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T-joins in infinite graphs

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Abstract

We characterize the class of infinite graphs G for which there exists a T-join for any choice of an infinite $T \subseteq V(G)$. We also show that the following well-known fact remains true in the infinite case. If G is connected and does not contain a T-join, then it will if we either remove an arbitrary vertex from T or add any new vertex to T.

1 Introduction

The graphs in this paper may have multiple edges although all of our results follow easily from their restrictions to simple graphs. A ***T*-join** in a graph G, where $T \subseteq V(G)$, is a system $\mathcal{P}$ of edge-disjoint paths in G such that the endvertices of the paths in $\mathcal{P}$ create a partition of T into two-element sets (in other words we match by edge-disjoint paths the vertices in T). This is a common tool in combinatorial optimization problems such as the well-known Chinese postman problem. For a detailed survey one can see [1]. For finite connected graphs the necessary and sufficient condition for the existence of a T-join is quiet simple, $|T|$ must be even. If $|T|$ is even but G is infinite, then we may simply take a connected, finite subgraph H of G for which $T \subseteq V(H)$ and apply the finite theorem to obtain a T-join. In this paper we investigate questions related to the existence of T-joins where T is infinite. For infinite T one can not guarantee in general the existence of a T-join in a connected graph. Consider for example an infinite star and subdivide all of its edges by a new vertex (see figure 1) and let T consist of the whole vertex set.

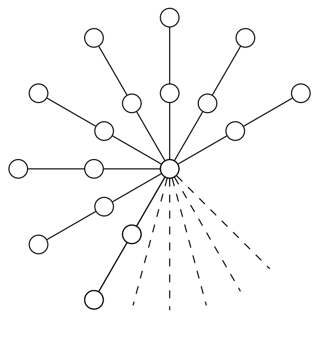

Figure 1: A subdivided infinite star. It has no T-join if T is the whole vertex set.

Our main result is the following characterization of infinite, connected graphs G that contain a T-join for every infinite $T \subseteq V(G)$.

Theorem 1.1. *The connected, infinite graph G contains a T-join for every infinite $T \subseteq V(G)$ iff there is no $A \subseteq V(G)$ for which every connected component of $G - A$ connects to A in G by a single edge and infinitely many of them are nontrivial (i.e. not consist of a single vertex).*

The "if" direction is straightforward since if such an A exists, then one can choose one vertex from A and 2 from each nontrivial connected component of $G - A$ to obtain a T for which there is no T-join in G.

Our other result describes the effect of finite modifications of T on the existence of a T-join. For finite T if G does not contain a T-join, then it contains a T'-join whenever $|(T \setminus T') \cup (T' \setminus T)| =: |T \vartriangle T'|$ is odd. (It is obvious, since in this case the existence of a T-join depends just on the parity of $|T|$.) Surprisingly this property remains true for infinite T as well. We have the following result about this.

Theorem 1.2. *Let G be a connected graph and $T \subseteq V(G)$. Let $N(T) = \{T' \subseteq V(G) : |T \vartriangle T'| < \infty\}$ furthermore let $N_e(T) = \{T' \in N(T) : |T \vartriangle T'|$ is even$\}$ and $N_o(T) := N(T) \setminus N_e(T)$. Then one of the cases below holds. (Any of the following cases can occur for a suitable choice of G and T)*

(A) G contains a T'-join for all $T' \in N(T)$,

(B) for $T' \in N(T)$ there is a T'-join in G iff $T' \in N_e(T)$,

(C) for $T' \in N(T)$ there is a T'-join in G iff $T' \in N_o(T)$.

To make the descriptions of the T-join-constructing processes more reader-friendly we introduce the following single player game terminology. There is an abstract set of tokens and every token is on some vertex of a graph G. (At the beginning typically we have exactly one token on each element of a prescribed vertex set T and none on the other vertices.) If two tokens are on the same vertex, then we may remove them (we say that we **match** them to each other). If we have a token t on the vertex u and uv is an edge of G, then we may move t from u to v but then we have to delete uv from G. A **gameplay** is a transfinite sequence of the steps above in which we move every token just finitely many times. Limit steps are defined by the earlier steps in a natural way. Indeed, we just delete all the edges that have deleted earlier, remove the tokens that have removed before, and put all the remaining tokens to their stabilized positions. We call a gameplay winning if we remove all the tokens eventually. To make easier to talk about the relevant part of the graph we also allow as a feasible step the deletion of vertices without tokens on them and deletion of edges. Clearly if there is a T-join in G, then there is a winning gameplay for the game on G where the initial token distribution is defined by T. On the other hand from a winning gameplay we can get a T-join.

