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Supermodularity in Unweighted Graph Optimization III: Highly Connected Digraphs

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Abstract

By generalizing a recent result of Hong, Liu, and Lai [13] on characterizing the degree-sequences of simple strongly connected directed graphs, a characterization is provided for degree-sequences of simple k-node-connected digraphs. More generally, we solve the directed node-connectivity augmentation problem when the augmented digraph is degree-specified and simple. As for edge-connectivity augmentation, we solve the special case when the edge-connectivity is to be increased by one and the augmenting digraph must be simple.

1 Introduction

There is an extensive literature of problems concerning degree sequences of graphs or digraphs with some prescribed properties such as simplicity or k-connectivity. For example, Edmonds [6] characterized the degree-sequences of simple k-edge-connected undirected graphs, while Wang and Kleitman [19] solved the corresponding problem for simple k-node-connected graphs.

In what follows, we consider only directed graphs for which the default understanding will be throughout the paper that loops and parallel arcs are allowed. A typical problem is as follows. Given an n-element ground-set V, decide for a specified integer-valued function $m_i : V \rightarrow \mathbf{Z}_+$ if there is a digraph $D = (V, A)$ with some prescribed properties realizing (or fitting) m_i, which means that $\varrho_D(v) = m_i(v)$ for every node $v \in V$ where $\varrho_D(v)$ denotes the number of arcs of D with head v. Often we call a function m_i an **in-degree specification** or **sequence** or **prescription**. An out-degree specification m_o is defined analogously, and a pair (m_o, m_i) of functions is simply called a **degree-specification**.

For any function $m : S \rightarrow \mathbf{R}$, the set-function $\widetilde{m}$ is defined by $\widetilde{m}(X) := \sum[m(v) : v \in X]$ for $X \subseteq S$, and we shall use this tilde-notation $\widetilde{m}$ throughout the paper. In

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order to realize (m_o, m_i) with a digraph, it is necessary to require that $\widetilde{m}_o(V) = \widetilde{m}_i(V)$ since both $\widetilde{m}_o(V)$ and $\widetilde{m}_i(V)$ enumerate the total number of arcs of a realizing digraph D. This common value will be denoted by γ, that is, our assumption throughout is that

$$\widetilde{m}_o(V) = \widetilde{m}_i(V) = \gamma. \tag{1}$$

The following result was proved by Ore [17] in a slightly different but equivalent form.

Theorem 1.1 (Ore)**.** A digraph $H = (V, F)$ has a subgraph fitting (m_o, m_i) if and only if

$$\widetilde{m}_i(X) + \widetilde{m}_o(Z) - d_H(Z, X) \leq \gamma \text{ for every } X, Z \subseteq V, \tag{2}$$

where $d_H(Z, X)$ denotes the number of arcs $uv \in F$ with $u \in Z, v \in X$.

This immediately implies the following characterization ([17], see also [8]).

Theorem 1.2 (Ore)**.** Let (m_o, m_i) be a degree-specification meeting ***(1)***.

(A) There always exists a digraph realizing (m_o, m_i).

(B) There is a loopless digraph realizing (m_o, m_i) if and only if

$$m_i(v) + m_o(v) \leq \gamma \text{ for every } v \in V. \tag{3}$$

(C) There is a simple digraph realizing (m_o, m_i) if and only if

$$\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + |X \cap Z| \leq \gamma \text{ for every } X, Z \subseteq V. \tag{4}$$

Moreover, it suffices to require the inequality in ***(4)*** only for its special case when X consists of the h largest values of m_i and Z consists of the j largest values of m_o $(h = 0, 1, \ldots, n,\ j = 0, 1, \ldots, n)$.

Note that (3) follows from (4) by taking $X = \{v\}$ and $Z = \{v\}$. We also remark that Part (A) can be proved directly by a simply greedy algorithm: build up a digraph by adding arcs uv one by one as long as there are (possibly not distinct) nodes u and v with $\varrho(v) < m_i(v)$ and $\delta(v) < m_o(v)$. Also, Part (B) immediately follows by the following **loop-reducing technique**. Let D be a digraph fitting (m_o, m_i) and suppose that there is a loop $e = vv$ sitting at v. Condition (3) implies that there must be an arc $f = xy$ with $x \neq v \neq y$ (but $x = y$ allowed). By replacing e and f with arcs xv and vy, we obtain a digraph fitting (m_o, m_i) that has fewer loops than D has. For later purposes, we remark that the loop-reducing procedure does not decrease the in-degree of any subset of nodes.

We call a digraph $D = (V, A)$ **strongly connected** or just **strong** if $\varrho_D(X) \geq 1$ whenever $\emptyset \subset X \subset V$. More generally, D is k**-edge-connected** if $\varrho_D(X) \geq k$ whenever $\emptyset \subset X \subset V$. D is k**-node-connected** or just k**-connected** if $k \leq |V| - 1$ and the removal of any set of less than k nodes leaves a strong digraph.

One may be interested in characterizing degree-sequences of k-edge-connected and k-node-connected digraphs. We will refer to this kind of problems as synthesis problems. The more general augmentation problem consists of making an initial digraph

$D_0 = (V, A_0)$ k-edge- or k-node-connected by adding a degree-specified digraph. Clearly, when A_0 is empty we are back at the synthesis problem. The augmentation problem was solved for k-edge-connectivity in [10] and for k-node-connectivity in [11], but in both cases the augmenting digraphs D were allowed to have loops or parallel arcs. The same approach rather easily extends to the case when D is requested to be loopless but treating simplicity is significantly more difficult.

The goal of the present paper is to investigate these degree-specified augmentation and synthesis problems when simplicity is expected. In the augmentation problem this means actually two possible versions depending on whether the augment**ing** digraph D or else the augment**ed** digraph $D_0 + D$ is requested to be simple. Clearly, when D_0 has no arcs (the synthesis problem) the two versions coincide.

An early result of this type is due to Beineke and Harary [1] who characterized degree-sequences of loopless strongly connected digraphs. In a recent work Hong, Liu, and Lai [13] characterized degree-sequences of simple strongly connected digraphs. In order to generalize conveniently their result, we formulate it in a slightly different but equivalent form.

Theorem 1.3 (Hong, Liu, and Lai)**.** Suppose that there is a simple digraph fitting the degree-specification (m_o, m_i). There is a strongly connected simple digraph fitting (m_o, m_i) if and only if

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{5}$$

holds for every pair of disjoint subsets $X, Z \subset V$ with $X \cup Z \neq \emptyset$. Moreover, it suffices to require ***(5)*** only in the special case when X consists of the h largest values of m_i and Z consists of the j largest values of m_o ($j = 0, 1, \ldots, n,\ h = 0, 1, \ldots, n,\ 1 \leq j + h \leq n$).

We are going to extend this result in two directions. In the first one, degree-specifications are characterized for which there is a simple realizing digraph D whose addition to an initial $(k-1)$-edge-connected digraph D_0 results in a k-edge-connected digraph $D_0 + D$. The general problem of augmenting an arbitrary initial digraph D_0 with a degree-specified simple digraph to obtain a k-edge-connected digraph remains open even in the special case when D_0 has no arcs at all. That is, the synthesis problem of characterizing degree-sequences of simple k-edge-connected digraphs remains open for $k \geq 2$.

Our second generalization of Theorem 1.3 provides a characterization of degree-sequences of simple k-node-connected digraphs. We also solve the more general degree-specified node-connectivity augmentation problem when the augmented digraph is requested to be simple. It is a bit surprising that node-connectivity augmentation problems are typically more complex than their edge-connectivity counterparts and yet an analogous characterization for the general k-edge-connected case, as indicated above, remains open.

In the proof of both extensions, we rely on the following general result of Frank and Jordán.

Theorem 1.4 (Supermodular arc-covering, [11])**.** Let p be a positively crossing supermodular bi-set function which is zero on trivial bi-sets. The minimum number of arcs of a loopless digraph covering p is equal to $\max\{\sum[p(B) : B \in \mathcal{I}]\}$ where the maximum is taken over all independent families $\mathcal{I}$ of bi-sets.

This theorem was earlier used to solve several connectivity augmentation problems. It should be emphasized, however, that even this general framework did not allow to handle simplicity. Even worse, there is no hope to extend Theorem 1.4 so as to characterize minimal *simple* digraphs covering p since this problem can be shown to include NP-complete special cases. In [2] and [3], we developed other applications of the supermodular arc-covering theorem when simplicity could be guaranteed.

The main novelty of the approach of the present paper is that, in the above-mentioned special cases when simplicity is an expectation, though Theorem 1.4 remains to form a fundamental starting point, relatively tedious additional work is needed. (The complications may be explained by the fact that some special cases are NP-complete while others are in NP∩co-NP.)

There are actually two issues here to be considered. The first one is to develop techniques for embedding special simplicity-requesting connectivity augmentation problems into the framework of Theorem 1.4. When this is successful, one has to resolve a second difficulty stemming from the somewhat complicated nature of an independent family of bi-sets in Theorem 1.4.

To demonstrate this second obstacle, consider the following digraph $D_0 = (V, A_0)$ with a particularly simple structure. Let $e = uv$ be an arc of D_0 if $u \in Z$ or $v \in X$, where Z and X are two specified disjoint subsets of V with $|X| = |Z| < k$. An earlier direct consequence of Theorem 1.4 (formulated in Section 3 as Theorem 3.2) does provide a formula for the minimum number of new arcs whose addition to any initial digraph results in a k-connected digraph. But to prove that this minimum for our special digraph D_0 is actually equal to the total out-deficiency $\sum[(k-\delta_0(v))^+ : v \in V]$ of the nodes of D_0 is rather tricky or tedious.

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1.1 Notions and notation

For a number x, let $x^+ = \max\{0, x\}$. For a function $m : V \to \mathbf{R}$ and for $X \subseteq V$, let $\widetilde{m}(X) := \sum[m(v) : v \in X]$. For a set-function p and a family $\mathcal{F}$ of sets, let $\widetilde{p}(\mathcal{F}) := \sum[p(Z) : Z \in \mathcal{F}]$.

