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# Countable Menger theorem with finitary matroid constraints on the ingoing edges

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**Abstract**

We present a strengthening of the countable Menger theorem [1] (edge version) of R. Aharoni (see also in [4] p. 217) by placing a finitary matroid on the ingoing edges of each vertex and demanding the path-system to use an independent set of ingoing edges at any vertex. Our result is the complementarity condition based generalization of the minimax theorem that one can prove in the finite case about the maximal possible size of the path-system.

## 1 Notations

The variables $\xi, \zeta$ denote ordinals and $\kappa$ stands for an infinite cardinal. We write $\omega$ for the smallest limit ordinal (i.e. the set of natural numbers).

The digraphs $D = (V, A)$ of this article could be arbitrarily large and may have multiple edges and loops (though the later is irrelevant). We write $D[X]$ for the subdigraph spanned by the vertex set $X$. If $e$ is an edge from vertex $u$ to vertex $v$, then we write $\mathsf{tail}(e) = u$ and $\mathsf{head}(e) = v$. The paths in this paper are assumed to be finite and directed. Repetition of vertices is forbidden in them (we say walk if we want to allow it). For a path $P$ we denote by $\mathsf{start}(P)$ and $\mathsf{end}(P)$ the first and the last vertex of $P$. If $X, Y \subseteq V$, then $P$ is a $X \to Y$ path iff $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$. For singletons we simplify the notation and write $x \to y$ instead of $\{x\} \to \{y\}$. For a path-system (set of paths) $\mathcal{P}$ we denote $\bigcup_{P \in \mathcal{P}} A(P)$ by $A(\mathcal{P})$ and we write for the set of the last edges of the elements of $\mathcal{P}$ simply $A_{last}(\mathcal{P})$. An $s$-arborescence is a directed tree in which every vertex is reachable (by a directed path) from its vertex $s$.

If $\mathcal{M}$ is a matroid and $S$ is a subset of its ground set, then $\mathcal{M}[S]$ stands for the restriction of $\mathcal{M}$ to $S$. We use $\bigoplus$ for the direct sum and $\mathcal{M}/J$ for the contraction of the set $J$. For the rank function we write $r$ with possibly subscript. Let us remind that a matroid is called finitary iff all of its circuits are finite. One can find a good survey about infinite matroids from the basics in [3].

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# 2 Main result

Let $D = (V, A)$ be a digraph with $s \neq t \in V$ and let $\mathcal{M}_v$ be a matroid on $\mathsf{in}_D(v)$ for each $v \in V$. A set $A_0 \subseteq A$ called **feasible** (with respect to the pair $\mathfrak{D} := (D, \{\mathcal{M}_v\}_{v \in V})$) iff for each $v \in V$ the set $A_0 \cap \mathsf{in}_D(v)$ is independent in $\mathcal{M}_v$. A path-system in $D$ considered feasible iff its edge set is feasible. For a vertex set $X$ we define

$$\mathcal{M}_X := \bigoplus_{v \in X} \mathcal{M}_v[\mathsf{in}_D(v) \cap \mathsf{in}_D(X)].$$

Assume for a moment that $D$ is finite. If $\mathcal{P}$ is a feasible system of edge-disjoint $s \to t$ paths and $X$ is a vertex set with $t \in X \subseteq V \setminus \{s\}$ (an $\overline{\boldsymbol{s}}\boldsymbol{t}$**-set** for short), then necessarily

$$|\mathcal{P}| \leq r_{\mathcal{M}_X}(\mathsf{in}_D(X)).$$

There is equality iff the following **complementarity conditions** hold

1. $A(\mathcal{P}) \cap \mathsf{out}_D(X) = \varnothing$,
2. $A(\mathcal{P}) \cap \mathsf{in}_D(X)$ spans $\mathsf{in}_D(X)$ in $\mathcal{M}_X$.

These complementarity conditions make sense even if $\mathcal{P}$ is infinite. We prove the following strengthening of the countable Menger theorem of Aharoni [1].

**Theorem 2.1.** *If $D$ is countable and $\mathcal{M}_v$ are finitary, then there is a feasible system $\mathcal{P}$ of edge-disjoint $s \to t$ paths and an $\overline{s}t$-set $X$ such that $\mathcal{P}$ and $X$ satisfy the complementarity conditions above.*

A pair $(\mathcal{W}, X)$ is called a **wave** (with respect to $\mathfrak{D}$) iff $X$ is an $\overline{s}t$-set and $\mathcal{W}$ is a feasible system of edge-disjoint $s \to X$ paths such that the second complementarity condition hold for $\mathcal{W}$ and $X$ (i.e. $A_{last}(\mathcal{W})$ is a base of $\mathcal{M}_X$).

