Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2016-09. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Supermodularity in Unweighted Graph Optimization I: Branchings and Matchings

Kristóf Bérczi and András Frank

July 2016

Supermodularity in Unweighted Graph Optimization I: Branchings and Matchings

Kristóf Bérczi* and András Frank†

Abstract

The main result of the paper is motivated by the following two, apparently unrelated graph optimization problems: (A) as an extension of Edmonds' disjoint branchings theorem, characterize digraphs comprising k disjoint branchings B_i each having a specified number μ_i of arcs, (B) as an extension of Ryser's maximum term rank formula, determine the largest possible matching number of simple bipartite graphs complying with degree-constraints. The solutions to these problems and to their generalizations will be obtained from a new min-max theorem on covering a supermodular function by a simple degree-constrained bipartite graph. A specific feature of the result is that its minimum cost extension is already **NP**-complete and therefore classical polyhedral tools do not help.

1 Introduction

Network flow theory provides a basic tool to treat conveniently various graph characterization and optimization problems such as the degree-constrained subgraph problem in a bipartite graph (or bigraph, for short) or the k edge-disjoint st-paths problem in a digraph. Another general framework in graph optimization is matroid theory. For example, the problem of extending k given subtrees of a graph to k disjoint spanning trees can be solved with the help of matroids, as well as the problem of finding a cheapest rooted k-edge-connected subgraph of a digraph.

A common generalization of these two big branches of combinatorial optimization is the theory of submodular flows, initiated by Edmonds and Giles [9]. This covers not only the basic results on maximum flows and min-cost circulations from network flow theory and weighted (poly)matroid intersection or matroid partition from matroid theory but also helps solving significantly more complex graph optimization problems such as the one of finding a minimum dijoin in a digraph (the classic theorem of Lucchesi and Younger) or finding a k-edge-connected orientation of a mixed graph.

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány P. s. 1/c, Budapest, Hungary, H-1117. e-mail: `berkri@cs.elte.hu` .

†MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány P. s. 1/c, Budapest, Hungary, H-1117. e-mail: `frank@cs.elte.hu` .

However general is the framework of submodular flows, it leaves open one of the most significant unsolved questions of matroid optimization concerning the existence of k (or just 2) disjoint common bases of two matroids. This is settled only in special cases, among them the most important one is a theorem of Edmonds [8] on the existence of disjoint spanning arborescences of common root in a digraph. This version is sometimes called the weak form of Edmonds' theorem while its strong form characterizes digraphs admitting k disjoint spanning branchings with prescribed root-sets. Due to the specific position of Edmonds' theorem within combinatorial optimization, it is particularly important to investigate its extensions and variations. For example, the problem of finding k disjoint spanning arborescences with no requirements on the location of their roots is a nicely tractable version [13], and even more generally, one may impose upper and lower bounds for each node v to constrain the number of arboresences rooted at v. By using analogous techniques, one can characterize digraphs comprising k disjoint spanning branchings each having μ arcs.

A characteristic feature of submodular flows is that the corresponding linear system is totally dual integral and therefore the weighted (or minimum cost) versions of the graph theoretic applications are typically also tractable. For example, not only the minimum cardinality dijoin problem can be solved in polynomial time but its minimum cost version as well. Or, via submodular flows, there is a polynomial time algorithm to find a cheapest k-edge-connected orientation of a $2k$-edge-connected graph.

More generally, a great majority of min-max theorems and good characterizations in combinatorial optimization has a polyhedral background that makes possible to manage weighted or min-cost versions (see, for example non-bipartite matchings) In this view, it is quite interesting that around the same time when submodular flows were introduced, pretty natural graph optimization problems showed up in which the minimum cardinality case was shown to be polynomially solvable while the weighted version turned out to be **NP**-complete. For example, Eswaran and Tarjan [10] found a min-max formula and an algorithm to make a digraph strongly connected by adding a minimum number of new arcs but the minimum cost version of the problem is clearly **NP**-complete as the directed Hamiltonian circuit problem is a special case. Therefore no polyhedral approach can exist for this augmentation problem. (Note that the original cardinality version of Eswaran and Tarjan has nothing to do with the problem of packing common bases of two matroids.)

Recently, it turned out that the roots of a somewhat similar phenomenon go back to as early as 1958 when Ryser [33] solved the maximum term rank problem (which is equivalent to finding a simple bipartite graph G with a specified degree sequence so that G has a matching with cardinality at least a specified number ℓ, or equivalently, the matching number $\nu(G)$ of G is as large as possible). The minimum cost version of this problem had not been settled for a long time. Ford and Fulkerson, for example, considered a natural attempt by using network flows but they concluded in their book [12] that the flow approach did not seem to work in this case. (For the exact citation, see Section 7.) Recently, however, it was shown ([25], [30], [31]) that this min-cost version of the maximum term rank problem is **NP**-complete.

Therefore the failure of using network flows to attack the maximum term rank problem was not by chance at all, and the same **NP**-completeness result shows that even submodular flows could not be able to help. The sharp borderline between the problem of finding a degree-specified simple bipartite graph and the problem of finding a degree-specified simple bipartite graph with matching number at least ℓ is best clarified by the fact that –though both problems are in **P**– the natural extension of the first problem, when a degree-specified subgraph of an initial bipartite graph is to be found, is still in **P**, while the analogous extension of the second problem, when a degree-specified subgraph with matching number at least ℓ of an initial bipartite graph is to be found, is already **NP**-complete.

In a paper by the second author [15], a min-max theorem was developed solve the general edge-connectivity augmentation problem of digraphs. It was shown in [16], that the digraph edge-connectivity augmentation problem could be embedded in an abstract framework concerning optimal arc-covering of supermodular functions. That min-max theorem seems to be the very first appearance of a min-max result on sub- or supermodular functions in which the weighted version included **NP**-complete problems.

Frank and Jordán [18] generalized this result further and proved a min-max theorem on optimally covering a so-called supermodular bi-set function by digraphs. We shall refer to the main result of [18] (and its equivalent reformulation, too) as the supermodular arc-covering theorem. It should be emphasized that this framework characteristically differs from previous models using sub- or supermodular functions, such as polymatroids or submodular flows, since it solves such cardinality optimization problems for which the corresponding weighted versions are **NP**-complete. One of the most important applications was a solution to the minimum directed node-connectivity augmentation problem but several other problems could be treated in this way. For example, with its help, the degree-sequences of k-edge-connected and k-node-connected digraphs could be characterized (without requiring simplicity of the realizing digraph). Also, it implied (an extension of) Győri's [22] beautiful theorem on covering a vertically convex polyomino by a minimum number of rectangles. Yet another application described a min-max formula for $K_{t,t}$-free t-matchings of a bipartite graph [14]. In a recent application, Soto and Telha [35] described an elegant extension of Győri's theorem.

One may consider analogous problems concerning simple digraphs covering supermodular functions. Unfortunately, it turned out recently that the problem of supermodular coverings with *simple* digraph includes **NP**-complete special cases. Therefore there is no hope to develop a general version of the min-max theorem of Frank and Jordán where the covering digraph is requested to be simple.

The present work is the first member of a series of three papers. Our general goal is to describe special cases where simplicity can successfully be treated. Here a new min-max theorem is developed on covering an intersecting supermodular function with a simple degree-constrained bipartite graph. One application is a new theorem on disjoint branchings which provides a necessary and sufficient condition for the existence of k disjoint spanning branchings $B_1, \dots, B_k$ in a digraph such that the cardinality of each $|B_i|$ lies between prescribed lower and upper bounds f_i and g_i and such the in-degree $\varrho_F(v)$ of each node $v \in V$ lies between specified lower and upper bounds $f_{in}(v)$ and $g_{in}(v)$, where $F = B_1 \cup \cdots \cup B_k$. As another consequence, we shall show that Ryser's maximum term rank problem nicely fits this new framework and not only the original maximum term rank formula can be derived but its extension as well to determine the maximum of the matching number of degree-constrained simple bigraphs.

In Part II [1] of the series, matroidal generalization of the new framework is described which gives rise to a matroidal extension of Ryser's maximum term rank theorem. We also develop the more general augmentation version of Ryser' max term rank formula, when some edges of the graph (correspondingly, some 1's of the matrix) are specified.

In Part III [2], yet another special case of the supermodular arc-covering theorem is analysed where simplicity of the covering digraph is tractable, and we derive there, among others, a characterization of degree-sequences of simple k-node-connected digraphs, providing in this way a straight generalization of a recent result of Hong, Liu, and Lai [24] on the characterization of degree-sequences of simple strongly connected digraphs.

Acknowledgements We are grateful to Richard Brualdi for the correspondence in which he always promptly provided us with extremely useful background information on the history of the topic.

The research was supported by the Hungarian Scientific Research Fund - OTKA, No K109240. The work of the first author was financed by a postdoctoral fellowship provided by the Hungarian Academy of Sciences.

1.1 Notions and notation

We close this introductory section by mentioning notions and notation.

For a number x, let $x^+ := \max\{x, 0\}$. For a function $m : V \to \mathbf{R}$, the set-function $\widetilde{m}$ is defined by $\widetilde{m}(X) = \sum[m(v) : v \in X]$. A set-function p can analogously extended to families $\mathcal{F}$ of sets by $\widetilde{p}(\mathcal{F}) = \sum[p(X) : X \in \mathcal{F}]$.

Two subsets X and Y of a ground-set V are **comparable** if $X \subseteq Y$ or $Y \subseteq X$, **intersecting** if $X \cap Y \neq \emptyset$, **properly intersecting** if they are non-comparable and intersecting, **crossing** if none of the sets $X - Y, Y - X, X \cap Y, V - (X \cup Y)$ is empty.

For two non-empty subsets S and T of V, the subsets X, Y are ST**-independent** if $X \cap Y \cap T = \emptyset$ or $S - (X \cup Y) = \emptyset$, ST**-crossing** if they are non-comparable, $X \cap Y \cap T \neq \emptyset$, and $S - (X \cup Y) \neq \emptyset$. X and Y are T**-intersecting** if $X \cap Y \cap T \neq \emptyset$, and **properly** T**-intersecting** if they are non-comparable and $X \cap Y \cap T \neq \emptyset$. Typically, we do not distinguish between a one-element set $\{v\}$, called a **singleton**, and its only element v.

For an arc $f = uv$, node v is the **head** of f and v is its **tail**. The arc uv **enters** or **covers** a subset $X \subset V$ if $u \in V - X$ and $v \in X$. Given a digraph $D = (V, A)$, the **in-degree** of a subset $X \subseteq V$ is the number of arcs entering X, denoted by $\varrho_D(X)$ or $\varrho_A(X)$. The **out-degree** $\delta_D(X) = \delta_A(X)$ is the number of arcs leaving X. An arc st is an ST**-arc** if $s \in S$ and $t \in T$.

An arc with coinciding head and tail is called a **loop**. Two arcs from s to t are called **parallel**. A digraph with no loops and parallel arcs is **simple**. Simplicity of an undirected graph is defined analogously.

A digraph $D = (V, A)$ **covers** a set-function p on V if $\varrho_D(X) \geq p(X)$ holds for every subset $X \subseteq V$.

Let $G = (S, T; E)$ a bipartite graph. For a subset $Y \subseteq T$, let

$$\Gamma_G(Y) = \{s \in S : \text{there is an edge } st \in E \text{ with } t \in Y\},$$

that is, $\Gamma_G(Y)$ is the set of neighbours of Y. We say that G **covers** a set-function p_T on T if

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{1}$$

Even if it is not mentioned explicitly, we assume throughout that each set-function is zero on the empty set. Also, the empty sum is defined to be zero. A set-function p on T is **monotone non-decreasing** if $p(X) \geq p(Y)$ whenever $\emptyset \subset X \subseteq Y \subseteq T$.

For a set-function b on ground-set V,

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{2}$$

is called the **submodular inequality** on $X, Y \subseteq V$.

The function b is **fully** (respectively, **intersecting, crossing**) **submodular** if (2) holds for each (resp., intersecting, crossing) sets X and Y. Fully submodular functions will often be mentioned simply as submodular. A set-function p is **supermodular** if $-p$ is submodular, **positively intersecting** (**crossing, ST-crossing**) supermodular if the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

holds for intersecting (crossing, ST-crossing) subsets for which $p(X) > 0$ and $p(Y) > 0$. The **complementary function** p of a set-function b with finite $b(V)$ is defined by

$$p(X) := b(V) - b(V - X).$$

Clearly, b is submodular if and only if p is supermodular. For a pair (p, b) of set-functions,

$$b(X) - p(Y) \geq b(X - Y) - p(X \cup Y) \tag{3}$$

is called the **cross-inequality** on $X, Y \subseteq V$. The pair is called **paramodular** (**intersecting paramodular**) if b is (intersecting) submodular, p is (intersecting) supermodular and the cross-inequality holds for every (properly intersecting) X and Y. For a paramodular pair (p, b), the polyhedron

$$Q(p, b) = \{x \in \mathbf{R}^V : p \leq \widetilde{x} \leq b\}$$

is called a **generalized polymatroid** or **g-polymatroid**. By convention, the empty set is also considered a g-polymatroid. For a submodular function b with $b(V)$ finite, the polyhedron $B(b) := \{x \in \mathbf{R}^\mathbf{V} : \widetilde{x} \leq b,\ \widetilde{x}(V) = b(V)\}$ is called a **base-polyhedron** and we speak of a **0-base-polyhedron** if $b(V) = 0$. For a supermodular function p with finite $p(V)$, the polyhedron $B'(p) := \{x \in \mathbf{R}^\mathbf{V} : \widetilde{x} \geq p,\ \widetilde{x}(V) = p(V)\}$ is also a base-polyhedron since $B'(p) = B(b)$ holds for the complementary function b of p.

All the notions, notation, and terminology not mentioned explicitly in the paper can be found in the book of the second author [17].

2 Background results

2.1 Degree-specified and degree-constrained bipartite graphs

2.1.1 Subgraph problems

Let S and T be two disjoint sets and $V := S \cup T$. Our starting point is the classic Hall theorem:

Theorem 2.1. A bigraph $G = (S, T; E)$ has a matching covering T if and only if

$$|\Gamma_G(Y)| \geq |Y| \text{ for every subset } Y \subseteq T. \tag{4}$$

G has a perfect matching if and only if $|S| = |T|$ and *(4)* holds.

For a given non-negative integer-valued function $m : V \to \mathbf{Z}_+$, its restrictions to S and to T are denoted by m_S and m_T, respectively. We also use the notation $m = (m_S, m_T)$. It is assumed throughout that $\widetilde{m}_S(S) = \widetilde{m}_T(T)$ and this common value will be denoted by γ. We say that m or the pair (m_S, m_T) is a **degree-specification** and that a bipartite graph $G = (S, T; E)$ **fits** or **meets** this degree-specification if $d_G(v) = m(v)$ holds for every node $v \in V$.

Theorem 2.2 (Ore [29])**.** Let $G_0 = (S, T; E_0)$ be a bipartite graph and $m = (m_S, m_T)$ a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a subgraph $G = (S, T; E)$ of G_0 fitting the degree-specification m if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) \leq \gamma \text{ whenever } X \subseteq S,\ Y \subseteq T \tag{5}$$

where $d_{G_0}(X, Y)$ denotes the number of edges connecting X and Y.

Let $g_S : S \to \mathbf{Z}_+$ and $g_T : T \to \mathbf{Z}_+$ be upper bound functions while $f_S : S \to \mathbf{Z}_+$ and $f_T : T \to \mathbf{Z}_+$ lower bound functions. Let $f_V = (f_S, f_T)$ and $g_V = (g_S, g_T)$ and assume that $f_V \leq g_V$. Call a bipartite graph $G = (S, T; E)$ (f_T, g_S)**-feasible** if

$$d_G(s) \leq g_S(s) \text{ for every } s \in S \text{ and } d_G(t) \geq f_T(t) \text{ for every } t \in T \tag{6}$$

and call G (f_V, g_V)**-feasible** if $f_S(s) \leq d_G(s) \leq g_S(s)$ for every $s \in S$ and $f_T(t) \leq d_G(t) \leq g_T(t)$ for every $t \in T$, or for short, $f_V \leq d_G \leq g_V$. $G = (S, T; E)$ (and its degree function d_G) is said to **comply with** or **degree-constrained by** (f_V, g_V) if $f_V(v) \leq d_G(v) \leq g_V(v)$ holds for every node $v \in V$.

Theorem 2.3 (Linking property, Ford and Fulkerson)**.** Let $G_0 = (S, T; E_0)$ be a bipartite graph. Let $g_S : S \to \mathbf{Z}_+$ and $g_T : T \to \mathbf{Z}_+$ be upper bound functions while $f_S : S \to \mathbf{Z}_+$ and $f_T : T \to \mathbf{Z}_+$ lower bound functions. There is an (f, g)-feasible subgraph G of G_0 if and only if there is an (f_S, g_T)-feasible subgraph G' of G_0 and there is an (f_T, g_S)-feasible subgraph G'' of G_0. •

With standard techniques, such as network flows or total unimodularity, the following theorem can also be derived.