Two subsets of a ground-set V are said to be **co-disjoint** if their complements are disjoint. By a **partition** of a ground-set V, we mean a family of disjoint subsets of

V whose union is V. A **subpartition** of V is a partition of a subset of V. A **co-partition** (resp., **co-subpartition**) of V is a family of subsets arising from a partition (subpartition) of V by complementing each of its member. Two subsets $X, Y \subseteq V$ are **crossing** if none of $X - Y, Y - X, X \cap Y, V - (X \cup Y)$ is empty. A family of subsets is **cross-free** if it contains no two crossing members. A family $\mathcal{F}$ of subsets is **crossing** if both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}$ whenever X and Y are crossing members of $\mathcal{F}$.

When X is a subset of V, we write $X \subseteq V$, while $X \subset V$ means that X is a proper subset. The standard notation $A \setminus B$ for set difference will be replaced by $A - B$. When it does not cause any confusion, we do not distinguish a one-element set $\{v\}$ (often called a **singleton**) from its only element v, and we use the notation v for the singleton as well. For example, we write $V - v$ rather than $V - \{v\}$, and $V + v$ stands for $V \cup \{v\}$. In some situations, however, the formally precise $\{v\}$ notation must be used. For example, an arc $e = uv$ in a digraph is said to enter a node v even if e is a loop (that is, $u = v$) while e enters the singleton $\{v\}$ only if $u \in V - v$. That is a loop sitting at v enters v but does not enter $\{v\}$. Therefore the in-degree $\varrho(v)$ (the number of arcs entering v) is equal to the in-degree $\varrho(\{v\})$ plus the number of loops sitting at v.

In a digraph $D = (V, A)$, an arc uv **enters** a subset $X \subseteq V$ or **leaves** $V - X$ if $u \in X,\ v \in V - X$. The in-degree $\varrho_D(X) = \varrho_A(X)$ of a subset $X \subseteq V$ is the number of arcs entering X while the out-degree $\delta_D(X) = \delta_A(X)$ is the number of arcs leaving X. Two arcs of a digraph are **parallel** if their heads coincide and their tails coincide. The oppositely directed arcs uv and vu are not parallel. We call a digraph **simple** if it has neither loops nor parallel arcs.

By the **complete digraph** $D^* = (V, A^*)$, we mean the digraph on V in which there is one arc from u to v for each ordered pair $\{u, v\}$ of distinct nodes, that is, D^* has $n(n-1)$ arcs. For two subsets $Z, X \subseteq V$, let $D^*[Z, X]$ denotes the subgraph of D^* consisting of those arcs uv for which $u \in Z$ or $v \in X$. Then $D^*[Z, X]$ has $|Z|(n-1) + (n - |Z|)|X| - |X - Z|$ arcs.

For two digraphs $D_0 = (V, A_0)$ and $D = (V, A)$ on the same node-set, $D_0 + D = (V, A_0 + A)$ denotes the digraph consisting of the arcs of D_0 and D. That is, $D_0 + D$ has $|A_0| + |A|$ arcs.

A digraph D **covers** a family $\mathcal{K}$ of subsets if $\varrho_D(K) \geq 1$ for every $K \in \mathcal{K}$. A digraph D **covers** a set-function p on V if $\varrho_D(K) \geq p(K)$ for every $K \subseteq V$.

By a **bi-set** we mean a pair $B = (B_O, B_I)$ of subsets for which $B_I \subseteq B_O$. Here B_O and B_I are the outer set and the inner set of B, respectively. A bi-set is **trivial** if $B_I = \emptyset$ or $B_O = V$. The set $W(B) := B_O - B_I$ is the **wall** of B, while $w(B) = |W(B)|$ is its **wall-size**. Two bi-sets B and C are **comparable** if $B_I \subseteq C_I$ and $B_O \subseteq C_O$ or else $B_I \supseteq C_I$ and $B_O \supseteq C_O$. The **meet** of two bi-sets B and C are defined by $B \sqcap C = (B_I \cap C_I, B_O \cap C_O)$ while their **join** is $B \sqcup C = (B_I \cup C_I, B_O \cup C_O)$. Note that $w(B)$ is a modular function in the sense that $w(B) + w(C) = w(B \sqcap C) + w(B \sqcup C)$. Two bi-sets B and C are **crossing** if $B_O \cup C_O \neq V$, $B_I \cap C_I \neq \emptyset$, and they are not comparable. A bi-set function p is **positively crossing supermodular** if

$$p(B) + p(C) \leq p(B \sqcap C) + p(B \sqcup C)$$

whenever $p(B) > 0, p(C) > 0$, B and C are crossing bi-sets.

An arc e **enters** (or **covers**) a bi-set B if e enters both B_O and B_I. The **in-degree** $\varrho(B)$ of a bi-set B is the number of arcs entering B. Two bi-sets are **independent** if no arc can cover both, which is equivalent to requiring that their inner sets are disjoint or their outer sets are co-disjoint. A family of bi-sets is **independent** if their members are pairwise independent. Given a digraph $D = (V, A)$, a bi-set $B = (B_O, B_I)$ is D**-one-way** or just **one-way** if no arc of D covers B.

2 Edge-connectivity

The degree-specified augmentation problem for k-edge-connectivity was shown by the second athour [10] to be equivalent to Mader's directed splitting off theorem [16].

Theorem 2.1 ([10])**.** An initial digraph $D_0 = (V, A_0)$ can be made k-edge-connected by adding a digraph $D = (V, A)$ fitting (m_o, m_i) if and only if $\widetilde{m}_i(X) + \varrho_{D_0}(X) \geq k$ and $\widetilde{m}_i(X) + \delta_{D_0}(X) \geq k$ hold for every subset $\emptyset \subset X \subset V$. If in addition *(3)* holds, then D can be chosen loopless. •

The second part immediately follows from the first one by applying the loop-reducing technique mentioned in Section 1 since loop-reduction never decreases the in-degree of a subset. Though the problem when simplicity of D is requested remains open even in the special case when D_0 has no arcs, we are able to prove the following straight extension of Theorem 1.3.

THEOREM 2.2. A digraph D_0 can be made strongly connected by adding a simple digraph fitting a degree-specification (m_o, m_i) if and only if *(4)* holds and

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{6}$$

holds for every pair of disjoint subsets $X, Z \subset V$ for which there is a non-empty, proper subset K of V so that $\varrho_{D_0}(K) = 0$ and $Z \subseteq K \subseteq V - X$.

This theorem is just a special case of the following.

THEOREM 2.3. A $(k-1)$-edge-connected digraph D_0 can be made k-edge-connected by adding a simple digraph fitting a degree-specification (m_o, m_i) if and only if *(4)* holds and

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{7}$$

holds for every pair of disjoint subsets $X, Z \subset V$ for which there is a non-empty, proper subset K of V so that $\varrho_{D_0}(K) = k - 1$ and $Z \subseteq K \subseteq V - X$.

Since the family of subsets of in-degree $k - 1$ in a $(k - 1)$-edge-connected digraph D_0 is a crossing family, the following result immediately implies Theorem 2.3.

THEOREM 2.4. Let $\mathcal{K}$ be a crossing family of non-empty proper subsets of V. Suppose that there is a simple digraph fitting the degree specification (m_o, m_i), that is, *(4)* holds. There is a simple digraph fitting (m_o, m_i) which covers $\mathcal{K}$ if and only if

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma \tag{8}$$

holds for every pair of disjoint subsets $X, Z \subseteq V$ for which there is a member $K \in \mathcal{K}$ with $Z \subseteq K \subseteq V - X$.

Proof. Suppose that there is a requested digraph D. By the simplicity of D, there are at most $|X||Z|$ arcs from Z to X. Therefore the total number of arcs with tail in Z or with head in X is at least $\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z|$. Moreover at least one arc enters K and such an arc neither leaves an element of Z nor enters an element of X, from which we obtain that $\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + 1 \leq \gamma$, that is, (8) is necessary.

To prove sufficiency, observe first that the theorem is trivial if $n := |V| \leq 2$ so we assume that $n \geq 3$. We need some further observations.

Claim 2.5.

$$\widetilde{m}_i(K) \geq 1 \text{ and } \widetilde{m}_o(V - K) \geq 1 \text{ holds for every } K \in \mathcal{K}. \tag{9}$$

In particular, if $\{v\} \in \mathcal{K}$ for some $v \in V$, then $m_i(v) \geq 1$, and if $V - u \in \mathcal{K}$ for some $u \in U$, then $m_o(u) \geq 1$.

Proof. $\widetilde{m}_i(K) \geq 1$ follows by applying (8) to $Z = \emptyset$ and $X = V - K$, while $\widetilde{m}_o(V - K) \geq 1$ follows with the choice $X = \emptyset$ and $Z = K$. •

This claim immediately implies the following.

Claim 2.6. $\mathcal{K}$ has at most γ pairwise disjoint and at most γ pairwise co-disjoint members. •

Claim 2.7. $m_o(v) \leq n - 1$ and $m_i(v) \leq n - 1$ for every $v \in V$.

Proof. By applying (4) to $X = \{v\}$ and $Z = V$, one gets $m_i(v) + \widetilde{m}_o(V) - 1 \cdot |V| + |\{v\}| \leq \gamma$, that is, $m_i(v) \leq n - 1$, and $m_o(v) \leq n - 1$ is obtained analogously by choosing $X = V$ and $Z = \{v\}$. •

Define a bi-set function p as follows. Let $p(Y_O, Y_I)$ be zero everywhere apart from the next three types of bi-sets.

Type 1: For $K \in \mathcal{K}$ with $1 < |K| < n - 1$, let $p(K, K) = 1$.

Type 2: For $u \in V$, let $p(V - u, V - u) = m_o(u)$.

Type 3: For $v \in Y \subset V$, let $p(Y, \{v\}) = m_i(v) - (|Y| - 1)$.

Note that the role of m_o and m_i is not symmetric in the definition of p. Since $n \geq 3$, each bi-set B with positive $p(B)$ belongs to exactly one of the three types.

Claim 2.8. The bi-set function p defined above is positively crossing supermodular.