**Remark 2.2.** By picking an arbitrary base $B$ of $\mathcal{M}_{V \setminus \{s\}}$ and taking $\mathcal{W} := B$ as a set of single-edge paths and $X := V \setminus \{s\}$ we obtain a wave $(\mathcal{W}, X)$ thus always exists some wave.

We say that the wave $(\mathcal{W}_1, X_1)$ extends the wave $(\mathcal{W}_0, X_0)$ and write $(\mathcal{W}_0, X_0) \leq (\mathcal{W}_1, X_1)$ iff $X_1 \subseteq X_0$ and $\mathcal{W}_1$ consists of the forward continuations of some of the paths in $\mathcal{W}_0$ including all the paths of $\mathcal{W}_0$ that meet $X_1$, such that all the new terminal segments are paths in $D[X_0]$. If in addition $\mathcal{W}_1$ contains a forward-continuation of all the elements of $\mathcal{W}_0$, then the extension is called **complete**. Note that $\leq$ is a partial ordering on the waves.

**Observation 2.3.** If $(\mathcal{W}_1, X_1)$ is an incomplete extension of $(\mathcal{W}_0, X_0)$, then $\mathcal{W}_1$ does not satisfy the second complementarity condition with $X_0$ (and hence $X_1 \subsetneq X_0$).

**Lemma 2.4.** *If a nonempty set $\mathcal{X}$ of waves is linearly ordered by $\leq$, then $\mathcal{X}$ has a unique, smallest upper bound* $\sup(\mathcal{X})$.

*Proof:* We may suppose that $\mathcal{X}$ has no maximal element. Let $\langle(\mathcal{W}_\xi, X_\xi) : \xi < \kappa\rangle$ be a cofinal sequence of $(\mathcal{X}, \leq)$. We define $X := \bigcap_{\xi<\kappa} X_\xi$ and

$$\mathcal{W} := \bigcup_{\zeta<\kappa} \bigcap_{\zeta<\xi} \mathcal{W}_\xi.$$

For $P \in \mathcal{W}$ we have $V(P) \cap X_\xi = \{\mathsf{end}(P)\}$ for all large enough $\xi < \kappa$ hence $V(P) \cap X = \{\mathsf{end}(P)\}$. The paths in $\mathcal{W}$ are pairwise edge-disjoint since $P_1, P_2 \in \mathcal{W}$ implies that $P_1, P_2 \in \mathcal{W}_\xi$ for all large enough $\xi$. Since the matroids $\mathcal{M}_v$ are finitary the same argument shows the feasibility of $\mathcal{W}$.

Let $e \in \mathsf{in}_D(X) \setminus A(\mathcal{W})$ be arbitrary. For a large enough $\xi < \kappa$ we have $e \in \mathsf{in}_D(X_\xi)$. Then the last edges of those elements of $\mathcal{W}_\xi$ that terminate in $\mathsf{head}(e)$ span $e$ in $\mathcal{M}_{\mathsf{head}(e)}$. These paths have to be elements of all the further waves of the sequence (because of the definition of $\leq$) and thus of $\mathcal{W}$ as well. Therefore $(\mathcal{W}, X)$ is a wave and clearly an upper bound.

Suppose that $(\mathcal{Q}, Y)$ is another upper bound for $\mathcal{X}$. Then $X_\xi \supseteq Y$ for all $\xi < \kappa$ hence $X \supseteq Y$. Let $Q \in \mathcal{Q}$ be arbitrary. We know that $\mathcal{W}_\xi$ contains an initial segment $Q_\xi$ of $Q$ for all $\xi < \kappa$ because $(\mathcal{Q}, Y)$ is an upper bound (see the definition of $\leq$). For $\xi < \zeta < \kappa$ the path $Q_\zeta$ is a (not necessarily proper) forward continuation of $Q_\xi$. From some index the sequence $\langle Q_\xi : \xi < \kappa\rangle$ needs to be constant, say $Q^*$, since $Q$ is a finite path. But then $Q^* \in \mathcal{W}$. Thus any $Q \in \mathcal{Q}$ is a forward continuation of a path in $\mathcal{W}$. Finally assume that some $P \in \mathcal{W}$ meets $Y$. Pick a $\xi < \kappa$ for which $P \in \mathcal{W}_\xi$. Then $(\mathcal{W}_\xi, X_\xi) \leq (\mathcal{Q}, Y)$ guarantees $P \in \mathcal{Q}$. Therefore $(\mathcal{W}, X) \leq (\mathcal{Q}, Y)$. ●

The Remark 2.2 and Lemma 2.4 imply via Zorn's Lemma the following.

