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Abstract

Gomory and Hu proved in [1] their well-known theorem which states that if G is a finite graph with non-negative weights on its edges, then there exists a tree T (called now Gomory-Hu tree) on $V(G)$ such that for all $u \neq v \in V(G)$ there is an $e \in E(T)$ such that the two components of $T - e$ determines an optimal (minimal valued) cut between u an v in G. In this paper we extend their result to countably infinite weighted graphs with finite total weight. Furthermore, we show by an example that one can not omit the condition of finiteness of the total weight.

1 Notions and notation

A weighted graph in this paper is a triple $G = (V, E, c)$ where (V, E) is a simple, countable graph and $c : E \to \mathbb{R}_+$ is a weight-function. We call the subsets of V cuts. We say X is an $u - v$ cut for some $u \neq v \in V$ if $u \in X$ and $v \notin X$. Cut X separates u and v if it is either a $u - v$ or a $v - u$ cut. Let us denote by $\boldsymbol{\delta(X)}$ the set of edges e whose endvertices are separated by X. For $X, Y \subseteq V$, $X \cap Y = \varnothing$ let $\boldsymbol{d_c(X,Y)} = \sum\{c(e) :\ e \text{ goes between } X \text{ and } Y\}$. We write $\boldsymbol{d_c(X)}$ for $d_c(X, V \setminus X) = \sum_{e \in \delta(X)} c(e)$. Let $\lambda_c : (V \times V) \setminus \{(v,v) :\ v \in V\} \to \mathbb{R}_+ \cup \{\infty\}$ where $\boldsymbol{\lambda_c(u,v)} := \inf\{d_c(X) :\ X \text{ is a } u - v \text{ cut }\}$. A cut X is an **optimal** $u - v$ cut if it is a $u - v$ cut with $d_c(X) = \lambda_c(u, v)$. The weighted graph G is **connected** iff λ_c is strictly positive and it is **finitely separable** iff λ_c has just finite values. All the weighted graphs considered in this paper are assumed to be connected and finitely separable even where we do not mention it explicitly. A tree $T = (V, F)$ is a **Gomory-Hu tree** for $G = (V, E, c)$ if for all $u \neq v \in V$ there is an $e \in F$ such that the **fundamental cuts correspond to** $\boldsymbol{e}$ (i.e. the vertex sets of the

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components of $T - e$) separate optimally u and v in G. Let $\mathcal{L}$ be a family of sets. Then $\bigcap \mathcal{L}$ ($\bigcup \mathcal{L}$) stands for the intersection (union) of the elements of $\mathcal{L}$. If $\mathcal{X}$ is a countable set (i.e. either countably infinite or finite), then $\mathcal{X} = \{X_n\}_{n\in\mathbb{N}}$ means that (X_n) is a sequence with range $\mathcal{X}$. If (X_n) is a sequence of sets, then let **$\liminf X_n$** $:= \cup_{m\in\mathbb{N}} \cap_{m<n} X_n$ and let **$\limsup X_n$** $:= \cap_{m\in\mathbb{N}} \cup_{m<n} X_n$. If $\liminf X_n = \limsup X_n =: X$, then we say that (X_n) **converges** to X and we write $X = \lim X_n$ or $(X_n) \to X$.

Gomory and Hu proved in [1] that for all finite weighted graph G there exists a Gomory-Hu tree. In this paper we extend their theorem for countably infinite weighted graphs with finite total weight. Since an infinite tree with arbitrary weights is a Gomory-Hu tree of itself, the finiteness of the total weight is just a sufficient and not a necessary condition for the existence of a Gomory-Hu tree. We also show by constructing a counterexample that a finitely separable weighted graph does not necessarily have a Gomory-Hu tree.

2 Basic tools

In this section we prove some basic facts that we need in order to prove our main result. Some of them are known for finite weighted graphs and remain true for countably infinite, finitely separable weighted graphs with essentially the same proof.

Claim 2.1.

1. *If (X_n) is a convergent sequence of cuts, then* $d_c(\lim X_n) \leq \liminf d_c(X_n)$.
2. *If in addition* $\sum_{e\in E} c(e) < \infty$, *then* $\lim d_c(X_n)$ *exists and* $\lim d_c(X_n) = d_c(\lim X_n)$ *holds.*

Proof: It is routine to check that $\delta(\lim X_n) = \lim \delta(X_n)$ holds. Consider the discrete measure space $(E, \mathcal{P}(E), \tilde{c})$ (where $\tilde{c}(F) = \sum_{e\in F} c(e)$ for $F \subseteq E$).