2 The 2-edge-connected case

We call a subgraph of G ***t*-infinite** iff it contains infinitely many tokens. We define the notions ***t*-finite**, ***t*-empty**, ***t*-odd** and ***t*-even** similarly.

Lemma 2.1. *If G is 2-edge-connected and t-infinite, then there is a winning gameplay.*

Claim 2.2. *Assume that G is 2-edge-connected and contains even number of tokens but at least 4 including $s \neq t$. Then there is a winning gameplay in which s and t are not matched with each other and t is not moved.*

Proof: We may assume that G is finite otherwise we may take a finite 2-edge-connected subgraph that contains all the tokens. Take an ear decomposition of G in which s and t are on the starting cycle. If the whole G is this cycle, then the claim is obvious. Otherwise let Q be the last ear. We match inside Q all but at most one tokens that are on some new vertex given by Q and move the possibly exception to the part of the G before addition of Q. Finally we delete the possibly remaining new vertices given by Q and use induction. ●

If R is a 2-edge-connected component of a connected graph G, then we denote by $\textsf{\textbf{tree}}(\boldsymbol{G};\boldsymbol{R})$ the tree of the 2-edge-connected components of G rooted at R. We usually pick such a root R arbitrarily without mentioning it explicitly. We write $\textsf{tree}(G)$ if it is not rooted. We do not distinguish strictly the subtrees of $\textsf{tree}(G;R)$ and the corresponding subgraphs of G.

Claim 2.3. *Let G be a connected graph with infinitely many tokens on it. Assume that G has only finitely many 2-edge-connected components. Then there is a finite gameplay after which all the components of the resulting G' are 2-edge-connected and contain infinitely many tokens. Furthermore the t-infinite 2-edge-connected components of G (and thus the positions of the tokens that are originally on them) are unchanged.*

Proof: By the pigeon hole principle there is a t-infinite, 2-edge-connected component R of G. We use induction on the number k of the 2-edge-connected components. For $k = 1$ we do nothing. If $k > 1$ we take a leaf C of $\textsf{tree}(G;R)$. If C is t-infinite we remove the unique outgoing edge of $V(C)$ in G and we use induction to the arising component other than C. If C is t-even, then we match the tokens on C with each other inside C and we delete the remaining part of C and its unique outgoing edge and use induction. In the t-odd case we match all but one tokens of C inside C and move one to the parent component. We delete the remaining part of C again and use induction. ●

Now we turn to the proof of Lemma 2.1. If t is a token and H is a subgraph of G, then we use the abbreviation $\boldsymbol{t \in H}$ to express the fact that t is on some vertex of H. Assume first that T is countable. Let t_0 be arbitrary. It is enough to show, that there is a finite sequence of steps of the game such that we remove t_0 and the resulting system still satisfies the following condition.

$$\text{All the connected components are 2-edge-connected and t-infinite.} \tag{1}$$

Let t_1 be a closest token to t_0 and let P be a shortest path between them. We use induction on $|E(P)|$. If $|E(P)| = 0$ we may just remove t_0 and t_1. Otherwise let vu be the first edge of P with respect to the $t_0 \rightarrow t_1$ direction. If $G - vu$ is still

2-edge-connected, then we move t_0 through this edge and use induction. Assume that it is not the case. Since G is 2-edge-connected $\mathsf{tree}(G - vu)$ necessarily forms a finite path $C_1, u_1v_1, C_2, u_2v_2, \ldots, u_{n-1}v_{n-1}, C_n$ where $v \in V(C_1)$ and $u \in V(C_n)$. Consider the smallest index l for which C_l contains a token other than t_0.

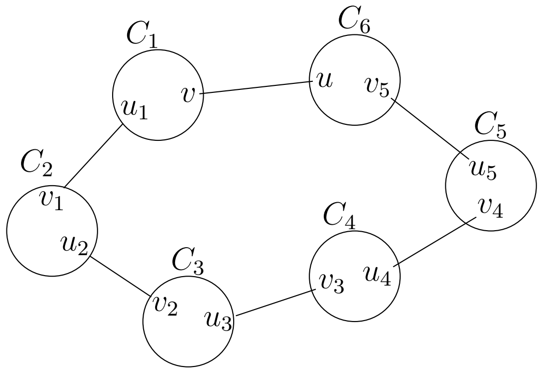

Figure 2: Structure of G in case $n = 6$, the subgraphs C_i are 2-edge-connected.

