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Packing countably many branchings with prescribed root-sets in digraphs without backward-infinite paths

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Abstract

We generalize an unpublished result of C. Thomassen. Let $D = (V, A)$ be a digraph without backward-infinite paths and let $\{V_i\}_{i\in\mathbb{N}}$ be a multiset of subsets of V. We show that if all $v \in V$ is simultaneously reachable from the sets V_i by edge-disjoint paths, then there exists a system of edge-disjoint spanning branchings $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ in D where the root-set of $\mathcal{B}_i$ is V_i.

1 Notations and background

The digraphs considered here may have multiple edges and arbitrary size. For $X \subseteq V$ let $\mathsf{in}_D(X)$ and $\mathsf{out}_D(X)$ be the set of ingoing and outgoing edges respectively of X in D, and let $\varrho_D(X)$, $\delta_D(X)$ be their respective cardinalities. The paths in this paper are directed, finite, simple paths. We say that the path ***P* goes from *X* to *Y*** if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$ ($\mathsf{start}(P) = \mathsf{end}(P)$ is allowed).

Let $\mathcal{V} = \{V_i\}_{i<\lambda}$ be a multiset of subsets of V, where λ is a cardinal, and let $D = (V, A)$ be a digraph. We say that $v \in V$ is **simultaneously reachable** from $\mathcal{V}$ in D if there is a system of edge-disjoint paths $\{P_i\}_{i<\lambda}$ in D such that P_i goes from V_i to v. The system $\mathcal{V}$ satisfies the **path condition** in D if all $v \in V$ is simultaneously reachable from $\mathcal{V}$. A digraph $\mathcal{B}_0 = (U, E)$ is a **branching** if for all $u \in U$ there is a unique path from the **root-set** $\{w \in U : \varrho_{\mathcal{B}_0}(w) = 0\}$ to u in $\mathcal{B}_0$. We call $\mathcal{B}$ a **multibranching** in D with respect to $\mathcal{V}$ if $\mathcal{B} = \{\mathcal{B}_i\}_{i<\lambda}$, where $\mathcal{B}_i$'s are edge-disjoint branchings in D and the root-set of $\mathcal{B}_i$ is V_i. If in addition all the $\mathcal{B}_i$'s are spanning branchings of D (i.e. their vertex-set is V), then we call $\mathcal{B}$ a **spanning multibranching** with respect to $\mathcal{V}$ in D. If there is a spanning multibranching $\mathcal{B}$ in D with respect to $\mathcal{V}$, then the system $\mathcal{V}$ obviously satisfies the path condition in D, since if $v \in V$, then $\mathcal{B}_i$ contains a path P_i from V_i to v ($i < \lambda$), and these paths are pairwise edge-disjoint because the branchings are pairwise edge-disjoint. If λ is finite, then by Menger's theorem one can formulate the path condition in the following equivalent form:

$$\forall X \subset V\, (X \neq \varnothing \Longrightarrow \varrho_D(X) \geq |\{i < \lambda : V_i \cap X = \varnothing\}|)\,. \tag{1}$$

The (strong form of) Edmonds' branching theorem (see [2] p. 349 Theorem 10.2.1) states that in the finite case (λ and D are finite) condition (1) is enough to assure the existence of a spanning multibranching. R. Aharoni and C. Thomassen proved by a construction (see [1]) that this theorem fails for infinite digraphs. Even so, one can weaken the finiteness condition for D in Edmonds' branching theorem, namely it is enough to assume that D does not contain backward-infinite paths as showed by C. Thomassen (unpublished). The exclusion of backward-infinite paths can be replaced by exclusion of forward-infinite paths as we proved in [4]. In this paper we prove the strengthening of Thomassen's result to countably many branchings, namely:

Theorem 1.1. *Let $\{V_i\}_{i\in\mathbb{N}}$ be a multiset of subsets of V that satisfies the path condition in the digraph $D=(V,A)$. Assume that D does not contain backward-infinite paths. Then there is a spanning multibranching in D with respect to $\{V_i\}_{i\in\mathbb{N}}$, i.e. there is a system of edge-disjoint spanning branchings $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ in D such that the root-set of $\mathcal{B}_i$ is V_i.*

Remark 1.2. Instead of excluding backward-infinite paths it is enough to assume that any backward-infinite path of D intersects all the V_i's. See Remark 4.3.

As we have already mentioned Thomassen proved the theorem above for finitely many V_i's (which motivated us to investigate this topic). The main idea of Thomassen's proof is the following: construct first a spanning subgraph $D'=(V,A')$ of D such that D' also satisfies condition (1) and all vertices of D' have finite in-degrees. After that, one can build the desired spanning multibranching in D' using Edmonds' branching theorem and compactness arguments. In the countably infinite case we have to use a very different approach.

2 Proof of the main Theorem

The key of the proof is the following lemma which actually works with arbitrary cardinal λ.

