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Non-TDI graph-optimization with supermodular functions (extended abstract)

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Non-TDI graph-optimization with supermodular functions (extended abstract)

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Abstract

Total dual integrality (TDI-ness) is a major concept in attacking various combinatorial optimization problems. Here we develop several new min-max theorems and good characterizations in graph theory where the minimum cost extension is already NP-complete, implying that such problems cannot be described by TDI linear systems. The main device is a min-max theorem of Frank and Jordán on covering a supermodular function by digraphs.

Keywords: supermodular functions, packing arborescences, graphical degree sequences, term rank

1 Introduction

A graph optimization problem aims at developing an algorithm to find a specified object in a graph which is optimal with respect to an objective function. Along the way, one also strives for finding a min-max relationship. A typical example is the Hungarian method to compute a maximum cardinality/weight matching in a bipartite graph (or for short, a **bigraph**) along with the min-max theorems of Kőnig and Egerváry. In a feasibility problem, one is to decide if a graph includes a specified object. Here the goal is again to find an algorithm and a good characterization for the existence. Feasibility problems can typically be reformulated as optimization problems and hence we consider them as part of graph optimization.

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An essential part of polynomially tractable graph optimization problems can be be solved via polyhedral approaches, and in particular, totally dual integral (TDI) linear systems play a central role. For example, matching problems, network flows, matroid optimization problems belong to this class. A basic feature of a problem describable by a TDI system is that not only the optimum cardinality version of the problem is tractable but its weighted (or minimum cost) extension as well. One of the most effective TDI frameworks is the one of submodular flows introduced by Edmonds and Giles [4].

There are however optimization problems where the cardinality case is nicely solvable but its min-cost extension is already NP-complete, and therefore a TDI-description in such a case is out of question. One of the first of this kind of results is due to Eswaran and Tarjan [5] who provided a min-max formula for the minimum number of new arcs necessary to make a digraph strongly connected.

Recently, however, it turned out that non-TDI min-max results showed up much earlier. In 1958, Ryser [21] solved the maximum term rank problem (which is equivalent to finding a a degree-specified simple bigraph G in which the the matching number $\nu(G)$ of G is as large as possible). The minimum cost version of this problem had not been settled for a long time but recently Pálvölgyi [19] proved its NP-completeness.

Since these two early appearances, several other non-TDI min-max theorems have been developed. For example, the second author of the present paper extended the result of Eswaran and Tarjan in [10] to minimal k-edge-connected augmentation. The paper [11] included an abstract generalization concerning the covering of a crossing supermodular set-function by a minimal digraph. Frank and Jordán [12] generalized the main result of [11] much further and proved a min-max theorem on optimally covering a so-called supermodular bi-set function by digraphs.

It should be emphasized that this framework characteristically differs from previous models such as submodular flows, since it solves such cardinality optimization problems for which the corresponding weighted versions are **NP**-complete. One of the most important applications was a solution to the minimum directed node-connectivity augmentation problem, but several other problems could also be treated in this way.

The analogous arc-covering problems using *simple* digraphs have been open so far. After pointing out that the problem of supermodular coverings with simple digraph includes **NP**-complete special cases, we describe situations for the supermodular arc-covering theorem when simplicity can successfully be treated. As a consequence, several new non-TDI min-max theorems will be derived.

All notions and notation not mentioned explicitly in the paper can be found in the book of the second author [8]. A set-function p is **positively ST-crossing supermodular** if if the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

holds whenever both $p(X)$ and $p(Y)$ are positive, X and Y are ST-crossing. A digraph $D = (V, A)$ **covers** a set-function p if $\varrho_D(X) \geq p(X)$ holds for every subset $X \subseteq V$.

Theorem 1.1 (Supermodular arc-covering, set-function version, [12])**.** *A positively ST-crossing supermodular function p can be covered by γ ST-edges if and only if $\widetilde{p}(\mathcal{I}) \leq \gamma$ holds for every ST-independent family $\mathcal{I}$ of subsets of V.*

Several applications of this theorem have been found but the question remained open if there is a min-max theorem on optimally covering p with a *simple* digraph. Such a result would be important since in many possible application simplicity is a natural requirement. However, based on a result of Dürr, Guinez, and Matamala [2], we observed the following.

Theorem 1.2. *Let S and T be two disjoint sets and p an ST-crossing supermodular function. Then the decision problem whether p can be covered by a degree-specified simple digraph D consisting of ST-arcs includes* **NP***-complete problems.*

The main goal of the present paper is to exhibit special situations when simplicity can be expected.