Assume first that C_l is t-finite. Move t_0 through $C_1, C_2, \ldots, C_{l-1}$ (and delete the remaining parts of them) to arrive C_l. If C_l is t-even after the arrival of t_0, then match the actual tokens of C_l and after the deletion of the remaining part of C_l apply Claim 2.3 to fix condition (1). Let C_l be t-odd now, then apply Claim 2.2 to graph C_l with the actual tokens on it including $t_0 =: s$ and one new fake token t on vertex u_l. We conclude that we are able to match all but one of the actual tokens of C_l and move the exception, which is not t_0, to the component C_{l+1} ($n > l$, since there is some t-infinite C_i and $C_1, C_2, \ldots, C_l$ are not t-infinite). After this delete the remaining part of C_l and use Claim 2.3 to fix condition (1).

Suppose now that C_l is t-infinite. Move t_0 through edge vu. Then the remaining graph is in the form $C_1, u_1v_1, C_2, u_2v_2, \ldots, u_{n-1}v_{n-1}, C_n$ where $t_0 \in C_n$ and $t_1 \in C_j$ for some $l \leq j \leq n$ and a terminal segment of P connects t_0 and t_1. We use induction on $k := n - j$. Consider the $k = 0$ (i.e. $t_1 \in C_n$) case. If C_n is t-finite, then we are able to match here all but at most one tokens and move the possible exception, which can be chosen not to be t_0, to C_{n-1}. After that we apply Claim 2.3 to fix (1). Assume now that C_n is t-infinite. The terminal segment P' of P which starts at u lies inside C_n. By applying Claim 2.3 the 2-edge-connected component C_n and the positions of its actual tokens (including t_0 and t_1) remain unchanged. Since $|E(P')| < |E(P)|$ holds we are done by induction.

Let now $k > 0$. Suppose first that C_n is t-finite. Match inside C_n all but at most one token and move the possible exception to C_{n-1}. Delete the remaining part of C_n. If t_0 has been matched, then we are done by applying Claim 2.3. Otherwise t_0 is what we moved to C_{n-1}. Note that a terminal segment of P still connects t_0 and t_1 and n (and therefore k) has reduced by one. Thus we are done by induction.

Let finally C_n be t-infinite. Since $t_1 \notin C_n$ the path P has a segment P' that goes from u to v_{n-1} inside C_n. Since $|E(P')| < |E(P)|$ we may use induction inside C_n by putting an extra fake token to v_{n-1}. Hence we are able to either match t_0 here or move it to v_{n-1} (and then to u_{n-1}) in such a way that the remaining part of C_n satisfies (1). If t_0 has been matched, then we use Claim 2.3 and we are done. Otherwise n has been reduced hence k as well and a terminal segment of P still connects t_0 and t_1 hence we are done by induction.

Consider now the general case where T can be arbitrary large. Pick a new vertex z and draw all the zv $(v \in T)$ edges to obtain H. Finding a T-join in G is clearly equivalent with covering in H all the edges incident with z by edge-disjoint cycles. To reduce the problem to the countable case it is enough to prove the following claim.

Claim 2.4. *There is a partition of $E(H)$ into countable sets E_i $(i \in I)$ in such a way, that for all $i \in I$ the graphs $H_i := (V(H), E_i)$ have the following property. The connected components of $H_i - z$ are 2-edge-connected and connect to z in H_i by either zero or infinitely many edges.*

Our proof to the Claim above is a basic application of the so called elementary submodel technique. One can find a detailed survey about this method with many combinatorial applications in [3]. We give here just the fundamental definitions and cite the results that we need. Let $\Sigma = \{\varphi_1, \dots, \varphi_n\}$ be a finite set of formulas in the language of set theory where the free variables of φ_i are $x_{i,1}, \dots, x_{i,n_i}$. A set M is a **Σ-elementary submodel** iff the formulas in Σ are absolute between $(M, \in)$ and the universe i.e.

$$[(M, \in) \models \varphi_i(a_1, \dots, a_{n_i})] \iff \varphi_i(a_1, \dots, a_{n_i})$$

holds whenever $1 \leq i \leq n$ and $a_1, \dots, a_{n_i} \in M$. By using Lévy's Reflection Principle and the Downward Löwenheim Skolem Theorem (see in [2] or any other set theory or logic textbook) one can derive the following fact.