Theorem 2.4. Suppose that a bigraph G_0 has a subgraph degree-constrained by (f_V, g_V). G_0 has a degree-constrained subgraph $G = (S, T; E)$:
(A) for which $\alpha \leq |E|$ if and only if

$$\widetilde{g}_S(S - X) + \widetilde{g}_T(T - Y) + d_{G_0}(X, Y) \geq \alpha \text{ for } X \subseteq S,\ Y \subseteq T, \tag{7}$$

(B) for which $|E| \leq \beta$ if and only if

$$\widetilde{f}_S(X) + \widetilde{f}_T(Y) - d_{G_0}(X, Y) \leq \beta \text{ for } X \subseteq S,\ Y \subseteq T, \tag{8}$$

(AB) for which $\alpha \leq |E| \leq \beta$ if and only if both *(7)* and *(8)* hold.

2.1.2 Synthesis problems

When the initial graph G_0 is the complete bipartite graph on S and T, the theorems can be simplified. Let $\mathcal{G}(m_S, m_T)$ denote the set of simple bipartite graphs fitting (m_S, m_T). Gale [21] and Ryser [32] found, in an equivalent form, the following characterization.

Theorem 2.5 (Gale and Ryser)**.** There is a simple bipartite graph G fitting the degree-specification m if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| \leq \gamma \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{9}$$

Moreover, *(9)* holds if the inequality is required only when X consists of the i elements of S having the i largest values of m_S and Y consists of the j elements of T having the j largest values of m_T ($i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|$).

Theorem 2.6. Let $g_S : S \to \mathbf{Z}_+$ be an upper bound function on S and let $f_T : T \to \mathbf{Z}_+$ be a lower bound function on T. There is a simple bipartite graph G for which $d_G(s) \leq g_S(s)$ for every $s \in S$ and $d_G(t) \geq f_T(t)$ for every $t \in T$ if and only if

$$\widetilde{g}_S(X) + \widetilde{f}_T(Y) - |X||Y| \leq \widetilde{g}_S(S) \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{10}$$

Moreover, *(10)* holds if the inequality is required only when X consists of elements with the i largest values of m_S and Y consists of elements with the j largest values of m_T ($i = 1, \ldots, |S|,\ j = 1, \ldots, |T|$).

The linking property formulated in Theorem 2.3 can also be specialized to the case when G_0 is the complete bipartite graph $G^* = (S, T; E^*)$.

Theorem 2.7. If there is a simple (f_T, g_S)-feasible bipartite graph and there is a simple (f_S, g_T)-feasible bipartite graph, then there is a simple (f_V, g_V)-feasible bipartite graph.

When G_0 is the complete bigraph on $S \cup T$, Theorem 2.4 specializes to the following synthesis-type problem.

Theorem 2.8. Suppose that there is simple bigraph degree-constrained by (f_V, g_V). There is a simple bigraph degree-constrained by (f_V, g_V):
(A) for which $\alpha \leq |E|$ if and only if

$$\widetilde{g}_S(S - X) + \widetilde{g}_T(T - Y) + |X||Y| \geq \alpha \text{ for } X \subseteq S,\ Y \subseteq T, \tag{11}$$

(B) for which $|E| \leq \beta$ if and only if

$$\widetilde{f}_S(X) + \widetilde{f}_T(Y) - |X||Y| \leq \beta \ \text{ for } X \subseteq S,\ Y \subseteq T, \tag{12}$$

(AB) for which $\alpha \leq |E| \leq \beta$ if and only if both *(11)* and *(12)* hold.

2.1.3 Synthesis versus subgraph problems

The synthesis problem of degree-constrained and degree-specified simple bigraphs is just a special case of the corresponding subgraph problems. It turns out, however, that several other synthesis problems cannot be attacked in this way since the more general subgraph problem is already **NP**-complete. For example, it is trivial to decide if there is a connected bigraph $G = (S, T; E)$ with degree-specification identically 2 since this is jut a bipartite Hamilton circuit and therefore the only requirement is $|S| = |T| \geq 2$. On the other hand, it is known to be **NP**-complete to decide if an initial bigraph G_0 includes a Hamilton circuit.

At other occasions the situation is more complicated. For example, one may consider the synthesis problem of finding a simple, perfectly matchable degree-specified bigraph. This problem is solvable but its subgraph version where a perfectly matchable degree-specified subgraph of an initial bigraph G_0 has to be found is already **NP**-complete ([25], [30], [31]).

2.2 Covering supermodular functions with digraphs and bigraphs

2.2.1 Covering by bigraphs

We call a set-function p on a ground-set T **element-subadditive** if $p(Y) + p(t) \geq p(Y + t)$ holds whenever $Y \subseteq T$ and $t \in T$. The following early result on bipartite graphs and supermodular functions is due to Lovász [27].

Theorem 2.9. Let $G_0 = (S, T; E_0)$ be a simple bipartite graph and p_T a positively intersecting supermodular function on T which is, in addition, element-subadditive. There is a subgraph G of G_0 covering p_T for which $d_G(t) = p(t)$ whenever $t \in T$ if and only if

$$|\Gamma_{G_0}(Y)| \geq p_T(Y) \text{ holds for every subset } Y \subseteq T. \tag{13}$$

This was extended by Frank and Tardos [20] as follows.

Theorem 2.10. Let $G_0 = (S, T; E_0)$ be a simple bipartite graph and p_T a positively intersecting supermodular function on T. Let $g_T : T \to \mathbf{Z}_+$ be an upper bound function. There is a subgraph G of G_0 covering p_T for which $d_G(t) \leq g_T(t)$ whenever $t \in T$ if and only if

$$|\Gamma_{G_0}(Z)| \geq p_T(Y \cup Z) - \widetilde{g}_T(Y) \text{ holds for disjoint subsets } Y, Z \subseteq T. \quad (14)$$

It should be noted that the problem in Theorem 2.9 can be formulated as a matroid intersection problem while the problem in Theorem 2.10 can be cast into the submodular flow framework. Therefore the minimum cost versions of both cases are also tractable. However, both problems become **NP**-complete if there is an upper-bound g_S, as well, for the degrees of G in S.

2.2.2 Covering by digraphs

Let p be a positively ST-crossing supermodular function. A basic tool in our investigations is the following general result of Frank and Jordán [18].

Theorem 2.11 (Supermodular arc-covering, set-function version)**.** Function p can be covered by γ ST-arcs if and only if $\widetilde{p}(\mathcal{I}) \leq \gamma$ holds for every ST-independent family $\mathcal{I}$ of subsets of V.

The theorem can be used [18] to describe characterizations for the existence of degree-specified (and even degree-constrained) digraphs covering p. It has a great many applications in graph optimization and it serves as the major tool for the present work. It significantly differs from the framework of Lovász above (or from submodular flows) in that its min-cost version includes **NP**-complete special cases such as the directed Hamilton circuit problem.

The existing applications give rise to a natural demand to develop a variation of Theorem 2.11 in which no parallel arcs of the covering digraph are allowed. Unfortunately, this is hopeless since the general problem includes **NP**-complete special cases, as we point out below. This fact underpins the significance and the difficulties of the present work that explores special cases of Theorem 2.11 where simplicity can be involved.

2.2.3 NP-completeness

THEOREM 2.12. (A) It is **NP**-complete to decide for two given degree specifications $m' \leq m$ on $V = S \cup T$ whether there exists a simple bigraph G fitting m which includes a subgraph fitting m'.

(B) The problem in Part (A) can be formulated as a special case of the problem of finding a minimal simple digraph covering an ST-crossing supermodular function.

Proof. (A) By choosing $m'' = m - m'$, Part (A) follows immediately from the following elegant **NP**-completeness result of Dürr, Guinez, and Matamala [7].

Lemma 2.13. It is **NP**-complete to decide whether, given two degree-specifications $m' = (m'_S, m'_T)$ and $m'' = (m''_S, m''_T)$, there is a simple bigraph $G = (S, T; E)$ which can be partitioned into two subgraphs $G' = (S, T; E')$ and $G'' = (S, T; E'')$ so that G' fits m' and G'' fits m''.

(B) Consider Theorem 2.2 with m' in place of m. This can be restated as follows.

Claim 2.14. A bipartite graph $G = (S, T; E)$ admits a subgraph G' fitting m' if and only if

$$\varrho_D(X \cup Y) \geq \widetilde{m}'_T(Y) - \widetilde{m}'_S(X) \text{ whenever } X \subseteq S,\ Y \subseteq T \tag{15}$$

where D is the digraph arising from G by orienting each arc from S toward T.

Let $\gamma' = \widetilde{m}'_S(S) = \widetilde{m}'_T(T)$ and $\gamma = \widetilde{m}_S(S) = \widetilde{m}_T(T)$ and define a set-function p on V as follows.

$$p(V') := \begin{cases} \widetilde{m}'_T(Y) - \widetilde{m}'_S(X) & \text{if } Y \subseteq T, X \subseteq S,\ V' = X \cup Y,\ 1 < |V'| < |V| - 1 \\ m_T(t) & \text{if } V' = \{t\} \text{ for some } t \in T \\ m_S(s) & \text{if } V' = V - s \text{ for some } s \in S. \end{cases} \tag{16}$$

Then p' is ST-crossing supermodular. Furthermore there is a simple digraph $D = (V, A)$ consisting of γ ST-arcs covering p if and only if there exists a simple bigraph $G = (S, T; E)$ fitting m so that (15) holds. By Claim 2.14, (15) in turn is equivalent to the solvability of the problem in Part (A). •

3 Bipartite graphs covering supermodular functions

3.1 Covering p_T with simple degree-specified bipartite graphs

Let p_T be a set-function p_T on T. Recall that a bipartite graph $G = (S, T; E)$ is said to covers p_T if

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{17}$$

For example, if $p_T(Y) = |Y|$ ($Y \subseteq T$), then (17) is the Hall-condition. Therefore Hall's theorem implies that $G = (S, T; E)$ covers p_T if and only if G has a matching covering T. Another special case is when $p_T(Y) := |Y| + 1$ ($\emptyset \subset Y \subseteq T$). By a theorem of Lovász [27], a bigraph $G = (S, T; E)$ covers this p_T if and only if G has a forest in which the degree of every node in T is 2. This result is a direct consequence of Theorem 2.9.

We are interested in finding simple bipartite graphs covering p_T which meet some degree-constraints (that is, upper and lower bounds) or exact degree-specifications. If no such constraints are imposed at all, then the existence of a bigraph covering p_T is obviously equivalent to the requirement that

$$p_T(Y) \leq |S| \text{ for each } Y \subseteq T. \tag{18}$$

Indeed, this condition is clearly necessary and it is also sufficient as the complete bipartite graph $G^* = (S, T; E^*)$ covers a set-function p_T meeting (18). Therefore we suppose throughout that (18) holds.

Our plan is the following. First we characterize the situation when there is a degree-prescription only on S. This is then used to settle the case when a degree-specification (m_S, m_T) is given on the whole node-set $V = S \cup T$. In Section 4.1, with the help of a novel construction, we introduce a base-polyhedron B and prove that (m_S, m_T) is realizable by a simple bigraph covering p_T precisely if the associated vector $(m_S, -m_T)$ is in B. As the intersection of a base-polyhedron with a box and with a plank is also a g-polymatroid whose non-emptiness is characterized in the literature, this result can finally be used to handle upper and lower bounds on the degrees of G and on its edge-number.

3.1.1 Degree-specification on S

Our first goal is to characterize the situation when there is a degree-specification only on S.

THEOREM 3.1. Let m_S be a degree-specification on S for which $\widetilde{m}_S(S) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. Suppose that

$$m_S(s) \leq |T| \text{ for every } s \in S. \tag{19}$$

The following statements are equivalent.

(A) There is a simple bipartite graph $G = (S, T; E)$ covering p_T and fitting the degree-specification m_S.

(B1)

$$\widetilde{m}_S(X) + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \gamma \text{ for } X \subseteq S \text{ and subpartition } \mathcal{T} \text{ of } T. \tag{20}$$

(B2)

$$\sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), q\} \text{ for every subpartition } \mathcal{T} = \{T_1, \dots, T_q\} \text{ of } T. \tag{21}$$

Proof. (A) ⇒ (B1) Suppose that there is a simple bipartite graph G meeting (17). We claim that the number $d_G(T_i, S - X)$ of edges between T_i and $S - X$ is at least $p_T(T_i) - |X|$. Indeed,

$$p_T(T_i) \leq |\Gamma_G(T_i)| = |\Gamma_G(T_i) \cap X| + |\Gamma_G(T_i) - X| \leq |X| + d_G(T_i, S - X),$$

that is, $d_G(T_i, S - X) \geq p_T(T_i) - |X|$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X) + \sum_i [p_T(T_i) - |X|]$ from which (20) follows.

(B1) ⇒ (B2) Suppose that (B2) is violated and there is a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of T for which $\sum_{i=1}^{q} p_T(T_i) > \sum_{s \in S} \min\{m_S(s), q\}$. Let $X := \{s \in S : m_S(s) > q\}$. Then

$$\sum_{i=1}^{q} p_T(T_i) > \sum_{s \in S} \min\{m_S(s), q\} = \sum [m_S(s) : s \in S - X] + q|X| =$$

$$\widetilde{m}_S(S - X) + q|X| = \gamma - \widetilde{m}_S(X) + q|X|$$

from which

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p_T(T_i) - |X|] > \gamma,$$

that is, (B1) is violated.

(B2) ⇒ (B1) Suppose that X and $\mathcal{T} = \{T_1, \dots, T_q\}$ violate (20), that is, $\widetilde{m}_S(X) + \sum_{i=1}^{q}[p_T(T_i) - |X|] > \gamma$. We can assume that $m_S(s) > q$ for every $s \in X$ for if $m_S(s) \leq q$ for some $s \in X$, then $X' := X - s$ and $\mathcal{T}$ would also violate (20). Furthermore, we can assume that $m_S(s) \leq q$ for every $s \in S - X$ for if $m_S(s) > q$ for some $s \in S - X$, then $X' := X + s$ would also violate (20).

Therefore

$$\sum_{s \in S} \min\{m_S(s), q\} = \widetilde{m}_S(S - X) + q|X| = \gamma - \widetilde{m}_S(X) + q|X|.$$

By combining this with $\widetilde{m}_S(X) + \sum_{i=1}^{q} p_T(T_i) - q|X| > \gamma$ we have

$$\sum_{i=1}^{q} p_T(T_i) > \gamma - \widetilde{m}_S(X) + q|X| = \sum_{s \in S} \min\{m_S(s), q\},$$

that is, (B2) is violated.

(B1) ⇒ (A) The following simple observation indicates that we need not concentrate on the simplicity of G.

Claim 3.2. If there is a not-necessarily simple bipartite graph $G = (S, T; E)$ covering p_T for which $d_G(s) \leq |T|$ for each $s \in S$, then there is a simple bipartite graph H covering p_T for which $d_G(s) = d_H(s)$ for each $s \in S$.

Proof. Suppose G has two parallel edges e and e' connecting s and t for some $s \in S$ and $t \in T$. Since $d_G(s) \leq |T|$, there is a node $t' \in T$ which is not adjacent with s. By replacing e' with an edge st', we obtain another bipartite graph G' for which $\Gamma_{G'}(Y) \supseteq \Gamma_G(Y)$ for each $Y \subseteq T$, $d_{G'}(s) = d_G(s)$ for each $s \in S$, and the number of parallel edges in G' is smaller than in G. By repeating this procedure, finally we arrive at a requested simple graph. •

A subset V' of $V := S \cup T$ is **ST-trivial** if no ST-arc enters it, which is equivalent to requiring that $T \cap V' = \emptyset$ or $S \subseteq V'$. We say that a subset $V' \subseteq V$ is **fat** if $V' = V - s$ for some $s \in S$ (that is, there are $|S|$ fat sets). The non-fat subsets of V will be called **normal**. An ST-independent family $\mathcal{I}$ of subsets is **strongly ST-independent** if any two of its normal members are T-independent, that is, the intersections of the normal members of $\mathcal{I}$ with T form a subpartition of T.

Define a set-function p_0 on V by

$$p_0(V') = p_T(Y) - |X| \text{ where } V' = X \cup Y \text{ for } X \subseteq S \text{ and } Y \subseteq T. \tag{22}$$

Note that p_0 is positively T-intersecting since if $p_0(V')$ is positive, then so is $p_T(Y)$. Furthermore, when (20) is applied to $X = S$, $q = 1$ and $T_1 = Y$, we obtain that $p_T(Y) \leq |X|$ and hence $p_0(V')$ can be positive only if $X \neq S$ and $Y \neq \emptyset$, that is, when V' is not ST-trivial.