By applying Fatou lemma to the characteristic functions of the sets $\delta(X_n)$ we obtain

$$d_c(\lim(X_n)) = \widetilde{c}(\delta(\lim X_n)) = \widetilde{c}(\lim \delta(X_n)) =$$
$$\widetilde{c}(\liminf \delta(X_n)) \leq \liminf \widetilde{c}(\delta(X_n)) = \liminf d_c(X_n).$$

At the 2. statement we have $\widetilde{c}(E) < \infty$, thus the constant 1 function is integrable. Therefore by using Lebesgue theorem to the characteristic functions of the sets $\delta(X_n)$ we obtain

$$d_c(\lim X_n) = \widetilde{c}(\lim \delta(X_n)) = \lim \widetilde{c}(\delta(X_n)) = \lim d_c(X_n). \bullet$$

Claim 2.2. *Every sequence (X_n) of cuts has a convergent subsequence.*

Proof: The proof is standard. Let $V = \{v_n\}_{n\in\mathbb{N}}$. We define recursively a sequence (I_k) of index sets. Let $I_0 = \mathbb{N}$ and

$$I_{k+1} \stackrel{\text{def}}{=} \begin{cases} \{n \in I_k : \ v_{k+1} \in X_n\} & \text{if this set is infinite,} \\ \{n \in I_k : \ v_{k+1} \notin X_n\} & \text{otherwise.} \end{cases}$$

Finally we choose $n_0 = 0$ and $n_{k+1} := \min\{n \in I_{k+1} : \ n_k < n\}$. From the construction it follows that $v_k \in X_{n_{k_1}} \iff v_k \in X_{n_{k_2}}$ whenever $k < k_1, k_2$ i.e. (X_{n_k}) is convergent. ●

Corollary 2.3. *We may write minimum instead of infimum at the definition of $\lambda_c(u, v)$ i.e. for all $u \neq v \in V$ there is an optimal $u - v$ cut.*

Proof: Let (X_n) be a sequence of $u - v$ cuts for which $\lim d_c(X_n) = \lambda_c(u, v)$. Pick an (X_{n_k}) convergent subsequence of it. Then $\lim X_{n_k}$ is a $u - v$ cut and it is optimal by Claim 2.1. ●

Claim 2.4. $d_c(X) + d_c(Y) \geq d_c(X \cup Y) + d_c(X \cap Y)$ *for all $X, Y \subseteq V$.*

Proof: If edge e goes between $X \setminus Y$ and $Y \setminus X$, then it contributes $2c(e)$ to the left side and 0 to the right side of the inequality. The contribution of any other kind of edge is the same for both sides. ●

Corollary 2.5. *The intersection and the union of finitely many optimal $u-v$ cuts is an optimal $u - v$ cut.*

Proof: Let X and Y be optimal $u - v$ cuts. Note that $d_c(X) + d_c(Y) < \infty$ since the weighted graphs we consider are finitely separable. On the one hand $d_c(X) \leq d_c(X \cup Y)$ and $d_c(Y) \leq d_c(X \cap Y)$ hold since $X \cup Y$ and $X \cap Y$ are $u - v$ cuts thus

$$d_c(X) + d_c(Y) \leq d_c(X \cup Y) + d_c(X \cap Y).$$

On the other hand

$$d_c(X) + d_c(Y) \geq d_c(X \cup Y) + d_c(X \cap Y)$$

by the Claim above. Hence equality holds and therefore necessarily $d_c(X) = d_c(X \cup Y)$ and $d_c(Y) = d_c(X \cap Y)$. ●

Corollary 2.6. *There is a $\subset$-smallest (largest) optimal $u-v$ cut $\boldsymbol{X_{u,v}}$ $(Y_{u,v})$ which is the intersection (union) of all optimal $u - v$ cuts.*

Proof: Let $V = \{v_n\}_{n\in\mathbb{N}}$ and

$$X'_n := \begin{cases} \text{an optimal } u-v \text{ cut } X \text{ for whic } v_n \notin X & \text{if there are any,} \\ V & \text{otherwise .} \end{cases}$$

Then $X_n := \cap_{m\le n} X'_m$ is on optimal $u-v$ cut again and by Claim 2.1 $\cap_{n\in\mathbb{N}} X_n = \lim X_n =: X_{u,v}$ as well. ●

It will be convenient to use an equivalent but formally weaker definition of Gomory-Hu trees.

Claim 2.7. *$T = (V, F)$ is a Gomory-Hu tree for $G = (V, E, c)$ if for all $uv \in F$ the fundamental cuts corresponding to uv in T separate optimally u and v.*

Proof: Let $u \neq v \in V$ be arbitrary and let $v_1, v_2, \dots, v_m$ be the vertices of the unique $u-v$ path in T numbered in the path order.