Lemma 2.1. *Let $\{V_i\}_{i<\lambda}$ be a multiset of subsets of V that satisfies the path condition in the digraph $D=(V,A)$. Assume that D does not contain backward-infinite paths. Then for any $j<\lambda$ and $v\in V\setminus V_j$ there is a path P from V_0 to v in D such that the path condition holds for $\{V_i'\}_{i<\lambda}$ in $D-A(P)$ where* $V_i'=\begin{cases} V_i\cup V(P) & \textit{if } i=j\\ V_i & \textit{otherwise.}\end{cases}$

We show first how Theorem 1.1 follows from Lemma 2.1. If $\mathcal{B}$ is a multibranching in D with respect to $\{V_i\}_{i<\lambda}$, then let $D\setminus\!\!\setminus\mathcal{B}=(V,A\setminus\cup_{i\in\mathbb{N}}A(\mathcal{B}_i))$. We say that the mutltibranching $\mathcal{B}$ satisfies the path condition if path condition holds in $D\setminus\!\!\setminus\mathcal{B}$ with respect to $\{V(\mathcal{B}_i)\}_{i\in\mathbb{N}}$. If $\mathcal{B}$ satisfies the path condition, then Lemma 2.1 makes possible to extend a prescribed $\mathcal{B}_i$ with a path in such a way that $\mathcal{B}_i$ reaches a prescribed vertex and the new multibranching still satisfies the path condition. The plan is to start with the multibranching $\{(V_i,\varnothing)\}_{i\in\mathbb{N}}$ and extend it to spanning multibranching by a transfinite recursion using Lemma 2.1 in every successor step and taking union in limit steps.

The only arising problem is that we may violate the path condition at limit steps. We can handle easily this problem. Since we have just countably many branchings, we can organize the recursion in such a way that if we extend a branching with a vertex v in some step, then before the next limit step we extend with v all the branchings. We claim that this ensures the path condition after limit steps.

Indeed, let $\mathcal{B}$ be the multibranching that we have after some limit step and let $u \in V$ be arbitrary. We may fix a system of edge-disjoint paths $\{P_i\}_{i\in\mathbb{N}}$ in D such that P_i goes from V_i to u. Let v_i be the first vertex on P_i such that the terminal segment P_i' of P_i that starts at v_i is still a path in $D \setminus\!\!\setminus \mathcal{B}$. It is enough to show that $v_i \in V(\mathcal{B}_i)$. If $v_i = \mathsf{start}(P_i)$, then it is clear since $\mathsf{start}(P_i) \in V_i \subset V(\mathcal{B}_i)$. If $v_i \neq \mathsf{start}(P_i)$, then by choice of v_i there is a successor step in which we extend some branching with v_i, but then before the next limit step we put v_i all the branchings thus $v_i \in V(\mathcal{B}_i)$. ■

Question 2.2. For uncountable λ the cheap trick that we use for the limit steps works no more and we do not know yet if Theorem 1.1 remains true with arbitrary cardinal instead of $\mathbb{N}$.

Without loss of generality it is enough to prove Lemma 2.1 for $j = 0$. Before the proof we have to generalize some phenomena that are well-known from finite branching-packing theorems.

3 Generalization of tight and dangerous sets

In context of Edmonds' branching theorem a set $\varnothing \neq X \subset V$ is usually called tight if $\varrho_D(X) = |\{i < \lambda : V_i \cap X = \varnothing\}|$. For infinite λ the literal, cardinality-based generalization of tightness is unusable. In this section we give another generalization that turns out to be useful and keeps all the nice properties of tight sets that are known from the finite case.

Suppose, that $\mathcal{V} = \{V_i\}_{i<\lambda}$ satisfies the path condition in a digraph $D = (V, A)$. We call the set $\varnothing \neq X \subset V$ **tight** (with respect to $\mathcal{V}$ in D) if whenever $\{P_i\}_{i<\lambda}$ is a system of edge-disjoint paths in D such that P_i goes from V_i to X, then the paths P_i necessarily use all the ingoing edges of X i.e. $\mathsf{in}_D(X) \subset \cup_{i<\lambda} A(P_i)$. (If $V_i \cap X \neq \varnothing$ then one may choose P_i as a path consisting of just one vertex, thus the definition is really about those i's for which $X \cap V_i = \varnothing$.) A $B \subset V$ is **dangerous** if it is tight and $V_0 \cap B \neq \varnothing$.

In this section we prove three simple consequences of dangerousness.

Corollary 3.1. *Let B be dangerous and let $v \in B$ arbitrary. If $\{P_i\}_{i<\lambda}$ is a system of edge-disjoint paths in D such that P_i goes from V_i to v, then $V(P_0) \subset B$. Furthermore if $V_i \cap B = \varnothing$, then P_i uses exactly one ingoing edge of B i.e. $|A(P_i) \cap \mathsf{in}_D(B)| = 1$.*

Proof: Assume, to the contrary, that $V(P_0) \not\subset B$. Then P_0 uses some edge $e \in \mathsf{in}_D(B)$. Replace P_0 with a path P_0' that consists of only one vertex u where $u \in V_0 \cap B$. The modified path-system no more uses e but all its members have a vertex in B hence by taking their appropriate initial segments we get a contradiction with the tightness of B.

For the second part of the proposition, if $|A(P_i) \cap \mathsf{in}_D(B)| > 1$ holds, then by replacing P_i by its appropriate initial segment we get a contradiction in similar way. ●

Corollary 3.2. *If $B_0, B_1 \subset V$ are dangerous sets with nonempty intersection, then $B_0 \cap B_1$ is also dangerous.*

Proof: Let $\{P_i\}_{i<\lambda}$ be a system of edge-disjoint paths in D such that P_i goes from V_i to $B_0 \cap B_1$. Suppose, to the contrary, that the paths do not use an edge $e \in \mathsf{in}_D(B_0 \cap B_1)$. By symmetry we may assume that $e \in \mathsf{in}_D(B_0)$. By taking appropriate initial segments of the paths we get a contradiction with tightness of B_0. Therefore $B_0 \cap B_1$ is tight. Let $v \in B_0 \cap B_1$ and let $\{Q_i\}_{i<\lambda}$ be a system of edge-disjoint paths in D such that Q_i goes from V_i to v. By using Corollary 3.1 with B_0 and with B_1 separately we get $V(Q_0) \subset B_0 \cap B_1$ hence $\mathsf{start}(Q_0) \in V_0 \cap B_0 \cap B_1$ thus $B_0 \cap B_1$ is dangerous.●

Remark 3.3. It is not too hard to prove that under the conditions of the corollary above $B_0 \cup B_1$ is also dangerous and there are no edges between $B_0 \setminus B_1$ and $B_1 \setminus B_0$ in any direction as it was the expectation from the finite case. In the current paper we do not need these facts.