Claim 3.3. $m_S(s) \geq p_0(V - s)$ holds for every $s \in S$.

Proof. By applying (9) to $X = S - s$ and $\mathcal{T} = \{T\}$, we obtain that $m_S(s) \geq p_T(T) - |S - s| = p_0(V - s)$. •

Define a set-function p_1 on V by modifying p_0 so as to lift its value on fat subsets $V - s$ from $p_0(V - s)$ to $m_S(s)$ $(s \in S)$, that is,

$$p_1(V') := \begin{cases} m_S(s) & \text{if } V' = V - s \text{ for some } s \in S, \\ p_0(V') & \text{otherwise.} \end{cases} \tag{23}$$

Note that the supermodular inequality

$$p_1(V_1) + p_1(V_2) \leq p_1(V_1 \cap V_2) + p_1(V_1 \cup V_2) \tag{24}$$

holds for T-intersecting normal sets with $p_1(V_1) > 0$ and $p_1(V_2) > 0$.

By Claim 3.3, $p_1 \geq p_0$. As p_0 is positively T-intersecting supermodular, p_1 is positively ST-crossing supermodular. Let ν_1 denote the maximum total p_1-value of a family of ST-independent sets. We call a family attaining the maximum a p_1-**optimizer.**

Claim 3.4. If $\mathcal{I}$ is a p_1-optimizer of minimum cardinality, then $\mathcal{I}$ is strongly ST-independent.

Proof. Clearly, $p_1(V') \geq 0$ for each $V' \in \mathcal{I}$ for otherwise $\mathcal{I}$ would not be a p_1-optimizer. Moreover, $p_1(V') > 0$ also holds for if we had $p_1(V') = 0$, then $\mathcal{I} - \{V'\}$ would also be a p_1-optimizer contradicting the minimality of $\mathcal{I}$.

Suppose indirectly that $\mathcal{I}$ has two properly T-intersecting normal members V_1 and V_2. Then (24) holds and since $\mathcal{I}$ is ST-independent, we must have $S \subseteq V_1 \cup V_2$ from which $p_1(V_1 \cup V_2) \leq 0$ follows. Then

$$p_1(V_1) + p_1(V_2) \leq p_1(V_1 \cap V_2) + p_1(V_1 \cup V_2) \leq p_1(V_1 \cap V_2).$$

Now $\mathcal{I}' = \mathcal{I} - \{V_1, V_2\} + \{V_1 \cap V_2\}$ is also ST-independent and $\widetilde{p}_1(\mathcal{I}') \geq \widetilde{p}_1(\mathcal{I})$, but we must have here equality by the optimality of $\mathcal{I}$, that is $\mathcal{I}'$ is also a p_1-minimizer, contradicting the minimality of $|\mathcal{I}|$. •

Claim 3.5. Let $\mathcal{I}$ be a strongly ST-independent p_1-optimizer. There exists a subset X and a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of T such that $\mathcal{I} = \{V - s : s \in X\} \cup \{X \cup T_i : i = 1, \dots, q\}$ for which

$$\nu_1 = p_1(\mathcal{I}) = \widetilde{m}_S(X) + \widetilde{p}_1(\mathcal{T}) - \mathcal{T}|X|. \tag{25}$$

Proof. Let $X := \{s \in S : V - s \in \mathcal{I}\}$ and let $\mathcal{I}_1 = \{V - s : V - s \in \mathcal{I}\}$. Let $\mathcal{I}_2 := \mathcal{I} - \mathcal{I}_1$ and let $V_1, \dots, V_q$ denote the members of $\mathcal{I}_2$. Furthermore, let $T_i := T \cap V_i$ and $X_i = S \cap V_i$ $(i = 1, \dots, q)$. By the strong ST-independence, the family $\mathcal{T} = \{T_1, \dots, T_q\}$ is a subpartition of T, and we also have $X \subseteq X_i$ for each i.

Define $V_i' := T_i \cup X$ for $i = 1, \dots, q$ and let $\mathcal{I}_2' = \{V_1', \dots, V_q'\}$. Then $\mathcal{I}' = \mathcal{I}_1 \cup \mathcal{I}_2'$ is also ST-independent. Since $p_1(V_i') = p_1(V_i) + |X_i - X|$ and $\mathcal{I}$ is a p_1-optimizer, we must have $X_i = X$ for each $i = 1, \dots, q$. The formula in (25) follows from

$$\nu_1 = \widetilde{p}_1(\mathcal{I}) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_1(\mathcal{I}_2) =$$

$$\sum [m_S(s) : V - s \in \mathcal{I}_1] + [\widetilde{p}_1(\mathcal{T}) - \mathcal{T}|X|] = \widetilde{m}_S(X) + \widetilde{p}_1(\mathcal{T}) - \mathcal{T}|X|. \bullet$$

Claim 3.6. $\nu_1 = \gamma$.

Proof. Since the family $\mathcal{L} = \{V - s : s \in S\}$ is ST-independent, $\nu_1 \geq \widetilde{p}_1(\mathcal{L}) = \widetilde{m}_S(S) = \gamma$ from which $\nu_1 \geq \gamma$. Let $\mathcal{I}$ be a strongly ST-independent p_1-optimizer for which $|\mathcal{I}|$ is minimum. It follows from (25) in Claim 3.5 and from the hypothesis (20) that $\nu_1 \leq \gamma$ and hence $\nu_1 = \gamma$. $\bullet$

By Theorem 2.11, there is a digraph $D = (V, A)$ on V with $\nu_1 = \gamma$ (possibly parallel) ST-arcs that covers p_1, that is, $\varrho_D(V') \geq p_1(V')$ for every subset $V' \subseteq V$. Let $G = (S, T; E)$ denote the underlying bipartite graph of D.

Claim 3.7. $d_G(s) = m_S(s)$ for every $s \in S$.

Proof. Since $d_G(s) = \delta_D(s) = \varrho_D(V - s) \geq p_1(V - s) = m_S(s)$ for every $s \in S$, we have $\gamma = |E| = \sum [d_G(s) : s \in S] \geq \widetilde{m}_S(S) = \gamma$, from which $d_G(s) = m_S(s)$ follows for every $s \in S$. $\bullet$

Claim 3.8. $|\Gamma_G(Y)| \geq p_T(Y)$ for every subset $Y \subseteq T$.

Proof. Let $X := \Gamma_G(Y)$ and $V' := X \cup Y$. Then $0 = \varrho_D(V') \geq p_1(V') \geq p_0(V') = p_T(Y) - |X| = p_T(Y) - |\Gamma_G(Y)|$, as required. $\bullet$

Therefore the bipartite graph G meets all the requirements of the theorem apart possibly from simplicity. By Lemma 3.2, G can be chosen to be simple. $\bullet \bullet$

3.2 Covering p_T with degree-specification on $S \cup T$

In the next problem we have degree-specification not only on S but on T as well. When the degree-specification was given only on S, we have observed that it sufficed to concentrate on finding a not-necessarily simple graph covering p_T because such a graph could easily be made simple. Based on this, it is tempting to conjecture that if there is a simple bipartite graph fitting a degree-specification $m_V = (m_S, m_T)$ and there is a (not-necessarily simple) one fitting m_V and covering p_T, then there is a simple bipartite graph fitting m_V and covering p_T. The following example shows, however, that this statement fails to hold.

Let $S = \{e, f, g, h\}$ and let the m_S-values on S, respectively, be $4, 4, 3, 2$. Let $T = \{a, b, c, d\}$ and let the m_T-values on T, respectively, be $4, 4, 3, 2$. Let $p_T(t) = m_T(t)$ for $t \in T$. Let $p_T(\{c, d\}) = 4$ and $p_T(\{y, z\} = 1$ whenever $\{y, z\} \neq \{c, d\}$, $\{y, z\} \subset T$. Let $p_T(\{a, c, d\}) = p_T(\{b, c, d\} = 3$, and $p_T(\{a, b, c\}) = p_T(\{a, b, d\}) = 2$. Finally, let $p_T(T) = 4$. Here there is a unique simple bipartite graph G fitting m_V, but G does not cover p_T since $|\Gamma_G(\{c, d\})| = |\{e, f, g\}| = 3 \not\geq 4 = p_T(\{c, d\})$. On the other hand the bipartite graph $G' = (S, T; E')$ with $E' = \{ae, ae, af, ag,\ be, bf, bf, bh,\ ce, cf, cg,\ dg, dh\}$ fits m_V and covers p_T.

THEOREM 3.9. Let S and T be disjoint sets and let $m_V = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. Let p_T be a positively intersecting supermodular function on T for which $p_T(\emptyset) = 0$ and $p_T(Y) \leq |S|$ for $Y \subseteq T$. There is a simple bigraph $G = (S, T; E)$ covering p_T and fitting the degree-specification m_V if and only if

$$\begin{aligned}&\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \gamma \\ &\text{for } X \subseteq S,\ Y \subseteq T, \text{ and subpartition } \mathcal{T} \text{ of } T - Y\end{aligned} \tag{26}$$

holds including the special case $\mathcal{T} = \emptyset$ *(when the condition is exactly (9))*. When p_T is fully supermodular, it suffices to require ***(26)*** for $|\mathcal{T}| \leq 1$. When p_T is fully supermodular and monotone non-decreasing, it suffices to require ***(26)*** only for $\mathcal{T} = \emptyset$ and $\mathcal{T} = \{T - Y\}$.

Proof. Necessity. Suppose that there is a requested bigraph G. Let $\mathcal{T} = \{T_1, \dots, T_q\}$ be a subpartition of $V - Y$. We claim that the number $d_G(T_i, S - X)$ of edges between T_i and $S - X$ is at least $p_T(T_i) - |X|$. Indeed,

$$p_T(T_i) \leq |\Gamma_G(T_i)| \leq |\Gamma_G(T_i) \cap X| + |\Gamma_G(T_i - X)| \leq |X| + |T_i - X| \leq |X| + d_G(T_i, S - X),$$

that is, $d_G(T_i, S - X) \geq p_T(T_i) - |X|$. Therefore the total number γ of edges is at least the number of edges between X and Y plus the number of edges between $\cup_i T_i$ and $S - X$. Here the first summand is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y|$ while the second one is at least $\sum_i [p_T(T_i) - |X|]$ from which (26) follows.

Sufficiency. Let t be an element of T. By applying (26) to $X = \emptyset, Y = T - t, q = 1, T_1 = \{t\}$, we obtain that $\widetilde{m}_T(T - t) + p_T(t) \leq \gamma$, that is, $p_T(t) \leq m_T(t)$.

Define a set-function p_T^+ on T by revising p_T so as to lift its value on each singleton $\{t\}$ to $m_T(t)$ $(t \in T)$. As $p_T(t) \leq m_T(t)$ and p_T is positively T-intersecting supermodular, so is p_T^+.

Let s be an element of S. By applying (26) to $X = \{s\}$, $Y = T$, and $q = 0$, we obtain that $m_S(s) + \widetilde{m}_T(T) - |T| \leq \gamma$, that is, $m_S(s) \leq |T|$, implying that (19) holds.

Claim 3.10. Condition ***(20)*** holds for p_T^+ in place of p_T.

Proof. Let $X \subseteq S$ and let $\mathcal{T}' = \{T_1, T_2, \dots, T_{q'}\}$ be a sub-partition of T. Let $T_1, T_2, \dots, T_q$ denote those members of $\mathcal{T}'$ for which $p_T^+(T_i) = p_T(T_i)$ and let $\mathcal{T} = \{T_1, T_2, \dots, T_q\}$. Then each of the remaining members T_j in $\mathcal{T}'$ is a singleton $\{z_j\}$ $(j = q + 1, \cdots, q')$ for which $p_T^+(T_j) = m_T(z_j)$. By letting $Y = \{z_{q+1}, \dots, z_{q'}\}$, we have $|Y| = q' - q$. By applying (26) to this choice of $(X, Y, \mathcal{T})$, we obtain that

$$\widetilde{m}_S(X) + \sum [p^+(T_i) - |X| : i = 1, \dots, q'] =$$

$$\widetilde{m}_S(X) + \sum [p_T(T_i) - |X| : i = 1, \dots, q] + \sum [m_T(z_j) - |X| : j = q + 1, \dots, q'] =$$

$$\widetilde{m}_S(X) + \sum [p_T(T_i) - |X| : i = 1, \dots, q] + \widetilde{m}_T(Y) - |X||Y| \leq \gamma,$$

that is, condition (20) holds indeed for p_T^+.

By applying Theorem 3.1 to p_T^+, we obtain that there is a simple bipartite graph fitting the degree-specification m_S for which $|\Gamma_G(Y)| \geq p_T^+(Y) \geq p_T(Y)$ for every subset $Y \subseteq T$. In particular, this implies for $Y = \{t\}$ that $d_G(t) = |\Gamma_G(t)| \geq p_T^+(t) = m_T(t)$. Therefore $\gamma = \sum[d_G(t) : t \in T] \geq \sum[m_T(t) : t \in T] = \widetilde{m}_T(T) = \gamma$ and hence we must have $d_G(t) = m_T(t)$ for every $t \in T$, making the proof of the main part of the theorem complete.

Suppose now that p_T is fully supermodular. Assume that X, Y, and $\mathcal{T}$ violate (26) and $\mathcal{T} = \{T_1, \ldots, T_q\}$ has a minimum number of members. If $q \geq 2$, then $p_T(T_1) - |X| + p_T(T_2) - |X| \leq p_T(T_1 \cup T_2) - |X|$ and hence the unchanged X, Y and the subpartition $\mathcal{T}' := \{T_1 \cup T_2, T_3, \ldots, T_q\}$ also violate (26), contradicting the minimal choice of q.

Finally, investigate the case when p_T is fully supermodular and monotone non-decreasing. If there are sets X, Y and a subpartition $\mathcal{T}$ of $T - Y$ violating (26) so that $\mathcal{T} = \{T_1\}$, then X, Y, and $\mathcal{T}' = \{T - Y\}$ also violates (26) since $p_T(T - Y) \geq p_T(T_1)$. • • •

Corollary 3.11. Let S, T, m_S, m_T, γ, and p_T be the same as in Theorem 3.9 and assume that p_T is non-decreasing and fully supermodular. There is a simple bigraph covering p_T and fitting the (m_S, m_T) if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| \leq \gamma \text{ for } X \subseteq S,\ Y \subseteq T \tag{27}$$

and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{p}_T(T - Y) - |X| \leq \gamma \text{ for } X \subseteq S,\ Y \subset T. \bullet \tag{28}$$

Proof. Recall that the members of $\mathcal{T}$ in (26) are non-empty, in particular, if $\mathcal{T} = \{T - Y\}$, then $Y \subset T$. By the last part of Theorem 3.9, the corollary follows.

In the example above, the subsets $X = \{e, f\}, Y = \{a, b\}$ and the subpartition $\mathcal{T} = \{\{c, d\}\}$ consisting of a single set (that is, $q = 1$) do violate the necessary condition (26) since $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \sum_{i=1}^{q}[p_T(T_i) - |X|] = 8 + 8 - 4 + [4 - 2] = 14 \not\leq 13 = 4 + 4 + 3 + 2 = \gamma$.

The essence of the next corollary of Theorem 3.1 is that it suffices to require (26) only for subsets $X \subseteq S$ with the j largest m_S-values. We leave out the straightforward proof which consists of pointing out the equivalence of (29) and (26).

Corollary 3.12. Let S, T, p_T, and $m_V = (m_S, m_T)$ be the same as in Theorem ***3.9***. There is a simple bipartite graph $G = (S, T; E)$ covering p_T and fitting m_V if and only if

$$\widetilde{m}_T(Y) + \sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), |Y| + q\} \tag{29}$$

holds for every subset $Y \subseteq T$ and subpartition $\{T_1, \ldots, T_q\}$ of $T - Y$, (including the special case when $q = 0$ or $Y = \emptyset$). •

3.2.1 An NP-complete extension

One may be wondering if the synthesis problem solved in Theorem 3.1 could possibly be extended to the corresponding subgraph problem, That is, the problem is to characterize the situation when the requested bigraph G (covering p_T) is a subgraph of an initial bipartite graph $G_0 = (S, T; E_0)$. However such an extension is unlikely to exist since it includes NP-complete problems.

To see this, let $G_0 = (S, T; E_0)$ be a bipartite graph in which $|S| = |T| + 1$. Define m_T to be identically 2 on T and m_S to be identically 2 on S apart from two specified nodes $s_1, s_2 \in S$ where $m_S(s_1) = m_S(s_2) = 1$. Define $p_T(Y) = |Y| + 1$ for each non-empty $Y \subseteq T$ and let $p_T(\emptyset) = 0$. Clearly, p_T is intersecting supermodular.

Lemma 3.13. A subgraph $G = (S, T; E)$ of G_0 covers p_T and fits $m_V = (m_S, m_T)$ if and only if G is a Hamilton path connecting s_1 and s_2.