Claim 2.5. *For any finite Σ set of formulas, set x and infinite cardinal κ there exists a Σ-elementary submodel $M \ni x$ with $\kappa = |M| \subseteq M$.*

Now we use some methods developed by L. Soukup in [3]. Call a class $\mathfrak{C}$ of graphs **well-reflecting** iff for all large enough finite set Σ of formulas, infinite cardinal κ, set x and $G \in \mathfrak{C}$ there is a Σ-elementary submodel M with $x, G \in M$ for which $\kappa = |M| \subseteq M$ and $(V(G), E(G) \cap M)$, $(V(G), E(G) \setminus M) \in \mathfrak{C}$. ("For large enough finite Σ" means here that there is some finite Σ_0 such that for all finite $\Sigma \supseteq \Sigma_0$.)

Theorem 2.6 (L. Soukup, Theorem 5.4 of [3])**.** *Assume that the graph-class $\mathfrak{C}$ is well-reflecting and $G \in \mathfrak{C}$. Then there is a partition of $E(G)$ into countable sets E_i $(i \in I)$ in such a way, that for all $i \in I$ we have $(V(G), E_i) \in \mathfrak{C}$.*

Remark 2.7. L. Soukup used a stricter notion of well-reflectingness but his proof still works with our weaker notion as well.

We apply the Theorem above to prove Claim 2.4. Let $\mathfrak{C}$ be the class of graphs G for which $z \in V(G)$, the connected components of $G - z$ are 2-edge-connected and connect to z in G either by infinitely many edges or send no edge to z at all. We need to show that $\mathfrak{C}$ is well-reflecting. Assume that Σ is a finite set of formulas that contains all the formulas of length at most 10^{10} with variables at most $x_1, \dots x_{10^{10}}$. Let κ, x and $G \in \mathfrak{C}$ be given. By Claim 2.4 we can find a Σ-elementary submodel $M \ni x, G, z$ with $\kappa = |M| \subseteq M$. We know that $(G-z) \in M$ by using the absoluteness of the formula "x_1 graph obtained by the deletion of vertex x_2 of graph x_3"$\in \Sigma$. The proof of $(V(G), E(G) \cap M) \in \mathfrak{C}$ is easy we just need the absoluteness of formulas such that "the local edge-connectivity between the vertices x_1 and x_2 in the graph x_3 is x_4" $\in \Sigma$. The hard part is to show $(V(G), E(G) \setminus M) \in \mathfrak{C}$. We use the following Proposition which is Lemma 5.3 of [3].

Proposition 2.8. *If M is a Σ-elementary submodel (for some large enough finite Σ) for which $G \in M$, $|M| \subseteq M$ and $x \neq y \in V(G)$ are in the same connected component of $(V(G), E(G) \setminus M)$ and $F \subseteq E(G) \setminus M$ separates them where $|F| \leq |M|$; then F separates x and y in G as well.*

If the local edge-connectivity between vertices $x \neq y$ would be one in the graph $(V(G-z), E(G-z) \setminus M)$, then we can separate them by the deletion of a single edge e. But then by applying Proposition 2.8 with $F = \{e\}$ we may conclude that the same separation is possible in $G - z$ which contradicts with the assumption $G \in \mathfrak{C}$.

Suppose, to the contrary, that z sends finitely many, but at least one, edges, say $e_1, \dots, e_k$, to a connected component of $(V(G-z), E(G-z) \setminus M)$ in $(V(G), E(G) \setminus M)$. Let s be the endvertex of e_1 other than z. Then $F := \{e_i\}_{i=1}^k$ separates s and z in $(V(G), E(G) \setminus M)$ and $|F| < \infty$ holds hence F separates them in G as well which is a contradiction.