Proposition 2.8. *For all pairwise distinct $u, v, w \in V : \lambda_c(u, w) \ge \min\{\lambda_c(u, v), \lambda_c(v, w)\}$.*

Proof: It follows from the fact that if a cut separates u and w, then it separates either u and v or v and w as well. ■

On one hand by applying the Proposition above repeatedly we obtain

$$\lambda_c(u, v) \ge \min\{\lambda_c(v_i, v_{i+1}) : \ 1 \le i < m\} =: \lambda_c(v_{i_0}, v_{i_0+1}) \text{ for some } 1 \le i_0 < m.$$

On the other hand the fundamental cuts corresponding to edge $v_{i_0}v_{i_0+1}$ separates u and v and have value $\lambda_c(v_{i_0}, v_{i_0+1})$ by assumption. Thus

$$\lambda_c(u, v) \le \lambda_c(v_{i_0}, v_{i_0+1}).$$

Hence equality holds and the fundamental cuts correspond to $v_{i_0}v_{i_0+1} \in F$ are optimal cuts between u and v. ●

Claim 2.9. *Let X be an optimal $s-t$ cut and let Y be an optimal $u-v$ cut.*

1. *Assume X is a $u-v$ cut. Then $Y \cup X$ is an optimal $u-v$ cut if $t \notin Y$ and $Y \cap X$ is an optimal $u-v$ cut if $t \in Y$.*
2. *Assume X is a $v-u$ cut. Then $Y \cup (V \setminus X)$ is an optimal $u-v$ cut if $s \notin Y$ and $Y \setminus X$ is an optimal $u-v$ cut if $s \in Y$.*
3. *Assume $u, v \in X$. Then $Y \cap X$ is an optimal $u-v$ cut if $t \notin Y$ and $Y \cup (V \setminus X)$ is an optimal $u-v$ cut if $t \in Y$.*

4. *Assume $u, v \notin X$. Then $Y \setminus X$ is an optimal $u-v$ cut if $s \notin Y$ and $Y \cup X$ is an optimal $u-v$ cut if $s \in Y$.*

Proof: It is enough to prove 1 and 3 since by using them by replacing X with the optimal $t-s$ cut $V \setminus X$ we obtain 2. and 4. To prove 1. assume first that $t \notin Y$. Since $X \cup Y$ is a $s-t$ cut and $X \cap Y$ is a $u-v$ cut we have $d_c(X \cup Y) \geq d_c(X)$ and $d_c(X \cap Y) \geq d_c(Y)$. Combining this with Claim 2.4 and finite separability we get

$$\infty > d_c(X) + d_c(Y) \geq d_c(X \cup Y) + d_c(X \cap Y) \geq d_c(X) + d_c(Y),$$

thus there must be equality in both non-strict inequalities. If $t \in Y$ and $s \in Y$, then $X \cup Y$ is a $u-v$ cut and $X \cap Y$ is a $s-t$ cut; therefore by arguing similarly as above we obtain that $X \cup Y$ must be on optimal $u-v$ cut. Finally if $t \in Y$ and $s \notin Y$, then on one hand Y separates t and s and X does this optimally therefore $d_c(X) \leq d_c(Y)$, on the other hand Y is an optimal $u-v$ cut and X is an $u-v$ cut hence $d_c(Y) \leq d_c(X)$. Thus $d_c(X) = d_c(Y)$ therefore X and Y both are optimal $u-v$ cuts thus $X \cup Y$ and $X \cap Y$ as well by Corollary 2.5. Proof of 3. is similar ●

Corollary 2.10. *If X is an optimal $s-t$ cut and $u \neq v \in X$, then either $X_{u,v} \subseteq X$ or $X_{v,u} \subseteq X$ (where $X_{x,y}$ stands for the $\subset$-smallest optimal $x-y$ cut).*

Proof: If $X_{u.v} \subseteq X$, then we are done. Assume $X_{u.v} \nsubseteq X$. By the minimality of $X_{u,v}$ the $u-v$ cut $X_{u,v} \cap X$ cannot be optimal therefore by Claim 2.9/3 $X_{u,v} \cup (V \setminus X)$ is an optimal $u-v$ cut. But then $V \setminus [X_{u,v} \cup (V \setminus X)] = X \setminus X_{u,v}$ is an optimal $v-u$ cut therefore we obtain $X_{v,u} \subseteq X \setminus X_{u,v} \subseteq X$. ●

3 Main results

3.1 Laminar system of optimal cuts

By putting together the original idea of Gomory and Hu and our basic observations about the limits of cuts we obtain the following Lemma.