Proof. Let $B = (B_O, B_I)$ and $C = (C_O, C_I)$ be two crossing bi-sets with $p(B) > 0$ and $p(C) > 0$. Then neither of B and C is of Type 2. Suppose first that both B and C are of Type 1. Observe that if $K = \{v\} \in \mathcal{K}$ for some $v \in V$, then (K, K) is of Type 3 and hence $p(K, K) = m_i(v) \geq 1$ by Claim 2.5. Similarly, if $K = V - u \in \mathcal{K}$ for some $u \in V$, then (K, K) is of Type 2 and hence $p(K, K) = m_o(v) \geq 1$. Therefore the supermodular inequality in this case follows from the assumption that the set-system $\mathcal{K}$ is crossing.

If both B and C are of Type 3, then $B_I = \{v\} = C_I$ for some $v \in V$, and in this case the supermodular inequality holds actually with equality.

Finally, let B be of Type 1 and let C be of Type 3. Then $B_I = K = B_O$ for some $K \in \mathcal{K}$ with $1 < |K| < n-1$ and $C_I = \{v\}$ for some $v \in V$. Observe that $B \sqcup C$ does not belong to any of the three types and hence $p(B \sqcup C) = 0$. Furthermore, since $B \sqcap C$ is of Type 3, we have $p(B \sqcap C) = m_i(v) - (|B_O \cap C_O| - 1)$.

Since (K, K) and $(C_O, \{v\})$ are not comparable, $|C_O \cap K| \leq |C_O| - 1$ and therefore $p(B) + p(C) = 1 + [m_i(v) - (|C_O| - 1)] \leq [m_i(v) - (|B_O \cap C_O| - 1)] + 0 = p(B \sqcap C) + p(B \sqcup C)$. •

It follows from the definition of p that every digraph covering p must have at least γ arcs.

Case 1. There is a loopless digraph $D = (V, A)$ with γ arcs covering p.

Now $\gamma = |A| = \sum[\varrho_A(v) : v \in V] \geq \sum[m_i(v) : v \in V] = \gamma$ from which $\varrho_A(v) = m_i(v)$ follows for every $v \in V$. Analogously, we get $\delta_A(v) = m_o(v)$ for every $v \in V$. By the definition of p, it also follows that D covers $\mathcal{K}$.

Claim 2.9. D is simple.

Proof. Suppose indirectly that D has two parallel arcs e and f from u to v. Consider the bi-set $(Y, \{v\})$ for $Y = \{u, v\}$. We have $\varrho(Y, \{v\}) \leq \varrho(v) - 2 = m_i(v) - 2 \leq p(Y, \{v\}) - 1$, a contradiction. •

We can conclude that in Case 1 the digraph requested by the theorem is indeed available.

Case 2. The minimum number of arcs of a loopless digraph covering p is larger than γ.

Theorem 1.4 implies that in Case 2 there is an independent family $\mathcal{I}$ of bi-sets for which $\widetilde{p}(\mathcal{I}) > \gamma$. Then $\mathcal{I}$ partitions into three parts according to the three possibly types its members belong to. Therefore we have a subset $\mathcal{F} \subseteq \mathcal{K}$, a subset $Z \subseteq V$, and a family $\mathcal{B} = \{(Y, \{v\})\}$ of bi-sets so that

$$\mathcal{I} = \{(K, K) : K \in \mathcal{F}\} \cup \{(V - z, V - z) : z \in Z\} \cup \mathcal{B}, \text{and}$$

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \sum[m_i(v) - (|Y| - 1) : (Y, \{v\}) \in \mathcal{B}] = \widetilde{p}(\mathcal{I}) > \gamma. \tag{10}$$

Claim 2.10. There are no two members $(Y, \{v\})$ and $(Y', \{v\})$ of $\mathcal{B}$ with the same inner set $\{v\}$.

Proof. If indirectly there are two such members, then $Y \cup Y' = V$ by the independence of $\mathcal{I}$. If we replace the two members $(Y, \{v\})$ and $(Y', \{v\})$ of $\mathcal{I}$ by the single bi-set $(Y \cap Y', \{v\})$, then the resulting family $\mathcal{I}'$ is also independent since any arc covering $(Y \cap Y', \{v\})$ covers at least one of $(Y, \{v\})$ and $(Y', \{v\})$. Furthermore,

$$p(Y, \{v\}) + p(Y', \{v\}) = m_i(v) - (|Y| - 1) + m_i(v) - (|Y'| - 1) \tag{11}$$

and

$$p(Y \cap Y', \{v\}) = m_i(v) - (|Y \cap Y'| - 1). \tag{12}$$

We claim that $p(Y, \{v\}) + p(Y', \{v\}) \le p(Y \cap Y', \{v\})$. Indeed, this is equivalent to $m_i(v) \le (|Y|-1) + (|Y'|-1) - (|Y \cap Y'|-1)$, that is, $m_i(v) \le |V|-1$, but this holds by Claim 2.7. •

Let $X := \{v : \text{there is a bi-set } (K_v, \{v\}) \in \mathcal{B}\}$. Now (10) transforms to

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \sum[m_i(v) - (|K_v|-1) : v \in X] = \widetilde{p}(\mathcal{I}) > \gamma. \tag{13}$$

We may assume that $\widetilde{p}(\mathcal{I})$ is as large as possible, and modulo this, $|\mathcal{F}|$ is minimal.

Claim 2.11. For $v \in X - Z$, one has $K_v = Z + v$, and for $v \in X \cap Z$, one has $K_v = Z$.

Proof. The independence of $\mathcal{I}$ implies $Z \subseteq K_v$. Suppose, indirectly, that there is an element $u \in (K_v - v) - Z$. Replace the member $(K_v, \{v\})$ of $\mathcal{I}$ by $(K_v - u, \{v\})$. Since $p(K_v - u, \{v\}) = p(K_v) + 1$, the resulting system $\mathcal{I}'$ is not independent, therefore there exists a member of $\mathcal{I}$ that is covered by arc uv. Since $u \notin Z$, this member must be in $\mathcal{F}$, that is, this member is of form (K, K) for some $K \in \mathcal{K}$. By leaving out (K, K) from $\mathcal{I}'$, we obtain an independent $\mathcal{I}''$ for which $\widetilde{p}(\mathcal{I}'') = \widetilde{p}(\mathcal{I})$, contradicting the minimal choice of $\mathcal{F}$. •

Due to these claims, Condition (10) reduces to $|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - (|X \cap Z|(|Z|-1) + |X - Z||Z| > \gamma$ which is equivalent to

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| > \gamma. \tag{14}$$

Claim 2.12. $X \cup Z \neq \emptyset$.

Proof. If $X \cup Z = \emptyset$, then (14) reduces to $|\mathcal{F}| > \gamma$. It is an easy observation that the members of the independent $\mathcal{F}$ are either pairwise disjoint or pairwise co-disjoint, contradicting Claim 2.6.

Claim 2.13. $Z \neq \emptyset$.

Proof. If indirectly $Z = \emptyset$, then $X \neq \emptyset$ in which case (14) reduces to

$$|\mathcal{F}| + \widetilde{m}_i(X) > \gamma. \tag{15}$$

By Claim 2.11 we have $K_v = \{v\}$ for every element $v \in X$. The independence of $\mathcal{I}$ implies that $K \cap X = \emptyset$ for every $K \in \mathcal{F}$. Hence $\mathcal{F}$ is a sub-partition $\{K_1, \dots, K_q\}$ and X is disjoint from $\cup_j K_j$. Furthermore we can assume that $X = V - \cup_j K_j$. Now (15) is equivalent to $q > \widetilde{m}_i(V - X)$ and by Claim 2.5 we have

$$q > \widetilde{m}_i(V - X) = \widetilde{m}_i(\cup_j K_j) = \sum_{j=1}^{q} \widetilde{m}_i(K_j) \ge q,$$

a contradiction. •

Claim 2.14. $X \neq \emptyset$.

Proof. Suppose, indirectly, that $X = \emptyset$. Then (14) reduces to

$$|\mathcal{F}| + \widetilde{m}_o(Z) > \gamma. \tag{16}$$

By the independence of $\mathcal{I}$, we have $Z \subseteq K$ for every $K \in \mathcal{K} = \{K_1, \ldots, K_q\}$. Hence $\{\overline{K}_1, \ldots, \overline{K}_q\}$ is a subpartition and $Z \cap (\cup_j \overline{K}_j) = \emptyset$. We may assume that $Z = V - \cup_j \overline{K}_j (= \cap_j K_j)$. By (16) and by Claim 2.5 we have

$$q = |\mathcal{F}| > \gamma - \widetilde{m}_o(Z) = \widetilde{m}_o(\cup_j \overline{K}_j) \geq \sum_{j=1}^{q} \widetilde{m}_o(K_j) \geq q,$$

a contradiction. •

We have concluded that $X \neq \emptyset$ and $Z \neq \emptyset$.

Case A $X \cap Z = \emptyset$.

On one hand, (4) reduces in this case to $\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| \leq \gamma$. On the other hand, (14) reduces to

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| > \gamma. \tag{17}$$

Therefore we must have $|\mathcal{F}| \geq 1$. For $K \in \mathcal{F}$, the independence of $\mathcal{I}$ implies that $Z \subseteq K$ and $X \subseteq V - K$. The independence of $\mathcal{I}$ also implies that $\mathcal{F}$ cannot have more than one member, that is, $|\mathcal{F}| = 1$, and hence K, X, Z violate (8).

Case B $X \cap Z \neq \emptyset$.

If $K \in \mathcal{F}$, then $Z \subseteq K$. Moreover, for an element $v \in X \cap Z$, the independence of (K, K) and $(Z, \{v\})$ implies that $v \in X \cap Z$ and $Z \cup K = V$, that is $K = V$, which is not possible. Hence $\mathcal{F} = \emptyset$ and (14) reduces to $\widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| > \gamma$ contradicting (4). This contradiction shows that Case 2 cannot occur, completing the proof of the theorem. • •

3 Node-connectivity

Let $H = (V, F)$ be a simple digraph on $n \geq k - 1$ nodes. Recall that a bi-set $B = (B_O, B_I)$ was called H-**one-way** or just **one-way** if $\varrho_H(B) = 0$, that is, if no arc of H enters both B_I and B_O. Recall the notation $w(B) := |B_O - B_I|$.