3 A simplification process

Lemma 2.1 makes possible to decide the existence of a T-join just investigating the structure of the 2-edge-connected components and the quantity of tokens on them. Let G be a connected graph, $T \subseteq V(G)$ and let R be a 2-edge-connected component of it. We define a graph $H = H(G, R, T)$ with a token-distribution on it. To obtain H we apply the following gameplay that we call **simplification process**. We denote by $\textbf{subt}(\boldsymbol{C}; \boldsymbol{G}, \boldsymbol{R})$ the subtree of the descendants of C rooted at C in $\mathsf{tree}(G; R)$. Delete all those 2-edge-connected componets C for which $\mathsf{subt}(C; G, R)$ does not contain any token. Then consider the t-finite leafs of the reminder of $\mathsf{tree}(G; R)$. (We do not consider the root as a leaf.) Match the tokens on any t-even leaf C inside C and for t-odd leafs C move one token to the parent and match the others inside. In both cases delete the remaining part of C. Iterate the steps above as long as possible and denote by H the resulting graph. Clearly either $H = R$ with some tokens on it or if $\mathsf{tree}(H)$ is nontrivial, then $\mathsf{subt}(C; H, R)$ must be t-infinite for all 2-edge-connected component C of H.

Claim 3.1. *There is a winning gameplay for the original system whenever H is t-even or t-infinite.*

Proof: In the t-even case it is obvious since H is connected. Assume that H is t-infinite. Then $\mathsf{subt}(C; H, R)$ is t-infinite for all 2-edge-connected component C of H.

For all 2-edge-connected components of H we fix a path P_C in in the tree $\mathsf{subt}(C; H, R)$ that starts at C and either terminates at some leaf of $\mathsf{tree}(H; R)$ or it is one-way infinite and meets infinitely many 2-edge-connected component of H which is not t-empty. After these preparations we do the following. If the root R is t-even or t-infinite, then we match all the tokens of it inside R (use Lemma 2.1 in the t-infinite case) and delete the remaining part of R. If it is t-odd we move one of its tokens, say it will turn to be t^*, to some child of R determined by the path P_R and we define $P_{t^*} := P_R$. We match the other tokens inside R and then delete the remaining part of R. At the next step we deal with the $\mathsf{subt}(C; H, R)$ trees where C is a child of R. In the cases where C is t-infinite or t-even we do the same as earlier. Assume that C is t-odd. If there is no token on C that comes from R, then we do the same as earlier. Suppose that there is, say t_0. If there is a token on C other than t_0, then we match here t_0 and send forward some other token t_1 in the direction defined by P_C and we let $P_{t_1} := P_C$. If t_0 is the only token of C_0, then we move t_0 in the direction P_{t_0} (it will prevent to move t_0 infinitely many times). Then we continue with the children of the children of R etc. ●

4 Proof of the theorems

Now we are able to prove the nontrivial direction of Theorem 1.1. Let G be an infinite, connected graph such that there is no $A \subseteq V(G)$ for which the connected components of $G - A$ connect to A in G by a single edge and infinitely many of them are nontrivial. Let $T \subseteq V(G)$ infinite. Consider the vertices v of degree one that are in T i.e. there is a token on them. Move these tokens to the only possible direction and then delete all the vertices of degree one or zero. We denote the resulting graph by G' and we fix a R 2-edge-connected component of it. If G' has a t-infinite 2-edge-connected component, then it cannot vanish during the simplification process, thus the resulting H will be t-infinite and we are done by Claim 3.1.

Assume there is no such a component. Degree of $C_0 := R$ in $\mathsf{tree}(G')$ must be finite otherwise $A := V(C_0)$ would violate the assumption about G. Since C_0 is t-finite by the pigeonhole principle there is a child C_1 of C_0 such that $\mathsf{subt}(C_1; G', R)$ contains infinitely many tokens. By recursion we obtain a one-way infinite path of $\mathsf{tree}(G')$ with vertices C_n ($n \in \mathbb{N}$) such that for every n the tree $\mathsf{subt}(C_n; G', R)$ contains infinitely many tokens. The set $A := \bigcup_{n=0}^{\infty} V(C_n)$ may not have infinitely many outgoing edges in G' otherwise A violates the condition about G. Thus for large enough n the tree $\mathsf{subt}(C_n; G', R)$ is just a terminal segment of the one-way infinite path we constructed. This implies that infinitely many of the C_n's contain at least one token. Since such a path cannot vanish during the simplification process it terminates with a t-infinite H again.

We turn to the proof of Theorem 1.2. Let a connected G and a 2-edge-connected R of it be fixed. Let $T \subseteq V(G)$. We will show that case (A) of Theorem 1.2 occurs iff the simplification process terminates with infinitely many tokens and (B)/(C) occurs iff it terminates with even/odd number of tokens respectively.