Lemma 3.1. *If $\mathcal{L}$ is a laminar system of optimal cuts and $u \neq v \in V$, then there is a cut X for which $\mathcal{L} \cup \{X\}$ is laminar and X separates optimally u and v.*

Proof: Let us partition $\mathcal{L}$ into four parts $\mathcal{L}_{u,\overline{v}} := \{X \in \mathcal{L} : \ u \in X \wedge v \notin X\}$, we define $\mathcal{L}_{\overline{u},v}, \mathcal{L}_{u,v}$ and $\mathcal{L}_{\overline{u},\overline{v}}$ similarly. If $X_{u,v} \subseteq \widehat{X}$ for some $\widehat{X} \in \mathcal{L}_{u,\overline{v}}$, then let $Y_0 = X_{u,v}$. Note that $\{Y_0\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{\overline{u},v}$ is laminar. Suppose that we have no such an $\widehat{X}$ even if we interchange u and v. By Corollary 2.10 we know that for all $W \in \mathcal{L}_{u,v}$ either $X_{u,v} \subseteq W$ or $X_{v,u} \subseteq W$. Hence by symmetry we may assume that $\bigcap \mathcal{L}_{u,v} \supseteq X_{u,v} =: Y_0$. We will show that $\{Y_0\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{\overline{u},v}$ is laminar in this case as well. Let $X \in \mathcal{L}_{\overline{u},v}$ be arbitrary. Then $X_{v,u} \not\subseteq X$ otherwise $\widehat{X} := X$ would be a bound. But then $X_{v,u} \cap X$ cannot be an optimal $v-u$ cut by the minimality of $X_{v,u}$. Therefore by Claim 2.9/1 $X_{v,u} \cup X$ is an optimal $v-u$ cut and hence $V \setminus (X_{v,u} \cup X)$ is an optimal $u-v$ cut. Thus $V \setminus (X_{v,u} \cup X) \supseteq X_{u,v}$ so $X \cap X_{u,v} = \varnothing$. Either way Y_0 has chosen. For $Y \subseteq V$ let

$$\begin{aligned}\mathcal{L}_{u,\overline{v}}(Y) &= \{X \in \mathcal{L}_{u,\overline{v}} :\ X \not\subseteq Y \wedge X \not\supseteq Y\} \\ \mathcal{L}_{\overline{u},\overline{v}}(Y) &= \{X \in \mathcal{L}_{u,\overline{v}} :\ X \not\subseteq Y \wedge X \cap Y \neq \varnothing\}.\end{aligned}$$

Then $\mathcal{L}_{u,\overline{v}}(Y_0) \cup \mathcal{L}_{\overline{u},\overline{v}}(Y_0)$ is the set of those elements of $\mathcal{L}$ that are not in laminar relation with Y_0. Our plan is to fix these "errors" of Y_0 via recursive modifications of it in such a way that we never ruin what we fixed once. Assume first that the sets $\mathcal{L}_{u,\overline{v}}(Y_0)$ and $\mathcal{L}_{\overline{u},\overline{v}}(Y_0)$ are just countable: $\mathcal{L}_{u,\overline{v}}(Y_0) = \{X_n\}_{n\in\mathbb{N}}$ and $\mathcal{L}_{\overline{u},\overline{v}}(Y_0) = \{X'_n\}_{n\in\mathbb{N}}$. Do recursively

$$Y_{n+1} := \begin{cases} Y_n & \text{if } Y_n \subseteq X_n \text{ or } X_n \subseteq Y_n, \\ Y_n \cap X_n & \text{if } Y_n \subsetneq X_n \text{ and } X_n \subsetneq Y_n \text{ and } Y_n \cap X_n \text{ is an optimal } u-v \text{ cut}, \\ Y_n \cup X_n & \text{otherwise.} \end{cases}$$

By Claim 2.9/1 the members of the sequence (Y_n) are optimal $u-v$ cuts thus by Claim 2.1 $Z_0 := \lim Y_{n_k}$ as well where (Y_{n_k}) is an arbitrary convergent subsequence of (Y_n) (exists by Claim 2.2). Note that $\{Z_0\} \cup (\mathcal{L} \setminus \mathcal{L}_{\overline{u},\overline{v}})$ is laminar. We apply another recursion

$$Z_{n+1} := \begin{cases} Z_n & \text{if } Z_n \cap X'_n \in \{\varnothing, X'_n\}, \\ Z_n \setminus X'_n & \text{if } Z_n \cap X'_n \notin \{\varnothing, X'_n\} \text{ and } Z_n \setminus X'_n \text{ is an optimal } u-v \text{ cut }, \\ Z_n \cup X'_n & \text{otherwise.} \end{cases}$$

By Claim 2.9/4 the members of the sequence (Z_n) are optimal $u-v$ cuts thus by Claim 2.1 $Z^* := \lim Z_{n_k}$ as well for an arbitrary convergent subsequence (Z_{n_k}). Finally $\{Z^*\} \cup \mathcal{L}$ is laminar hence we are done.