For multisets $\mathcal{V}$ and $\mathcal{T}$ we denote by $\mathcal{V} \uplus \mathcal{T}$ the multiset where the multiplicity of an element is the sum of its multiplicities in $\mathcal{V}$ and $\mathcal{T}$.

Corollary 3.4. *Let B be a dangerous set in D with respect to $\mathcal{V} = \{V_i\}_{i<\lambda}$. Then the multiset $\mathcal{V}[B] \overset{def}{=} \{\{\mathsf{end}(e)\} : e \in \mathsf{in}_D(B)\} \uplus \{V_i \cap B : i < \lambda,\ V_i \cap B \neq \varnothing\}$ satisfies the path condition in $D[B]$. Furthermore a set $X \subset B$ is dangerous with respect to $\mathcal{V}$ in D iff X is dangerous with respect to $\mathcal{V}[B]$ in $D[B]$.*

Proof: Let $v \in B$ be arbitrary. The system $\mathcal{V}$ satisfies the path condition in D thus we may fix a system of edge-disjoint paths $\{P_i\}_{i<\lambda}$ in D such that P_i goes from V_i to v. The definition of tightness and Proposition 3.1 shows that the appropriate terminal segments of paths $\{P_i\}_{i<\lambda}$ certify that v is simultaneously reachable from $\mathcal{V}[B]$ in $D[B]$.

Assume that $X \subset B$ is not dangerous with respect to $\mathcal{V}$ in D and the path-system $\{Q_i\}_{i<\lambda}$ shows it. Cut the initial segments of the Q_i's that are not in B. The resulting system shows that X is not dangerous with respect to $\mathcal{V}[B]$ in $D[B]$. The other direction is similar. ●

4 Proof of the key-Lemma

Assume, seeking contradiction, that Lemma 2.1 is false and $D_0, \mathcal{V}_0, u_0$ witness it (where $\mathcal{V}_0 = \{V_i\}_{i<\lambda}$) i.e. there is no path P from V_0 to u_0 in D_0 such that the path condition holds for $\{V_0 \cup V(P)\} \cup \{V_i\}_{0 \neq i<\lambda}$ in $D - A(P)$.

Let $\{P_i\}_{i<\lambda}$ be a system of edge-disjoint paths in D_0 such that P_i goes from V_i to u_0. Consider the first edge e of P_0.

In the first case, assume that $\{V_0 \cup \{\mathsf{end}(e)\} \cup \{V_i\}_{0\neq i<\lambda}$ satisfies the path condition in $D_0 - e$. Then let $\mathcal{V}_1 = \{V_0 \cup \{\mathsf{end}(e)\}\} \cup \{V_i\}_{0\neq i<\lambda}$ and $D_1 = D_0 - e$ and continue the process with the following edge of P_0. Otherwise extending with e violates the path condition. We need here a claim that we will prove in the next section.

Claim 4.1. *Let $\mathcal{V} = \{V_i\}_{i<\lambda}$ be a multiset of subsets of V that satisfies the path condition in the digraph $D = (V, A)$. Let $e \in \mathsf{out}_D(V_0)$ and let $\mathcal{V}' = \{V_i'\}_{i<\lambda}$ where*
$$V_i' = \begin{cases} V_i \cup \{\mathsf{end}(e)\} & \text{if } i = 0 \\ V_i & \text{otherwise .} \end{cases}$$
Then $\mathcal{V}'$ violates the path condition in $D' = D - e$ iff there exists a $B \subset V$ such that B is dangerous with respect to $\mathcal{V}$ in D and $e \in \mathsf{in}_D(B)$.

Hence by Claim 4.1 there is a set B such that $e \in \mathsf{in}_{D_0}(B)$, and B is dangerous with respect to $\mathcal{V}_0$ in D_0. Note that $V(P_0) \not\subset B$, thus by Corollary 3.1 $u_0 \notin B$. Let u_1 be the last vertex of P_0 which is in B.

In the second case, we assume that the triple $D_0[B], \mathcal{V}_0[B], u_1$ (see Corollary 3.4) is not a counterexample to Lemma 2.1. Let $\mathcal{V}_1 = \{V_0 \cup V(P)\} \cup \{V_i\}_{0\neq i<\lambda}$ and $D_1 = D_0 - A(P)$, where P is a path in $D_0[B]$ that goes from $V_0 \cap B$ to u_1, and the extension does not violate the path path condition with respect to $\mathcal{V}_0[B]$ in $D_0[B]$. We claim that extending with path P does not violate the path condition with respect to $\mathcal{V}_0$ in D_0 either (i.e. $\mathcal{V}_1$ satisfies the path condition in D_1). Suppose, to the contrary, that it does. Assume first that P consists of only one edge f. Then by Claim 4.1 there is a set B' such that B' is dangerous with respect to $\mathcal{V}_0$ and $f \in \mathsf{in}_{D_0}(B')$. But then $f \in \mathsf{in}_{D_0}(B \cap B')$, where $B \cap B'$ is dangerous with respect to $\mathcal{V}_0$ in D_0 by Corollary 3.2, thus by Corollary 3.4 $B \cap B'$ is dangerous with respect to $\mathcal{V}_0[B]$ in $D_0[B]$ too, therefore by Claim 4.1 extending with P does not keep the path condition with respect to $\mathcal{V}_0[B]$ in $D_0[B]$ which contradicts the choice of P. If $1 < |A(P)|$ then we may decompose the extension with P to a finite sequence of one-edge extensions and arguing in a similar way. Let P_0' be the terminal segment of P_0 that starts at u_1. Unfortunately path P may use edges from the P_i's, that is why we need the following Proposition.