Let S and T be two disjoint sets and $V := S \cup T$. We are given a non-negative integer-valued function $m : V \to \mathbf{Z}_+$ whose restrictions to S and to T are denoted by m_S and m_T, respectively. We also use the notation $m = (m_S, m_T)$. It is assumed throughout that $\widetilde{m}_S(S) = \widetilde{m}_T(T)$ and this common value will be denoted by γ. We say that m or the pair (m_S, m_T) is a **degree-specification** and that a bigraph $G = (S, T; E)$ **fits** this degree-specification if $d_G(v) = m(v)$ holds for every node $v \in V$.

Let $g_S : S \to \mathbf{Z}_+$ and $g_T : T \to \mathbf{Z}_+$ be upper bound functions while $f_S : S \to \mathbf{Z}_+$ and $f_T : T \to \mathbf{Z}_+$ lower bound functions. Let $f = (f_S, f_T)$ and $g = (g_S, g_T)$. Call a bipartite graph $G = (S, T; E)$ (f_T, g_S)**-feasible** if

$$d_G(s) \le g_S(s) \text{ for every } s \in S \text{ and } d_G(t) \ge f_T(t) \text{ for every } t \in T$$

and call G (f, g)**-feasible** if $f_S(s) \le d_G(s) \le g_S(s)$ for every $s \in S$ and $f_T(t) \le d_G(t) \le g_T(t)$ for every $t \in T$, or for short, $f \le d_G \le g$.

Let $\mathcal{G}(m_S, m_T)$ denote the set of simple bipartite graphs fitting (m_S, m_T). Gale [13] and Ryser [20] found, in an equivalent form, the following characterization.

Theorem 1.3 (Gale and Ryser)**.** *There is a simple bipartite graph G fitting the degree-specification m if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| \le \gamma \textit{ whenever } X \subseteq S,\ Z \subseteq T. \qquad (1)$$

Moreover, (1) *holds if the inequality is required only when X consists of elements with the i largest values of m_S and Z consists of elements with the j largest values of m_T $(i = 1, \dots, |S|,\ j = 1, \dots, |T|)$.*

The following linking property can be derived with network flow techniques but is is actually a special instance of a general linking property concerning g-polymatroids.

Theorem 1.4. *If there is a simple (f_T, g_S)-feasible bipartite graph and there is a simple (f_S, g_T)-feasible bipartite graph, then there is a simple (f, g)-feasible bipartite graph.*

2 Bipartite graphs covering supermodular functions

2.1 Covering p_T with simple degree-specified bipartite graphs

Suppose now that we are given a set-function p_T on T. We say that a bipartite graph $G = (S, T; E)$ **covers** p_T if

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{2}$$

We are interested in finding simple bipartite graphs covering p_T which meet some degree constraints. If no such degree-constraints are imposed at all then the existence of such a G is obviously equivalent to the requirement that

$$p_T(Y) \leq |S| \text{ for every } Y \subseteq T. \tag{3}$$

Indeed this condition is clearly necessary and it is also sufficient since the complete bipartite graph $G^* = (S, T; E^*)$ covers a p_T meeting (3). Therefore we suppose throughout that (3) holds.

2.1.1 Degree-specification on S

Our first goal is to characterize the situation when there is a degree-specification only on S.

Theorem 2.1. *Let m_S be a degree-specification on S for which $\widetilde{m}_S(S) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. Suppose that*

$$m_S(s) \leq |T| \textit{ for every } s \in S. \tag{4}$$

The following statements are equivalent.

(A) *There is a simple bipartite graph $G = (S, T; E)$ covering p_T and fitting the degree-specification m_S on S.*

(B1)

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p_T(T_i) - |X|] \leq \gamma \tag{5}$$

holds for every $X \subseteq S$ and subpartition $\mathcal{T} = \{T_1, \ldots, T_q\}$ of T.

(B2)

$$\sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), q\} \tag{6}$$

holds for every subpartition $\mathcal{T} = \{T_1, \ldots, T_q\}$ of T.

Proof. **(A) $\Rightarrow$ (B1)** Suppose that there is a simple bipartite graph G meeting (2). We claim that the number $d_G(T_i, S - X)$ of edges between T_i and $S - X$ is at least $p_T(T_i) - |X|$. Indeed,

$$p_T(T_i) \leq |\Gamma_G(T_i)| = |X \cap \Gamma_G(T_i)| + |\Gamma_G(T_i) - X| \leq |X| + d_G(T_i, S - X),$$

that is, $d_G(T_i, S - X) \geq p_T(T_i) - |X|$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X) + \sum_i [p_T(T_i) - |X|]$ from which (5) follows.