Lemma 3.1. The following are equivalent.
(A1) $H = (V, F)$ is k-connected.
(A2) $\varrho_H(B) + w(B) \geq k$ for every non-trivial bi-set B.
(A3) $w(B) \geq k$ for every non-trivial one-way bi-set B.
(B) There are k openly disjoint st-paths in H for every ordered pair of nodes s, t.

Proof. (A1) implies (B) by the directed node-version of Menger's theorem.

(B)⇒(A2). Let $B = (B_O, B_I)$ be a non-trivial bi-set, let $s \in V - B_O$ and $t \in B_I$. By (B), there are k openly disjoint st-paths. Each of them uses an arc entering B

or an element of the wall $W(B)$ of B, from which $\varrho_H(B) + w(B) \geq k$, that is, (A2) holds.

(A2)⇒(A3). Indeed, (A3) is just a special case of (A2).

(A3)⇒(A1). If (A1) fails to hold, then there is a subset Z of less than k nodes so that $H - Z$ is not strongly connected. Let B_I be a non-empty proper subset of $V - Z$ with no entering arc of $H - Z$ and let $B_O := Z \cup B_I$. Then $B = (B_O, B_I)$ is a non-trivial bi-set with $W(B) = Z$ for which $\varrho_H(B) = 0$ and $w(B) = |Z| < k$, that is, (A3) also fails to hold. •

3.1 Connectivity augmentation: known results

Let $D_0 = (V, A_0)$ be a starting digraph on $n \geq k + 1$ nodes. The in-degree and out-degree functions of D_0 will be abbreviated by ϱ_0 and δ_0, respectively. In the connectivity augmentation problem we want to make D_0 k-connected by adding new arcs. Since parallel arcs and loops do not play any role in node-connectivity, we may and shall assume that D_0 is simple.

In one version of the connectivity augmentation problem, one strives for minimizing the number of arcs to be added. In this case, the optimal augmenting digraph is automatically simple. The following result is a direct consequence of Theorem 1.4.

Theorem 3.2 (Frank and Jordán, [11])**.** A digraph D_0 can be made k-connected by adding a simple digraph with at most γ arcs if and only if

$$\sum[k - w(B) : B \in \mathcal{I}] \leq \gamma \tag{18}$$

holds for every independent family $\mathcal{I}$ of non-trivial D_0-one-way bi-sets.

In what follows, $\mathcal{B}_0$ denotes the family of non-trivial D_0-one-way bi-sets. For any bi-set B, let $p_1(B) := k - w(B)$. With these terms, (18) requires that $\widetilde{p}_1(\mathcal{I}) \leq \gamma$ for every independent $\mathcal{I} \subseteq \mathcal{B}_0$.

In a related version of the connectivity augmentation problem, the goal is to find an augmenting digraph D fitting a degree-specification (m_o, m_i) (meeting (1)) so that the augmented digraph $D_0^+ := D_0 + D$ is k-connected. The paper [11] described a characterization for the existence of such a D, but in this case the augmenting digraph is not necessarily simple. This characterization can also be derived from Theorem 1.4.

Theorem 3.3. There exists a digraph D fitting (m_o, m_i) such that $D_0 + D$ is k-connected if and only if

$$\varrho_0(v) + m_i(v) \geq k \text{ and } \delta_0(v) + m_o(v) \geq k \text{ for each } v \in V, \tag{19}$$

$$\widetilde{m}_i(Z) \geq \sum[k - w(B) : B \in \mathcal{I}] \tag{20}$$

holds for every independent family $\mathcal{I}$ of non-trivial D_0-one-way bi-sets B with $B_I \subseteq Z$, and

$$\widetilde{m}_o(Z) \geq \sum[k - w(B) : B \in \mathcal{I}] \tag{21}$$

holds for every independent family $\mathcal{I}$ of non-trivial one-way bi-sets B with $B_O \cup Z = V$.

Note that in this theorem both parallel arcs and loops are allowed in the augmenting digraph D. By using Theorem 1.4 and some standard steps, one can derive the following variation when loops are excluded.

THEOREM 3.4. There exists a loopless digraph D fitting (m_o, m_i) such that D_0+D is k-connected if and only if each of ***(3)***, ***(19)***, ***(20)***, and ***(21)*** hold.

Corollary 3.5. If there is a loopless digraph fitting (m_o, m_i) and if there is a digraph D fitting (m_o, m_i) for which $D_0 + D$ is k-connected, then D can be chosen loopless.

Note that an analogous statement for k-edge-connectivity in Theorem 2.1 follows immediately by applying the loop-reducing technique, but this approach does not seem to work here since a loop-reducing step may destroy k-node-connectivity.

3.2 Degree-specified connectivity augmentation preserving simplicity

Our present goal is to solve the degree-specified node-connectivity augmentation problem when simplicity of the augmented digraph $D_0 + D$ is requested. With the help of a similar approach another natural variant, when only the simplicity of the augmenting digraph D is requested, can also be managed. Corollary 2.3 provided a complete answer to this latter problem in the special case $k = 1$.

Let $\overline{D}_0 = (V, \overline{A}_0)$ denote the complementary digraph of D_0 arising from the complete digraph $D^* = (V, A^*)$ by removing A_0, that is, $\overline{A}_0 := A^* - A_0$. In these terms, our goal is to find a degree-specified subgraph D of $\overline{D}_0$ for which $D_0 + D$ is k-connected. Note that in the case when an arbitrary digraph H for possible new arcs is specified instead of $\overline{D}_0$, the problem becomes NP-complete even in the special case $k = 1$ and $A_0 = \emptyset$ since, for the degree specification $m_o \equiv 1 \equiv m_i$, it is equivalent to finding a Hamilton circuit of H.

We will show that the problem can be embedded into the framework of Theorem 1.4 in such a way that the augmented digraph $D_0 + D$ provided by this theorem will automatically be simple. In this way, we shall obtain a good characterization for the genaral case when the initial digraph D_0 is arbitrary. This characterization, however, will include independent families of bi-sets, and in this sense it is more complicated than the one given in Theorem 1.3 for the special case $k = 1$.

In the special case when D_0 has no arcs at all, that is, when the goal is to find a degree-specified k-connected digraph, the general characterization will be significantly simplified in such a way that the use of independent bi-set families is completely avoided, and we shall arrive at a characterization which is a straight extension of the one in Theorem 1.3 concerning the special case $k = 1$. Recall the definition of $D^*[Z, X]$.

THEOREM 3.6. Let $D_0 = (V, A_0)$ be a simple digraph with in- and out-degree functions ϱ_0 and δ_0, respectively. There is a digraph $D = (V, A)$ fitting (m_o, m_i) for which $D_0^+ := D_0 + D$ is simple and k-connected if and only

$$\widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) \leq \gamma \tag{22}$$

for subsets $X, Z \subseteq V$ and for independent familiy $\mathcal{F}$ of non-trivial bi-sets which are one-way with respect to $D_0 + D^*[Z, X]$, where $p_1(B_O, B_I) = k - w(B)$ for $B \in \mathcal{F}$ and $d_{\overline{A}_0}(Z, X)$ denotes the number of arcs $a = zx \in \overline{A}_0$ for which $z \in Z, x \in X$.

Proof. Note that the requirement for the members $B = (B_O, B_I)$ of $\mathcal{F}$ to be one-way with respect to $D_0 + D^*[Z, X]$ is equivalent to require that $B \in \mathcal{B}_0$, $Z \subseteq B_O$, and $X \cap B_I = \emptyset$.

For proving necessity, assume that D is a digraph meeting the requirements of the theorem. To see (22), observe that there are $\widetilde{m}_o(Z)$ arcs of D with tail in Z and there are $\widetilde{m}_o(X)$ arcs with head in X. Since D can have at most $d_{\overline{A}_0}(Z, X)$ arcs with tail in Z and head in X, the number of arcs of D with tail in Z or head in X is at least $\widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X)$. Moreover, since $D_0 + D$ is k-connected, D contains at least $k - w(B)$ arcs covering any D_0-one-way bi-set. Therefore D contains at least $\widetilde{p}_1(\mathcal{F})$ arcs covering $\mathcal{F}$. Since the members of $\mathcal{F}$ are one-way with respect to $D^*[Z, X]$, the tail of these arcs are not in Z and the heads of these arcs are not in X. Therefore the total number γ of arcs of D is at least $\widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X)$, and (22) follows.

Sufficiency. Assume that (22) holds. Define

$$N_0^+(u) := \{u\} \cup \{v : uv \in A_0\}, \;\; N_0^-(u) := \{u\} \cup \{v : vu \in A_0\}.$$

Claim 3.7.

$$k \leq m_o(v) + \delta_0(v) \leq n - 1, \quad k \leq m_i(v) + \varrho_0(v) \leq n - 1 \tag{23}$$

for every node $v \in V$.

Proof. We prove only the first half of (23) since the second half follows analogously.

By choosing $\mathcal{F} = \emptyset$, $Z = \{v\}$, and $X = V$, (22) gives rise to $m_o(u) + \widetilde{m}_i(V) - d_{\overline{A}_0}(Z, X) \leq \gamma$. Since $\widetilde{m}_i(V) = \gamma$ and $d_{\overline{A}_0}(Z, X) = n - 1 - \delta_0(v)$, we obtain that $m_o(v) + \delta_0(v) \leq n - 1$.