**Corollary 2.5.** *There exists a maximal wave. Furthermore there is a maximal wave which is greater or equal to an arbitrary prescribed wave.*

Let $(\mathcal{W}, X)$ be a maximal wave. To prove Theorem 2.1 it is enough to show that there is a feasible system of edge-disjoint $s \to t$ paths $\mathcal{P}$ that consists of the forward-continuation of all the paths in $\mathcal{W}$. Indeed, condition $A(\mathcal{P}) \cap \mathsf{out}_D(X) = \varnothing$ will be true automatically (otherwise $\mathcal{P}$ would violate feasibility, when the violating path "comes back" to $X$) and hence $\mathcal{P}$ and $X$ will satisfy the complementarity conditions.

We need a method developed by Lawler and Martel in [5] for the augmentation of polymatroidal flows in the finite networks which works in infinite case as well.

**Lemma 2.6.** *Let $\mathcal{P}$ be a feasible system of edge-disjoint $s \to t$ paths. Then either there is a feasible system of edge-disjoint $s \to t$ paths $\mathcal{P}'$ with $\mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P})) \subsetneq \mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P}'))$ or there is a $\overline{s}t$-set $X$ such that the complementarity conditions hold for $\mathcal{P}$ and $X$.*

*Proof:* Call $W$ an augmenting walk (with respect to $\mathfrak{D}$ and $\mathcal{P}$) iff

1. $W$ is a directed walk with respect to the digraph that we obtain from $D$ by changing the direction of edges in $A(\mathcal{P})$,

2. $\mathsf{start}(W) = s$ and $W$ meets no more $s$,
3. $A(W)\triangle A(\mathcal{P})$ is feasible,
4. if for some initial segment $W'$ of $W$ the set $A(W')\triangle A(\mathcal{P})$ is not feasible, then for the one edge longer initial segment $\mathcal{W}'' = \mathcal{W}'e$ the set $A(W'')\triangle A(\mathcal{P})$ is feasible again.

If there is an augmenting walk terminating in $t$, then let $W$ be a shortest such a walk. Build $\mathcal{P}'$ from the edges $A(W)\triangle A(\mathcal{P})$ in the following way. Keep untouched those $P \in \mathcal{P}$ for which $A(W) \cap A(P) = \varnothing$ and replace the remaining finitely many paths, say $\mathcal{Q} \subseteq \mathcal{P}$ where $|\mathcal{Q}| = k$, by $k+1$ new $s \to t$ paths constructed from the edges $A(W)\triangle A(\mathcal{Q})$ by the greedy method. Obviously $\mathcal{P}'$ is a feasible system of edge-disjoint $s \to t$ paths. We need to show that

$$\mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P})) \subsetneq \mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P}')).$$

If only the last vertex of $W$ is $t$, then it is clear. Let $f_1, e_1, \dots, f_n, e_n, f_{n+1}$ be the edges of $W$ incident with $t$ enumerated with respect to the direction of $W$. The initial segments of $W$ up to an inner appearances of $t$ may not be augmenting walks (since $W$ is a shortest that terminates in $t$) hence by condition 4 the one edge longer and the one edge shorter segments are. It follows that for any $1 \leq i \leq n$ there is a $\mathcal{M}_t$-circuit $C_i$ in

$$A_i := \mathcal{A}(P) \cap \mathsf{in}_D(t) + f_1 - e_1 + f_2 - e_2 + \cdots + f_i.$$

Furthermore $f_i \notin A(\mathcal{P})$ and $e_i \in C_i \cap A(\mathcal{P})$. It implies by induction that $A_i \setminus \{e_i\}$ spans the same set in $\mathcal{M}_t$ as $\mathcal{A}(P) \cap \mathsf{in}_D(t)$ whenever $1 \leq i \leq n$ and hence $A_n \cup \{f_{n+1}\}$ spans a strictly larger set.