Proposition 4.2. *The set $\{P_0'\}$ can be extended to a system of edge-disjoint paths $\{P_i'\}_{i<\lambda}$ in D_1 such that P_i' goes from V_i to u_0 for $0 \neq i < \lambda$.*

Proposition 4.2 (that we will prove in the end of this section) makes possible to continue the process with the following edge i.e. the outgoing edge of u_1 in P_0'.

In the remaining third case, the triple $D_0[B], \mathcal{V}_0[B], u_1$ is a counterexample to Lemma 2.1. Note that this third case necessarily occurs after at most $|A(P_0)|$ steps otherwise we would reach u_0 (since in the first two cases we reach a new vertex along P_0) and then the deleted edges would contain a unique path P from V_0 to u_0 that satisfies what Lemma 2.1 demands contradicting with the counterexample nature of the triple $D_0, \mathcal{V}_0, u_0$.

To simplify the notations we may assume that the third case arise at the first step. Let us summarize that we get. We have a nontrivial path (namely P_0') in D_0 from the unreachable vertex of the new counterexample (u_1) to the unreachable vertex of the original counterexample (u_0) such that the vertex-set of the new counterexample (B) contains just the first vertex of this path. Now we may forget the process above and just concentrate on the fact that we are able to find such a new counterexample from an arbitrary counterexample. By iterating it recursively we can find a sequence of counterexample triples $D_i, \mathcal{V}_i, u_i$ ($i \in \mathbb{N}$) where D_{i+1} is a subgraph of D_i and $u_i \in V(D_i) \setminus V(D_{i+1})$ (see figure 1). We also find a sequence of paths Q_i ($i \in \mathbb{N}$) such that Q_i is a path in D_i and it goes from u_{i+1} to u_i in such a way that $V(Q_i) \cap V(D_{i+1}) = \{u_{i+1}\}$. By uniting the paths Q_i we get a backward-infinite path contradicting with the assumption about D_0. ■

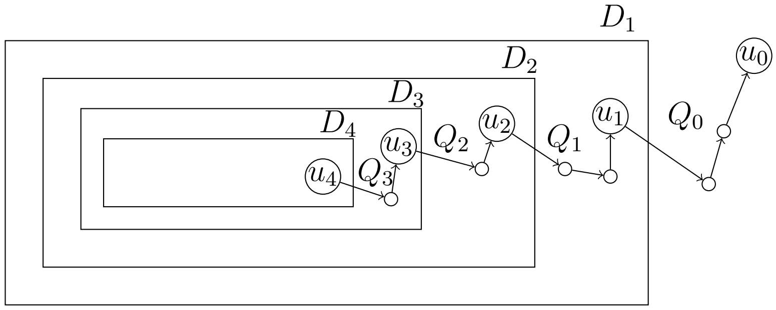

Figure 1: The sequence of the counterexample triples and the construction of the backward-infinite path in D_0.

Remark 4.3. If D_0 may contain backward-infinite paths but all of them intersects all the V_i's, then the proof above still works because we also find a backward-infinite path in $D_1 = D_0[B]$ and it is still a contradiction since B is disjoint from some V_i (B is dangerous and $e \in \mathsf{in}_{D_0}(B)$).

To prove Proposition 4.2 we need the following version of a well-known technique developed by L. R Ford and D. R Fulkerson (see [3]).

Proposition 4.4. *Let $\{P_i\}_{i \in I}$ be a system of edge-disjoint $s \to t$ paths in a digraph D (where $s \neq t \in V(D)$) and denote the first and the last edge of P_i by e_i and by f_i respectively. Let $\overleftarrow{D}$ be the digraph that we get by change the direction of edges $\cup_{i \in I} A(P_i)$ in D. We call these edges* ***backward-edges*** *of $\overleftarrow{D}$ and we call* ***forward-edges*** *the others. Denote by U the set of the vertices in $V(D)$ that are reachable in $\overleftarrow{D}$ from s. If $t \in U$ and path R certifies it, where the first edge of R is e and the last edge of R is f, then there is a system of edge-disjoint paths $\{Q_j\}_{j \in J}$ in D such that the set of the first and the set of the last edges of paths Q_j ($j \in J$) are $\{e_i\}_{i \in I} \cup \{e\}$ and $\{f_i\}_{i \in I} \cup \{f\}$ respectively.*

If $t \notin U$, then there is a system of edge-disjoint paths $\{Q_h : h \in \mathsf{out}_D(U)\}$ from U to t in D such that the first edge of Q_h is h.

Proof: Consider a finite subgraph D' of D that contains R and all the P_i's, say $P_1, \ldots, P_k$, that give a backward-edge to R. The $P_1, \ldots, P_k$ paths determinate a flow in D' with amount k with respect to the constant 1 upper bound on the edges. By the technique of Ford and Fulkerson one can get by using R a flow in D' that has amount $k+1$. By decomposing this flow to $k+1$ edge-disjoint $s \to t$ paths and keeping the untouched P_i's we get the desired system $\{Q_j\}_{j\in J}$. For the second part of the Proposition the terminal segments of the paths $\overleftarrow{P_i}$ that go from U to t will be suitable, it follows immediately from the construction of $\overleftarrow{D}$ and from the definition of U.●

Now we are able to prove Proposition 4.2.