Let $B = (V - v, V - N_0^+(v))$, $\mathcal{F} = \{B\}$, $Z = V - v$, and $X = \emptyset$. Then $\widetilde{p}_1(\mathcal{F}) = k - \delta_0(v)$ and $d_{\overline{A}_0}(Z, X) = 0$ from which (22) implies that $k - \delta_0(v) + \widetilde{m}_o(V - v) \leq \gamma$ from which $k \leq m_o(v) + \delta_0(v)$, as required. $\bullet$

Let $\mathcal{B}_0$ denote the set of non-trivial D_0-one-way bi-sets. Observe that $\mathcal{B}_0$ is crossing. We introduce the following four functions defined on $\mathcal{B}_0$. For $B = (B_O, B_I) \in \mathcal{B}_0$, let

$$p_1(B) := k - w(B) \; (= k + |B_I| - |B_O|), \tag{24}$$

$$p_2(B) := \begin{cases} m_o(u) & \text{if } B_O = V - u, \; B_I = V - N_0^+(u) \text{ for some } u \in V \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

$$p_3(B) := \begin{cases} m_i(v) + |N_0^-(v)| - |B_O| & \text{if } B_I = \{v\} \text{ for some } v \in V \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

$$p(B) := \max\{p_1(B), \; p_2(B), \; p_3(B), \; 0\}. \tag{27}$$

Here p_2 and p_3 are to encode the out-degree and the in-degree specifications, respectively. Note, however, that the definitions of p_2 and p_3 are not symmetric since a node u determines a single bi-set B with outer set $V-u$ for which $p_2(B)$ is positive, while a node v may determine several bi-sets with inner set $\{v\}$ for which $p_3(B)$ is positive. The reason of this apparently undue asymmetry is that both the supermodularity of p and the simplicity of the augmented digraph can be ensured only in this way.

Claim 3.8. Let B and C be two crossing bi-sets. Then *(a)* $p_1(B)+p_1(C)=p_1(B\sqcap C)+p_1(B\sqcup C)$ and *(b)* $p_2(B)=0=p_2(C)$. *(c)* If $p_3(B)>0$ and $p_3(C)>0$, then $p_3(B)+p_3(C)=p_3(B\sqcap C)+p_3(B\sqcup C)$.

Proof. Part (a) is immediate from the definition of p_1. A bi-set B in $\mathcal{B}_0$ with $p_2(B)$ cannot cross any member of $\mathcal{B}_0$ from which (b) follows. In Case (c), $|B_I|=1=|C_I|$ and hence $B_I=\{v\}=C_I$ for some $v\in V$. Therefore $B\sqcap C=(B_O\cap C_O,\{v\})$ and $B\sqcup C=(B_O\cup C_O,\{v\})$ from which $p_3(B)+p_3(C)=p_3(B\sqcap C)+p_3(B\sqcup C)$ follows. •

Lemma 3.9. The bi-set function p on $\mathcal{B}_0$ is positively crossing supermodular.

Proof. Let B and C be two crossing members of $\mathcal{B}_0$ with $p(B)>0, p(C)>0$. By Part (b) of Claim 3.8, $p(B)=\max\{p_1(B),p_3(B)\}$ and $p(C)=\max\{p_1(C),p_3(C)\}$. If $p(B)=p_1(B)$ and $p(C)=p_1(C)$, then, by Part (a) of Claim 3.8, $p(B)+p(C)=p_1(B)+p_1(C)=p_1(B\sqcap C)+p_1(B\sqcup C)\leq p(B\sqcap C)+p(B\sqcup C)$. The supermodular inequality follows analogously from Part (C) of Claim 3.8 when $p(B)=p_3(B)$ and $p(C)=p_3(C)$.

Finally, suppose that $p(B)=p_3(B)=m_i(v)+|N_0^+(v)|-|B_O|$ where $B_I=\{v\}$ for some $v\in V$, and $p(C)=p_1(C)=k+|C_I|-|C_O|$. Now $C_I\cap B_I=\{v\}$ from which $p_3(B\sqcap C)=m_i(v)+|N_0^+(v)|-|B_O\cap C_O|$. Furthermore, $C_I\cap B_I=C_I$ from which $p_1(B\sqcup C)=k-|C_I|+|B_O\cup C_O|$. Hence we have

$$p(B)+p(C)=[m_i(v)+|N_0^+(v)|-|B_O|]+[k+|C_I|-|C_O|]=$$

$$[m_i(v)+|N_0^+(v)|-|B_O\cap C_O|]+[k-|C_I|+|B_O\cup C_O|]=$$

$$p_3(B\sqcap C)+p_1(B\sqcup C)\leq p(B\sqcap C)+p(B\sqcup C).\ \bullet$$

Lemma 3.10. A loopless digraph $D=(V,A)$ covering p has at least γ arcs. If D has exactly γ arcs, then *(a)* D fits (m_o,m_i), *(b)* D_0+D is k-connected, and *(c)* D_0+D is simple.

Proof. Since D covers p and $p\geq p_2$, we obtain that D has at least $m_o(u)$ arcs leaving u for every node u, and therefore

$$|A|=\sum[\delta_D(u):u\in V]=\sum[\varrho_D(V-u):u\in V]\geq\widetilde{m}_o(V)=\gamma. \tag{28}$$

Suppose now that D has exactly γ arcs. Then (28) implies $\delta_D(u)=u$ for every node $u\in V$. Furthermore, $p_3(B)=m_i(v)$ for $B=(N_0^-(v),\{v\})$, and thus $\varrho_D\geq p\geq p_3$ implies that $\varrho_D(B)=\varrho_D(D)\geq m_i(v)$, from which $\varrho_D(v)=m_i(v)$ for every $v\in V$. That is, D fits (m_o,m_i).

By Lemma 3.1, in order to see that $H := D_0 + D$ is k-connected, it suffices to show that $w(B) \geq k$ for every non-trivial H-one-way bi-set B. But this follows from $0 = \varrho_D(B) \geq p(B) \geq p_1(B) = k - w(B)$.

Finally, we prove that $D_0 + D$ is simple. Let v be any node. For $B = (N_0^-(v), \{v\})$ we have $\varrho_D(v) \geq \varrho_D(B) \geq p(B) \geq p_3(B) = m_i(v) = \varrho_D(v)$ from which one has equality throughout. But $\varrho_D(v) = \varrho_D(B)$ implies that every arc $e = uv$ of D must enter $N_0^-(v)$ as well, impyling that e cannot be parallel to any arc of D_0. Suppose now that there is an arc in D from u to v. We have just seen that e is not parallel to any arc of D_0, that is, $u \notin N_0^-(v)$. Let $B = (N_0^-(v)+u, \{v\})$. Then $p_3(B) = m_i(v)-1$ and hence

$$\varrho_D(v) \geq \varrho_D(B) + 1 \geq p(B) + 1 \geq p_3(B) + 1 = m_i(v) - 1 + 1 = \varrho_D(v)$$

from which one has equality throughout. In particular, $\varrho_D(v) = \varrho_D(B) + 1$, implying that there is at most one arc in D from u to v. •

Lemma 3.11. There is a loopless digraph $D = (V, A)$ with γ arcs covering p.

Proof. Suppose indirectly that no such a digraph exists. Theorem 1.4 implies that there is an independent family $\mathcal{I}$ of non-trivial D_0-one-way bi-sets for which $\widetilde{p}(\mathcal{I}) > \gamma$. We may assume that $\widetilde{p}(\mathcal{I})$ is as large as possible, modulo this, $|\mathcal{I}|$ is minimal, and within this,

$$\sum[|B_I| : (B_O, B_I) \in \mathcal{I}] \text{ is as small as possible.} \tag{29}$$

The minimality of $\mathcal{I}$ implies $p(B) > 0$ for every $B \in \mathcal{I}$.

Claim 3.12. There are no two members $B = (B_O, \{v\})$ and $C = (C_O, \{v\})$ of $\mathcal{I}$ (with the same inner set $\{v\}$) for which $p(B) = p_3(B)$ and $p(C) = p_3(C)$.

Proof. If on the contrary there are two such members, then $B_O \cup C_O = V$ by the indpendence of $\mathcal{I}$. If we replace the two members B and C of $\mathcal{I}$ by the single bi-set $B \sqcap C = (B_O \cap C_O, \{v\})$, then the resulting family $\mathcal{I}'$ is also independent since any arc covering $B \sqcap C$ covers at least one of B and C.

Recall that $p_3(B \sqcap C) = m_i(v) + |N_0^-(v)| - |B_O \cap C_O|$. By Claim 3.7, $m_i(v) + |N_0^-(v)| \leq n - 1$ and hence

$$p(B) + p(C) = p_3(B) + p_3(C) = m_i(v) + |N_0^-(v)| - |B_O| + m_i(v) + |N_0^-(v)| - |C_O| =$$

$$m_i(v) + |N_0^-(v)| - |B_O \cap C_O| + m_i(v) + |N_0^-(v)| - |B_O \cup C_O| =$$

$$p_3(B \sqcap C) + m_i(v) + \varrho_0(v) + 1 - |V| \leq p_3(B \sqcap C) + 0 \leq p(B \sqcap C).$$

By the maximality of $\widetilde{p}(\mathcal{I})$ we must have $p(B) + p(C) = p(B \sqcap C)$ and hence $\widetilde{p}(\mathcal{I}) = \widetilde{p}(\mathcal{I}')$ but this contradicts the minimality of $|\mathcal{I}|$. •

Let

$$\mathcal{I}_1 := \{B \in \mathcal{I} : p(B) = p_1(B) > \max\{p_2(B), p_3(B)\}\},$$

$$\mathcal{I}_2 := \{B \in \mathcal{I} : p(B) = p_2(B)\},$$

$$\mathcal{I}_3 := \{B \in \mathcal{I} : p(B) = p_3(B) > p_2(B)\}.$$

Note that if $p_2(B) = p_3(B) \geq p_1(B)$ for $B \in \mathcal{I}$, then $B \in \mathcal{I}_2$. It follows that $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ form a partition of $\mathcal{I}$.

Let Z consist of those nodes u for which the bi-set $(V - u, V - N_0^+(u))$ is in $\mathcal{I}_2$. Let X consist of those nodes v for which there is a set K_v such that $(K_v, \{v\}) \in \mathcal{I}_3$. By Claim 3.12, there is at most one such K_v.

Claim 3.13. For $v \in X$, one has $K_v = Z \cup N_0^+(v)$.

Proof. $N_0^+(v) \subseteq K_v$ holds since $(K_v, \{v\}) \in \mathcal{I}_3$. If an element $u \in Z - N_0^+(v)$ would not be in K_v, then uv would cover both $(V - u, V - N_0^+(u)) \in \mathcal{I}_2$ and $(K_v, \{v\}) \in \mathcal{I}_3$, contradicting the independence of $\mathcal{I}$. Therefore $Z \cup N_0^+(v) \subseteq K_v$.