(B1) ⇒ (B2) Suppose that (B2) is violated and there is a subpartition $\mathcal{T} = \{T_1, \ldots, T_q\}$ of T for which $\sum_{i=1}^q p_T(T_i) > \sum_{s \in S} \min\{m_S(s), q\}$. Let $X := \{s \in S : m_S(s) > q\}$. Then

$$\sum_{i=1}^{q} p_T(T_i) > \sum_{s \in S} \min\{m_S(s), q\} = \sum [m_S(s) : s \in S - X] + q|X| =$$

$$\widetilde{m}_S(S - X) + q|X| = \gamma - \widetilde{m}_S(X) + q|X|$$

from which

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p_T(T_i) - |X|] > \gamma,$$

that is, (B1) is violated.

(B2) ⇒ (B1) Suppose that X and $\mathcal{T} = \{T_1, \ldots, T_q\}$ violate (5), that is, $\widetilde{m}_S(X) + \sum_{i=1}^q [p_T(T_i) - |X|] > \gamma$. We can assume that $m_S(s) > q$ for every $s \in X$ for if $m_S(s) \leq q$ for some $s \in X$, then $X' := X - s$ and $\mathcal{T}$ would also violate (5). Furthermore, we can assume that $m_S(s) \leq q$ for every $s \in S - X$ for if $m_S(s) > q$ for some $s \in S - X$, then $X' := X + s$ would also violate (5).

Therefore

$$\sum_{s \in S} \min\{m_S(s), q\} = \widetilde{m}_S(S - X) + q|X| = \gamma - \widetilde{m}_S(X) + q|X|.$$

By combining this with $\widetilde{m}_S(X) + \sum_{i=1}^q p_T(T_i) - q|X| > \gamma$ we have

$$\sum_{i=1}^{q} p_T(T_i) > \gamma - \widetilde{m}_S(X) + q|X| = \sum_{s \in S} \min\{m_S(s), q\},$$

that is, (B2) is violated.

(B1) ⇒ (A) The proof is presented in the Appendix.

□

We remark that Theorem 2.1 cannot be extended for the case when the graph G is requested to be a subgraph of an initial bigraph since this formulation includes NP-complete problems.

2.1.2 Covering p_T with degree-specification on S and T

In the next problem we have degree-specification not only on S but on T as well. When the degree-specification was given only on S, we have observed that it sufficed to concentrate on finding a not-necessarily simple graph covering p_T because such a graph could easily be made simple. Based on this, it is tempting to conjecture that if there is a simple bipartite graph fitting a degree-specification $m = (m_S, m_T)$ and there is a (not-necessarily simple) one fitting m and covering p_T, then there is a simple bipartite graph fitting m and covering p_T. This kind of linking property however can be shown to fail. The next theorem includes the right answer.

Theorem 2.2. *Let S and T be disjoint sets and let $m = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. There is a simple bipartite graph $G = (S, T; E)$ fitting the degree-specification m such that*

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T \tag{7}$$

if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + \sum_{i=1}^{q}[p_T(T_i) - |X|] \leq \gamma \tag{8}$$

holds for every pair of subsets $X \subseteq S$ and $Z \subseteq T$ and for subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of $T - Z$.

The proof consists of building m_T into p_T and applying Theorem 2.1.

Corollary 2.3. *Let S and T be disjoint sets and let $m = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. There is a simple bipartite graph $G = (S, T; E)$ fitting the degree-specification m and covering p_T if and only if*

$$\widetilde{m}_T(Z) + \sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), |Z| + q\} \tag{9}$$

holds for every subset $Z \subseteq T$ and subpartition $\{T_1, \dots, T_q\}$ of $T - Z$.

2.2 Upper and lower bounds

Instead of a degree-specification $m = (m_S, m_T)$, one may impose more generally a lower bound $f = (f_S, f_T)$ and an upper bound $g = (g_S, g_T)$ for the degrees of the simple bipartite graph covering p_T. We assume that $f \leq g$. Also, since the graph G is requested to be simple, we may assume in advance that $g_S(s) \leq |T|$ for every $s \in S$ and $g_T(t) \leq |S|$ for every $t \in T$.

Theorem 2.4. *There is a simple bipartite graph $G = (S, T; E)$ covering p_T for which*

$$f(v) \leq d_G(v) \leq g(v) \text{ holds for every } v \in S \cup T \tag{10}$$

if and only if

$$\widetilde{f}_T(Z) - |X||Z| + \sum[p_T(T_i) - |X| : i = 1, \dots, q] \leq \widetilde{g}_S(S - X) \tag{11}$$

holds whenever $X \subseteq S$, $Z \subseteq T$, and $\{T_1, \dots, T_q\}$ is a subpartition of $T - Z$, and

$$\widetilde{f}_S(X) - |X||Z| + \sum[p_T(T_i) - |X| : i = 1, \dots, q] \leq \widetilde{g}_S(T - Z) \tag{12}$$

holds whenever $X \subseteq S$, $Z \subseteq T$, and $\{T_1, \dots, T_q\}$ is a subpartition of $T - Z$.