Proof of Proposition 4.2. If $A(P_i) \cap \mathsf{span}_D(B) = \varnothing$ for some $0 \neq i < \lambda$, then let $P_i' = P_i$. Consider now $I = \{0 \neq i < \lambda : A(P_i) \cap \mathsf{span}_D(B) \neq \varnothing\}$. All but at most $|A(P)|$ from the paths $\{P_i\}_{i\in I}$ are still paths in D_1. To simplify the notations we assume that the problematic paths (i.e. paths that has edge in $A(P)$) are $P_1, \dots, P_k$. For $i \in I$ let u_i denote the first and v_i the last intersection of P_i with B. We construct a digraph G starting with $D_1[B]$ (see figure 2). If for some $w, z \in B$ there is an $i \in I$ such that P_i has a $w \to z$ segment such that the interior vertices of the segment are not in B, then draw a new e_{iwz} edge from w to z. Pick a new vertex t and draw an edge f_i from v_i to t ($i \in I$). For all $i \in I$ pick a new vertex s_i. If path P_i starts inside B, then draw one edge from s_i to all the elements of $V_i \cap B$, otherwise draw an edge from s_i to u_i. Construction of G is complete.

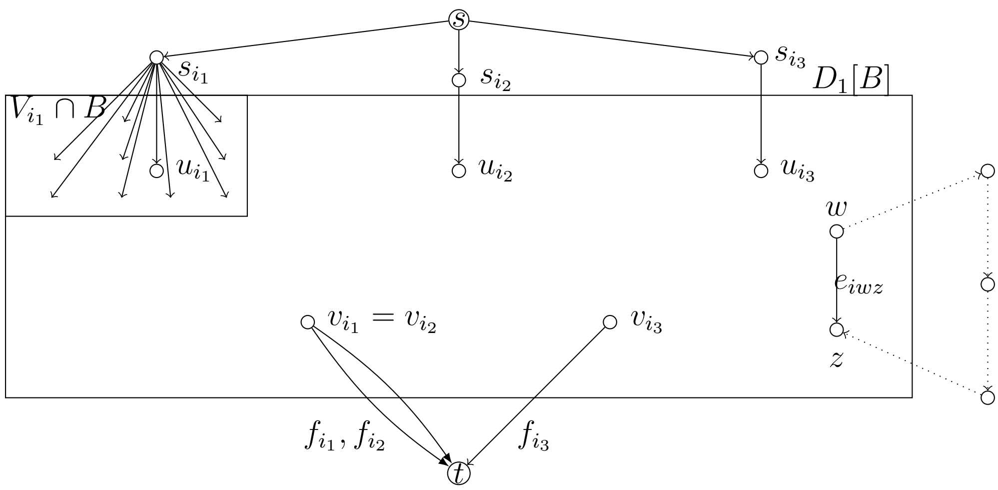

Figure 2: Construction of G from $D_1[B]$. Here $B \cap V_{i_1} \neq \varnothing$ but $B \cap V_{i_2} = B \cap V_{i_3} = \varnothing$.

It is enough to show that there is a system of edge-disjoint $s \to t$ paths $\{Q_i\}_{i\in I}$ in G such that Q_i's first edge is ss_i because then we can construct the desired paths $\{P_i'\}_{i\in I}$ in a natural way. Indeed, let $i \in I$ arbitrary and assume first that P_i starts in $B \cap V_i$. Let the last edge of Q_i is f_j for some $j \in I$. Let Q_i' the path that we get from Q_i by deleting the first two (s and s_i) and the last (t) vertices and replace the edges in the form e_{lwz} with the $w \to z$ segment of P_l which instead we put e_{lwz} before. We get P_i' by uniting Q_i' with the terminal segment of P_j that starts at v_j. If P_i starts outside B we do the same except we have to use also the initial segment of P_i that ends in u_i to the construction of P_i'.

For $k < i \in I$ we can construct from P_i a $s \to t$ path R_i in G in a natural way. Indeed, take the segment of P_i between u_i ad v_i and replace the segments of it that leave B with the appropriate edges e_{lwz}, finally give two new initial and a new last vertex to it namely s, s_i and t. Then $\{R_i\}_{k<i\in I}$ is a system of edge-disjoint $s \to t$ paths in G such that the first edge of R_i is ss_i and $f_1, \ldots, f_k$ are unused by the path-system. Try to extend this system by using Proposition 4.4. If it succeeds, then iterate this with the resulting path-system. Assume that it do not. Then $\mathsf{start}(f_1) \notin U$ (we use here the notations of Proposition 4.4) since f_1 is a forward edge of $\overleftarrow{G}$. The vertex $\mathsf{start}(f_1)$ is simultaneously reachable from $\mathcal{V}_1$ in $D_1[B]$ which implies by construction of G that there is a system of edge-disjoint $s \to \mathsf{start}(f_1)$ paths in G that uses all the outgoing edges of s. Take the initial segments of these paths that goes from s to $V(G) \setminus U$ and extend them by using the second part of Proposition 4.4 to a system of edge-disjoint $s \to t$ paths that uses all the outgoing edges of s.