To see the reverse inclusion suppose indirectly that there is an element $u \in K_v - (Z \cup N_0^-(v)$. Replace the member $B = (K_v, v)$ of $\mathcal{I}$ by $B' = (K_v - u, v)$. Since $p(B') \geq p_3(B') = p_3(B) + 1 = p(B) + 1$, the maximality of $\widetilde{p}(\mathcal{I})$ implies that the resulting system $\mathcal{I}'$ is not independent. Therefore there is a member C of $\mathcal{I}$ that is covered by arc uv.

Since $u \notin Z$, the bi-set C cannot be in $\mathcal{I}_2$. Since C and B are distinct, Claim 3.12 implies that C cannot be in $\mathcal{I}_3$ either. Therefore $C \in \mathcal{I}_1$. We claim that $|C_I| \geq 2$. Indeed, if $|C_I| = 1$, then $C_I = \{v\}$. Since $p_1(C) > p_3(C)$, $p_1(C) = k - |C_O| + |C_I| = k - |C_O| + 1$, and $p_3(C) = m_i(v) - |C_O| + |N_0^+(v)|$, we obtain that $k + 1 > m_i(v) + |N_0^+(v)| = m_i(v) + \varrho_0(v) + 1$, contradicting the first half of (23).

By replacing the member C of $\mathcal{I}'$ with $C' := (C_O, C_I - v)$, we obtain an independent family $\mathcal{I}''$. Since $p(C') \geq p_1(C') = p_1(C) - 1 = p(C) - 1$, we must have $\widetilde{p}(\mathcal{I}'') = \widetilde{p}(\mathcal{I})$, but this contradicts the minimality property given in (29). •

Claim 3.14. $p_3(\mathcal{I}_3) = \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X)$.

Proof. Let $B = (K_v, \{v\}) \in \mathcal{I}_3$. By Claim 3.13, $K_v = Z \cup N_0^+(v)$ and hence $p_3(B) = m_i(v) + |N_0^-(v)| - |B_O| = m_i(v) - |Z - N_0^-(v)| = m_i(v) - d_{\overline{A}_0}(Z, \{v\})$, and therefore $p_3(\mathcal{I}_3) = \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X)$. •

Since $p_2(\mathcal{I}_2) = \widetilde{m}_o(Z)$, we can conclude that

$$\gamma < \widetilde{p}(\mathcal{I}) = \widetilde{p}(\mathcal{I}_1) + \widetilde{p}(\mathcal{I}_2) + \widetilde{p}(\mathcal{I}_3) =$$

$$\widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_2(\mathcal{I}_2) + \widetilde{p}_3(\mathcal{I}_3) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X),$$

and, as $\mathcal{F} := \mathcal{I}_1$ consists of bi-sets which are one-way with respect to $D_0 + D^*[Z, X]$, this inequality contradicts the hypothesis (22) of the theorem. • •

Remark With a similar technique, it is possible to solve the degree-specified node-connectivity augmentation problem when the augmenting digraph is required to be simple. Even more, we may prescribe a subset F of the starting digraph D_0 and request for the degree-specified augmenting digraph D to be found to be simple and have no parallel arcs with the elements of F. If F is empty, then this requires that the augmenting digraph be simple, while if F is the whole A_0, then this requires that the augmented digraph be simple.

3.3 Simplified characterization for $k = 1$

In Theorem 2.2 we considered the augmentation problem in which an initial digraph D_0 was to be made strongly connected by adding a degree-specified simple digraph D. In Theorem 3.6, the general degree-specified node-connectivity augmentation problem was solved when the augmented digraph was required to be simple. The characterization in Theorem 3.6 had, however, a slight aesthetic drawback in the sense that it included independent families of bi-sets. The goal of the present section is to show that in the special case of $k = 1$ this drawback can be eliminated.

THEOREM 3.15. Let (m_o, m_i) be a degree-specification with $\widetilde{m}_o(V) = \widetilde{m}_i(V) = \gamma$ and let $D_0 = (V, A_0)$ be a simple digraph. There is a digraph $D = (V, A)$ fitting (m_o, m_i) for which $D_0^+ := D_0 + D$ is simple and strongly connected if and only

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) \leq \gamma \tag{30}$$

holds for every $X, Z \subseteq V$ and

$$\widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) + 1 \leq \gamma \tag{31}$$

holds for every disjoint subsets $X, Z \subseteq V$ such that Z is not reachable from X in D_0, where $d_{\overline{A}_0}(Z, X)$ denotes the number of arcs $a = zx \in \overline{A}_0$ for which $z \in Z, x \in X$.

Proof. Necessity. Since $D_0 + D$ is simple, D_0 must be a subgraph of $\overline{D}_0$, and Condition (30) is a special case of (2) when $H = \overline{D}_0$.

Let ϱ_0 denote the in-degree function of D_0. To see the necessity of (31), let D be a requested digraph. Since Z is not reachable from X in D_0, there is a subset K for which $Z \subseteq K \subseteq V - X$ and $\varrho_0(K) = 0$. Digraph D must have at least $\widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X)$ arcs having tail in Z or having head in X, and D has at least one more arc entering K.

Sufficiency. Let $\mathcal{K} = \{K : \emptyset \subset K \subset V, \varrho_0(K) = 0\}$. Theorem 3.6, when applied in the special case $k = 1$, states that the requested digraph D exists if and only if

$$\widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) \leq \gamma \tag{32}$$

for subsets $X, Z \subseteq V$ and for independent families $\mathcal{F}$ of non-trivial bi-sets which are one-way with respect to $D_0 + D^*[Z, X]$, where $p_1(B_O, B_I) = 1 - w(B)$ for $B \in \mathcal{F}$.

Now $p_1(B_O, B_I)$ can be positive only if $B_O = B_I$. The requirement that a (B_O, B_I) is D_0-one-way is equivalent to require that no arc of D_0 enters B_I. Therefore the condition can be restated as follows.

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) \leq \gamma \tag{33}$$

where $\mathcal{F}$ is an independent family of sets $K \in \mathcal{K}$ such that no arc of $D^*[Z, X]$ enters K. This last property requires that $Z \subseteq K$ and that $K \subseteq V - X$. The independence of $\mathcal{F}$ means that $\mathcal{F}$ consists of pairwise disjoint or pairwise co-disjoint sets.

Our goal is to prove that (33) follows from (30) and (31). When $\mathcal{F}$ is empty, (33) is just (30). Suppose now that $\mathcal{F}$ is non-empty.

If neither Z nor X is empty, then $\mathcal{F}$ has exactly one member, and in this case (33) and (31) coincide.

Claim 3.16. $\widetilde{m}_i(K) \geq 1$ and $\widetilde{m}_o(V-K) \geq 1$ for each $K \in \mathcal{K}$.

Proof. When (31) is applied to $Z = 0$ and $X = V - K$, one obtains that $\widetilde{m}_i(X) + 1 \leq \gamma = \widetilde{m}_i(V) = \widetilde{m}_i(K) + \widetilde{m}_i(X)$, that is, $\widetilde{m}_i(K) \geq 1$.

When (31) is applied to $Z = K$ and $X = \emptyset$, one obtains that $\widetilde{m}_o(Z) + 1 \leq \gamma = \widetilde{m}_o(V) = \widetilde{m}_o(V-K) + \widetilde{m}_o(Z)$, that is, $\widetilde{m}_O(V-K) \geq 1$. •

Suppose now that $X = \emptyset = Z$. When $\mathcal{F}$ is a sub-partition, $|\mathcal{F}| \leq \sum[m_i(K) : K \in \mathcal{F}] \leq \widetilde{m}_i(V) = \gamma$. When the members of $\mathcal{F}$ are pairwise co-disjoint, $|\mathcal{F}| \leq \sum[m_i(V - K) : K \in \mathcal{F}] \leq \widetilde{m}_i(V) = \gamma$ Therefore in this case (33) holds.

Suppose now that $Z = \emptyset$ and $X \neq \emptyset$. Then $\mathcal{F}$ is a sub-partition and

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) = |\mathcal{F}| + \widetilde{m}_i(X) \leq$$

$$\sum[m_i(K) : K \in \mathcal{F}] + \widetilde{m}_i(X) \leq \widetilde{m}_i(V) = \gamma,$$

that is, (33) holds.

Finally, consider the remaining case when $Z \neq \emptyset$ and $X = \emptyset$. Then the members of $\mathcal{F}$ include Z and are pairwise co-disjoint. Hence

$$|\mathcal{F}| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - d_{\overline{A}_0}(Z, X) = |\mathcal{F}| + \widetilde{m}_o(Z) \leq$$

$$\sum[m_o(V - K) : K \in \mathcal{F}] + \widetilde{m}_o(Z) \leq \widetilde{m}_o(V) = \gamma,$$

that is, (33) holds.

Theorem 3.6 ensures that the requested digraph D exists. • •

Remark Theorem 3.15 implies Theorem 2.2 as follows. Subdivide each arc of D_0 by a node and define $m_o(z) = 0$ and $m_i(z) = 0$ for each subdividing node, and apply Theorem 3.15 to the resulting digraph D_0'.

4 Degree-sequences of simple k-connected digraphs

In 1972, Wang and Kleitman [19] characterized the degree-sequences of simple k-connected undirected graphs. The goal of this section is to solve the analogous problem for directed graphs. Before formulating the main result, we start with some preparatory work. Assume throughout that $1 \leq k < n = |V|$. A node v_f of a digraph H is said to be **full** if both $v_f u$ and $u v_f$ are arcs of H for every node $u \neq v_f$.

4.1 Preparations

Let Z and X be two proper (but possibly empty) subsets of V. Let $D_1 = (V, A_1)$ denote the simple digraph $D^*[Z, X]$ in which uv $(u \neq v)$ is an arc if $u \in Z$ or $v \in X$. Note that each node in $X \cap Z$ is full. Let $\mathcal{B}_1$ denote the set of non-trivial D_1-one-way bi-sets. Clearly, $Z \subseteq B_O$ and $B_I \cap X = \emptyset$ hold for each $B \in \mathcal{B}_1$.