If $\mathcal{L}_{u,\overline{v}}(Y_0)$ or $\mathcal{L}_{\overline{u},\overline{v}}(Y_0)$ is uncountable, then take a countable dense subposets of them (with respect to the order-topology where inclusion is the ordering) and do the recursions above with these first. For the resulting Y'_0 we know that $\mathcal{L}_{u,\overline{v}}(Y'_0) \cup \mathcal{L}_{\overline{u},\overline{v}}(Y'_0)$ is countable since it cannot contain from a chain of $\mathcal{L}_{u,\overline{v}}(Y_0) \cup \mathcal{L}_{\overline{u},\overline{v}}(Y_0)$ more than two elements and $\mathcal{L}_{u,\overline{v}}(Y_0) \cup \mathcal{L}_{\overline{u},\overline{v}}(Y_0)$ can be partitioned into countably many chains (sets that contain a certain vertex form a chain because of laminarity). Do the recursions above again starting with Y'_0 and using $\mathcal{L}_{u,\overline{v}}(Y'_0)$ and $\mathcal{L}_{\overline{u},\overline{v}}(Y'_0)$ to get Z^*. ●

3.2 Gomory-Hu trees of countably infinite weighted graphs

Theorem 3.2. *If $G = (V, E, c)$ is a countable weighted graph where $\sum_{e\in E} c(e) < \infty$, then there is a Gomory-Hu tree for G.*

Let G be as in the theorem fixed. Note that connectedness of G is not a real restriction here since otherwise we may construct Gomory-Hu trees for the connected components and connect these trees to obtain a tree on V in arbitrary way.

A sequence (X_n) of optimal cuts is **essential** if all of its members separate optimally a vertex pair that the earlier members do not.

Lemma 3.3. *Assume $\sum_{e\in E} c(e) < \infty$. If (X_n) is a monotone sequence of optimal cuts and $\lim X_n =: X \notin \{\varnothing, V\}$, then (X_n) has no essential subsequence.*

Proof: Assume, to the contrary, that (X_n) is a counterexample. By symmetry we may suppose that (X_n) is increasing. By trimming (X_n) we may assume that it is essential witnessed by s_n, t_n i.e. X_n is an optimal $s_n - t_n$ cut but X_m is not whenever $m < n \in \mathbb{N}$.

Claim 3.4. *$t_n \notin X$ holds for all large enough $n \in \mathbb{N}$.*

Proof: Note that by Claim 2.1/2, $d_c(X \setminus X_n) \to 0$ since $(X \setminus X_n) \to \varnothing$, and thus

$$\begin{aligned}\lim d_c(V \setminus X, X_n) &= \lim(d_c(V \setminus X, X) - d_c(X \setminus X_n)) \\ &= d_c(V \setminus X) - \lim d_c(X \setminus X_n) = d_c(V \setminus X).\end{aligned}$$

Pick an $n_1 \in \mathbb{N}$ for which $d_c(V \setminus X, X_n) > d_c(V \setminus X)/2 > 0$ holds whenever $n \geq n_1$. For $n \geq n_1$ we know that $t_n \notin X$ otherwise $X_n \cup (V \setminus X)$ would be a better $s_n - t_n$ cut than the optimal X_n. ●

By trimming (X_n) we may assume that $t_n \notin X$ for all $n \in \mathbb{N}$. It implies that $d_c(X_n) \leq d_c(X_{n+1})$ for all $n \in \mathbb{N}$ because X_{n+1} is an $s_n - t_n$ cut and X_n is an optimal $s_n - t_n$ cut. But then $s_{n+1} \notin X_n$ for all $n \in \mathbb{N}$ otherwise X_n would be at least as good $s_{n+1} - t_{t_{n+1}}$ cut as the optimal one but X_{n+1} is the first optimal $s_{n+1} - t_{n+1}$ cut of the sequence by the choice of (X_n).