Assume first that the result H of the simplification process initialized by T (T-process from now on) is t-infinite. Let $T' \in N(T)$. Call T'-process the simplification process with the initial tokens given by T' and denote by H' the result of it. If G has a 2-edge-connected, t-infinite component C with respect to T, then C is t-infinite with respect to T' as well. Observe that such a C remains untouched during the simplification process. Thus H' is t-infinite and therefore there is a T'-join in G by Claim 3.1.

We may suppose that there is no 2-edge-connected, t-infinite component in G. If such a component C arises during the T-process, then C receives a token from infinitely many child D of C. The token-structure of $\mathsf{subt}(D; G, R)$ is the same for all but finitely many D at the case of T'. Thus C will receive infinitely many tokens during the T'-process as well.

Finally we suppose that such a component does not arise i.e. H has no t-infinite component. Then $\mathsf{tree}(H)$ must be an infinite tree since H is t-infinite. Furthermore, $\mathsf{subt}(C; H, R)$ must contain at least one token for all 2-edge-connected component C of H otherwise we may erase $\mathsf{subt}(C; H, R)$ to continue the simplification process. Fix a one-way infinite path P in $\mathsf{tree}(H)$ with vertices $C_n (n \in \mathbb{N})$, where $C_0 = R$ and infinitely many C_n contains at least one token. For a large enough n_0 the token-distribution of $\mathsf{subt}(C_{n_0}; G, R)$ is the same at the T and at the T' cases. Hence the T-process and T'-process runs identical on the subgraph of G corresponds to $\mathsf{subt}(C_{n_0}; G, R)$ thus $\mathsf{subt}(C_{n_0}; H, R) = \mathsf{subt}(C_{n_0}; H', R)$ holds and the tokens on them are the same. But then the terminal segment of path P in $\mathsf{tree}(H')$ shows that H' is t-infinite as well.

Claim 4.1. *There is no T-join in G if the simplification process terminates with odd number of tokens.*

Proof: Remember that no t-infinite, 2-edge-connected component may arise during the simplification process in this case. Assume, to the contrary, that there is a T-join $\mathcal{P}$ in G. We play a winning gameplay induced by $\mathcal{P}$ i.e. every step we move some token along the appropriate $P \in \mathcal{P}$ towards its partner. If for a component C the subgraph $\mathsf{subt}(C; G, R)$ does not contain any vertex from T, then clearly no $P \in \mathcal{P}$ comes here hence we may delete these parts of the graph. If C is a leaf of (the remaining part of) $\mathsf{tree}(G; R)$ with $|T \cap V(C)|$ even, then the corresponding paths are inside C. In the odd case exactly one $P \in \mathcal{P}$ uses the unique outgoing edge of C and some other paths match inside C the other T-vertices of C. Thus along P we may move one token to the parent component and match the others along the other paths. Therefore after we do these steps the quantity of the tokens on the 2-edge-connected components will be the same as after the first step of the simplification process. Similar arguments show that it remains true after successor steps of the simplification process as well. Since the first difference may not arise at a limit step for all step of the simplification process we have a corresponding position of the gameplay induced by $\mathcal{P}$ where the token quantities on the components are the same. On the other hand it cannot be true for the terminating position since the play induced by $\mathcal{P}$ is a winning gameplay and hence it cannot arise a system with odd number of tokens during the play which is a contradiction. ●

Claim 4.2. *If the simplification process for T terminates with even (odd) number of tokens and $|T \triangle T'| = 1$, then the simplification process for T' terminates with odd (even) number of tokens.*

Proof: By symmetry we may let $T' = T \cup \{v\}$. If $v \in V(R)$, then the simplification process for T' runs in the same way as for T except that at the end we have the extra token on v which changes the parity of the remaining tokens as we claimed. If v is not in the root R, then it is in $\mathsf{subt}(C; G, R)$ for some child C of R. This C is closer in $\mathsf{tree}(G)$ to the 2-edge-connected component that contains v than R. On the one hand we know by induction that the parity of the number of tokens on C will be different when C will become a leaf in the case of T'. On the other hand for the other children of R this parity will be clearly the same, thus the parity of the number of the remaining tokens changed again. ●

The remaining part of Theorem 1.2 follows from Claim 4.1 and from the repeated application of Claim 4.2. ■

References

[1] Frank, A. A survey on t-joins, t-cuts, and conservative weightings.

[2] Kunen, K. *Set theory.* College Publ., 2011.

[3] Soukup, L. Elementary submodels in infinite combinatorics. *Discrete Mathematics 311*, 15 (2011), 1585–1598.