Suppose now that none of the augmenting walks terminate in $t$. Let us denote the set of the last vertices of the augmenting walks by $Y$. We show that $\mathcal{P}$ and $X := V \setminus Y$ satisfy the complementarity conditions. Obviously $X$ is an $\overline{s}t$-set. Suppose that $e \in A(\mathcal{P}) \cap \mathsf{out}_D(X)$. Pick an augmenting walk $W$ terminating in $\mathsf{head}(e)$ (considering the original edge directions). If $e \notin A(W)$, then $We$ is an augmenting walk contradicting to the definition of $X$. Otherwise consider the initial segment of $W'$ of $W$ for which the following edge is $e$. Then $W'e$ is an augmenting walk (if $W'$ itself is not, then it is because of condition 4) which leads to the same contradiction.

Assume that $f \in \mathsf{in}_D(X) \setminus A(\mathcal{P})$. Choose an augmenting walk $W$ that terminates in $\mathsf{tail}(f)$. We may suppose that $f \notin A(W)$ otherwise we consider the initial segment $W'$ of $W$ for which the following edge is $f$ (it is an augmenting walk, otherwise $W'f$ would be by applying condition 4). The initial segments of $Wf$ that terminate in $\mathsf{head}(f)$ may not be augmenting walks. Let $f_1, e_1, \dots, f_n, e_n$ be the ingoing-outgoing edge pairs of $\mathsf{head}(f)$ in $W$ enumerating with respect to the direction of $W$ and let $f_{n+1} = f$. Then for any $1 \leq i \leq n+1$ there is a $\mathcal{M}_{\mathsf{head}(f)}$-circuit $C_i$ in

$$\mathcal{A}(P) \cap \mathsf{in}_D(\mathsf{head}(f)) + f_1 - e_1 + f_2 - e_2 + \cdots + f_i.$$

It follows by using condition 4 and the definition of $X$ that for $1 \leq i \leq n$

1. $f_i \notin A(\mathcal{P})$ and $e_i \in C_i \cap A(\mathcal{P})$,
2. $\mathsf{tail}(e_i), \mathsf{tail}(f_i) \in Y$ (tail with respect to the original direction),
3. $C_i \subseteq \mathsf{in}_D(X)$.

Assume that we already know for some $1 \leq i \leq n$ that $f_j$ spanned by $F := \mathcal{A}(P) \cap \mathsf{in}_D(X) \cap \mathsf{in}_D(\mathsf{head}(f))$ in $\mathcal{M}_{\mathsf{head}(f)}$ whenever $j < i$. Any element of $C_i \setminus \{f_i\}$ which is not in $F$ has a form $f_j$ for some $j < i$ thus by the induction hypothesis it is spanned by $F$ and hence we obtain that $f_i \in \mathsf{span}_{\mathcal{M}_{\mathsf{head}(f)}}(F)$. By induction it is true for $i = n+1$ as well. ●

**Proposition 2.7.** *Assume that $(\mathcal{W}, X)$ and $(\mathcal{Q}, Y)$ are waves where $Y \subseteq X$ and $\mathcal{Q}$ consists of the forward-continuation of some of the paths in $\mathcal{W}$ inside $X$. Let $\mathcal{W}_Y := \{P \in \mathcal{W} : \mathsf{end}(P) \in Y\}$. Then for an appropriate $\mathcal{Q}' \subseteq \mathcal{Q}$ the pair $(\mathcal{W}_Y \cup \mathcal{Q}', Y)$ is a wave with $(\mathcal{W}, X) \leq (\mathcal{W}_Y \cup \mathcal{Q}', Y)$.*

*Proof:* The path-system $\mathcal{W}_Y \cup \mathcal{Q}$ (not necessarily disjoint union) is edge-disjoint since the edges in $A(\mathcal{Q}) \setminus A(\mathcal{W})$ lie in $X$. For the same reason it may violate feasibility only at the vertices $\{\mathsf{end}(P) : P \in \mathcal{W}_Y\} \subseteq Y$. Pick a base $B$ of $\mathcal{M}_Y$ for which

$$A_{last}(\mathcal{W}_Y) \subseteq B \subseteq A_{last}(\mathcal{W}_Y) \cup A_{last}(\mathcal{Q}).$$

It is routine to check that the choice $\mathcal{Q}' = \{P \in \mathcal{Q} : A(P) \cap B \neq \varnothing\}$ is suitable. ●