Corollary 2.5. *There is a simple bipartite graph G covering p_T such that*

(A) $f_T(t) \leq d_G(t)$ *for every* $t \in T$ *and* $d_G(s) \leq g_S(s)$ *for every* $s \in S$ *if and only if* (11) *holds,*

(B) $f_S(s) \leq d_G(s)$ *for every* $s \in S$ *and* $d_G(t) \leq g_T(t)$ *for every* $t \in T$ *if and only if* (12) *holds,*

(AB) $f(v) \leq d_G(v) \leq g(v)$ *holds for every* $v \in S \cup T$ *if and only if both* (11) *and* (12) *hold whenever* $X \subseteq S$, $Z \subseteq T$, *and* $\{T_1, \dots, T_q\}$ *is a subpartition of* T.

The proof of Theorem 2.4 relies on Theorem 2.2 and it requires an additional, rather difficult reduction technique. It should be noted that the linking property formulated in the Corollary does not seem to follow from the general linking property of g-polymatroids.

2.3 Matroidal extension

By using a similar proof technique, we obtain a matroidal extension of Theorem 2.2.

Theorem 2.6. *Let S and T be disjoint sets and let $m = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. Let $M = (S, r)$ be a matroid on S with rank function r. There is a simple bipartite graph $G = (S, T; E)$ fitting the degree-specification m such that*

$$r(\Gamma_G(Y)) \geq p_T(Y) \text{ for every subset } Y \subseteq T \tag{13}$$

if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + \sum_{i=1}^{q} [p_T(T_i) - r(X)] \leq \gamma \tag{14}$$

holds for every pair of subsets $X \subseteq S$ and $Z \subseteq T$ and for subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of $T - Z$.

3 Packing branchings and arborescences

The starting point is the classic result of Edmonds [3].

Theorem 3.1 (Edmonds)**.** *Let $D = (V, A)$ be a digraph.* (A: weak form) *Let r_0 be a specified root-node of D. There are k disjoint spanning arborescences of root r_0 if and only if $\varrho(X) \geq k$ holds for every non-empty subset $X \subseteq V - r_0$.* (B: strong form) *Let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of non-empty subsets of V. There are k disjoint branchings $B_1, B_2, \dots, B_k$ with root-sets $R_1, \dots, R_k$, respectively, if and only if $\varrho(X) \geq p_{\mathcal{R}}(X)$ holds for every non-empty subset $X \subseteq V$ where $p_{\mathcal{R}}(X)$ denotes the number of root-sets disjoint from X.*

There are several variations and extensions of Edmonds' theorem (for surveys, see [8], [16]). For example, with the help of Lovász' proof [17] of Edmonds' theorem, one can show that the k arborescences in the weak form can be chosen in such a way that they have essentially the same number of root-edges. When this result is combined with a min-max formula [9] on the minimum number of root-edges of a rooted k-edge-connected subgraph of a digraph one arrives at the following consequence.

Theorem 3.2. *Let $\mu \leq |V|-1$ be a positive integer. A digraph $D=(V,A)$ comprises k disjoint branchings of μ arcs if and only if*

$$\sum[\varrho_D(V_i) : i=1,\dots,k] \geq k[q-(|V|-\mu)] \tag{15}$$

holds for every subpartition $\{V_1,\dots,V_q\}$ of V.

Our new contribution is a generalization of this result when the prescribed sizes of the k branchings may be different.

Theorem 3.3. *Let $\mu_i \leq |V|-1$ be positive integers for $i=1,\dots,k$. A digraph $D=(V,A)$ comprises k disjoint branchings $B_1,\dots,B_k$ for which $|B_i|=\mu_i$ for $i=1,\dots,k$ of μ arcs if and only if*

$$\sum[\varrho_D(V_i) : i=1,\dots,k] \geq \sum[q-(|V|-\mu_i) : i=1,\dots,k] \tag{16}$$

holds for every subpartition $\{V_1,\dots,V_q\}$ of V.

The proof follows immediately by combining the strong form of Edmonds' theorem with Theorem 2.1. It is interesting to remark that a question on bipartite matchings analogous to Theorem 3.2 was answered by Folkman and Fulkerson [7] who characterized bipartite graphs comprising k disjoint matchings of size μ. On the other hand, there is no hope to find a matching-counterpart of Theorem 3.3 since the problem of packing k matchings with prescribed cardinalities in a bipartite graph was recently shown, by Pálvölgyi [19], to be NP-complete even for $k=2$.

By applying the more general result of Theorem 2.4, we derived characterizations for digraphs including k disjoint branchings $B_1,\dots,B_k$ so that $\varphi_i \leq |B_i| \leq \gamma_i$, where φ_i and γ_i $(i=1,\dots,k)$ are specified lower and upper bounds, respectively, and so that each node $v \in V$ belongs to at least $f(v)$ and at most $g(v)$ root-sets of the k branchings, where f and g are lower and upper bounds on V.