Proof. A Hamilton path G contains a matching covering T and hence $|\Gamma_G(Y)| \geq |Y|$ for every $Y \subseteq T$. If indirectly G does not cover p_T, then there is a non-empty subset Y of T for which $|\Gamma_G(Y)| = |Y|$. But then the subgraph of G induced by $Y \cup \Gamma_G(Y)$ has exactly $2|Y| = |Y \cup \Gamma_G(Y)|$ edges, contradicting the assumption that G is a path.

Suppose now that G covers p_T and fits m_V. Then G has $2|T| = |S \cup T| - 1$ edges. It cannot comprise a circuit C since then we would have $|\Gamma_G(Y)| = |Y|$ for $Y = T \cap C$ contradicting the assumption that G covers p_T. Therefore G is a spanning tree, and since G fits m_V, it must be a Hamilton circuit connecting s_1 and s_2. •

Since the Hamilton path problem is NP-complete, so is the equivalent problem of finding a subgraph of G_0 that covers p_T and fits m_V.

4 The master base-polyhedron associated with realizable degree-specifications

As before, S and T are two disjoint non-empty sets, $V := S \cup T$, and $m = (m_S, m_T)$ is a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. Let p_T be a positively intersecting supermodular set-function on T for which

$$p_T(Y) \leq |S| \text{ for every subset } Y \subseteq T. \tag{30}$$

This implies that the complete bipartite graph $(S, T; E^*)$ is a simple bigraph covering p_T. Recall Theorem 3.9 which stated that there is a simple bigraph covering p_T and fitting m if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \gamma$$
$$\text{for } X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y. \tag{31}$$

We allow throughout the empty subpartition with the convention $\widetilde{p}_T(\emptyset) = 0$. For brevity we call such a degree-specification realizable (with respect to p_T). In this section, we investigate the problem when, rather than an exact degree specification m, lower and upper bounds are prescribed for the degrees of the requested simple bigraph covering p_T. Instead of attacking the problem directly, we exhibit first a novel construction for a submodular function b_0 and show that there is a simple one-to-one correspondence between the realizable degree-specifications and the integral elements of the base-polyhedron $B_0 = B(b_0)$. Because of its central role, we call B_0 the **master base-polyhedron** associated with p_T and S.

Recall that for a submodular function b with $b(V)$ finite, the polyhedron $B(b) := \{x \in \mathbf{R^V} : \widetilde{x} \leq b,\ \widetilde{x}(V) = b(V)\}$ is called a **base-polyhedron**, and we speak of a **0-base-polyhedron** if $b(V) = 0$. Given this correspondence at hand, we can apply some known characterizations for the non-emptiness of the intersection of a g-polymatroid with a box and with a plank. This approach enables us to treat situations when, in addition to degree-constraints, upper and lower bounds for the total number of edges can also be prescribed.

4.1 A new submodular function

With each vector $m = (m_S, m_T)$, we associate the vector $m' = (m_S, -m_T)$. Note that the property $\widetilde{m}_S(S) = \widetilde{m}_T(T)$ is equivalent to $\widetilde{m}'(V) = 0$. The condition (31) for the realizability of m is equivalent to the following.

$$\widetilde{m}'(X \cup Z) \leq |T - Z||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X| \quad \text{for } X \subseteq S, Z \subseteq T,\ \mathcal{T} \text{ a subpartition of } Z. \tag{32}$$

Define a set-function b_0 on V as follows. For $X \subseteq S$ and $Z \subseteq T$, let

$$b_0(X \cup Z) := \min\{|T - Z||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X| :\ \mathcal{T} \text{ a subpartition of } Z\}. \tag{33}$$

Clearly, (32) is equivalent to

$$\widetilde{m}'(U) \leq b_0(U) \text{ whenever } U \subseteq V. \tag{34}$$

Claim 4.1. $b_0(\emptyset) = 0$ and $b_0(V) = 0$.

Proof. When $Z = \emptyset$, a subpartition of Z is also empty, and hence $b_0(\emptyset)$ is indeed zero.

For $X = S$ and $Z = T$, we have $b_0(V) = \min\{-\widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||S| : \mathcal{T}$ a subpartition of $T\}$. By choosing $\mathcal{T}$ to be empty, we see that the minimum is at most 0. On the other hand $-\widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||S| \geq 0$ holds for every subpartition $\mathcal{T}$ of T since (30) implies that $\widetilde{p}_T(\mathcal{T}) \leq |\mathcal{T}||S|$. Therefore $b_0(V) = 0$. •

THEOREM 4.2. b_0 is fully submodular.

Proof. Let $V_1 = X_1 \cup Z_1$ and $V_2 = X_2 \cup Z_2$ be two subsets of V with $X_i \subseteq S$ and $Z_i \subseteq T$ $(i = 1, 2)$. Let $\mathcal{T}_i$ define the optimizer subpartition in the definition of $b_0(V_i)$, that is,

$$b_0(V_i) = |T - Z_i||X_i| - \widetilde{p}_T(\mathcal{T}_i) + |\mathcal{T}_i||X_i|.$$

Let $\mathcal{F}_0$ denote the multi-union of $\mathcal{T}_1$ and $\mathcal{T}_2$, that is, each member of $\mathcal{T}_1$ and $\mathcal{T}_2$ occurs in $\mathcal{F}_0$, and if X is in both $\mathcal{T}_1$ and $\mathcal{T}_2$, then two copies of X occur in $\mathcal{F}_0$. Hence $|\mathcal{T}_1| + |\mathcal{T}_2| = |\mathcal{F}_0|$.

An uncrossing step consists of replacing two properly intersecting members A and B with $p_T(A) > 0$, $p_T(B) > 0$ by their union and intersection. The uncrossing procedure starts with $\mathcal{F}_0$ and repeatedly performs uncrossing steps. It is known that the uncrossing procedure is finite (as the number of sets does not change while to total sum of the squares of cardinalities strictly increases). Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_q$ denote the subsequent families, that is, $\mathcal{F}_{j+1}$ arises by applying the uncrossing step to two properly intersecting members of $\mathcal{F}_j$.

Claim 4.3. Every family $\mathcal{F}_j$ (j=0,…,q) covers each element of $Z_1 \cap Z_2$ at most twice, each element of the symmetric difference $Z_1 \ominus Z_2$ at most once, and no element outside $Z_1 \cup Z_2$.

Proof. The property clearly holds for $j = 0$ and it is maintained throughout since an uncrossing step does not affect the number of sets containing any given element of T. •

Claim 4.4. If the family $\mathcal{F}_h$ for some $h = 0, \dots, q$ contains two copies of a set W, then each family $\mathcal{F}_j$ $(j = 0, \dots, q)$ contains two copies of W. In particular, $W \in \mathcal{T}_1$ and $W \in \mathcal{T}_2$.

Proof. By induction, it suffices to show that both $\mathcal{F}_{h+1}$ and $\mathcal{F}_{h-1}$ contain two copies of W.

By Claim 4.3, no member of $\mathcal{F}_h$ can intersect properly W, and therefore both copies of W belong to $\mathcal{F}_{h+1}$. Similarly, Claim 4.3 implies that both copies of W must be in $\mathcal{F}_{h-1}$ since if the second copy of W in $\mathcal{F}_h$ arises as the intersection or the union of two properly intersecting members A and B of $\mathcal{F}_{h-1}$, then the elements of $A \cap B$ would belong to A, B, and W. •

Claim 4.5. Let W be a member of $\mathcal{F}_{j+1}$ arising as the intersection of two properly intersecting members A and B of $\mathcal{F}_j$, and let Y be any member of $\mathcal{F}_{j+1} \cup \dots \cup \mathcal{F}_q$ intersecting W. Then $W \subset Y$.

Proof. We say that a pair of elements of T is **non-separated** by a family of sets if no member of the family contains exactly one of the two elements. Clearly, if a pair is non-separated, then it remains so after an uncrossing step.

By Claim 4.4, W does not occur in two copies and hence $Y \neq W$. By Claim 4.3, any two elements of $A \cap B$ are non-separated by $\mathcal{F}_j$ and hence by each of $\mathcal{F}_{j+1}, \dots, \mathcal{F}_q$, as well. Therefore, as Y intersects W, it must properly include W. •

Claim 4.6. Let W be a member of $\mathcal{F}_{j+1}$ arising as the union of two properly intersecting members A and B of $\mathcal{F}_j$. Then W has a subset belonging to $\mathcal{T}_1$ and W has a subset belonging to $\mathcal{T}_2$.

Proof. Suppose the claim fails to hold and let j be the smallest index occurring in a counter-example. If both A and B would belong to $\mathcal{F}_0$, then one of them is in $\mathcal{T}_1$

while the other one in $\mathcal{T}_2$, as these families are subpartitions. But in this case the pair (W, j) would not be a counter-example.

Therefore at least one of A and B, say A, is not in $\mathcal{F}_0$. By Claim 4.5, A could not arise as an intersection at an uncrossing step, that is, A arose as the union of two sets. By the minimality of j, A has a subset belonging to $\mathcal{T}_1$ and A has a subset belonging to $\mathcal{T}_2$. As W is a superset of A, W also has a subset belonging to $\mathcal{T}_1$ and a subset belonging to $\mathcal{T}_2$. •

Let $\mathcal{L}$ denote the subfamily $\mathcal{F}_q$ consisting of those members W for which $p_T(W) > 0$. Clearly, $\mathcal{L}$ is laminar. Let $\mathcal{P}_1$ consist of the minimal members of $\mathcal{L}$ which are subsets of $Z_1 \cap Z_2$, with the convention that if two copies of a set $W \subseteq Z_1 \cap Z_2$ belong to $\mathcal{L}$, then one of them is placed in $\mathcal{P}_1$. Let $\mathcal{P}_2$ consist of the members of $\mathcal{L}$ which are not in $\mathcal{P}_1$.

Claim 4.7. $\mathcal{P}_1$ is a subpartition of $Z_1 \cap Z_2$ and $\mathcal{P}_2$ is a subpartition of $Z_1 \cup Z_2$.

Proof. Since $\mathcal{L}$ is laminar, its minimal members are disjoint and hence $\mathcal{P}_1$ is indeed a subpartition.

To see that $\mathcal{P}_2$ is also a subpartition, assume indirectly that two members A and B of $\mathcal{P}_2$ are not disjoint. Then the laminarity of $\mathcal{L}$ implies that one of A and B includes the other, say, $A \subseteq B$. We must have $A \subset B$ for if we had $A = B$, then one of A and B would belong to $\mathcal{P}_1$ by the definition of $\mathcal{P}_1$. Because each element of $Z_1 \ominus Z_2$ belongs to at most one member of $\mathcal{L}$, we have $A \subseteq Z_1 \cap Z_2$. But A is not in $\mathcal{P}_1$, that is, A is not a minimal member of $\mathcal{L}$, contradicting the property that each element of T belongs to at most two members of $\mathcal{L}$. •

Claim 4.8. Let W be a member of $\mathcal{P}_2$. If $W \subseteq Z_i$ $(i = 1, 2)$, then W has a subset belonging to $\mathcal{T}_i$.

Proof. Since the indices 1 and 2 play a symmetric role, we prove the claim only for $i = 1$. That is, we assume that $W \subseteq Z_1$ and will show that there is a subset of W belonging to $\mathcal{T}_1$. If W is in $\mathcal{P}_1$, as well, that is, if two copies of W occur in $\mathcal{L}$, then we are done by Claim 4.4. Therefore, we can assume that $W \notin \mathcal{P}_1$.

By Claim 4.6, we are done if W has arisen as a union during the uncrossing procedure. Suppose now that W arises as an intersection of A and B during the uncrossing procedure. Then Claim 4.3 implies that $W = A \cap B \subseteq Z_1 \cap Z_2$. Since W is not in $\mathcal{P}_1$, there must be a set $Y \in \mathcal{L}$ for which $Y \subset W$, contradicting Claim 4.5.

In the remaining case, W belongs each of the families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_q$. In particular, W is in $\mathcal{F}_0$. Since we are done if $W \in \mathcal{T}_1$, we can assume that $W \in \mathcal{T}_2$. In this case, $W - Z_2 = \emptyset$, that is, $W \subseteq Z_1 \cap Z_2$. Since W is not in $\mathcal{P}_1$, there must be a set $Y \in \mathcal{L}$ for which $Y \subset W$. Since W belongs to each $\mathcal{F}_j$, Y could not arise as an intersection or a union during the uncrossing procedure, and therefore Y is also a member of $\mathcal{F}_0$. Since $\mathcal{T}_2$ is a subpartition, Y cannot be in $\mathcal{T}_2$, that is, $Y \in \mathcal{T}_1$. •

For simplifying calculations, we introduce the following four parameters.

$$\tau_1 := |T - Z_1| + |\mathcal{T}_1| \quad \text{and} \quad \tau_2 := |T - Z_2| + |\mathcal{T}_2|,$$

$$\pi_1 := |T - (Z_1 \cap Z_2)| + |\mathcal{P}_1| \quad \text{and} \quad \pi_2 := |T - (Z_1 \cup Z_2)| + |\mathcal{P}_2|.$$

Claim 4.9. $\pi_2 \leq \tau_1$ and $\pi_2 \leq \tau_2$.

Proof. Since the role of τ_1 and τ_2 is symmetric, we prove only the first inequality. Since $\mathcal{P}_2$ is a subpartition, $\mathcal{P}_2$ has at most $|Z_1 - Z_2|$ members intersecting $Z_1 - Z_2$, and, by Claim 4.8, $\mathcal{P}_2$ has at most $\mathcal{T}_1$ members not intersecting $Z_1 - Z_2$. Therefore $|\mathcal{P}_2| \leq |Z_2 - Z_1| + |\mathcal{T}_1|$. By adding this to the identity $|T - (Z_1 \cup Z_2)| = |T - Z_1| - |Z_2 - Z_1|$, we obtain the required $\pi_2 \leq \tau_1$. •

Claim 4.10.

$$\tau_1 + \tau_2 \geq \pi_1 + \pi_2$$

and

$$\widetilde{p}_T(\mathcal{T}_1) + \widetilde{p}_T(\mathcal{T}_2) \leq \widetilde{p}_T(\mathcal{P}_1) + \widetilde{p}_T(\mathcal{P}_2).$$

Proof. Clearly, $|\mathcal{T}_1| + |\mathcal{T}_2| = |\mathcal{F}_0| = |\mathcal{F}_q| \geq |\mathcal{L}| = |\mathcal{P}_1| + |\mathcal{P}_2|$. By adding this to $|T - Z_1| + |T - Z_2| = |T - (Z_1 \cap Z_2)| + |T - (Z_1 \cup Z_2)|$, the first inequality follows.

Since p_T is positively intersecting supermodular, an uncrossing step cannot decrease the p_T-sum of the current family. Hence $\widetilde{p}_T(\mathcal{T}_1) + \widetilde{p}_T(\mathcal{T}_2) = \widetilde{p}_T(\mathcal{F}_0) \leq \widetilde{p}_T(\mathcal{F}_q) \leq \widetilde{p}_T(\mathcal{L}) = \widetilde{p}_T(\mathcal{P}_1) + \widetilde{p}_T(\mathcal{P}_2)$. •

For $i = 1, 2$, we have:

$$b_0(V_i) = |T - Z_i||X_i| - \widetilde{p}_T(\mathcal{T}_i) + |\mathcal{T}_i||X_i| \ = \ \tau_i|X_i| - \widetilde{p}_T(\mathcal{T}_i). \tag{35}$$

Since $\mathcal{P}_1$ is a subpartition of $Z_1 \cap Z_2$, we have

$$b_0(V_1 \cap V_2) \leq |T - (Z_1 \cap Z_2)||X_1 \cap X_2| - \widetilde{p}_T(\mathcal{P}_1) + |\mathcal{P}_1||X_1 \cap X_2| \ = \ \pi_1|X_1 \cap X_2| - \widetilde{p}_T(\mathcal{P}_1). \tag{36}$$

Since $\mathcal{P}_2$ is a subpartition of $Z_1 \cup Z_2$, we have

$$b_0(V_1 \cup V_2) \leq |T - (Z_1 \cup Z_2)||X_1 \cup X_2| - \widetilde{p}_T(\mathcal{P}_2) + |\mathcal{P}_2||X_1 \cup X_2| \ = \ \pi_2|X_1 \cup X_2| - \widetilde{p}_T(\mathcal{P}_2). \tag{37}$$

By combining these inequalities, we obtain:

$$b_0(V_1) + b_0(V_2) = [\tau_1|X_1| - \widetilde{p}_T(\mathcal{T}_1)] + [\tau_2|X_2| - \widetilde{p}_T(\mathcal{T}_2)] =$$

$$\tau_1|X_1 - X_2| + \tau_2|X_2 - X_1| + (\tau_1 + \tau_2)|X_1 \cap X_2| - \widetilde{p}_T(\mathcal{T}_1) - \widetilde{p}_T(\mathcal{T}_2) \geq$$

$$\pi_2|X_1 - X_2| + \pi_2|X_2 - X_1| + (\pi_1 + \pi_2)|X_1 \cap X_2| - \widetilde{p}_T(\mathcal{P}_1) - \widetilde{p}_T(\mathcal{P}_2) =$$

$$[\pi_1|X_1 \cap X_2| - \widetilde{p}_T(\mathcal{P}_1)] + [\pi_2|X_1 \cup X_2| - \widetilde{p}_T(\mathcal{P}_2)] \geq$$

$$b_0(V_1 \cap V_2) + b_0(V_1 \cup V_2),$$

that is, the function b_0 is indeed fully submodular. • •

Corollary 4.11. An integral vector $m = (m_S, m_T)$ is the degree-vector of a simple bigraph covering p_T if and only if the associated vector $m' = (m_S, -m_T)$ belongs to the 0-base-polyhedron $B(b_0) := \{x \in \mathbf{R^V} : \widetilde{x} \leq b_0,\ \widetilde{x}(V) = 0\}$. •

Corollary 4.12. There is an integral g-polymatroid Q_S in $\mathbf{R}^S$ so that a vector $m_S : S \to \mathbf{Z}_+$ belongs to Q_S if and only if there is a simple bigraph covering p_T for which $d_G(s) = m_S(s)$ for every $s \in S$.