□

5 Characterization of infeasible single-edge extensions

Our only debt is the proof of Claim 4.1. For finite λ one can use the equivalent formulation of path condition (condition (1)). In this case tightness of a set X means that X satisfies with equation the inequality at (1). Consider first Claim 4.1 with finite λ. Assume that by extending with edge e we violate condition (1) and let B be a witness of it. Then necessarily $\varrho_{D'}(B) < \varrho_D(B)$ (thus $e \in \mathsf{in}_D(B)$) and $|\{i < \lambda : V_i \cap B = \varnothing\}| = |\{i < \lambda : V_i' \cap B = \varnothing\}|$ hence B intersects V_0. Finally $\varrho_D(B) = |i < \lambda : V_i \cap B = \varnothing|$ (because the extension can worsen the inequality (1) at most by one) which implies the dangerousness of B.
In the general case where the cardinal λ can be infinite the Claim is less trivial. We present a proof of it in the rest of the section.

Proof of Claim 4.1. For the easy direction assume, seeking contradiction, that $\mathcal{V}'$ does not violate the path condition in D', but there exists a $\mathcal{V}$-dangerous set B such that $e \in \mathsf{in}_D(B)$. Then there is a $\{P_i\}_{i<\lambda}$ system of edge-disjoint paths in D' such that P_i goes from V_i' to B. We may assume that P_0 consists of one vertex u where $u \in B \cap V_0$ hence P_i goes from V_i to B in D for all $i < \lambda$. The edge $e \in \mathsf{in}_D(B)$ is unused by the paths P_i which contradicts with the tightness of B.

For the other direction assume that t certifies that $\mathcal{V}'$ violates the path condition in D'. We have to show that there is a $B \subset V$ such that $e \in \mathsf{in}_D(B)$ and B is dangerous with respect to $\mathcal{V}$.

Definition 5.1. Let $\mathcal{T} = \{V_i\}_{i\in I}$ be a multiset of nonempty subsets of V and let $D = (V, A)$ be a digraph. Pick a new vertex s_i for all $i \in I$ and draw one edge from s_i to all the elements of V_i. Finally pick a new vertex s and draw one edge from s to s_i $(i \in I)$. We denote the resulting digraph by $D_{\mathcal{T}}$.

Note that we can get $D'_{\mathcal{V}'}$ from $D_{\mathcal{V}}$ by deleting e and adding the edge $s_0\mathsf{end}(e)$.

Let $\mathcal{P} = \{P_i\}_{i<\lambda}$ be a system of edge-disjoint $s \to t$ paths in $D_{\mathcal{V}}$ such that the first edge of P_i is ss_i (see Figure 3). Note that, there must be a $0 \neq i_{\mathcal{P}} < \lambda$ such that $e \in A(P_{i_{\mathcal{P}}})$ otherwise t would be simultaneously reachable from $\mathcal{V}'$ in D'.

Observation 5.2. There is no system of edge-disjoint $s \to t$ paths in $D'_{\mathcal{V}'}$ that uses all the outgoing edges of s (since t is not simultaneously reachable from $\mathcal{V}'$ in D').

For $i_{\mathcal{P}} \neq i < \lambda$ path P_i is a path in $D'_{\mathcal{V}'}$ too and it does not use edge $s_0\mathsf{end}(e)$ since $s_0\mathsf{end}(e)$ is not an edge of $D_{\mathcal{V}}$. We denote by $U = \boldsymbol{U(\mathcal{P})}$ the set of vertices in $V(D'_{\mathcal{V}'})$ that are reachable from s in the digraph $H = \boldsymbol{H(\mathcal{P})}$ what we get by changing the direction of edges $\cup_{i_{\mathcal{P}}\neq i<\lambda} A(P_i)$ in $D'_{\mathcal{V}'}$. On one hand, $\mathsf{start}(e) \in U$, it is certified by the initial segment of $P_{i_{\mathcal{P}}}$ that ends at $\mathsf{start}(e)$. On the other hand, $t \notin U$ because of Observation 5.2 and Proposition 4.4. Furthermore the vertices of the terminal segment of $P_{i_{\mathcal{P}}}$ that starts at $\mathsf{end}(e)$ are disjoint from U otherwise $t \in U$ would also hold. It implies that $s_0 \notin U$ since $s_0\mathsf{end}(e)$ is a forward edge of H. It follows $V_0 \setminus U \neq \varnothing$ since the third vertex, say v, of P_0 cannot be in U because vs_0 is a backward-edge of H.

We claim that $s_i \in U$ iff $V'_i \subset U$. For $i_{\mathcal{P}}$ it is obviously holds since $\{s_{i_{\mathcal{P}}}\} \cup V'_{i_{\mathcal{P}}} \subset U$. Let $i_{\mathcal{P}} \neq i < \lambda$. Assume that $s_i \in U$. All but one edges in $\{s_i v : v \in V'_{i_{\mathcal{P}}}\}$ are forward edges of H, and we necessarily reach from s in H the exception too, because the only ingoing edge of s_i in H comes from there. For the other direction if $s_i \notin U$, then the third vertex, say v, of the path P_i cannot be in U because then vs_i would be an outgoing edge of U in H which contradicts with the definition of U.

Let $\mathcal{Q} = \boldsymbol{\mathcal{Q}(\mathcal{P})} = \{Q_f : f \in \mathsf{out}(U)\}$ where Q_f is the terminal segment of the unique P_i that contains f such that Q_f's first edge is f. We claim that it is well defined and $V(Q_f) \cap U = \{\mathsf{start}(f)\}$. On the one hand for e we have $e \in A(P_{i_{\mathcal{P}}})$ and we have already observed that the terminal segment of $P_{i_{\mathcal{P}}}$ that starts at $\mathsf{end}(e)$ is disjoint from U. On the other hand for an $e \neq f \in \mathsf{out}_{D_{\mathcal{V}}}(U)$: since $\mathsf{out}_H(U) = \varnothing$ there is an $i < \lambda$ such that $f \in A(P_i)$; and $A(Q_f) \cap \mathsf{in}_{D_{\mathcal{V}}}(U) = \varnothing$ also holds since $\mathsf{in}_{D_{\mathcal{V}}}(U) = \mathsf{in}_{D'_{\mathcal{V}'}}(U)$ (because neither e in $D_{\mathcal{V}}$ nor $s_0\mathsf{end}(e)$ in $D'_{\mathcal{V}'}$ is an ingoing edge of U) and $\mathsf{out}_H(U) = \varnothing$.