4 Maximum term rank problems

Let S and T be disjoint sets and $m=(m_S,m_T)$ a degree specification with $\widetilde{m}_S(S)=\widetilde{m}_T(T)=\gamma$ so that there is a simple bipartite graph fitting m. Let $\mathcal{G}(m_S,m_T)$ denote the set of all such graphs.

Members of $\mathcal{G}(m_S,m_T)$ can be identified with $(0,1)$-matrices of size $|S||T|$ with row sum vector m_S and column sum vector m_T. Let $\mathcal{M}(m_S,m_T)$ denote the set of these matrices. Ryser [21] defined the **term rank** of a $(0,1)$-matrix M by the maximum number of independent 1's which is the matching number of the bipartite graph corresponding to M. Ryser developed a formula for the maximum term rank of matrices in $\mathcal{M}(m_S, m_T)$. The maximum term rank problem is equivalent to finding a bipartite graph G in $\mathcal{G}(m_S, m_T)$ whose matching number $\nu(G)$ is as large as possible. Ryser's theorem is equivalent to the following.

Theorem 4.1 (Ryser)**.** *Let $\ell \leq |T|$ be an integer. Suppose that $\mathcal{G}(m_S, m_T)$ is non-empty, that is,* (1) *holds. Then $\mathcal{G}(m_S, m_T)$ has a member G with matching number $\nu(G) \geq \ell$ if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + (\ell - |X| - |Z|) \leq \gamma \text{ whenever } X \subseteq S,\ Z \subseteq T. \qquad (17)$$

Moreover, (17) *holds if the inequality is required only when X consists of the i largest values of m_S and Z consists of the j largest values of m_T $(i = 1, \ldots, |S|, j = 1, \ldots, |T|)$.*

4.1 Degree-constrained max term rank

Our first goal is to extend Ryser's theorem for the case when upper or lower bounds are given for the degrees rather than exact prescriptions. Let $f = (f_S, f_T)$ and $g = (g_S, g_T)$ be lower and upper bound functions respectively for which $f \leq g$. Assume that $g_S(s) \leq |T|$ for every $s \in S$ and $g_T(t) \leq |S|$ for every $t \in T$.

Theorem 4.2. *Let $\ell \leq |T|$ be an integer. Let $g_S : S \to \mathbf{Z}_+$ be an upper bound function on S for which $g_S(s) \leq |T|$ holds for every $s \in S$, and let $f_T : T \to \mathbf{Z}_+$ be a lower bound function on T. There is a simple bipartite graph $G = (S, T; E)$ with matching number $\nu(G) \geq \ell$ so that $d_G(s) \leq g_S(s)$ for every $s \in S$ and $d_G(t) \geq f_T(t)$ for every $t \in T$ if and only if*

$$\widetilde{f}_T(Z) - |X||Z| + (\ell - |X| - |Z|)^+ \leq \widetilde{g}_S(S - X) \text{ whenever } X \subseteq S,\ Z \subseteq T. \qquad (18)$$

Theorem 4.3. *Suppose that there is a simple (f, g)-feasible bipartite graph* (as characterized by a theorem of Gale and Ryser)*. If there is a simple (f_T, g_S)-feasible bipartite graph with matching number at least ℓ and there is a simple (f_S, g_T)-feasible bipartite graph with matching number at least ℓ, then there is a simple (f, g)-feasible bipartite graph with matching number at least ℓ.*

We have two independent proofs of these results. The first one is based on the application of Theorem 2.4 to the special supermodular function p_T defined by $p_T(Y) = |Y| - (|T| - \ell)$ $(Y \subseteq T)$. Since p_T is fully supermodular and monotone increasing, the conditions in Theorem 2.4 simplify significantly and give rise to the condition in Theorem 4.3. Our other proof is a direct algorithm to compute either a degree-constrained bigraph with $\nu(G) \geq \ell$ or a pair of subsets violating the conditions described in the theorem.

4.2 Matroidal max term rank

Our next goal is to prove a matroidal extension of Ryser's theorem. We need the following equivalent form of a theorem of Brualdi.

Theorem 4.4 (Brualdi, [1]). *Let $G = (S, T; E)$ be a bipartite graph with a matroid $M_S = (S, r_S)$ on S and with a matroid $M_T = (T, t_T)$ on T for which $r_S(S) = r_T(T) = \ell$, where r_S denotes the rank function of M_S while t_T is the co-rank function of M_T* (that is, $t_T(Y) = \ell - r_T(T - Y)$). *There is a matching of G with ℓ edges covering bases of M_S and M_T if and only if*

$$r_S(X) \geq t_T(\Gamma_G(X)) \tag{19}$$

holds for every $X \subseteq S$.