Let $\mathcal{F} \subseteq \mathcal{B}_1$ be an independent family meaning that each arc of the complete digraph $D^* = (V, A^*)$ covers at most one member of $\mathcal{F}$. Let $p_1(B) = k - w(B)$ where

$w(B) = |B_O - B_I|$. In proving our characterization of degree-sequences realizable by simple k-connected digraphs, we shall need a simple upper bound for $\widetilde{p}_1(\mathcal{F})$. Note that if A is a set of arcs for which $D_1^+ = (V, A_1 + A)$ is k-connected, then $p_1(\mathcal{F}) \leq |A|$.

Lemma 4.1. For an independent family $\mathcal{F} \subseteq \mathcal{B}_1$,

$$\widetilde{p}_1(\mathcal{F}) \leq 0 \text{ when } |X \cap Z| \geq k, \tag{34}$$

$$\widetilde{p}_1(\mathcal{F}) \leq k - |X \cap Z| \text{ when } |X \cap Z| < k \text{ and } |X|, |Z| \geq k. \tag{35}$$

Proof. If $|X \cap Z| \geq k$, then D_1 has k full nodes and hence D_1 is k-connected, implying that $\widetilde{p}_1(\mathcal{F}) \leq 0$.

The second part follows once we show that D_1 can be made k-connected by adding $k' = k - |X \cap Z|$ new arcs. By symmetry, we may assume that $|Z| \geq |X|$. Let $x_1, \ldots, x_{k'}$ be nodes in $X - Z$ and let $z_1, \ldots, z_{k'}$ be nodes in $Z - X$. Let $A = \{x_1 z_1, \ldots, x_{k'} z_{k'}\}$ be a set of k' disjoint arcs.

We claim that $D^+ = (V, A_1 + A)$ is k-connected. Indeed, if $D^+ - K$ is not strong for some $K \subset V$, then K contains every full node and hence $X \cap Z \subseteq K$. Moreover, K must hit every arc in A since if $\{x_i, z_i\} \cap K = \emptyset$ for some i, then $D^+ - K$ would be strong as there is an arc $z_i u$ and an arc $u x_i$ for each node $u \in V - K - \{x_i, z_i\}$. Therefore $|K| \geq k$ and hence D^+ is k-connected. This means that D_1 has been made k-connected by adding k' new arcs, from which $\widetilde{p}_1(\mathcal{F}) \leq k' = k - |X \cap Z|$. •

The total out-deficiency of the nodes is defined by $\sigma_o = \sum[(k - \delta_{D_1}(v))^+ : v \in V]$. Clearly, if $|X| < k$, then

$$\sigma_o = (n - |Z|)(k - |X|) + |X - Z|.$$

Lemma 4.2. Let $\mathcal{F} \subseteq \mathcal{B}_1$ be an independent family. If $|X| < k$ and $|Z| \geq |X|$, then

$$\widetilde{p}_1(\mathcal{F}) \leq \sigma_o. \tag{36}$$

Proof. The number of arcs of D_1 can be expressed as follows.

$$|A_1| = |Z|(n-1) + (n - |Z|)|X| - |X - Z|. \tag{37}$$

(36) is equivalent to

$$\sum[(k - w(B) : B \in \mathcal{F}] \leq (n - |Z|)(k - |X|) + |X - Z|. \tag{38}$$

Let $q := |\mathcal{F}|$. We distinguish two cases.

Case 1 $q \geq n - |Z|$.

Claim 4.3.

$$\sum[(n-1) - w(B) : B \in \mathcal{F}] \leq (n - |Z|)(n - 1 - |X|) + |X - Z|. \tag{39}$$

Proof. Since $\mathcal{F}$ is independent, the total number of those arcs of the complete digraph $D^* = (V, A^*)$ which cover a member B of $\mathcal{F}$ is $\sum[|B_I|(n - |B_O|) : B \in \mathcal{F}]$. Since A_1 covers no member of $\mathcal{F}$, we conclude that

$$\sum[|B_I|(n - |B_O|) : B \in \mathcal{F}] + |Z|(n-1) + (n - |Z|)|X| - |X - Z| =$$

$$\sum[|B_I|(n - |B_O|) : B \in \mathcal{F}] + |A_1| \leq |A^*| = n(n-1).$$

By observing that

$$|B_I|(n - |B_O|) \geq 1 \cdot [n - |B_O| + (|B_I| - 1)] = n - 1 - w(B),$$

we obtain

$$\sum[(n-1) - w(B) : B \in \mathcal{F}] \leq \sum[|B_I|(n - |B_O|) : B \in \mathcal{F}] \leq$$

$$n(n-1) - [|Z|(n-1) + (n - |Z|)|X| - |X - Z|] = (n - |Z|)(n - 1 - |X|) + |X - Z|,$$

as required for (39). •

As we are in Case 1, $q(n-1-k) \geq (n - |Z|)(n-1-k)$. By subtracting this inequality from (39), we obtain (38), proving the lemma in Case 1.

Case 2 $q < n - |Z|$. Recall that we have assumed $|Z| \geq |X|$ and $|X| < k$.

Claim 4.4.

$$\sum[n - w(B) : B \in \mathcal{F}] \leq q(n - |X|) + |X - Z|. \tag{40}$$

Proof. Let $h := |X - Z|$. Let $x_1, \ldots, x_h$ be the elements of $X - Z$ and let $\{z_1, \ldots, z_h\}$ be a subset of $Z - X$. Consider the set $A = \{x_1z_1, \ldots, x_hz_h\}$ of disjoint arcs. For $B \in \mathcal{F}$, let $\alpha(B)$ denote the number of arcs in A entering B_O but not B_I, and $\beta(B)$ the number of arcs in A entering both B_O and B_I (that is, covering B). Note that $X \cap Z \subseteq B_O - B_I$.

Clearly, $\alpha(B) + \beta(B) = |X - B_O|$. The definition of $\alpha(B)$ immediately shows that $\alpha(B) \leq w(B) - |X \cap B_O|$. It follows that

$$\beta(B) = |X - B_O| - \alpha(B) \geq |X - B_O| - [w(B) - |X \cap B_O|] = |X| - w(B).$$

Since $\mathcal{F}$ is independent, every arc in A covers at most one member of $\mathcal{F}$ and hence

$$\sum[\beta(B) : B \in \mathcal{F}] \leq |A| = |X - Z|.$$

By combining these observations, we obtain

$$|X - Z| \geq \sum[\beta(B) : B \in \mathcal{F}] \geq \sum[|X| - w(B) : B \in \mathcal{F}] =$$

$$\sum[n - w(B) : B \in \mathcal{F}] - q(n - |X|),$$

that is, $\sum[n - w(B) : B \in \mathcal{F}] \leq q(n - |X|) + |X - Z|$, and the claim follows. •

By subtracting first the equality $\sum[n - k : B \in \mathcal{F}] = q(n-k)$ from (40) and applying then the assumption $q < n - |Z|$, one obtains

$$\sum[k - w(B) : B \in \mathcal{F}] \leq q(k - |X|) + |X - Z| < (n - |Z|)(k - |X|) + |X - Z|,$$

that is, (38) holds, completing the proof of the lemma. • •

4.2 The characterization

THEOREM 4.5. Let V be a set of n nodes and let $1 \leq k \leq n-1$. Suppose that (m_o, m_i) is a degree-specification meeting *(4)*. There exists a simple k-connected digraph fitting (m_o, m_i) if and only if

$$\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + k \leq \gamma \text{ for distinct subsets } X, Z \subset V. \tag{41}$$

Moreover, it suffices to require the inequality in *(41)* only to its special case when $|X \cap Z| < k$, X consists of the h largest values of m_i and Z consists of the j largest values of m_o.

Proof. Necessity. Suppose that there is a digraph D with the requested properties, and let X and Z be two distinct, proper subsets of V. If $|X \cap Z| \geq k$, then (4) implies $\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + k \leq \widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + |X \cap Z| \leq \gamma$ and (41) holds. This argument also implies that, given (4), it suffices to require the inequality in (41) only for X and Z for which $|X \cap Z| < k$.

Assume now that $|X \cap Z| \leq k-1$. By the simplicity of D, there are at most $|X||Z| - |X \cap Z|$ arcs with tail in Z and head in X. Therefore the total number of arcs with tail in Z or head in X is at least $\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + |X \cap Z|$. Moreover, the k-connectivity of D implies for distinct subsets $X, Y \subset V$ that there are at least $k - |X \cap Z|$ arcs from $V - Z$ to $Z - X$ or there are at least $k - |X \cap Z|$ arcs from $X - Z$ to $V - X$. Since the tails of these arcs are not in Z and the heads of these arcs are not in X, we can conclude that $\widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + k = \widetilde{m}_i(X) + \widetilde{m}_o(Z) - |X||Z| + |X \cap Z| + (k - |X \cap Z|) \leq \gamma$, that is, the inequality in (41) holds in this case, too.

Sufficiency. Suppose indirectly that the requested digraph does not exist. By applying Theorem 3.6 to the empty digraph $D_0 = (V, \emptyset)$ and observing that in this case $d_{\overline{D}_0}(Z, X) = |Z||X| - |Z \cap X|$, we obtain that there exists an independent family $\mathcal{F}$ of bi-sets and subsets Z and X for which $Z \subseteq B_O, X \cap B_I = \emptyset$, and

$$\widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| > \gamma. \tag{42}$$

If we reorient a digraph by reversing each of its arcs, then the k-connectivity is preserved and the in-degrees and out-degrees transform into each other. Therefore the roles of m_o and m_i in Theorem 3.6 are symmetric and therefore we may and shall assume that $|Z| \geq |X|$ in (42).

We can also assume that $\mathcal{F}$ is minimal and hence $p_1(B) > 0$ for each $B \in \mathcal{F}$. Since (m_o, m_i) is required to meet (4), $\mathcal{F}$ is non-empty. Lemma 4.1 implies $|X \cap Z| < k$.

Suppose first that $|X|, |Z| \geq k$. Since $|X \cap Z| < k$ we have $X \neq Z$, $X \neq V$, and $Z \neq V$. By Lemma 4.1, $\widetilde{p}_1(\mathcal{F}) \leq k - |X \cap Z|$. From (42), we have

$$\gamma < \widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| \leq$$

$$k - |X \cap Z| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| = k + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z|,$$

contradicting (41).