Claim 3.5. *X_n is an optimal $s_n - s_{n+1}$ cut for all $n \in \mathbb{N}$.*

Proof: By Corollary 2.10 there is an $Y \in \{X_{s_n,s_{n+1}}, X_{s_{n+1},s_n}\}$ for which $Y \subseteq X_{n+1}$. If $d_c(Y) < d_c(X_n)$ would hold, then Y would be either a better $s_n - t_n$ cut than X_n or a better $s_{n+1} - t_{n+1}$ cut than X_{n+1} which is impossible. ●

Since $s_n \in X$ for al $n \in \mathbb{N}$ the Claim above contradicts Claim 3.4 with the choices $s_n := s_n$ and $t_n := s_{n+1}$. ■

Let X be an optimal cut. For $u \neq v \in X$ we write $\boldsymbol{u \prec_X v}$ iff $X_{u,v} \nsubseteq X$.

Claim 3.6. *Relation $\prec_X$ is a strict partial ordering on X.*

Proof: It is irreflexive by definition. For transitivity assume $u \prec_X v \prec_X w$. If $u = w$, then we have $u \prec_X v$ and $v \prec_X u$ which contradicts Corollary 2.10. Thus we may assume that u, v, w are pairwise distinct. Suppose, to contrary, that $u \prec_X w$ does not hold i.e. $X_{u,w} \subseteq X$. Assume first that $v \in X_{u,w}$. By Corollary 2.10 either $X_{u,v} \subseteq X_{u,w}$ or $X_{v,u} \subseteq X_{u,w}$. Since $u \prec_X v$, necessarily $X_{v,u} \subseteq X_{u,w}$. But then $X_{u,w}$ and $X_{v,u}$ are both $v - w$ cuts and

$$\lambda_c(v, w) \geq \min\{\lambda_c(v, u),\ \lambda_c(u, w)\} = \min\{d_c(X_{v,u}),\ d_c(X_{u,w})\}$$

shows that one of them is optimal which contradicts $v \prec_X w$.

Hence $v \notin X_{u,w}$ holds. $X_{u,w}$ is not an optimal $u - v$ cut since $u \prec_X v$. Therefore $d_c(X_{u,w}) > d_c(X_{v,u})$. (Note that $X_{v,u} \subseteq X$ by Corollary 2.10 and by $u \prec_X v$). Hence $w \notin X_{v,u}$ otherwise $X_{v,u}$ would be a better cut between w and u than the optimal. On the other hand $X_{v,u}$ is not an optimal $v - w$ cut since $v \prec_X w$ hence $X_{w,v} \subseteq X$ and $d_c(X_{w,v}) < d_c(X_{v,u})$ hold. Necessarily $u \in X_{w,v}$, otherwise $X_{w,v}$ separates better w and u than $X_{u,w}$, but then $X_{w,v}$ separates better u and v than $X_{v,u}$, which is a contradiction. ●

Lemma 3.7. *If X is an optimal $u_0 - v_0$ cut, then X has a $\prec_X$-minimal element u_1. For all such a u_1, cut X it is an optimal $u_1 - v_0$ cut.*

Proof: Let $B = \{b \in X : \lambda_c(b, v_0) = \lambda_c(u_0, v_0)\}$ and $A := \{a \in X : \lambda_c(a, v_0) < \lambda_c(u_0, v_0)\}$. Then $A \cup B$ is a partition of X.

Claim 3.8. *For all $a \in A$ and $b \in B$: $b \prec_X a$ holds.*

Proof: If $a \in A$ and $b \in B$, then $\lambda_c(a, b) < \lambda_c(u_0, v_0)(= \lambda_c(b, v_0))$, otherwise

$$\lambda_c(a, v_0) \geq \min\{\lambda_c(a, b), \lambda_c(b, v_0)\} = \lambda_c(b, v_0) = \lambda_c(u_0, v_0)$$

contradicts $a \in A$. Therefore if $X_{b,a} \subseteq X$ would hold, then since $X_{b,a}$ is a $b - v_0$ cut

$$\lambda_c(b, v_0) \leq \lambda_c(a, b) < \lambda_c(u_0, v_0)$$

which is impossible since $b \in B$. ●

By the claim above, it is enough to find a minimal element for the poset $(B, \prec_X)$. The existence of such an element follows immediately from the following claim.

Claim 3.9. *The set B is finite.*

Proof: Assume, to seeking a contradiction, that B is infinite. By applying Lemma 3.1 repeatedly starting with $\mathcal{L} = \{X\}$ we are able to build an essential sequence (X_n), where $X_n \subseteq X$ for all $n \in \mathbb{N}$ and all of its members separate optimally two vertices from B. Note that $d_c(X_n) \geq \lambda_c(u_0, v_0)$ holds for all $n \in \mathbb{N}$ because X_n separates some $b \in B$ from v_0.

