ (by Corollary 6.3), in this second case v is proper. Let $A_i = I \cap Z_i$ and $W(I) = \Gamma_W(I)$ and $E'' = \{wx \in E \mid w \in W(I),\ x \in I\}$. First we claim that if we take the subgraph defined by E'', and next we contract each A_i to a vertex a_i, then the resulting bipartite graph is a forest. Suppose not, i.e., it contains a cycle C' with $W' = V(C') \cap W = \{w_1, \ldots, w_k\}$. Let C be its "pre-image" in G, a cycle $w_1, v_1, x_1, u_2, w_2, v_2, x_2, u_3, w_3, \ldots, u_k, w_k, v_k, x_k, u_{k+1} = u_1$, where $w_i \in W(I)$ and $v_i, x_i, u_{i+1} \in Z_i$. Note that v_i, x_i, u_{i+1} are not necessarily distinct vertices, so two subsequent vertices of this sequence are either identical or connected by an edge. By the definition of E'' each $u_i, v_i \in I$. As every w_i is connected to two contracted vertices, we have u_i and v_i are all proper vertices, otherwise, e.g., $I(v_i) = I$ and edge $w_i v_i$ is not a unique edge from w_i that enters $I(v_i)$. Using Claim 6.4, the sets $B_i = I(v_i) \cup I(u_{i+1}) \subseteq A_i$ are tight sets. By repeatedly using Corollary 6.3 we get that $D_j = \cup_{i=1}^{j} B_j$ are tight sets for $j = 1, 2, \ldots, k-1$; note that $D_{k-1} \subseteq A_1 \cup \ldots \cup A_{k-1}$. Finally we get a contradiction to Corollary 6.3 for disjoint tight sets D_{k-1} and B_k as $w_k, w_1 \in U(D_{k-1}, B_k)$.

Now we are able to finish the proof of Lemma 6.5. By our assumptions $b(A_1) \geq 2$ and $b(A_i) \geq 1$ for $i = 2 \ldots r$, so we have $\sum_{i=1}^{r} b(A_i) \geq r + 1$. We claim that $|W(I)| + r \leq |W(I)| + \sum_{i=1}^{r} b(A_i) - b(I) = |E''|$, this would give the required contradiction. As I is tight, $b(I) = 1$. $\Gamma_S(A_i) \subseteq Z_i$ are pairwise disjoint, $I = \cup_{i=1}^{r} A_i$, so $\Gamma_S(I)$ is the disjoint union of sets $\Gamma_S(A_i)$, and $c_G(I) = \sum_{i=1}^{r} c_G(A_i)$. Thus $\sum_{i=1}^{r} b(A_i) - b(I) = \sum_{i=1}^{r} |\Gamma_S(A_i)| + \sum_{i=1}^{r} |\Gamma_W(A_i)| - |\Gamma_S(I)| - |\Gamma_W(I)| = \sum_{i=1}^{r} |\Gamma_W(A_i)| - |\Gamma_W(I)| = \sum_{i=1}^{r} d(A_i, W(I)) - |W(I)|$ by the second statement of Claim 6.4, and $\sum_{i=1}^{r} d(A_i, W(I)) = |E''|$, so our last claim is proved, finishing the proof of the lemma. □

As for a component Z_i of $G[S]$ we have $Z_i = \cup_{u \in Z_i} I(u)$ by Lemma 6.5, repeatedly usage of Claim 6.4 shows that $\cup_{u \in Z_i} I(u)$ is a tight set.

Corollary 6.6. *For every component Z_i of $G[S]$, the set Z_i is tight.*

We prove Theorem 6.1 by induction on the number of vertices. If G' has an isolated vertex, then it is either $w \in W$ (we simply delete w and use induction), or a vertex z_j, where we can use induction separately for $G[Z_j]$ and for $G - G[Z_j]$.

If G' has a vertex of degree one, then its either $w \in W$ or a vertex z_j. If w has degree one we take its neighbor $u \in S$, delete w and reset $f(u) = f(u) - 1$. Now we can use induction, the assumption of the theorem is not violated (it may be the case that u gets outside of S – if $f(u)$ becomes 1 – but $G[S - u]$ remains a cograph). Suppose z_j has degree one in G' and uw is the unique edge leaving Z_j in G, where $u \in Z_j$ and $w \in W$. We delete edge uw, reset $f(u) = f(u) - 1$, and then we can use induction separately for $G[Z_j]$ and for $G - G[Z_j]$, finally edge uw can be put back safely to the union of the two resulting forests.

Otherwise we have a cycle in G' with vertices $w_1, z_1, w_2, z_2 \ldots, w_k, z_k$. We repeat the arguments above to show that this assumption leads to a contradiction. In this case let $D_j = \cup_{i=1}^{j} Z_j$, these are tight sets for each $j = 1, 2, \ldots, k-1$, using Corollaries 6.3 and 6.6. Finally we get a contradiction for tight sets D_{k-1} and Z_k, as w_k and w_1 are both in $U(D_{k-1}, Z_k)$. □ □

6.1 Algorithmic aspects

Egawa and Ozeki already observed that their proof is "almost" algorithmic but they were not able to give a polynomial algorithm because they used the induction hypothesis for exponentially many subgraphs.

Our proof does not do this, so it amounts to a polynomial algorithm, at least using general strongly polynomial submodular function minimization of either Iwata, Fleischer and Fujishige [6] or of Schrijver [10]. These algorithms can calculate for every vertex $v \in S$ the minimum of $b(A)$ among the sets $A \subseteq S$ containing v, and also a minimal set $I(v)$ with $b(I(v))$ having the minimum value. So we can use them for checking the assumption of Theorem 6.1 and finding the sets $I(v)$ for all v. Then the induction on the number of vertices can be implemented in polynomial time, as we either make a recursive call for one graph with one vertex less, or two graphs with total number $|V|$ of the vertices.

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