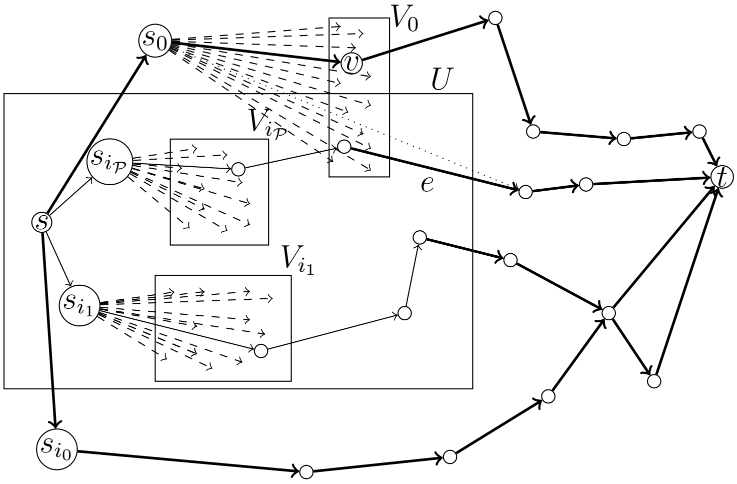

Figure 3: Some parts of $D_\mathcal{V}$ that are relevant with respect to the construction above. The dotted edge from s_0 to $\mathsf{end}(e)$ is not in $D_\mathcal{V}$ but in $D'_{\mathcal{V}'}$. The figure contains four paths from $\mathcal{P}$. Their thickened terminal segments are elements of $\mathcal{Q}$.

We define by transfinite recursion two sequences U_α and $\mathcal{Q}_\alpha$. We demand the following properties for all $i < \lambda$ and for all $\beta < \alpha$:

1. (a) $s_0, t \notin U_\beta \subset V(D_\mathcal{V}),\ e \in \mathsf{out}_{D_\mathcal{V}}(U_\beta),\ V_0 \setminus U_\beta \neq \varnothing$,
   (b) $s_i \in U_\beta \iff V'_i \subset U_\beta$ ($\iff V_i \subset U_\beta$ because of $1(a)$),
   (c) $U_\beta \supsetneq U_\alpha$,
2. (a) $\mathcal{Q}_\alpha = \{Q^\alpha_f : f \in \mathsf{out}_{D_\mathcal{V}}(U_\alpha)\}$ is a system of edge-disjoint paths from U_α to t in $D_\mathcal{V}$ such that the first edge of Q^α_f is f.
   (b) $\mathsf{in}_{D_\mathcal{V}}(U_\beta) \cap A(Q^\alpha_f) = \varnothing$,
   (c) the terminal segment of Q^α_f that goes from U_β to t is an element of $\mathcal{Q}_\beta$.

Let $\mathcal{P} = \{P_i\}_{i<\lambda}$ be a system of edge-disjoint $s \to t$ paths in $D_\mathcal{V}$ such that the first edge of P_i is ss_i and let $U_0 = U(\mathcal{P})$ and $\mathcal{Q}_0 = \mathcal{Q}(\mathcal{P})$. We have already showed that it satisfies properties $1(a, b), 2(a)$ (for $\alpha = 0$ that is all what we need). Suppose that U_β and $\mathcal{Q}_\beta$ have already defined if $\beta < \alpha$ for some $0 < \alpha$ in such a way that it satisfies the properties $1(a, b, c), 2(a, b, c)$.

Assume first that α is a limit ordinal. Let $U_\alpha = \cap_{\beta<\alpha} U_\beta$. The properties $1(a, b, c)$ obviously hold. If $f \in \mathsf{out}_{D_\mathcal{V}}(U_\alpha)$, then there is a $\delta < \alpha$ such that $f \in \mathsf{out}_{D_\mathcal{V}}(U_\delta)$, so we may let $Q^\alpha_f \overset{\text{def}}{=} Q^\delta_f$. Here $V(Q^\alpha_f) \cap U_\alpha = \{\mathsf{start}(f)\}$ since $V(Q^\delta_f) \cap U_\delta = \{\mathsf{start}(f)\}$ and $U_\delta \supset U_\alpha$ thus Q^α_f goes from U_α to t .

Observe that $Q^\delta_f = Q^\gamma_f$ for all $\delta < \gamma < \alpha$, it is implied by property $2(c)$ for δ and γ and by the fact that the terminal segment of Q^γ_f that goes from U_δ to t is Q^γ_f itself. We claim that the paths in $\mathcal{Q}_\alpha$ are edge-disjoint. Indeed, let $Q^\beta_f, Q^\gamma_g \in \mathcal{Q}_\alpha$ arbitrary where $\beta \leq \gamma < \alpha$ and assume that $h \in A(Q^\beta_f) \cap A(Q^\gamma_g)$. By our observation above $Q^\beta_g = Q^\gamma_g$ thus $h \in A(Q^\gamma_f) \cap A(Q^\gamma_g)$ hence by property $2(a)$ necessarily $Q^\gamma_g = Q^\gamma_f$

therefore $Q_g^\gamma = Q_f^\beta$. To prove $2(b,c)$ pick a γ such that $\beta < \gamma < \alpha$ and $Q_f^\alpha = Q_f^\gamma$ and apply property $2(b,c)$ to γ and β.