Our matroidal extension of Ryser's max term rank theorem is as follows.

Theorem 4.5. *Let S and T be two disjoint sets and $m = (m_S, m_T)$ a degree-specification on $S \cup T$ for which $m_S(S) = m_T(T) = \gamma$ and $\mathcal{G}(m_S, m_T)$ is non-empty. Let $M_S = (S, r_S)$ and $M_T = (T, r_T)$ be matroids for which $r_S(S) = t_T(T) = \ell$. There is a simple graph fitting m that includes a matching covering bases of M_S and M_T if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + (\ell - r_S(X) - r_T(Z)) \leq \gamma \tag{20}$$

holds for every $X \subseteq S$ and $Z \subseteq T$.

The proof consists of applying Theorem 2.6 to the matroid M_S and to the supermodular function p_T determined by the co-rank function of M_T. Since p_T in this case is fully supermodular and monotone increasing, the necessary condition in Theorem 2.6 simplifies to the special case when $q = 1$ and $T_1 = T - Z$. This form in turn is equivalent to (20).

5 Degree-specified higher connectivity of simple digraphs

We are given an out-degree specification $m_o : V \to \mathbf{Z}$ and an in-degree specification $m_i : V \to \mathbf{Z}$ for which $\widetilde{m}_o(V) = \widetilde{m}_i(V) = \gamma$. A digraph $D = (V, F)$ **fits** this degree-specification if $\varrho_D(v) = m_i(v)$ and $\delta_D(v) = m_o(v)$ for every node $v \in V$. The following characterization (in a simpler but equivalent form) is due to D.R. Fulkerson [6].

Theorem 5.1. *Let $m_o : V \to \mathbf{Z}$ and $m_i : V \to \mathbf{Z}$ be out- and in-degree specifications. There exists a simple digraph fitting (m_o, m_i) if and only if*

$$\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + |X \cap Z| \leq \gamma \text{ for every } X, Z \subseteq V. \tag{21}$$

The approach of [12] allows one to characterize degree sequences of digraphs D whose addition to a starting digraph D_0 results in a k-edge- or k-node-connected digraph $D_0 + D$. Jordán [15] pointed out that the problem becomes NP-complete if the resulting $D_0 + D$ is requested to be simple. Therefore we only require the simplicity of the augmenting digraph D.

Recently, Hong, Liu, and Lai [14] found a characterization for simple strongly connected digraphs. For a convenient generalization, we formulate it in a slightly more redundant though equivalent form.

Theorem 5.2 (Hong, Liu, and Lai). *Suppose that there is a simple digraph fitting the degree-specification (m_o, m_i). There is a strongly connected simple digraph fitting (m_o, m_i) if and only if*

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) + 1 - |X||Z| \leq \gamma \text{ for every disjoint } X, Z \subset V. \tag{22}$$

We extend this result in two directions.

Theorem 5.3. *Let $\mathcal{K}$ be a crossing family of non-empty proper subsets of V. Suppose that there is a simple digraph fitting the degree specification (m_o, m_i), that is, (21) holds. There is a simple digraph fitting (m_o, m_i) which covers $\mathcal{K}$ if and only if*

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{23}$$

holds for every pair of disjoint subsets $X, Z \subseteq V$ for which there is a member $K \in \mathcal{K}$ with $Z \subseteq K \subseteq V - X$.

Corollary 5.4. *Let $D_0 = (V, A_0)$ be a $(k-1)$-edge-connected digraph $(k \geq 0)$. Suppose that there is a simple digraph fitting the degree specification (m_o, m_i), There is a simple digraph fitting (m_o, m_i) for which $D_0 + D$ is k-edge-connected if and only if*

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{24}$$

holds for every pair of disjoint subsets $X, Z \subseteq V$ for which there is a subset K with $Z \subseteq K \subseteq V - X$ and with $\varrho_{D_0}(K) = k - 1$.

The more general degree-specified edge-connectivity augmentation problem when the starting digraph D_0 is not supposed to be $(k-1)$-edge-connected remains open. As for node-connectivity is concerned, we have the following straight extension of Theorem 5.3 to k-node-connected simple digraphs.

Theorem 5.5. *Let V be a set with $|V| \geq k + 1$ and let $m = (m_o, m_i)$ be a degree-specification for which there is a simple digraph fitting it. There exists a simple k-node-connected digraph fitting m if and only if*

$$\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + k \leq \gamma \text{ for } X, Z \subseteq V \text{ with } X \neq Z,\ X \neq V \neq Z. \tag{25}$$

Let $D_0 = (V, A_0)$ be a starting digraph on $|V| \geq k + 1$ nodes and consider the problem of finding a simple digraph $D = (V, F)$ fitting a degree specification (m_o, m_i) for which $D_0 + D$ is k-node-connected. This problem can be embedded into the framework of the supermodular arc-covering theorem which provides a rather complex characterization of degree sequences of the augmenting simple digraphs. However, if the starting digraph has no arc, then this characterization can be reduced (with rather heavy work) to condition (25).