Proof. Take Q_S to be the projection of $B(b_0)$ to S. •

5 Degree and edge-number constraints

5.1 Basic properties of generalized polymatroids

In what follows, we make use of some basic notions and theorems of the theory of generalized polymatroids. (For a background, see for example [19] or Chapter 14 in book [17].) Let (p, b) be a fully paramodular (or, for short, paramodular) pair of set-functions p and b defined on a ground-set V. By definition, this means that

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X)$$

holds for every pair of subsets X, Y of V. The polyhedron $Q(p, b) := \{x \in \mathbf{R^V} : p \leq \widetilde{x} \leq b\}$ is called a g-polymatroid and (p, b) is its border pair. Here we consider only integer-valued functions p and b. The empty set is also considered as a g-polymatroid, though it cannot be defined with the help of a paramodular pair. A special g-polymatroid is a box $T(f, g) = \{x \in \mathbf{R}^V : f \leq x \leq g\}$ where $f : V \rightarrow \mathbf{Z} \cup \{-\infty\}$, $g : V \rightarrow \mathbf{Z} \cup \{+\infty\}$ with $f \leq g$. Another special g-polymatroid is a plank $K(\alpha, \beta) = \{x \in \mathbf{R}^V : \alpha \leq \widetilde{x}(V) \leq \beta\}$ where $\alpha \in \mathbf{Z} \cup \cup\{-\infty\}$, $\beta \in \mathbf{Z} \cup \{\infty\}$ with $\alpha \leq \beta$.

With a submodular function b with finite $b(V)$, we can associate the complementary set-function p defined for $U \subseteq V$ by $p(U) := b(V) - b(V - U)$. We list some basic properties.

Claim 5.1. If p is the complementary function of a submodular function b, then (p, b) is paramodular and $B(b) = Q(p, b)$.

Claim 5.2. A g-polymatroid defined by an integral paramodular pair is a non-empty integral polyhedron.

Claim 5.3. A non-empty g-polymatroid Q uniquely determines its defining paramodular pair (p, b), namely,

$$p(U) = \min\{\widetilde{x}(U) : x \in Q\} \text{ and } b(U) = \max\{\widetilde{x}(U) : x \in Q\}.$$

Claim 5.4. The intersection of two integral g-polymatroids is an integral polyhedron. $Q(p_1, b_1) \cap Q(p_2, b_2)$ is non-empty if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.

Claim 5.5. The intersection of a g-polymatroid, a box, and a plank is a g-polymatroid.

Claim 5.6. The intersection Q' of a g-polymatroid $Q = Q(p, b)$ and a box $T = T(f, g)$ is non-empty if and only if $\widetilde{f} \leq b$ and $p \leq \widetilde{g}$. When Q' is non-empty, its unique border pair (p', b') is given by

$$p'(U) = \max\{p(U') - \widetilde{g}(U' - U) + \widetilde{f}(U - U') : U' \subseteq V\}, \tag{38}$$

$$b'(U) = \min\{b(U') - \widetilde{f}(U' - U) + \widetilde{g}(U - U') : U' \subseteq V\}. \tag{39}$$

Claim 5.7 (Linking property of g-polymatroids)**.** If a g-polymatroid $Q = Q(p, b)$ has an element x' with $x' \geq f$, and Q has an element x'' with $x'' \leq g$, then Q has an element x with $f \leq x \leq g$. In addition, x can be chosen to be integral if p, b, f, g are integral.

Claim 5.8. The intersection Q' of g-polymatroid $Q = Q(p, b)$ and a plank $K(\alpha, \beta)$ is non-empty if and only if $\alpha \leq b(S)$ and $p(S) \leq \beta$. In particular, if Q has an element x' with $\widetilde{x}'(V) \geq \alpha$ and Q has an element x'' with $\widetilde{x}''(V) \leq \beta$, then Q has an element x with $\alpha \leq \widetilde{x}(V) \leq \beta$. Moreover, if p, b, α, β are integral, then Q' is an integral polyhedron.

Claim 5.9. Given a non-empty subset $S \subset V$, the projection Q' of a g-polymatroid $Q = Q(p, b)$ to $\mathbf{R}^S$ (or, for short, to S) is the g-polymatroid $Q(p|_S, b|_S)$ where $p|_S$ and $b|_S$ are the restriction of p and b, respectively, on S. Each integral element of Q' is the projection of an integral element of Q.

5.2 Degree constraints

We are given a lower bound function $f_V = (f_S, f_T)$ and an upper bound function $g_V = (g_S, g_T)$ on $V = S \cup T$ for which $-\infty \leq f_V \leq g_V \leq +\infty$.

THEOREM 5.10. Let p_T be an intersecting supermodular function on T for which $p_T(Y) \leq |S|$ for every $Y \subseteq T$. There is a simple bigraph $G = (S, T; E)$ covering p_T and degree-constrained by (f, g) if and only if

$$\widetilde{f}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \widetilde{g}_S(S - X)$$
$$\text{for } X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y \tag{40}$$

and

$$\widetilde{f}_S(X) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \widetilde{g}_T(T - Y)$$
$$\text{for } X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y. \tag{41}$$

If p_T is fully supermodular, then it suffices to require the two conditions only for subpartitions $\mathcal{T}$ having at most one member. If p_T is fully supermodular and monotone non-decreasing, then it suffices to require the two conditions only for $\mathcal{T} = \{\emptyset\}$ and $\mathcal{T} = \{T - Y\}$.

Proof. Let

$$f' := (f_S, -g_T) \text{ and } g' := (g_S, -f_T). \tag{42}$$

Recall the submodular function b_0 and let p_0 denote its complementary function (that is, $p_0(U) = -b_0(V - U)$). Then $B(b_0) = Q(p_0, b_0)$ and, by Corollary 4.11, the requested bigraph exists if and only if the intersection $Q' = Q(p_0, b_0) \cap T(f', g')$ is non-empty. By Claim 5.6, Q' is non-empty precisely if $\widetilde{f'} \leq b_0$ and $p_0 \leq \widetilde{g'}$. We are going to show that $\widetilde{f'} \leq b_0$ is equivalent to (41) and that $p_0 \leq \widetilde{g'}$ is equivalent to (40).

By (33), $\widetilde{f}' \leq b_0$ is equivalent to requiring the following inequality for every pair of subsets $X \subseteq S,\ Z \subseteq T$:

$$\widetilde{f}'(X \cup Z) \leq |T - Z||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X| \text{ whenever } \mathcal{T} \text{ is a subpartition of } Z.$$

By taking $Y := T - Z$ and observing that $\widetilde{f}'(X \cup Z) = \widetilde{f}_S(X) - \widetilde{g}_T(Z) = \widetilde{f}_S(X) - \widetilde{g}_T(T - Y)$, we conclude that $\widetilde{f}' \leq b_0$ is equivalent to

$$\widetilde{f}_S(X) - \widetilde{g}_T(T - Y) \leq |Y||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X|$$
$$\text{for } X \subseteq S, Y \subseteq T,\ \ \mathcal{T} \text{ a subpartition of } T - Y,$$

which is the same as (41).

Let us prove not the equivalence $p_0 \leq \widetilde{g}'$ and (40). By taking $Y := T - Z$ and $X' := S - X$, we have $\widetilde{g}'(X' \cup Y) = \widetilde{g}_S(X') - f_T(Y) = \widetilde{g}_S(S - X) - \widetilde{f}_T(Y)$ and

$$p_0(X' \cup Y) = -b_0(X \cup Z) =$$
$$-\min\{|T - Z||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X| : \mathcal{T} \text{ a subpartition of } Z\}.$$

Condition $g' \geq p_0$ means that $g'(X' \cup Y) \geq p_0(X' \cup Y)$ for every pair of sets $X' \subseteq S, Y \subseteq T$, and this is equivalent to requiring

$$\widetilde{g}_S(S - X) - \widetilde{f}_T(Y) \geq -[|Y||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X|]$$

for every subpartition $\mathcal{T}$ of $T - Y$, and this inequality is the same as the one in (40).

The last part of the theorem concerning fully supermodular p_T follows exactly the same way how the analogous statement was derived in the proof of Theorem 3.9. •

Corollary 5.11. Let p_T be an intersecting supermodular function on T for which $p_T(Y) \leq |S|$ for $Y \subseteq T$.

(A) There is a simple bigraph G' covering p_T and degree-constrained by (f_T, g_S) if and only if *(40)* holds.

(B) There is a simple bigraph G'' covering p_T and degree-constrained by (f_S, g_T) if and only if *(41)* holds.

(AB) There is a simple bigraph G covering p_T and degree-constrained by (f_V, g_V) if and only if both G' and G'' exist *(that is, both (40) and (41) hold).*

When p_T is fully supermodular, it suffices to require the two conditions only for subpartitions $\mathcal{T}$ having at most one member. If p_T is fully supermodular and monotone non-decreasing, then it suffices to require the two conditions only for $\mathcal{T} = \{\emptyset\}$ and $\mathcal{T} = \{T - Y\}$.

Proof. (A) Define $f_S :\equiv -\infty$ and $g_T :\equiv +\infty$, and observe that (41) automatically holds when $X \neq \emptyset$ or $Y \subset T$. If $X = \emptyset$ and $Y = T$, then $\mathcal{T}$ is empty and the requirement in (41) becomes void. Hence Theorem 5.10 implies Part (A).

(B) Define $f_T :\equiv -\infty$ and $g_S :\equiv +\infty$, and observe that (40) automatically holds when $X \subset S$ or $Y \neq \emptyset$. If $X = S$ and $Y = \emptyset$, then (41) reduces to $\widetilde{p}_T(\mathcal{T}) \leq |\mathcal{T}||S|$ for every subpartition $\mathcal{T}$ of T, but this follows from the hypothesis that $p_T(Y) \leq |S|$ for every $Y \subseteq T$. Hence Theorem 5.10 implies Part (B).

(AB) Theorem 5.10 implies immediately Part (AB). •

Corollary 5.12. Let S and T be disjoint sets and let m_S be a degree-specification on S for which $\widetilde{m}_S(S) = \gamma$. Let $g_T : T \to \mathbf{Z}_+$ be an upper bound function for which $g_T(t) \leq |S|$ for every $t \in T$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. There is a simple bigraph covering p_T and fitting m_S for which

$$d_G(t) \leq g_T(t) \text{ whenever } t \in T \tag{43}$$

if and only if

$$\widetilde{m}_S(X) + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \gamma \text{ for } X \subseteq S \text{ and subpartition } \mathcal{T} \text{ of } T \tag{44}$$

and

$$\begin{aligned} &\widetilde{m}_S(X) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \widetilde{g}_T(T - Y) \\ &\text{for } X \subseteq S,\ Y \subseteq T, \text{ and subpartition } \mathcal{T} \text{ of } T - Y \end{aligned} \tag{45}$$

where each of X, Y, and $\mathcal{T}$ may be empty.

Proof. (outline) Define $f_S := m_S\ g_S := m_S$, $f_T := -\infty$, and apply Theorem 5.10. •

Corollary 5.13. Let p_T be an intersecting supermodular function on T for which $p_T(Y) \leq |S|$ for $Y \subseteq T$. There is a simple bigraph $G = (S, T; E)$ covering p_T and degree-constrained by (f_S, g_S) if and only if

$$f_S(s) \leq |T| \text{ for } s \in S. \tag{46}$$

and

$$\widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| \leq \widetilde{g}_S(S - X) \text{ for } X \subseteq S,\ \mathcal{T} \text{ a subpartition of } T \tag{47}$$

Proof. Define $f_T(t) \equiv -\infty$ and $g_T(t) \equiv +\infty$ and apply Theorem 5.10. Observe that condition (40) automatically holds when $Y \neq \emptyset$. When $Y = \emptyset$, (40) is just (47). Similarly, condition (41) automatically holds when $Y \neq T$. When $Y = T$, then $\mathcal{T} = \emptyset$ and (41) requires $f_S(X) \leq |X||T|$ for every $X \subseteq S$ but this is equivalent to (46). •

Corollary 5.14. Let p_T be an intersecting supermodular function on T for which $p_T(Y) \leq |S|$ for $Y \subseteq T$. Let $g_T : T \to \mathbf{Z}_+$ be a function for which $g_T(t) \leq |S|$ for $t \in T$. There is a simple bigraph $G = (S, T; E)$ covering p_T and degree-constrained by g_T if and only if

$$p_T(Y) \leq \widetilde{g}_T(Y) \text{ for } Y \subseteq T. \tag{48}$$

Proof. Define $f_V :\equiv -\infty$ and $g_S :\equiv +\infty$. Then (40) holds automatically (as we showed this in the proof of Part (B) of Corollary 5.11). Condition (41) holds automatically when $X \neq \emptyset$. If $X = \emptyset$, then (41) transforms to

$$\widetilde{p}_T(\mathcal{T}) \leq \widetilde{g}_T(T - Y) \text{ for } Y \subset T,\ \mathcal{T} = \{V_1, \dots, V_q\} \text{ a subpartition of } T - Y.$$

By Condition (48), $p_T(V_i) \leq \widetilde{g}(V_i)$ from which $\widetilde{p}_T(\mathcal{T}) \leq \sum_{i=1}^q \widetilde{g}_T(V_i) \leq \widetilde{g}_T(T - Y)$. Therefore the conditions of Theorem 5.10 hold and hence the required degree-constrained bigraph exists. •

Remark Corollary 5.14 is not particularly exciting since it can actually be formulated in a more general form when G is a subgraph of an initial bipartite graph G_0. That was the content of Theorem 2.10. To derive Corollary 5.14, choose G_0 to be the complete bipartite graph $G^* = (S, T, E^*)$ and observe that (14) holds automatically when $Z \neq \emptyset$. For $Z = \emptyset$, (14) is just (48).

5.3 Edge-number constraints

Suppose now that there exists a simple bigraph covering p_T and constrained by (f, g), that is, conditions (40) and (41) hold. Our next goal is to characterize the situation when, in addition to the degree constraints (f, g), there are lower and upper bounds $\alpha \leq \beta$ for the number of edges, as well, where α and β are non-negative integers.

THEOREM 5.15. Suppose that conditions *(40)* and *(41)* hold. There is simple bigraph $G = (S, T; E)$ covering p_T and degree-constrained by (f, g) for which **(A)** $\alpha \leq |E|$ if and only if

$$\widetilde{g}_S(S - X) + \widetilde{g}_T(T - Y) + |X||Y| - [\widetilde{p}_T(\mathcal{T}) - |X||\mathcal{T}|] \geq \alpha \quad \text{for } X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y, \tag{49}$$

(B) $|E| \leq \beta$ if and only if

$$\widetilde{f}_S(X) + \widetilde{f}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |X||\mathcal{T}| \leq \beta \quad \text{for } X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y, \tag{50}$$

(AB) $\alpha \leq |E| \leq \beta$ if and only if both *(49)* and *(50)* hold.

When p_T is fully supermodular, it suffices to require the two conditions only for subpartitions $\mathcal{T}$ having at most one member. If p_T is fully supermodular and monotone non-decreasing, then it suffices to require the two conditions only for $\mathcal{T} = \{\emptyset\}$ and $\mathcal{T} = \{T - Y\}$.

Proof. Consider the functions f' and g' defined in (42). As we proved above, there is a simple bigraph covering p_T and constrained by (f, g) if and only if the g-polymatroid $Q' = Q(p_0, b_0) \cap T(f', g')$ is non-empty. By our hypothesis Q' is non-empty. Let (p', b') denote the unique border pair of Q' which can be obtained by applying Claim 5.6 to p_0, b_0, f', g'.