Let now $\alpha = \gamma + 1$. If $V \setminus U_\gamma$ is tight in D with respect to $\mathcal{V}$, then by property $1(a)$ we know that $e \in \mathsf{in}_D(V \setminus U_\gamma)$ and $V_0 \setminus U_\gamma \neq \varnothing$ therefore the set $B \stackrel{\text{def}}{=} V \setminus U_\alpha$ satisfies the conditions of Claim 4.1 and we are done. Otherwise $V \setminus U_\gamma$ is not tight in D with respect to $\mathcal{V}$, thus let $I = \{i < \lambda : V_i \subset U_\gamma\}$ and we may fix a system of edge-disjoint paths $\{P_i'\}_{i \in I}$ in D such that P_i' goes from V_i to $V \setminus U_\gamma$ and there is an edge-set $\varnothing \neq F \subset \mathsf{out}_D(U_\gamma)$ such that the elements of F are unused by the paths P_i'. Let $P_i^* = ss_iP_i'$ if $i \in I$ and ss_i otherwise. By property $1(b)$ $s_i \in U_\gamma$ iff $V_i \subset U_\gamma$ thus $\{P_i^*\}_{i<\lambda}$ is a system of edge-disjoint paths in $D_\mathcal{V}$ such that P_i^* goes from s to $V \setminus U_\gamma$ and the first edge of P_i^* is ss_i. For all $i < \lambda$ unite P_i^* with the unique element of $\mathcal{Q}_\gamma$ whose first edge is the last edge of P_i^* and denote the result by P_i^γ. Let $\mathcal{P}_\gamma = \{P_i^\gamma\}_{i<\lambda}$, $U_{\gamma+1} = U(\mathcal{P}_\gamma)$ and $\mathcal{Q}_{\gamma+1} = \mathcal{Q}(\mathcal{P}_\gamma)$ (see Figure4).

Properties $1(a,b)$ and $2(a)$ hold as we have seen it for arbitrary $\mathcal{P}$. From the construction is clear that paths P_i^γ have no ingoing edges to U_γ hence $\mathsf{out}_{H(\mathcal{P}_\gamma)}(U_\gamma) = F$. If for some $f \in F$ the vertex $\mathsf{start}(f)$ would be in $U_{\gamma+1}$, then Q_f^γ shows that $t \in U_{\gamma+1}$ which is false for arbitrary $\mathcal{P}$. This proves $U_\gamma \supsetneq U_{\gamma+1}$.

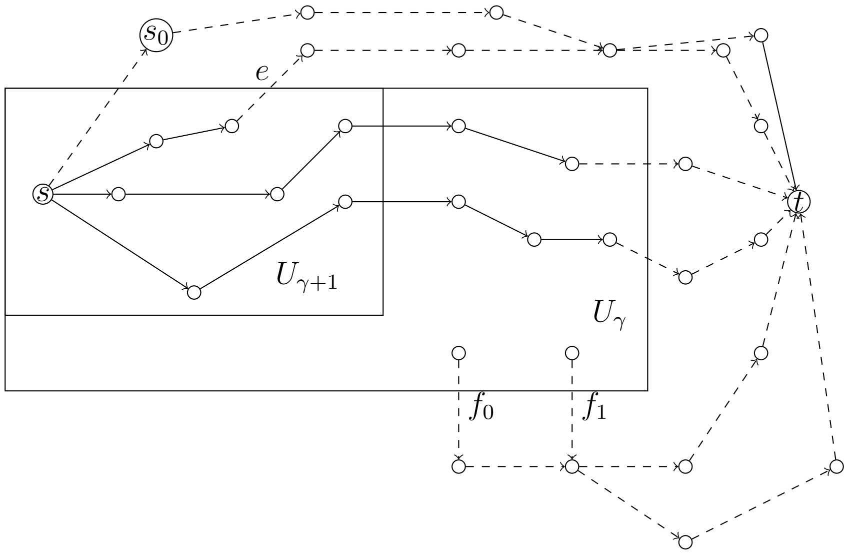

Figure 4: There are four paths of $\mathcal{P}_\gamma$ in the figure. The dashed paths are elements of $\mathcal{Q}_\gamma$, two of them are not used in the construction of $\mathcal{P}_\gamma$ and $f_0, f_1 \in F$.

To show $2(b)$ for $\gamma + 1$ let $\beta < \gamma + 1$ and assume (reducatio ad absurdum) that $h \in A(Q_f^{\gamma+1}) \cap \mathsf{in}_{D_\mathcal{V}}(U_\beta)$. Since $\mathsf{in}_{D_\mathcal{V}}(U_\beta) \cap \mathsf{span}_{D_\mathcal{V}}(U_\gamma) = \varnothing$ (because $U_\beta \supset U_\gamma$) from the construction of $\mathcal{Q}_{\gamma+1}$ it follows that $h \in Q_g^\gamma$ for some $Q_g^\gamma \in \mathcal{Q}_\gamma$. But then β and γ violates property $2(b)$ if $\beta < \gamma$ and γ violates $2(a)$ if $\beta = \gamma$. The proof of $2(c)$ for $\gamma + 1$ is similar. This concludes the description of the recursion.

On the one hand, the recursion stops iff we find the desired B in some successor step. On the other hand property $1(c)$ ensures that it will happen after less than $|V(D_\mathcal{V})|^+$ steps since there is no sequence of strictly decreasing subsets of $V(D_\mathcal{V})$ with length $|V(D_\mathcal{V})|^+$. □

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