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Appendix

Proof of **(B1)** $\Rightarrow$ **(A)** *in Theorem 2.1*

The following simple observation indicates that we need not concentrate on the simplicity of G.

Lemma 5.6. *If there is a not-necessarily simple bipartite graph $G = (S, T; E)$ covering p_T for which $d_G(s) \leq |T|$ for each $s \in S$, then there is a simple bipartite graph H covering p_T for which $d_G(s) = d_H(s)$ for each $s \in S$.*

Proof. Suppose G has two parallel edges e and e' connecting s and t for some $s \in S$ and $t \in T$. Since $d_G(s) \leq |T|$, there is a node $t' \in T$ which is not adjacent with s. By replacing e' with an edge st', we obtain another bipartite graph G' for which $\Gamma_{G'}(Y) \supseteq \Gamma_G(Y)$ for each $Y \subseteq T$, $d_{G'}(s) = d_G(s)$ for each $s \in S$, and the number of parallel edges in G' is smaller than in G. By repeating this procedure, finally we arrive at a requested simple graph. $\square$

A subset V' of $V = S \cup T$ is ST**-trivial** if no ST-arc enters it, which is equivalent to requiring that $T \cap V' = \emptyset$ or $S \subseteq V'$. We say that a subset $V' \subseteq V$ is **fat** if $V' = V - s$ for some $s \in S$ (that is, there are $|S|$ fat sets). The non-fat subsets of V will be called **normal**. An ST-independent family $\mathcal{I}$ of subsets is **strongly** ST**-independent** if any two of its normal members are T-independent, that is, the intersections of the normal members of $\mathcal{I}$ with T form a subpartition of T.

Define a set-function p_0 on V by

$$p_0(V') = p_T(Y) - |X| \text{ where } V' = X \cup Y \text{ for } X \subseteq S \text{ and } Y \subseteq T. \tag{26}$$

Note that p_0 is positively T-intersecting since if $p_0(V')$ is positive, then so is $p_T(Y)$. Furthermore, when (5) is applied to $X = S$, $q = 1$ and $T_1 = Y$, we obtain that $p_T(Y) \leq |X|$ and hence $p_0(V')$ can be positive only if $X \neq S$ and $Y \neq \emptyset$, that is, when V' is not ST-trivial.

Claim 5.7. *$m_S(s) \geq p_0(V - s)$ holds for every $s \in S$.*

Proof. By applying (2) to $X = S - s$ and $\mathcal{T} = \{T\}$, we obtain that $m_S(s) \geq p_T(T) - |S - s| = p_0(V - s)$. $\square$

Define a set-function p_1 on V by modifying p_0 so as to lift to its value on fat subsets $V - s$ from $p_0(V - s)$ $m_S(s)$ $(s \in S)$, that is,

$$p_1(V') := \begin{cases} m_S(s) & \text{if } V' = V - s \text{ for some } s \in S, \\ p_0(V') & \text{otherwise.} \end{cases} \tag{27}$$

Note that the supermodular inequality

$$p_1(V_1) + p_1(V_2) \leq p_1(V_1 \cap V_2) + p_1(V_1 \cup V_2) \tag{28}$$

holds for T-intersecting normal sets with $p_1(V_1) > 0$ and $p_1(V_2) > 0$.

By the Claim, $p_1 \geq p_0$. As p_0 is positively T-intersecting supermodular, p_1 is positively ST-crossing supermodular. Let ν_1 denote the maximum total p_1-value of a family of ST-independent sets. Such a family $\mathcal{I}$ will be called a p_1-**optimizer** if $\widetilde{p}_1(\mathcal{I}) = \nu_1$.

Claim 5.8. *If $\mathcal{I}$ is a p_1-optimizer of minimum cardinality, then $\mathcal{I}$ is strongly ST-independent.*

Proof. Clearly, $p_1(V') \geq 0$ for each $V' \in \mathcal{I}$ for otherwise $\mathcal{I}$ would not be a p_1-optimizer. Moreover, $p_1(V') > 0$ also holds for if we had $p_1(V') = 0$, then $\mathcal{I} - \{V'\}$ would also be a p_1-optimizer contradicting the minimality of $\mathcal{I}$.