Let Q'_S denote the projection of Q' to S. By Claim 5.9 the unique border pair of Q'_S is $(p'|_S, b'|_S)$, and any integral element of Q'_S is the projection of an integral element of Q'. Therefore the requested bigraph exists if and only of the intersection of Q'_S and the plank $K_S(\alpha, \beta)$ in $\mathbf{R}^S$ is non-empty. By Claim 5.8 this intersection is non-empty if and only if $p'(S) \leq \beta$ and $\alpha \leq b'(S)$.

By applying (39) to $U = S$ and $U' = X \cup Z$ (where $X \subseteq S,\ \ Z \subseteq T$), we obtain that $\alpha \leq b'(S)$ is equivalent to requiring

$$\alpha \leq [|T - Z||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X|] - \widetilde{f}'(Z) + \widetilde{g}'(S - X)$$

$$\text{for } X \subseteq S, Z \subseteq T,\ \mathcal{T} \text{ a subpartition of } Z.$$

By letting $Y = T - Z$ and observing that $-\widetilde{f}'(Z) + \widetilde{g}'(S - X) = \widetilde{g}_T(T - Y) + \widetilde{g}_S(S - X)$, we conclude that $\alpha \leq b'(S)$ is equivalent to (49).

Let $U = S,\ Y = T - Z,\ X = S - X',\ U' = X' \cup Y$. Then $V - U' = X \cup Z,\ U' - S = Y,\ S - U' = X$, and $p_0(U') = -b_0(V - U') = -b_0(X \cup Z) = -b_0(X \cup (T - Y))$. Furthermore

$$p'(S) = \max\{p_0(U') - \widetilde{g}'(U' - S) + \widetilde{f}'(S - U') :\ U' \subseteq V\} =$$

$$= \max\{-b_0(X \cup (T - Y)) + \widetilde{f}_T(Y) + \widetilde{f}_S(X) :\ X \subseteq S, Y \subseteq T\}.$$

Hence $\beta \geq p'(S)$ is equivalent to

$$\beta \geq -[|Y||X| - \widetilde{p}_T(\mathcal{T}) + |\mathcal{T}||X|] + \widetilde{f}_T(Y) + \widetilde{f}_S(X)$$

$$\text{for every } X \subseteq S, Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y,$$

and this is just (50). •

Corollary 5.16. Provided that there is simple bigraph covering p_T and degree-constrained by (f, g), the minimum number of edges of such a bigraph is

$$\max\{\widetilde{f}_S(X) + \widetilde{f}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |X||\mathcal{T}| :\ X \subseteq S,\ Y \subseteq T,\ \mathcal{T} \text{ a subpartition of } T - Y\}. \tag{51}$$

Analogous theorem can be formulated for the minimum number of edges, as well.

6 Packing branchings and arborescences

Let $D = (V, A)$ be a digraph on n nodes. An **arborescence** is a directed tree in which one node, its root-node, has no entering arc and the in-degree of all other nodes is 1. A **branching** (V, B) of D is a directed forest consisting of arborescences. Its **root-set** $R(B)$ is the set of nodes of in-degree zero. By the **size** of a branching we mean the number of its arcs while the **root-size** is $|R(B)|$. Obviously, $|B| + |R(B)| = n$. In what follows the same term B will be used for a branching and its set of arcs.

D is called **rooted k-edge-connected** with respect to a root-node r_0 if $\varrho_D(X) \geq k$ for every $\emptyset \subset X \subseteq V - r_0$. By Menger, this is equivalent to requiring that there are k edge-disjoint paths from r_0 to v for every node $v \in V$.

6.1 Background

A major open problem in combinatorial optimization is to find a good characterization for the existence of k disjoint common bases of two matroids. This is solved only in special cases. For example, μ-element matchings of a bipartite graph form the common bases of two matroids. Folkman and Fulkerson [11] proved the following.

Theorem 6.1. A bigraph $G = (S, T; E)$ includes k disjoint matchings of size μ if and only if

$$k(\mu + |Z| - |S \cup T|) \leq i_G(Z) \text{ for } Z \subseteq S \cup T$$

where $i_G(Z)$ denotes the number of edges induced by Z.

As the branchings of a digraph form the common independent sets of two matroids, the problems of finding k disjoint spanning arborescences or k disjoint branchings of size μ can also be viewed as special cases of the disjoint common bases problem. This matroidal aspect particularly underpins the significance of the following fundamental result of Edmonds [8].

Theorem 6.2 (Edmonds)**.** Let $D = (V, A)$ be a digraph. **(Weak form)** D includes k disjoint spanning arborescences with a specified root-node r_0 if and only if D is rooted k-edge-connected. **(Strong form)** D includes k disjoint branchings with specified root-sets $R_1, R_2, \ldots, R_k$ if and only if $\varrho_D(X) \geq k - p_R(X)$ for $X \subseteq V$ where $p_R(X)$ denotes the number of root-sets disjoint from X when $X \neq \emptyset$ and $p_R(\emptyset) = 0$.

Though Lovász [28] found a short proof relying on submodular functions and also a great number of variations and generalizations have been developed (see the book of Schrijver [34] or a recent survey by Kamiyama [26]), Edmonds' theorem and the topic of disjoint branchings remained rather isolated from general frameworks like the one of submodular flows. Due to its specific position within combinatorial optimization, it is particularly important to investigate extensions and variations.

An early variation of the weak form was proved in [13].

Theorem 6.3. A digraph D has k disjoint spanning arborescences with unspecified roots *(that is, k disjoint branchings of size $|V| - 1$)* if and only of

$$\textstyle\sum_{i=1}^q \varrho_D(V_i) \geq k(q-1) \text{ for every subpartition } \{V_1, \ldots, V_q\} \text{ of } V.$$

The following extension is due to Cai [6] and Frank [13] (see also Theorem 10.1.11 in the book [17]).

Theorem 6.4. Let $f : V \to \mathbf{Z}_+$ and $g : V \to \mathbf{Z}_+$ be lower and upper bounds for which $f \leq g$. A digraph $D = (V, A)$ includes k disjoint spanning arborescences so that each node v is the root of at least $f(v)$ and at most $g(v)$ of these arborescences if and only if $\widetilde{f}(V) \leq k$,

$$\textstyle\sum_{i=1}^q \varrho_D(V_i) \geq k(q-1) + \widetilde{f}(V_0) \text{ for every partition } \{V_0, V_1, \ldots, V_q\} \text{ of } V$$

where $q \geq 1$ and only V_0 can be empty, and

$$\widetilde{g}(X) \geq k - \varrho_D(X) \text{ for every subset } \emptyset \subset X \subseteq V.$$

With similar techniques, the following generalization of Theorem 6.3 can also be derived (though, to our best knowledge, it was not explicitly formulated earlier.)

THEOREM 6.5. A digraph D has k disjoint branchings of size μ if and only if

$$\textstyle\sum_{i=1}^q \varrho_D(V_i) \geq k[q - (n-\mu)] \text{ for every subpartition } \{V_1, \ldots, V_q\} \text{ of } V.$$

6.2 Packing branchings with prescribed sizes

The following possible extension emerges naturally for branchings and matchings, as well. What is a necessary and sufficient condition for the existence of k disjoint branchings in a digraph (respectively, k disjoint matchings in a bigraph) having prescribed sizes $\mu_1, \mu_2, \dots, \mu_k$? A bit surprisingly, the answer in the two cases is quite different. For bipartite matchings the problem was shown to be **NP**-complete even for $k = 2$ ([25], [30], [31]). On the other hand, for branchings we have the following straight generalization of Theorem 6.5.

THEOREM 6.6. Given k positive integers $\mu_1, \mu_2, \dots, \mu_k$ ($\mu_j \leq n-1$), a digraph $D = (V, A)$ on n nodes has k disjoint branchings $B_1, \dots, B_k$ of sizes $|B_j| = \mu_j$ ($j = 1, \dots, k$) if and only if

$$\textstyle\sum_{i=1}^{q} \varrho_D(V_i) \geq \sum_{j=1}^{k} [q - (n - \mu_j)]^+ \text{ for every subpartition } \mathcal{P} = \{V_1, \dots, V_q\} \text{ of } V. \tag{52}$$

Proof. Necessity. The root-set R_j of a branching B_j of size μ_j has $m_j := n - \mu_j$ elements. If B_j has no arc entering V_i, then R_j has an element in V_i, therefore there are at least $(q - m_j)^+$ arcs of B_j entering a member of the subpartition $\mathcal{P} = \{V_1, \dots, V_q\}$, implying that the total number $\sum_{i=1}^{q} \varrho_D(V_i)$ of arcs entering some members of $\mathcal{P}$ is at least $\sum_{j}^{k} [q - (n - \mu_j)]^+$. (Note that the assumption ($\mu_j \leq n-1$) is actually superfluous since (52), when applied to $q = 1$ and $\mathcal{P} = \{V\}$, implies that $0 = \varrho_D(V) \geq \sum_{j=1}^{k} [1 - (n - \mu_j)]^+$ from which each summand $[1 - (n - \mu_j)]^+$ must be zero, that is, $1 \leq n - \mu_j$.)

Sufficiency. Let $S = \{s_1, s_2, \dots, s_k\}$ be a set of k elements. We may consider S as the index set of the k branchings to be found. Define $m_S : S \to \mathbf{Z}_+$ by $m_S(s_j) := m_j$ ($j = 1, \dots, k$). Let $T := V$ and define a set-function p_T on T as follows.

$$p_T(Y) := \begin{cases} k - \varrho_D(Y) & \text{if } \emptyset \subset Y \subseteq T \\ 0 & \text{if } Y = \emptyset. \end{cases} \tag{53}$$

Then p_T is intersecting supermodular. From (52), we have

$$\sum_{i=1}^{q} \varrho_D(V_i) \geq \sum_{j=1}^{k} (q - m_j)^+ = \sum_{j=1}^{k} \max\{q - m_j, 0\} =$$

$$kq + \sum_{j=1}^{k} \max\{-m_j, -q\} = kq - \sum_{j=1}^{k} \min\{m_j, q\}$$

from which

$$\sum_{i=1}^{q} p_T(V_i) = \sum_{i=1}^{q} [k - \varrho_D(V_i)] \leq \sum_{j=1}^{k} \min\{m_j, q\} = \sum_{s \in S} \min\{m_S(s), q\}.$$

Therefore (21) holds and Theorem 3.1 implies that there is a simple bigraph $G = (S, T; E)$ covering p_T for which $d_G(s) = m_S(s)$ for every $s \in S$.

For $s_j \in S$ let R_j denote the set of neighbours of s_j in G. Then $|R_j| = m_j$ for $j = 1, \dots, k$. Since each non-empty subset Y of V has at least $p_T(Y) = k - \varrho_D(Y)$ neighbours, the number of non-neighbours is at most $\varrho_D(Y)$, that is, the number of sets R_j's disjoint form Y is at most $\varrho_D(Y)$. The strong form of Edmonds' theorem implies that there are k disjoint branchings $B_1, \dots, B_k$ with root sets $R_1, \dots, R_k$, respectively. By the definition of m_j, we have $|B_j| = n - |R_j| = n - m_j = \mu_j$. •

With a similar approach, we can characterize the situation when not only the sizes of the k disjoint branchings are specified but the indegree of each node in their union, as well.

THEOREM 6.7. Let $D = (V, A)$ be a digraph on n nodes, $m_{\text{in}} : V \to \mathbf{Z}_+$ an in-degree prescription with $0 \le m_{\text{in}}(v) \le \varrho_D(v)$ and $m_{\text{in}}(v) \le k$ for each $v \in V$. Let $\mu_1, \mu_2, \dots, \mu_k$ be k positive integers such that $\mu_1 + \cdots + \mu_k = \widetilde{m}_{\text{in}}(V)$. There is a subgraph (V, F) of D which is the union of k disjoint branchings $B_1, \dots, B_k$ of sizes $|B_j| = \mu_j \quad (j = 1, \dots, k)$ and for which

$$\varrho_F(v) = m_{\text{in}}(v) \text{ for each } v \in V$$

if and only if

$$\widetilde{m}_{\text{in}}(Y) + \sum_{i=1}^{q} \varrho_D(V_i) \ge \sum_{j=1}^{k} [q + |Y| - (n - \mu_j)]^+ \tag{54}$$

for every subset $Y \subseteq V$ and every subpartition $\{V_1, \dots, V_q\}$ of $V - Y$.

Proof. Necessity. Suppose that the requested k branchings $B_1, \dots, B_k$ exist and let $F = B_1 \cup \cdots \cup B_k$. Let $Y \subseteq V$ and $\mathcal{P} = \{V_1, \dots, V_q\}$ a subpartition of $V - Y$. As before, $m_j = n - \mu_j$ is the cardinality of the root-set R_j of B_j. Therefore the number of non-root nodes in Y $(= |Y - R_j|)$ plus the number of V_i's disjoint from R_j is at least $|Y| + q - m_j$, and hence the number of arcs of B_j entering a node of Y plus the number of arcs of B_j entering a member of $\mathcal{P}$ is at least $(|Y| + q - m_j)^+$. Hence

$$\widetilde{m}_{\text{in}}(Y) + \sum_{i=1}^{q} \varrho_D(V_i) \ge \widetilde{m}_{\text{in}}(Y) + \sum_{i=1}^{q} \varrho_F(V_i) =$$

$$\sum_{j=1}^{k} [\sum_{v \in Y} \varrho_{B_j}(v) + \sum_{i=1}^{q} \varrho_{B_j}(V_i)] \ge \sum_{j=1}^{k} (|Y| + q - m_j)^+,$$

and (54) follows.

Sufficiency. Let S, T, and m_S be the same as in the preceding proof. Define a set-function p_T on T as follows.

$$p_T(Y) := \begin{cases} k - \varrho_D(Y) & \text{if } Y \subseteq T,\ |Y| \ge 2 \\ k - m_{\text{in}}(v) & \text{if } Y = \{v\},\ v \in V\} \\ 0 & \text{if } Y = \emptyset. \end{cases} \tag{55}$$

The hypothesis $m_{\rm in}(v) \le \varrho_D(v)$ implies that $k - m_{\rm in}(v) \ge k - \varrho_D(v)$ and hence p_T is intersecting supermodular. Let $\mathcal{T} = \{V_1, \dots, V_q, V_{q+1}, \dots, V_{q'}\}$ be a subpartition of V so that the first q members are of cardinalities at least two while the subsequent members are singletons. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ and let Y denote the union of $V_{q+1}, \dots, V_q'$ (that is, $|Y| = q' - q$).

By (54), we have

$$\widetilde{m}_{\rm in}(Y) + \sum_{i=1}^{q} \varrho_D(V_i) \ge \sum_{j=1}^{k} [q' - (n - \mu_j)]^+ = \sum_{j=1}^{k} \max\{q' - m_j, 0\} =$$

$$kq' + \sum_{j=1}^{k} \max\{-m_j, -q\} = kq' - \sum_{j=1}^{k} \max\{m_j, q'\}$$

from which

$$\sum_{i=1}^{q'} p_T(V_i) = \sum_{v\in Y} [k - m_{\rm in}(v)] + \sum_{i=1}^{k} [k - \varrho_D(V_i)] = k(|Y| + q) - \widetilde{m}_{\rm in}(Y) - \sum_{i=1}^{k} \varrho_D(V_i) =$$

$$kq' - \widetilde{m}_{\rm in}(Y) - \sum_{i=1}^{k} \varrho_D(V_i) \le \sum_{j=1}^{k} \max\{m_j, q'\} = \sum_{j=1}^{k} \max\{m_S(s_j), q'\}.$$

Therefore (21) holds with q' in place of q and with V_i in place of T_i, and Theorem 3.1 implies that there is a simple bigraph $G = (S, T; E)$ covering p_T for which $d_G(s) = m_S(s)$ for every $s \in S$. Since G covers p_T, it follows that $d_G(v) \ge p_T(v) = k - m_{\rm in}(v)$. Hence

$$\sum_{v\in V} [k - m_{\rm in}(v)] \le \sum_{v\in V} d_G(v) = \sum_{s\in S} d_G(s) = \sum_{s\in S} m_S(s) = \sum_{j=1}^{k} (k - \mu_j)$$

But here we must have equality since we assumed that $\mu_1 + \dots + \mu_k = \widetilde{m}_{\rm in}(V)$. This implies that $d_G(v) = k - m_{\rm in}(v)$ for each $v \in V$.

For $s_j \in S$ let R_j denote the set of neighbours of s_j in G. Then $|R_j| = m_j$ for $j = 1, \dots, k$. Since each non-empty subset Y of V has at least $p_T(Y) = k - \varrho_D(Y)$ neighbours, the number of non-neighbours is at most $\varrho_D(Y)$, that is, the number of sets R_j's disjoint form Y is at most $\varrho_D(Y)$. The strong form of Edmonds' theorem implies that there are k disjoint branchings $B_1, \dots, B_k$ with root sets $R_1, \dots, R_k$, respectively. By the definition of m_j, we have $|B_j| = n - |R_j| = n - m_j = \mu_j$.