Proposition 3.10. *If $\mathcal{A}$ is an antichain of $(\{X_n\}_{n\in\mathbb{N}}, \subset)$, then*

$$|\mathcal{A}| \leq \left\lfloor \frac{2 \cdot \sum_{e\in E} c(e)}{\lambda_c(u_0, v_0)} \right\rfloor.$$

Proof: $\mathcal{A}$ is a subpartition of X thus

$$2\sum_{e\in E} c(e) \geq \sum_{Y\in\mathcal{A}} d_c(Y) \geq |\mathcal{A}| \cdot \lambda_c(u_0, v_0).■$$

By Dilworth's theorem we can partition $(\{X_n\}_{n\in\mathbb{N}}, \subset)$ into finitely many chains. At least one of them will be infinite. Choose from that chain a monotone subsequence (X_{n_k}) of (X_n). Since $\liminf d_c(X_{n_k}) \geq \lambda_c(u_0, v_0) > 0$, we know by Claim 2.1/2 that $\lim X_n \neq \varnothing$ (and obviously $\lim X_n \neq V$) but then the existence of (X_{n_k}) contradicts Lemma 3.3. ●

For the second part of Lemma 3.7 let u_1 be a $\prec_X$-minimal element of X. Since $B \neq \varnothing$, Claim 3.8 implies that necessarily $u_1 \in B$ and we are done. ■

Lemma 3.11. *Let X be an optimal $u_0 - v_0$ cut, where u_0 is a $\prec_X$-minimal element of X. Then there is a partition $X = \{u_0\} \cup \bigcup_{i\in I} X_i$ of X such that X_i is an optimal $x_i - u_0$ cut for some $x_i \in X_i$ $(i \in I)$.*

Proof: Let $X \setminus \{u_0\} = \{x_n\}_{n\in\mathbb{N}}$. We build a (possibly finite) sequence (X_n) recursively. In the case $x_n \notin \cup_{i<n} X_i$ we want to X_n be an optimal $x_n - u_0$ cut for which $\{X_i\}_{i\leq n}$ is laminar. To find such an X_n use Lemma 3.1 with x_n, u_0 and $\mathcal{L} = X \cup \{X_i\}_{i<n}$. By the $\prec_X$-minimality of u_0 the resulting $X =: X_n$ must be a $x_n - u_0$ cut. If $x_n \in \cup_{i<n} X_i$, then let $X_n := X_{n-1}$. The recursion is done.

The system $\{X_n\}_{n\in\mathbb{N}}$ is laminar and $\cup_{n\in\mathbb{N}} X_n = X \setminus \{u_0\}$ holds. Furthermore, (X_n) does not contain a monotone strictly increasing subsequence since such a subsequence would be essential and would have nontrivial limit by the construction, which contradicts Lemma 3.3. Finally the maximal elements of $\{X_n\}_{n\in\mathbb{N}}$ is a desired partition. ■

By using the Lemma above repeatedly we are able the build up a Gomory-Hu tree for G. For a tree T we denote by $\mathsf{leaf}(T)$ the set $\{v \in V(T) : \ d(v) = 1\}$ and let $\mathsf{int}(T) = V(T) \setminus \mathsf{leaf}(T)$. We build recursively sequences T_n and g_n such that

1. T_n is a tree with $V(T) \subseteq V$,
2. $\{g_n(t)\}_{t\in\mathsf{leaf}(T_n)}$ is a partition of $V \setminus \mathsf{int}(T_n)$ in which $g_n(t)$ is an optimal $t - t'$ cut, where t' is the unique neighbour of t in T_n and t is a $\prec_{g_n(t)}$-minimal element of $g_n(t)$,
3. T_n is a sub-tree of T_{n+1} and all the elements of $V(T_{n+1}) \setminus V(T_n)$ are adjacent with some leaf of T_n,
4. if $l \in \mathsf{leaf}(T_n)$ and S is the set of its new neighbours in T_{n+1}, then $\{g_{n+1}(t)\}_{t\in S}$ is a partition of $g_n(l) \setminus \{l\}$,
5. if $uv \in E(T_n)$, and C is a component of $T_n - uv$, then $\bigcup\{g_n(t) : \ t \in V(C) \cap \mathsf{leaf}(T_n)\} \cup V(C)$ is an optimal $u - v$ or $v - u$ cut in G. We call such a cut a fundamental cut of T_n corresponding to uv.