If $A_0 \subseteq A$ is feasible with respect to $\mathfrak{D}$, then let $\boldsymbol{\mathfrak{D}(A_0)} := (D(A_0), \{\mathcal{M}_v(A_0)\}_{v \in V})$ where

$$\mathcal{M}_v(A_0) = \mathcal{M}_v / \mathsf{span}_{\mathcal{M}_v}(A_0 \cap \mathsf{in}_D(v)) \quad (v \in V)$$
$$D(A_0) = D - \bigcup_{v \in V} \mathsf{span}_{\mathcal{M}_v}(A_0 \cap \mathsf{in}_D(v)).$$

**Observation 2.8.** The matroids $\mathcal{M}_v(A_0)$ have no loops (for any choice of $A_0$).

**Observation 2.9.** If $(\mathcal{W}, X)$ is $\mathfrak{D}$-wave and for some $A_0 \subseteq A \setminus A(\mathcal{W})$ the set $A_0 \cup A(\mathcal{W})$ is $\mathfrak{D}$-feasible, then $(\mathcal{W}, X)$ is a $\mathfrak{D}(A_0)$-wave as well.

**Lemma 2.10.** *If $(\mathcal{W}, X)$ is a maximal wave with respect to $\mathfrak{D}$ and $e \in A \setminus A(\mathcal{W})$ for which $A(\mathcal{W}) \cup \{e\}$ is $\mathfrak{D}$-feasible, then all the extensions of $(\mathcal{W}, X)$ with respect to $\mathfrak{D}(e)$ are complete extensions.*

*Proof:* Seeking a contradiction, assume that we have an incomplete extension $(\mathcal{Q}, Y)$ of $(\mathcal{W}, X)$ with respect to $\mathfrak{D}(e)$. Observe that $e \in \mathsf{in}_D(Y)$ and $r(\mathcal{M}_Y / A_{last}(\mathcal{Q})) = 1$ necessarily hold; furthermore, $Y \subsetneq X$ by Observation 2.3.

We show that $(\mathcal{W}, X)$ has a proper extension with respect to $\mathfrak{D}$ as well contradicting with its maximality. Without loss of generality we may assume that $\mathsf{in}_D(X) = A_{last}(\mathcal{W})$. Indeed, otherwise we delete the edges $\mathsf{in}_D(X) \setminus A(\mathcal{W})$ from $D$ and from the corresponding matroids. It is routine to check that after the deletion

$(\mathcal{W}, X)$ is still a wave and a proper extension of it remains a proper extension after putting back these edges.

Contract $V \setminus X$ to $s$ and contract $Y$ to $t$. Associate to them the matroids $\mathcal{M}_{V\setminus X}$ and $\mathcal{M}_Y$ respectively. Apply the augmenting walk method in the resulting system with the $V \setminus X \to Y$ terminal segments of the elements of $\mathcal{Q}$. If the augmentation is possible, then the assumption $\mathsf{in}_D(X) = A_{last}(\mathcal{W})$ ensures that the first edges of any of the resulting paths $\mathcal{R}$ is a last edge of some path in $\mathcal{W}$. By uniting the elements of $\mathcal{R}$ with the corresponding paths from $\mathcal{W}$ we can get a new feasible system of edge-disjoint $s \to Y$ paths $\mathcal{Q}'$ (with respect to $\mathfrak{D}$). Furthermore $r(\mathcal{M}_Y/A_{last}(\mathcal{Q})) = 1$ guarantees that $A_{last}(\mathcal{Q}')$ is a base of $\mathcal{M}_Y$ and hence $(\mathcal{Q}', Y)$ is a wave. Thus by Proposition 2.7 we get an extension of $(\mathcal{W}, X)$ and it is proper because $Y \subsetneq X$ which is impossible.

Thus the augmentation must be unsuccessful which implies by Lemma 2.6 that there is some $Z$ with $Y \subseteq Z \subseteq X$ such that $Z$ and $\mathcal{Q}$ satisfy the complementarity conditions. By Proposition 2.3 we know that $Z \subsetneq X$. For the initial segments $\mathcal{Q}_Z$ of the paths in $\mathcal{Q}$ up to $Z$ the pair $(\mathcal{Q}_Z, Z)$ forms a wave. Thus applying Proposition 2.7 with $(\mathcal{W}, X)$ and $(\mathcal{Q}_Z, Z)$ we obtain an extension of $(\mathcal{W}, X)$ which is proper because $Z \subsetneq X$ contradicting to the maximality of $(\mathcal{W}, X)$. ●