Suppose indirectly that $\mathcal{I}$ has two properly T-intersecting normal members V_1 and V_2. Then (28) holds and since $\mathcal{I}$ is ST-independent, we must have $S \subseteq V_1 \cup V_2$ from which $p_1(V_1 \cup V_2) \leq 0$ follows. Then

$$p_1(V_1) + p_1(V_2) \leq p_1(V_1 \cap V_2) + p_1(V_1 \cup V_2) \leq p_1(V_1 \cap V_2).$$

Now $\mathcal{I}' = \mathcal{I} - \{V_1, V_2\} + \{V_1 \cap V_2\}$ is also ST-independent and $\widetilde{p}_1(\mathcal{I}') \geq \widetilde{p}_1(\mathcal{I})$, but we must have here equality by the optimality of $\mathcal{I}$, that is $\mathcal{I}'$ is also a p_1-minimizer, contradicting the minimality of $|\mathcal{I}|$. □

Lemma 5.9. *Let $\mathcal{I}$ be a strongly ST-independent p_1-optimizer. Then there exists a subset X and a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of T such that*

$$\mathcal{I} = \{V - s \in s \in X\} \cup \{X \cup T_i : i = 1, \dots, q\}$$

for which

$$\nu_1 = p_1(\mathcal{I}) = \widetilde{m}_S(X) + \sum [p_1(T_i) - |X| : i = 1, \dots, q]. \tag{29}$$

Proof. Let $X := \{s \in S : V - s \in \mathcal{I}\}$ and let $\mathcal{I}_1 = \{V - s : V - s \in \mathcal{I}\}$. Let $\mathcal{I}_2 := \mathcal{I} - \mathcal{I}_1$ and let $V_1, \dots, V_q$ denote the members of $\mathcal{I}_2$. Furthermore, let $T_i := T \cap V_i$ and $X_i = S \cap V_i$ $(i = 1, \dots, q)$. By the strong ST-independence, the family $\mathcal{T} = \{T_1, \dots, T_q\}$ is a subpartition of T, and we also have $X \subseteq X_i$ for each i.

Define $V_i' := T_i \cup X$ for $i = 1, \dots, q$ and let $\mathcal{I}_2' = \{V_1', \dots, V_q'\}$. Then $\mathcal{I}' = \mathcal{I}_1 \cup \mathcal{I}_2'$ is also ST-independent. Since $p_1(V_i') = p_1(V_i) + |X_i - X|$ and $\mathcal{I}$ is a p_1-optimizer, we must have $X_i = X$ for each $i = 1, \dots, q$. The formula in (29) follows from

$$\nu_1 = \widetilde{p}_1(\mathcal{I}) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_1(\mathcal{I}_2) =$$

$$\sum [m_S(s) : V - s \in \mathcal{I}_1] + \sum [p_T(T_i) - |X_i| : i = 1, \dots, q] =$$

$$\widetilde{m}_S(X) + \sum [p_T(T_i) - |X| : i = 1, \dots, q].$$

□

Claim 5.10. $\nu_1 = \gamma$.

Proof. Since the family $\mathcal{L} = \{V - s : s \in S\}$ is ST-independent, $\nu_1 \geq \widetilde{p}_1(\mathcal{L}) = \widetilde{m}_S(S) = \gamma$ from which $\nu_1 \geq \gamma$. Let $\mathcal{I}$ be a strongly ST-independent p_1-optimizer for which $|\mathcal{I}|$ is minimum. It follows from (29) in Lemma 5.9 and from the hypothesis (5) that $\nu_1 \leq \gamma$ and hence $\nu_1 = \gamma$. □

By Theorem 1.1, there is a digraph $D = (V, A)$ on V with $\nu_1 = \gamma$ (possibly parallel) ST-arcs that covers p_1, that is, $\varrho_D(V') \geq p_1(V')$ for every subset $V' \subseteq V$. Let $G = (S, T; E)$ denote the underlying bipartite graph of D.

Claim 5.11. *$d_G(s) = m_S(s)$ for every $s \in S$.*

Proof. Since $d_G(s) = \delta_D(s) = \varrho_D(V - s) \geq p_1(V - s) = m_S(s)$ for every $s \in S$, we have $\gamma = |E| = \sum[d_G(s) : s \in S] \geq \widetilde{m}_S(S) = \gamma$, from which $d_G(s) = m_S(s)$ follows for every $s \in S$. □

Claim 5.12. *$|\Gamma_G(Y)| \geq p_T(Y)$ for every subset $Y \subseteq T$.*

Proof. Let $X := \Gamma_G(Y)$ and $V' := X \cup Y$. Then $0 = \varrho_D(V') \geq p_1(V') \geq p_0(V') = p_T(Y) - |X| = p_T(Y) - |\Gamma_G(Y)|$, as required. □

Therefore the bipartite graph G meets all the requirements of the theorem apart possibly from simplicity. By Lemma 5.6, G can be chosen to be simple.