To define T_0, pick an optimal $u_0 - v_0$ cut X for some $u_0 \neq v_0 \in V$, where u_0 is a $\prec_X$-minimal element of X and v_0 is a $\prec_{X\setminus V}$-minimal element of $X \setminus V$ (it is possible by Lemma 3.7). Let T_0 be the tree consisting of u_0, v_0 and an edge between them. Finally let $g_0(u_0) = X$ and let $g_0(v_0) = V \setminus X$. Assume that T_k has already been defined if $k \leq n$ for some $n \in \mathbb{N}$ and it satisfies the properties above. For all $l \in \mathsf{leaf}(T_n)$ do the following. Partition $g_n(l) \setminus \{l\}$ into optimal $x_i - l$ cuts $\{X_i\}_{i\in I}$ by using Lemma 3.11, where x_i is a $\prec_{X_i}$-minimal element of X_i. Let the new neighbours of l be these x_i vertices and let $g_{n+1}(x_i) = X_i$. The construction is done.

Claim 3.12. *For all $y \in V$ there is an $n \in \mathbb{N}$ such that $y \in V(T_n)$.*

Proof: Assume, seeking contradiction, that y witnesses that the claim is false. Let t_n be the unique leaf of T_n for which $y \in g_n(t_n) =: X_n$. Then X_{n+1} is an optimal $t_{n+1} - t_n$ cut and the existence of the decreasing sequence (X_n) contradicts Lemma 3.3 since it is essential (t_n and t_{n+1} are not even separated by X_m if $m < n + 1$) and $y \in \cap_{n\in\mathbb{N}} X_n = \lim X_n$. ●

Let $T = (\cup_{n\in\mathbb{N}} V(T_n), \cup_{n\in\mathbb{N}} E(T_n))$. It is a tree on V by the construction and by the Claim above. To show that T is a Gomory-Hu tree for G, we use Claim 2.7. Pick an $uv \in E(T)$. Then $uv \in E(T_n)$ for some large enough n. The fundamental cuts corresponding to uv in T_n separates u and v optimally by property 5. On the other hand, the fundamental cuts corresponding to uv in T are the same cuts by the construction. $\square$

3.3 A counterexample

We construct a finitely separable, countably infinite weighted graph such that it has no Gomory-Hu tree. Let $V = \{v_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ and let $E = \{v_\infty v_n, v_n v_{n+1}\}_{n\in\mathbb{N}}$. Finally $c(v_\infty v_n) := 1$ for all $n \in \mathbb{N}$ and

$$c(e_n) := \begin{cases} 2 & \text{if } n = 0 \\ c(e_{n-1}) + n + 1 & \text{if } n > 0. \end{cases}$$

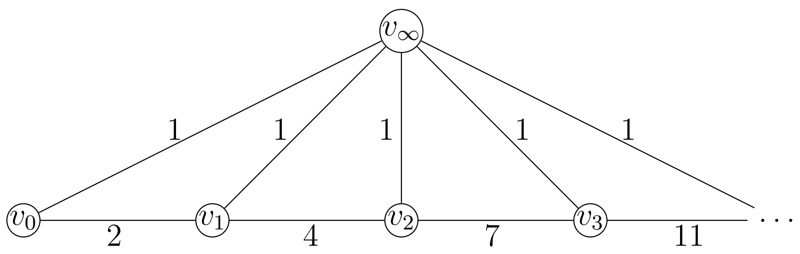

Figure 1: A finitely separable weighted graph without Gomory-Hu tree

Claim 3.13. *If $n < m \in \mathbb{N} \cup \{\infty\}$, then $\{v_0, v_1, \ldots, v_n\} =: V_n$ is the only optimal $v_n - v_m$ cut.*

Proof: Pick an optimal $v_n - v_m$ cut X. Since $d_c(V_n) < c(e_k)$ whenever $k > n$, cut X may not separate the endvertices of such an e_k. Then $v_\infty \notin X$ otherwise $d_c(X) = \infty$. Thus we have $X \subseteq V_n$. Suppose, for contradiction, that $v_l \notin X$ for some $l < n$ and l the largest such index. Then

$$d_c(X) - d_c(V_n) \geq c(e_l) - l > 0,$$

which contradicts the optimality of X. $\blacksquare$

Claim 3.14. *G has no Gomory-Hu tree.*

Assume, to the contrary, that T is a Gomory-Hu tree of G. For all $e \in E(T)$ pick the fundamental cut X_e that corresponds to e and does not contain v_∞. On the one hand $\mathcal{L} := \{X_e\}_{e\in E(T)}$ is a laminar system of optimal cuts that contains at least one maximal element (if e incident with v_∞, then X_e is a maximal element). On the other hand $\mathcal{L} = \{V_n\}_{n\in\mathbb{N}}$ since the optimal cuts are unique up to complementation and the additional condition "does not contain v_∞" makes them unique. This is a contradiction since (V_n) is strictly increasing.

References

[1] R. E. Gomory and T. C. Hu, *Multi-terminal network flows*, Journal of the Society for Industrial and Applied Mathematics, 9 (1961), pp. 551–570.