**Proposition 2.11.** *If $(\mathcal{W}, X)$ is a maximal wave and $v \in X$, then there is a $v \to t$ path $Q$ such that $A(\mathcal{W}) \cap A(Q) = \varnothing$ and $A(\mathcal{W}) \cup A(Q)$ is feasible.*

*Proof:* It is equivalent to show that there exists an $v \to t$ path in $D(A(\mathcal{W}))$. Suppose, to the contrary, that it is not the case. Let $X' \subsetneq X$ be the set of those vertices in $X$ that are unreachable from $v$ in $D(A(\mathcal{W}))$ (note that $t \in X'$ by the indirect assumption). Let $\mathcal{W}'$ be consist of the paths in $\mathcal{W}$ that meet $X'$. If we prove that $(\mathcal{W}', X')$ is a wave, then we are done since it would be a proper extension of the maximal wave $(\mathcal{W}, X)$. Assume that $f \in \mathsf{in}_D(X') \setminus A(\mathcal{W}')$ is a non-loop element in $\mathcal{M}_{\mathsf{head}(f)}$. Then by the definition of $X'$ we have $\mathsf{tail}(f) \in V \setminus X$ thus $f \in \mathsf{in}_D(X)$. Then $f$ is spanned by the last edges of the paths in $\mathcal{W}$ terminating in $\mathsf{head}(f)$ and all these paths are in $\mathcal{W}'$ as well. Therefore $(\mathcal{W}', X')$ is a wave which contradicts to the maximality of $(\mathcal{W}, X)$. ●

**Lemma 2.12.** *Let $(\mathcal{W}, X_0)$ be a maximal wave with respect to $\mathfrak{D}$ and assume that $P \in \mathcal{W}$ and let $\mathcal{W}_0 = \mathcal{W} \setminus \{P\}$. Then there is an $s$-arborescence $\mathcal{A}$ such that*

1. *$A(P) \subseteq A(\mathcal{A})$,*
2. *$A(\mathcal{A}) \cap A(\mathcal{W}_0) = \varnothing$,*
3. *$A(\mathcal{A}) \cup A(\mathcal{W}_0)$ is $\mathfrak{D}$-feasible,*
4. *$t \in V(\mathcal{A})$,*
5. *there is a maximal wave with respect to $\mathfrak{D}(A(\mathcal{A}))$ which is a complete extension of the $\mathfrak{D}(A(\mathcal{A}))$-wave $(\mathcal{W}_0, X_0)$.*

*Proof:* Fix a well-ordering of $A$ with order type $|A| \leq \omega$.

**Proposition 2.13.** *The pair $(\mathcal{W}_0, X_0) := (\mathcal{W} \setminus \{P\}, X_0)$ is a maximal wave with respect to $\mathfrak{D}(A(P))$.*

*Proof:* It is clearly a wave thus we show just the maximality. Suppose that $(\mathcal{Q}, Y)$ is a proper extension of $(\mathcal{W} \setminus \{P\}, X_0)$ with respect to $\mathfrak{D}(A(P))$. Necessarily $\mathsf{end}(P) \in Y$ otherwise it would be a wave with respect to $\mathfrak{D}$ which properly extends $(\mathcal{W}, X_0)$. Let $e$ be the last edge of $P$. We know that $A_{last}(\mathcal{Q})$ is a base of $\mathcal{M}_Y/e$. Since $A(\mathcal{Q})$ is $\mathfrak{D}(A(P))$-feasible it follows that $(\mathcal{Q} \cup \{P\}, Y)$ is a wave with respect to $\mathfrak{D}$. But then it properly extends $(\mathcal{W}, X_0)$ which is a contradiction. ●

We build the arborescence $\mathcal{A}$ by recursion. Let $\mathcal{A}_0 := P$. Assume that $\mathcal{A}_m, \mathcal{W}_m$ and $X_m$ has already defined for $m \leq n$ in such a way that

1. $A(\mathcal{A}_m) \cap A(\mathcal{W}_m) = \varnothing$,
2. $A(\mathcal{A}_m) \cup A(\mathcal{W}_m)$ is $\mathfrak{D}$-feasible,
3. $(\mathcal{W}_m, X_m)$ is a maximal wave with respect to $\mathfrak{D}_m := \mathfrak{D}(A(\mathcal{A}_m))$ and a complete extension of the $\mathfrak{D}_m$-wave $(\mathcal{W}_k, X_k)$ whenever $k < m$,
4. $\mathcal{A}_{m+1} = \mathcal{A}_m + e_m$ for some $e_m \in \mathsf{out}_D(V(\mathcal{A}_m))$.