Let $F := B_1 \cup \dots \cup B_k$. As $d_G(v)$ is the number of R_j's containing v, the indegree $\varrho_F(v)$ is $k - d_G(v) = m_{\rm in}(v)$, as required. •

Note that the indegree $\varrho_F(v)$ in the union F of k disjoint branchings is exactly k minus the number of root-sets not containing v. Therefore Theorem 6.7 could be described in an equivalent form when, instead of the indegree of each node v in the union of k branchings with specified sizes, the number of root-sets containing v is prescribed.

6.3 Packing branchings with bounds on sizes, on total indegrees, and on total size

Suppose now that, instead of exact prescription μ_j for the size of the branchings B_j, we are given a lower bound φ_j and an upper bound γ_j with $0 \leq \varphi_j \leq \gamma_j \leq n-1$ $(j = 1, \dots, k)$. Furthermore, instead of the exact prescription $m_{in}(v)$ for the indegree $\varrho_F(v)$ $(v \in V)$, where F denotes the union of the k branchings, we are given a lower bound $f_{in}(v)$ and an upper bound $g_{in}(v)$ for which $0 \leq f_{in}(v) \leq g_{in}(v) \leq k$. Moreover, we impose a lower bound α_F and an upper bound β_F for the cardinality of the union of the k branchings.

The proof of Theorem 6.6 relied on a one-to-one correspondence between simple bigraphs $G = (S, T; E)$ covering the function p_T defined in (53) (where $T = V$ and $S = \{s_1, \dots, s_k\}$ is a k-element index set of the k branchings to be found) and the families $\mathcal{R} = \{R_1, \dots, R_k\}$ of k root-sets satisfying the necessary condition in the strong form of Edmonds' theorem (which required that $\varrho_D(Y)$ is at least the number of R_i's disjoint from Y for each non-empty $Y \subseteq V$). Let $B_1, \dots, B_k$ denote the k disjoint branchings ensured by Edmonds' theorem for which $R(B_j) = R_j$, and let $F = B_1 \cup \dots \cup B_k$.

In this correspondence, the degree of a node $s_j \in S$ is the cardinality of R_j, that is,

$$d_G(s_j) = |R_j| = n - |B_j|.$$

Furthermore, the degree of a node $v \in V = T$ is the number of root-sets R_j's containing v, that is,

$$d_G(v) = k - \varrho_F(v).$$

Finally, for the total number of edges of G, we have

$$|E| = \sum_{j=1}^{k} d_G(s_j) = \sum_{j=1}^{k} |R_j| = \sum_{j=1}^{k} (n - |B_j|) = nk - |F|.$$

Define

$$f_S(s_j) := n - \gamma_j \text{ and } g_S(s_j) = n - \varphi_j \quad \text{for } s_j \in S,$$

$$f_T(v) := k - g_{\text{in}}(v) \text{ and } g_T(v) := k - f_{\text{in}}(v) \quad \text{for } v \in T = V,$$

$$\alpha := kn - \beta_F \text{ and } \beta := kn - \alpha_F.$$

By this vocabulary, $\varphi_j \leq |B_j| \leq \gamma_j$ if and only if $f_S(s_j) \leq d_G(s_j) \leq g_S(s_j)$ $(s_j \in S)$. Furthermore, $f_{\text{in}}(v) \leq \varrho_F(v) \leq g_{\text{in}}(v)$ if and only if $f_T(v) \leq d_G(v) \leq g_T(v)$ $(v \in T)$. Finally, $\alpha_F \leq |B_1 \cup \dots \cup B_k| \leq \beta_F$ if and only if $\alpha \leq |E| \leq \beta$. By aggregating Theorems 5.10 and 5.15, we obtain the following.

THEOREM 6.8. In a digraph $D = (V, A)$ on n nodes, there are k disjoint branchings $B_1, \dots, B_k$ for which $\varphi_j \leq |B_j| \leq \gamma_j$ $(j = 1, \dots, k)$, for which $f_{\text{in}}(v) \leq \varrho_F(v) \leq g_{\text{in}}(v)$ $(v \in V)$, and for which $\alpha_F \leq |F| \leq \beta_F$, where $F = B_1 \cup \dots \cup B_k$, if and only if the conditions ***(40)***, ***(41)***, ***(49)***, and ***(50)*** hold for the choice of $f_T, g_T, f_S, g_S, \alpha, \beta$ defined above. •

7 Maximum term rank problems

7.1 Degree-specified max term rank

The members of $\mathcal{G}(m_S, m_T)$ (that is, simple bigraphs fitting the degree-specification (m_S, m_T)) can be identified with $(0,1)$-matrices of size $|S||T|$ with row sum vector m_S and column sum vector m_T. Let $\mathcal{M}(m_S, m_T)$ denote the set of these matrices. Ryser [33] defined the **term rank** of a $(0,1)$-matrix M by the maximum number of independent 1's which is the matching number of the bipartite graph corresponding to M. Ryser developed a formula for the maximum term rank of matrices in $\mathcal{M}(m_S, m_T)$. The maximum term rank problem is equivalent to finding a bipartite graph G in $\mathcal{G}(m_S, m_T)$ whose matching number $\nu(G)$ is as large as possible. Although, we use graph terminology, the original name "term rank" for the problem will be kept throughout. In graphical terms, Ryser's theorem is equivalent to the following.

Theorem 7.1 (Ryser)**.** Let $\ell \leq |T|$ be an integer. Suppose that $\mathcal{G}(m_S, m_T)$ is non-empty, that is, Condition ***(9)*** holds. Then $\mathcal{G}(m_S, m_T)$ has a member G with matching number $\nu(G) \geq \ell$ if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + (\ell - |X| - |Y|) \leq \gamma \text{ whenever } X \subseteq S,\ Y \subseteq T. \quad (56)$$

Moreover, ***(56)*** holds if the inequality in it is required only when X consists of the i largest values of m_S and Y consists of the j largest values of m_T ($i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|$).

Observe that the conditions (56) and (9) in Theorem 7.1 can be united as follows.

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + (\ell - |X| - |Y|)^+ \leq \gamma \text{ whenever } X \subseteq S,\ Y \subseteq T, \quad (57)$$

that is, assuming this inequality, we do not need to impose explicitly the non-emptiness of $\mathcal{G}(m_S, m_T)$.

Note that the strengthening formulated in the second part of the theorem is nothing but a straightforward observation. Beyond the aesthetic joy, a practical advantage is that such simplified condition can easily be checked in polynomial time since there are only a few $((|S| + 1)(|T| + 1))$ inequalities to be checked. This will be crucial in the algorithm described below for the degree-constrained max term rank problem. Note that the original proof of Ryser gives rise to a polynomial time algorithm to compute the matrix itself. Also, Brualdi and Ross [5] described a simpler proof and this results in a simple algorithm.

We also remark that there is a characterization given by Haber [23] for the minimum term rank of the graphs in $\mathcal{G}(m_S, m_T)$ but we deal only with the maximum term rank problem.

7.1.1 Relation to network flows

As the bipartite matching problem and the more general degree-prescribed subgraph problem can be treated with network flow technique, one may be wondering if Ryser's theorem could also be derived via network flows. Ford and Fulkerson, for example, remarked in their classic book ([12], p. 89) that:

"Neither term rank problem appears amenable to flow approach".

Such a link could help solving the weighted and the subgraph version of the max term rank problem. But recently it turned out that the failure of the attempt of Ford and Fulkerson was was not just by chance. It was proved ([25], [30], [31]) that the problem of deciding whether an initial bigraph G_0 has a perfectly matchable degree-specified subgraph is **NP**-complete. Therefore both the weighted and the subgraph versions of the max term rank problem is **NP**-complete, showing that even the theory of submodular flows cannot help. The first goal of this section is to show that Ryser's theorem immediately follows from the general result on covering a supermodular function by simple bigraphs developed in Section 3.

Unfortunately not only weighted, but quite natural unweighted extensions also turned out to be **NP**-complete. For example, finding a member of $\mathcal{G}(m_S, m_T)$ in which there is a subgraph with specified degrees is equivalent to finding two disjoint simple bipartite graphs on the same node-set, and this latter problem was shown to be **NP**-complete by Dürr, Guinez, and Matamala [7] (see Proposition 2.13).

In the light of the **NP**-complete problems in the close neighbourhood, it is pleasing to realize that there are nicely tractable extensions of the max term rank problem. In the present section, we shall extend Ryser's theorem to the case when the bigraph with high matching number is degree-constrained and edge-number constrained, not just degree-specified.

In paper [1], we shall develop an augmentation and a matroidal generalization. In the first one, a given initial bigraph is to be augmented to get a simple degree-specified bigraph with matching number at least ℓ. In matrix terms, this means that some of the entries of the $(0,1)$-matrix are specified to be 1. This is in sharp contrast with the **NP**-completeness of that version when some entries of the matrix are specified to be 0. In the matroidal extension of Ryser's theorem, there are matroids on S and on T and we want to find a degree-specified simple bigraph including a matching that covers bases in both matroids.

7.1.2 Proof of Ryser's theorem

Proof. Necessity. Let G be a bipartite graph with the requested properties. Since G is simple, it has at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y|$ edges having at least one end-node in $X \cup Y$. Moreover, since G has a matching of ℓ edges, there are at least $(|X \cup Y| - \ell)$ edges connecting $S - X$ and $S - Y$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - |X \cup Y|$, that is, (56) is indeed necessary.

Sufficiency. We need the following deficiency form of Hall's theorem.

Lemma 7.2 (Hall and Ore)**.** Let $G = (S, T; E)$ be a bipartite graph and $\ell \leq |T|$ an integer. The matching number $\nu(G)$ is at least ℓ *(that is, there is a matching of ℓ edges)* if and only if

$$|\Gamma(Y)| \geq \ell - |T - Y| \text{ holds for every } \ Y \subseteq T. \tag{58}$$

Define a set-function p_T on T by

$$p_T(Y) := \begin{cases} \ell - (|T-Y|) & \text{if } \emptyset \subset Y \subseteq T \\ 0 & \text{if } Y = \emptyset. \end{cases} \tag{59}$$

Then p_T is fully supermodular and monotone non-decreasing. If there is a simple bipartite graph $G = (S, T; E)$ covering p_T and fitting (m_S, m_T), then G has a matching of size ℓ by Lemma 7.2, and we are done. Suppose now that no such a G exists. Since $\mathcal{G}(m_S, m_T)$ assumed to be non-empty, (27) holds. By Corollary 3.11, there are subsets $X \subseteq S$ and $Y \subset T$ violating (27), that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{p}_T(T-Y) - |X| > \gamma.$$

By $\widetilde{p}_T(T-Y) = \ell - |Y|$, we have $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - |Y| - |X| > \gamma$, contradicting (56). •

7.2 Degree and edge-number constrained max term rank

Our goal is to extend Ryser's theorem for the case when upper or lower bounds are given for the degrees rather than exact prescriptions. Bounds for the total number of edges can also be incorporated. Let $f_V = (f_S, f_T)$ and $g_V = (g_S, g_T)$ be lower and upper bound functions with $0 \leq f_V \leq g_V$. As we are interested in simple bigraphs, we may suppose that $g_S(s) \leq |T|$ for every $s \in S$ and $g_T(t) \leq |S|$ for every $t \in T$.

Ryser's theorem was derived above by applying Corollary 3.11 to the set-function p_T defined in (59). By applying Corollary 5.11 to the same p_T, we obtain the following extension.

THEOREM 7.3. Let $\ell \leq |T|$ be an integer, $f_V = (f_S, f_T)$ and $g_V = (g_S, g_T)$ bounds with $f_V \leq g_V$.

(A) There is a simple bigraph $G' = (S, T; E')$ with matching number $\nu(G) \geq \ell$ and degree-constraints (f_T, g_S) if and only if

$$\widetilde{f}_T(Y) - |X||Y| + (\ell - |X| - |Y|)^+ \leq \widetilde{g}_S(S-X) \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{60}$$

Moreover, ***(60)*** holds if the inequality in it is required only when X consists of the i largest values of g_S and Y consists of the j largest values of f_T $(i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|)$.

(B) There is a simple bigraph $G'' = (S, T; E'')$ with matching number $\nu(G) \geq \ell$ and degree-constraints (f_S, g_T) if and only if

$$\widetilde{f}_S(X) - |X||Y| + (\ell - |X| - |Y|)^+ \leq \widetilde{g}_T(T-Y) \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{61}$$

Moreover, ***(61)*** holds if the inequality in it is required only when X consists of the i largest values of f_S and Y consists of the j largest values of g_T $(i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|)$.

(AB) There is a simple bigraph $G = (S, T; E)$ with matching number $\nu(G) \geq \ell$ and degree-constraints (f_V, g_V) if and only if both G' and G'' exist *(that is, both (60) and (61) hold)*. •

By applying Theorem 5.15 to the same p_T defined in (59), we obtain the following extension.

THEOREM 7.4. Suppose that there is a simple bigraph with matching number at least ℓ which is degree-constrained by (f_V, g_V) *(that is, conditions (60) and (61) hold)*. There is simple bigraph $G = (S, T; E)$ with matching number at least ℓ which is degree-constrained by (f_V, g_V):

(A) for which $\alpha \leq |E|$ if and only if

$$\widetilde{g}_S(S - X) + \widetilde{g}_T(T - Y) + |X||Y| - (\ell - |X| - |Y|)^+ \geq \alpha \text{ for } X \subseteq S,\ Y \subseteq T, \quad (62)$$

Moreover, ***(62)*** holds if the inequality in it is required only when X consists of the i largest values of g_S and Y consists of the j largest values of g_T $(i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|)$.

(B) $|E| \leq \beta$ if and only if

$$\widetilde{f}_S(X) + \widetilde{f}_T(Y) - |X||Y| + (\ell - |X| - |Y|)^+ \leq \beta \text{ for } X \subseteq S,\ Y \subseteq T, \quad (63)$$

Moreover, ***(63)*** holds if the inequality in it is required only when X consists of the i largest values of f_S and Y consists of the j largest values of f_T $(i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|)$.

(AB) $\alpha \leq |E| \leq \beta$ if and only if both ***(62)*** and ***(63)*** hold. •

7.2.1 Algorithmic aspects

As already indicated above, the original proof of Ryser is algorithmic. Using this as a subroutine, we describe an algorithm to find a degree-constrained bigraph with matching number at least ℓ. A specific feature of the algorithm is that it makes use of Theorem 7.3 (and does not re-prove it). Another basic constituent is the observation that conditions (60) and (61) can easily be checked in polynomial time, as stated in the theorem, since it suffices to check the inequalities in question only for $(|S|+1)(|T|+1)$ cases. The algorithm starts by checking (60) and (61), and terminates if anyone of them fails to hold. Suppose now that both conditions do hold.

Assume that there is a **loose** node v meaning that $f_V(v) < g_V(v)$. We can check in polynomial time whether $f_V(v)$ can be increased by 1 without destroying (60) and (61), and if it can, increase $f_V(v)$ by 1. By repeating this operation as long as possible, we arrive at a situation where $f_V(v)$ cannot be increased any more at any loose node.

By Theorem 7.3, there is a simple bigraph G with $\nu(G) \geq \ell$ and degree-constrained by (f_V, g_V). Then $d_G(v) = f_V(v)$ clearly holds for a node with $f_V(v) = g_V(v)$, but $d_G(v) = f_V(v)$ holds for a loose node v, as well, since if we had $f_V(v) < d_G(v)$, then $f_V(v)$ could be increased without destroying the conditions. We can conclude that $m_V := f_V$ and $\gamma := f_S(S)$ satisfy (56) and therefore Ryser's algorithm (or the simpler algorithm by Brualdi and Ross) can be applied to construct the requested G.

The same approach works in the case when, in addition to the degree-constraints (f_V, g_V), there is a lower bound α and an upper bound β for the number of edges.

First, we can check in polynomial time if each of conditions (60), (61), (62), and (63) holds. If any of them is violated, the algorithm terminates. Suppose that these conditions hold. We can also check in polynomial time if there is a loose node v for which $f_V(v)$ can be increased by 1 without violating any of these conditions, and we make these liftings of f_T as long as possible. Therefore the final f_V and g_V continue to meet the four conditions. By Theorem 7.3, there is a bigraph G satisfying the requirements.

By Theorem 7.4, there is a simple bigraph $G = (S, T; E)$ with $\nu(G) \geq \ell$ and $\alpha \leq |E| \leq \beta$ which is degree-constrained by (f_V, g_V). Then $d_G(v) = f_V(v)$ clearly holds for a node with $f_V(v) = g_V(v)$, but $d_G(v) = f_V(v)$ holds for a loose node v, as well, since if we had $f_V(v) < d_G(v)$, then $f_V(v)$ could be increased without destroying the conditions. We can conclude that $m_V := f_V$ and $\gamma := f_S(S)$ satisfy (56).