In the remaining case $|Z| \geq |X|$ and $|X| < k$. Lemma 4.2 implies that $\widetilde{p}_1(\mathcal{F}) \leq \sigma_o = (n - |Z|)(k - |X|) + |X - Z|$. By applying the inequality in (41) to $X = \emptyset$ and $Z = V - v$, one gets $\widetilde{m}_o(v) \geq k$ from which $\widetilde{m}_o(V - Z) \geq k|V - Z|$. By using again (42), we obtain

$$\gamma < \widetilde{p}_1(\mathcal{F}) + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| \leq$$

$$(k - |X|)(n - |Z|) + |X - Z| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X \cap Z| =$$
$$k(|V| - |Z|) - |X||V| + |X||Z| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) - |X||Z| + |X| \leq$$
$$\widetilde{m}_o(V - Z) - |X||V| + \widetilde{m}_o(Z) + \widetilde{m}_i(X) + |X| =$$
$$\widetilde{m}_o(V) + \widetilde{m}_i(X) - |X||V| + |X \cap V|,$$

and this contradicts the inequality (4) with V in place of Z. •

5 Degree-sequences of k-elementary bipartite graphs

As an application of Theorem 4.5, we extend a characterization of degree-sequences of elementary bipartite graphs (bigraphs, for short), due to R. Brualdi [4], to k-elementary bigraphs. A simple bipartite graph $G = (S, T; E)$ is called **elementary** if G is perfectly matchable and the union of its perfect matchings is a connected subgraph. It is known (see, [15], p. 122) that G is elementary if and only if it is either just one edge or $|S| = |T| \geq 2$ and the Hall-condition holds with strict inequality for every non-empty proper subset of T. Another equivalent formulation requires that either G is just one edge or $G - s - t$ has a perfect matching for each $s \in S, t \in T$. Yet another characterization states that G is elementary if and only if it is connected and every edge belongs to a perfect matching. It should be noted that there is a one-to-one correspondence between elementary bipartite graphs and fully indecomposable $(0, 1)$-matrices [5].

Let $k \leq n - 1$ be a positive integer. We call a simple bigraph $G = (S, T; E)$ with $|S| = |T| = n$ k-**elementary** if the removal of any j-element subset of S and any j-element subset of T leaves a perfectly matchable graph for each $0 \leq j \leq k$. Equivalently, $|\Gamma(X)| \geq |X| + k$ for every subset $X \subseteq S$ with $|X| \leq n - k$. Obviously this last property is equivalent to requiring $|\Gamma(X)| \geq |X| + k$ for every subset $X \subseteq T$ with at most $n-k$ elements. Note that for $k = n-1$ the complete bigraph $K_{n,n}$ is the only k-elementary bigraph. For $n \geq 2$, a bigraph is 1-elementary if and only if it is elementary, while for $n = 1$ the bigraph $K_{1,1}$ consisting of a single edge is considered elementary but not 1-elementary.

There is a natural correspondence between k-connected digraphs and k-elementary bigraphs with a specified perfect matching. Let $M = \{e_1, \ldots, e_n\}$ be a perfect matching of a k-elementary bigraph $G = (S, T; E)$. Let $V = \{v_1, \ldots, v_n\}$ be a node-set where v_i is associated with e_i. For each edge e of G, let s_e and t_e denote the end-nodes of e belonging to S and T, respectively. We associate a digraph $D = (V, A)$ with the pair (G, M) as follows. For each edge $e \in E - M$ of G, let $s_e t_e$ be an arc of D. It can easily be observed that the digraph D obtained this way is k-connected if and only if G is k-elementary. By using network flow techniques, a digraph can be checked in polynomial time whether it is k-connected or not, and therefore a bipartite graph can also be checked for being k-elementary.

Conversely, one can associate a perfectly matchable bigraph with a digraph D on node-set $\{v_1, \dots, v_n\}$, as follows. Define $G = (S, T; E)$ so that $s_i t_i$ belongs to E for $i = 1, \dots, n$ and, for every arc $v_j v_h$ of D, let $t_j s_h$ be an edge of G. This G is a simple k-elementary graph.

Let $m = (m_S, m_T)$ be a degree-specification for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$ and assume that the set $\mathcal{G}(m_S, m_T)$ of simple bipartite graphs fitting (m_S, m_T) is non-empty, that is, by a theorem of Gale and Ryser [12], [18]

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| \leq \gamma \text{ whenever } \emptyset \subseteq X \subseteq S,\ \emptyset \subseteq Z \subseteq T. \tag{43}$$

Brualdi [4] characterized the degree-sequences of elementary bipartite graphs in terms of fully indecomposable $(0,1)$-matrices. Here we extend his results (apart from its trivial special case when $n = 1$) to k-elementary bipartite graphs.

THEOREM 5.1. There is a k-elementary member G of $\mathcal{G}(m_S, m_T)$ if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + (n - |X| - |Z| + k) \leq \gamma \text{ whenever } X \subset S,\ Z \subset T. \tag{44}$$

Furthermore, it suffices to require *(44)* only for $X \subset S$ consisting of the h largest m_S-valued elements and for $Z \subset T$ consisting of the j largest m_T-valued elements $(h, j < n)$. Moreover, if $m_S(s_1) \leq \cdots \leq m_S(s_n)$ and $m_T(t_1) \geq \cdots \geq m_T(t_n)$, the graph G can be chosen in such a way that $s_1 t_1, \dots, s_n t_n$ is a perfect matching of G.

Proof. Necessity. Suppose that there is a requested bigraph G. Since G is k-elementary, the degree of each node is at least $k + 1$. For $X \subset S$ and $Z \subset T$, let γ_1 denote the number of edges incident to a node in $X \cup Z$ while γ_2 is the number of the remaining edges.

Then $\gamma_1 \geq \widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z|$. If $|Z| \geq k$, then $T - Z$ has at least $|T - Z| + k$ neighbours and hence $T - Z$ has at least $|T - Z| + k - |X| = n - |X| - |Z| + k$ neighbours in $S - X$, from which $\gamma_2 \geq |T - Z| + k - |X|$ implying that

$$\gamma = \gamma_1 + \gamma_2 \geq \widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + n - |X| - |Z| + k,$$

that is, the inequality in (44) holds for this case.

If $|Z| < k$, then each node in $S - X$ has at least $k + 1 - |Z|$ neighbours in $T - Z$ from which $\gamma_2 \geq |S - X|(k + 1 - |Z|) = (n - |X|)(k - |Z|) + n - |X| \geq k - |Z| + n - |X|$. Therefore

$$\gamma = \gamma_1 + \gamma_2 \geq \widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + k - |Z| + n - |X|$$

implying the inequality in (44) for this case, too.

Sufficiency. We may suppose that the elements of S and T are ordered in such a way that $m_S(s_1) \leq \cdots \leq m_S(s_n)$ and $m_T(t_1) \geq \cdots \geq m_T(t_n)$. The inequality in (44), when applied to $X := V - s$ and to $Z = \emptyset$, implies that $m_S(s) \geq k + 1$ for $s \in S$

and $m_T(t) \geq k+1$ follows analogously for $t \in T$. Let $V = \{v_1, \dots, v_n\}$ be a set. Let $m_o(v_j) := m_T(t_j) - 1$ and $m_i(v_j) := m_S(s_j) - 1$ for $j = 1, \dots, n$. Let $\gamma_D = \gamma - n$.

Let Z_j denote the first j $(0 \leq j \leq n)$ elements of V and let X_h denote the last h $(0 \leq h \leq n)$ elements of V. The (possibly empty) subset of T corresponding to Z_j is denoted by Z while the subset of S corresponding to X_j is denoted by X. Due to the assumption made on the ordering of the elements of V, Z_j consists of the j elements of V with largest m_o-values while X_h consists of the h element of V with largest m_o-values.

Claim 5.2. Suppose that $j < n$ and $h < n$, that is, $X \subset S$ and $Z \subset T$. The inequality *(41)* with γ_D in place of γ holds for X and Z defined above.

Proof. $\widetilde{m}_o(Z_j) + \widetilde{m}_i(X_h) - jh + k = \widetilde{m}_T(Z) - |Z| + \widetilde{m}_S(X) - |X| - |X||Z| + k \leq \gamma - n = \gamma_D$. •

Claim 5.3. Suppose that $j \leq n$ and $h \leq n$, that is, $X \subseteq S$ and $Z \subseteq T$. The inequality *(4)* with γ_D in place of γ holds for X and Z defined above.

Proof. Suppose first that $h + j \geq n$, that is, $|X_h \cup Z_j| = n$. Then

$$\widetilde{m}_i(X_h)+\widetilde{m}_o(Z_j)-|X_h||Z_j|+|X_h \cap Z_j| = \widetilde{m}_S(X)-|X|+\widetilde{m}_T(Z)-|Z|-|X||Z|+|X_h \cap Z_j| =$$

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| - |X_h| - |Z_j| + |X_h \cap Z_j| =$$

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| - |X_h \cup Z_j| \leq \gamma - |X_h \cup Z_j| = \gamma - n = \gamma_D.$$

Second, suppose that $h + j < n$, that is, $X_h \cup Z_j \subset V$ implying that $X_h \cap Z_j = \emptyset$. By Claim 5.2, $\widetilde{m}_o(Z_j) + \widetilde{m}_i(X_h) - |X_h||Z_j| + k \leq \gamma_D$ and hence $\widetilde{m}_i(X_h) + \widetilde{m}_o(Z_j) - |X_h||Z_j| + |X_h \cap Z_j| = \widetilde{m}_i(X_h) + \widetilde{m}_o(Z_j) - |X_h||Z_j| < \gamma_D$. •

Consider the bigraph $G = (S, T; E)$ associated with D (in which $s_i t_i$ belongs to E for $i = 1, \dots, n$ and, for every arc $v_j v_h$ of D, let $t_j s_h$ be an edge of G.) This G is a simple k-elementary bigraph fitting (m_S, m_T). •

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