With a little care, it can be shown that the complexity of the algorithm above to construct the degree-specification m_V satisfying (56) for which $f_V \leq m_V \leq g_V$ and $\alpha \leq \widetilde{m}_S(S) \leq \beta$ is $O(n^2 \log n)$.

7.3 Further matching-type requirements

A special case of the max term rank problem characterizes degree-specifications which can be realized by a perfectly matchable bipartite graph. Brualdi [4] characterized degree-specifications which can be realized by elementary bipartite graphs. (A perfectly matchable bigraph is **elementary** if it has a perfect matching after removing any one of its edges.) His result is extended in the [2] to so-called k-elementary bigraphs.

In this section, we describe yet another model for degree-specified bigraphs. By a T_2**-forest** we mean a bigraph $(S, T; F)$ which is a forest with $d_F(t) = 2$ for every $t \in T$. Lovász originally developed Theorem 2.9 to characterize bigraphs $G_0 = (S, T; E_0)$ including a T_2-forest.

Theorem 7.5. In a bigraph $G = (S, T; E)$, there exists a T_2-forest if and only if

$$|\Gamma_G(Y)| \geq |Y| + 1 \text{ for } \emptyset \neq Y \subseteq T. \tag{64}$$

Lovász used this result to prove a conjecture of Erdős on 2-colourability of hypergraphs with the strong Hall inequality. Here we show another application.

THEOREM 7.6. Let S and T be disjoint sets with $|S| \geq |T| + 1$ and let $V = S \cup T$. Let $m_V = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$ and $m_T(t) \geq 2$ for every $t \in T$. There exists a simple bigraph $G = (S, T; E)$ fitting m_V and including a T_2-forest if and only if ***(9)*** holds and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| - |X| - |Y| + |T| + 1 \leq \gamma \text{ for } \emptyset \neq X \subseteq S,\ Y \subseteq T. \tag{65}$$

Proof. Necessity. Theorem 2.5 stated that (9) was the necessary and sufficient condition for the realizability of (m_S, m_T).

Suppose that there is a simple bigraph $G = (S, T; E)$ realizing m_V and including a T_2-forest F. The graph has at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y|$ edges with at least one end in $X \cup Y$. Forest F has exactly $2|T - Y|$ edges ending in $T - Y$. Among these edges, at most $|X| + |T - Y| - 1$ are induced by $X \cup (T - Y)$ since F is a forest (and X is non-empty). Therefore F has at least

$$2|T - Y| - (|X| + |T - Y| - 1) = |T| - |X| - |Y| + 1$$

edges connecting $T - Y$ and $T - X$. By combining these observations, we conclude that the left hand side of the inequality in (65) is indeed a lower bound for the number γ of edges of G.

Sufficiency. Define a set-function p_T on T by

$$p_T(Y) := \begin{cases} |Y| + 1 & \text{if } \emptyset \subset Y \subseteq T \\ 0 & \text{if } Y = \emptyset. \end{cases} \tag{66}$$

Then p_T is intersecting supermodular and monotone non-decreasing. If there is a simple bigraph G covering p_T and fitting m_V, then Theorem 7.5 implies that G has a T_2-forest and we are done. If no such G exists, then Theorem 3.9 implies that there are subsets $X \subseteq S$, $Y \subseteq T$ and a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$ of $T - Y$ for which

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}||X| > \gamma,$$

that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \sum_{i=1}^{q}(|T_i| + 1) - q|X| > \gamma \tag{67}$$

We cannot have $q = 0$, that is, $\mathcal{T}$ cannot be empty because of (9).

We cannot have $X = \emptyset$, for otherwise

$$\widetilde{m}_T(Y) + |T - Y| + |T - Y| \geq \widetilde{m}_T(Y) + \sum_{i=1}^{q}(|T_i| + 1) =$$

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \sum_{i=1}^{q}(|T_i| + 1) - q|X| > \gamma,$$

from which $2|T - Y| > \gamma - \widetilde{m}(T) = \widetilde{m}_T(T - Y)$, contradicting the hypothesis that $m_T(t) \geq 2$ for each $t \in T$. Therefore $X \neq \emptyset$. Since $q \leq |T - Y|$ and $|X| \geq 1$, we have $q(|X| - 1) \leq |T - Y|(|X| - 1)$ from which

$$-|X| - |Y| + |T| + 1 = |T - Y| - (|X| - 1) \geq |T - Y| - q(|X| - 1) =$$

$$|T - Y| + q - q|X| \geq \sum_{i=1}^{q}(|T_i| + 1) - q|X|.$$

This and (67) imply

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| - |X| - |Y| + |T| + 1 \geq$$

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \sum_{i=1}^{q}(|T_i| + 1) - q|X| > \gamma,$$

contradicting (65), •

Actually, Theorem 2.9 implies the following more general form of Theorem 7.5 in which the forest has a specified degree (not necessarily identically 2) at each node in T.

Theorem 7.7. Let $m_F : T \to \mathbf{Z}_+$ be a degree specification. In a bigraph $G = (S, T; E)$, there exists a forest F with $d_F(t) = m_F(t) \quad (t \in T)$ if and only if

$$|\Gamma_G(Y)| \geq \widetilde{m}_F(Y) - |Y| + 1 \text{ for } \emptyset \neq Y \subseteq S. \tag{68}$$

Consequently, Theorem 7.6 can also be generalized in such a way that the simple bigraph should fit a degree specification m_V and should include a forest with specified degrees in the nodes in T.

THEOREM 7.8. Let S and T be disjoint sets and $V := S \cup T$. Let $m_V = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$, and let $m_F : T \to \mathbf{Z}_+$ be a degree specification on T for which $m_F \leq m_T$. There exists a simple bigraph $G = (S, T; E)$ fitting m_V and including a forest F with $d_F(t) = m_F(t) \quad (t \in T)$ if and only if ***(9)*** holds and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \widetilde{m}_F(T-Y) - |T-Y| - |X| + 1 \leq \gamma \text{ for } \emptyset \neq X \subseteq S,\ Y \subseteq T. \bullet \tag{69}$$

We also remark that the results of Section 4 can be used in a similar way to generalize Theorem 7.6 so as to have upper and lower bounds for the degrees of the nodes.

7.4 Wooded hypergraphs

A hypergraph is called **wooded** if it can be trimmed to a graph which is a forest, that is, if it is possible to select two distinct elements from each hyperedge in such a way that the selected pairs, as graph edges, form a forest. Suppose we have a hypergraph $H = (S, \mathcal{T})$ on node-set S. It is well known that H can be represented with a simple bipartite graph $G_H = (S, T; E)$ where the elements of T corresponds to the hyperedges and the set of neighbours of $t \in T$ in G_H is just the hyperedge corresponding to t. Obviously, H is wooded precisely if the associate bipartite graph G_H has a T_2-tree. In this terminology, Theorem 7.5 assets that a hypergraph is wooded if and only if the union of any $j > 0$ hyperedges has at least $j + 1$ elements.

Theorem 7.6 can also be reformulated in terms of wooded hypergraphs but here we do this only for the special case when the hypergraph is ℓ-uniform where $\ell \geq 2$.

Corollary 7.9. Let m_S be a degree-specification on S with $\widetilde{m}_S(S) = \gamma$ and let $\ell \geq 2$ be an integer. There is an ℓ-uniform wooded hypergraph fitting m_S if and only if $\tau := \gamma/\ell$ is an integer and

$$m_S(s) \leq \tau \leq |S^+| - 1 \text{ for } s \in S^+ \tag{70}$$

where $S^+ = \{s \in S : m_S(s) > 0\}$.

Proof. As the necessity of the conditions is straightforward, we consider only sufficiency. Since nodes $s \in S$ with $m_S(s) = 0$ will not belong to any hyperedge, we can delete them, and thus assume that $S^+ = S$. Note that (70) implies that $\widetilde{m}_S(X) \leq \tau|X|$ for every $X \subseteq S$.

Let T be a set of τ elements. Define $m_T(t) := \ell$ for each $t \in T$ and let p_T be a set-function on T defined in (66). If there is a simple bigraph $G = (S, T; E)$ covering p_T and complied with (m_S, m_T), then G is wooded and the hypergraph on S associated with G is an ℓ-uniform wooded hypergraph, in which case we are done.

Suppose that the requested bigraph does not exist. Then one of the conditions in Theorem 7.6 fails to hold. Suppose first that there are sets $X \subseteq S, Y \subseteq T$ violating (9), that is, $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| > \gamma$, implying

$$\widetilde{m}_S(X) + |Y|(\ell - |X|) = \widetilde{m}_S(X) + \ell|X| - |X||Y| > \gamma = \tau\ell.$$

If $\ell \geq |X|$, then

$$\widetilde{m}_S(X) + \tau(\ell - |X|) = \widetilde{m}_S(X) + |T|(\ell - |X|) \geq \widetilde{m}_S(X) + |Y|(\ell - |X|) > \tau\ell,$$

from which $\widetilde{m}_S(X) > \tau|X|$, a contradiction.

If $\ell < |X|$, then

$$\gamma = \widetilde{m}_S(S) \geq \widetilde{m}_S(X) \geq \widetilde{m}_S(X) + |Y|(\ell - |X|) > \gamma,$$

a contradiction again, showing that (9) holds.

Consider now the case when there are sets $\emptyset \neq X \subseteq S,\ Y \subseteq T$ violating (65), that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| - |X| - |Y| + \tau + 1 > \gamma$$

from which

$$\widetilde{m}_S(X) + |Y|(\ell - |X| - 1) - |X| + \tau + 1 = \widetilde{m}_S(X) + \ell|Y| - |X||Y| - |X| - |Y| + \tau + 1 > \gamma = \tau\ell.$$

If $\ell > |X| + 1$, then

$$\widetilde{m}_S(X) + \tau(\ell - |X| - 1) - |X| + \tau + 1 \geq \widetilde{m}_S(X) + |Y|(\ell - |X| - 1) - |X| + \tau + 1 > \tau\ell$$

from which $\widetilde{m}_S(X) - \tau|X| - |X| + 1 > 0$, and hence $\tau|X| \geq \widetilde{m}_S(X) > \tau|X| + |X| - 1$, implying that $X = \emptyset$, a contradiction.

Suppose now that $\ell \leq |X| + 1$. Since $m_S(s)$ is positive for every $s \in S$, we have $\widetilde{m}_S(S) - |S| \geq \widetilde{m}_S(X) - |X|$. Hence

$$\widetilde{m}_S(S) - |S| + \tau + 1 \geq \widetilde{m}_S(X) - |X| + \tau + 1 \geq \widetilde{m}_S(X) + |Y|(\ell - |X| - 1) - |X| + \tau + 1 > \tau\ell$$

from which

$$\tau\ell - |S| + 1 = \widetilde{m}_S(S) - |S| + 1 > \tau\ell - \tau,$$

that is, $\tau > |S| - 1$, contradicting (70). •

References

[1] K. Bérczi and A. Frank, *Supermodularity in unweighted graph optimization II: Matroidal term rank augmentation*, EGRES Technical Report TR-2016-10 (July 2016) (`http://www.cs.elte.hu/egres`. ISSN 1587-4451)

[2] K. Bérczi and A. Frank, *Supermodularity in unweighted graph optimization III: Highly connected digraphs*, EGRES Technical Report TR-2016-11 (July 2016) (`http://www.cs.elte.hu/egres`. ISSN 1587-4451)

[3] R.A. Brualdi, *Admissible mappings between dependence spaces*, Proceedings of the London Mathematical Society, Vol 3, No 2 (1970) 296–312.

[4] R.A. Brualdi, *Matrices of* 0*'s and* 1*'s with total support*, J. Combinatorial Theory, Ser. A, Vol. 28, 3 (1980) 249–256.

[5] R.A. Brualdi and J.A. Ross, *On Ryser's Maximum Term Rank Formula*, Linear algebra and its applications, 29 (1980) pp. 33–38.

[6] M.-C. Cai, *Arc-disjoint arborescences of digraphs*, Journal of Graph Theory, 7 (1983) 235–240.

[7] C. Dürr, F. Guinez, and M. Matamala, *Reconstructing 3-colored grids from horizontal and vertical projections is NP-hard*, SIAM J. on Discrete Mathematics, Vol. 26, No. 1 (2012) 330–352.

[8] J. Edmonds, *Edge-disjoint branchings,* in: Combinatorial Algorithms (B. Rustin, ed.), Acad. Press, New York, (1973), 91–96.

[9] J. Edmonds and R. Giles, *A min-max relation for submodular functions on graphs*, Annals of Discrete Mathematics 1, (1977), 185-204.

[10] K.P. Eswaran and R.E. Tarjan, *Augmentation problems,* SIAM J. Computing **5** No. 4 (1976) 653–665.

[11] J. Folkman and D.R. Fulkerson, *Edge-colorings in bipartite graphs*, in: Combinatorial Mathematics and its Applications (Proc. Conference Chapel Hill, North Carolina, 1967). The University of North Carolina Press, 1969, (R.C. Bose, T.A. Dowling, eds.) 561-577.

[12] L.R. Ford and D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton NJ., 1962.

[13] A. Frank, *On disjoint trees and arborescences,* in: Algebraic Methods in Graph Theory, Colloquia Mathematica Soc. J. Bolyai, 25 (1978) 159-169. North-Holland.

[14] A. Frank, *Restricted t-matchings in bipartite graphs,* in: Submodularity, (guest editor S. Fujishige) Discrete Applied Mathematics, Vol. 131, Issue 2. (September 2003). pp. 337–346.

[15] A. Frank, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. on Discrete Mathematics, (1992 February), Vol. 5, No 1, pp. 22-53. A preliminary version appeared in the proceedings of the FOCS meeting held in October, 1990.

[16] A. Frank, *Connectivity augmentation problems in network design,* in: Mathematical Programming: State of the Art 1994, (J.R. Birge and K.G. Murty, eds.), The University of Michigan, pp. 34–63.

[17] A. Frank, Connections in Combinatorial Optimization, Oxford University Press, 2011 (ISBN 978-0-19-920527-1). Oxford Lecture Series in Mathematics and its Applications, 38.

[18] A. Frank and T. Jordán, *Minimal edge-coverings of pairs of sets*, J. Combinatorial Theory, Ser. B. Vol. 65, No. 1 (1995, September) pp. 73-110.

[19] A. Frank and É. Tardos, *Generalized polymatroids and submodular flows*, Mathematical Programming, Ser. B. 42 (1988) 489-563.

[20] A. Frank and É. Tardos, *An application of submodular flows*, Linear Algebra and its Applications, 114/115, (1989) 329-348.

[21] D. Gale, *A theorem on flows in networks*, Pacific J. Mathematics, 10 (1957), 1073–1082.

[22] E. Győri, *Covering simply connected regions by rectangles,* Combinatorica, 5 (1), (1985) 53–55.

[23] R.M. Haber, *Term rank of* $0,1$ *matrices*, Rendiconti del Seminario Matematico della Universita di Padova, tome 30 (1960) pp. 24–51.

[24] Yanmei Hong, Qighai Liu, and Hon-Jian Lai, *Degree sequence of strongly connected digraphs*, Journal of Graph Theory, submitted.

[25] R.R. Kamalian and V.V Mkrtchyan, *On complexity of special maximum matchings constructing*, Discrete Mathematics, Vol. 308, No. 10, (2008) 1792–1800.

[26] N. Kamiyama, *Arborescence Problems: Theorems and Algorithms*, Interdisciplinary Information Sciences, Vol. 20 No. 1 (2014) 51–70.

[27] L. Lovász, *A generalization of Kőnig's theorem,* Acta. Math. Acad. Sci. Hungar. 21 (1970), 443–446.

[28] L. Lovász, *On two minimax theorems in graph theory,* J. Combinatorial Theory (B) 21 (1976) 96-103.

[29] O. Ore, *Studies on directed graphs, I*, Annals of Mathematics, 63 (1956) 383–406.

[30] D. Pálvölgyi, *Partitioning to three matchings of given size is NP-complete for bipartite graphs*, Acta Universitatis Sapientiae, Informatica, Vol. 6, No. 2 (2014) 206–209.

[31] G.J. Puleo, *Complexity of a disjoint matching problem on bipartite graphs* Information Processing Letters, Vol. 116, Iss. 10, (October 2016) 649–652.

[32] H.J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canadian. J. Math., 9. (1957) 371–377.

[33] H.J. Ryser, *The term rank of a matrix*, Canadian J. of Mathematics, 10 (1958) 57–65.

[34] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer, 2003. Vol 24. of the series Algorithms and Combinatorics.

[35] J.A. Soto and C. Telha, *Independent sets and bicolored rectangular families*, preprint, http://arxiv.org/abs/1411.2311