If $t \in V(\mathcal{A}_n)$, then $\mathcal{A}_n$ satisfies the requirements of Lemma 2.12 thus we are done. Hence we may assume that $t \notin V(\mathcal{A}_n)$.

**Proposition 2.14.** $\mathsf{out}_{D(A(\mathcal{W}_n))}(V(\mathcal{A}_n)) \neq \varnothing$.

*Proof:* Since $(\mathcal{W}_n, X_n)$ is a maximal wave with respect to $\mathfrak{D}(\mathcal{A}_n)$ but not a wave with respect to $\mathfrak{D}$ the $s$-arborescence $\mathcal{A}_n$ need to have an edge $e \in \mathsf{in}_D(X)$. Apply Proposition 2.11 with $(\mathcal{W}_n, X_n)$ and $\mathsf{head}(e)$ (in system $\mathfrak{D}_n$) to obtain a $\mathfrak{D}_n(A(\mathcal{W}_n))$-feasible $\mathsf{head}(e) \to t$ path $Q$. Consider the last vertex $v$ of $Q$ which is in $V(\mathcal{A}_n)$. Since $v \neq t$ there is an outgoing edge $f$ of $v$ in $Q$ and hence $f \in \mathsf{out}_{D(A(\mathcal{W}_n))}(V(\mathcal{A}_n))$. ●

Pick the smallest element $e_n$ of $\mathsf{out}_{D(A(\mathcal{W}_n))}(V(\mathcal{A}_n))$ and let $\mathcal{A}_{n+1} := \mathcal{A}_n + e_n$. By Observation 2.8 we know that $e_n$ is not a loop in $\mathcal{M}_{\mathsf{head}(e_n)}(A(\mathcal{W}))$. Let $(\mathcal{W}_{n+1}, X_{n+1})$ be a maximal wave with respect to $\mathfrak{D}(A(\mathcal{A}_{n+1})) = \mathfrak{D}(A(\mathcal{A}_n))(e)$ which extends $(\mathcal{W}_n, X_n)$ (exists by Corollary 2.5 ). Lemma 2.10 ensures that it is a complete extension.

Suppose to the contrary that the recursion never stops. Let

$$\mathcal{A}_\infty := \left( \bigcup_{n=0}^{\infty} V(\mathcal{A}_n), \bigcup_{n=0}^{\infty} A(\mathcal{A}_n) \right).$$

Note that $A(\mathcal{A}_\infty)$ is feasible because $\mathcal{M}_v$ are finitary. Then $\langle (\mathcal{W}_n, X_n) : n < \omega \rangle$ is an $\leq$-increasing sequence of waves with respect to $\mathfrak{D}(A(\mathcal{A}_\infty))$. Let $(\mathcal{W}_\infty, X_\infty)$ be a maximal wave with respect to $\mathfrak{D}(A(\mathcal{A}_\infty))$ which extends $\sup_n(\mathcal{W}_n, X_n)$ (see Lemma 2.4).

It may not be a wave with respect to $\mathfrak{D}$. Indeed, otherwise $\mathsf{end}(P) \notin X_\infty$ (since $A_{last}(\mathcal{W}_\infty)$ does not span the last edge $e$ of $P$) and therefore it would extend $(\mathcal{W}, X_0)$ properly with respect to $\mathfrak{D}$. Hence the $s$-arborescence $\mathcal{A}$ contains an edge $e \in \mathsf{in}_D(X_\infty)$. Apply Proposition 2.11 with $(\mathcal{W}_\infty, X_\infty)$ and $\mathsf{head}(e)$ (in system $\mathfrak{D}(A(\mathcal{A}_\infty))$) to obtain a $\mathfrak{D}(A(\mathcal{W}_\infty) \cup A(\mathcal{A}_\infty))$-feasible $\mathsf{head}(e) \to t$ path $Q$. Consider the last vertex

$v$ of $Q$ which is in $V(\mathcal{A}_\infty)$. Since $v \neq t$ by assumption there is an outgoing edge $f$ of $v$ in $Q$. Then $f \in \mathsf{out}_{D(A(\mathcal{W}_\infty)}(V(\mathcal{A}_\infty))$ which implies that for some $n_0 < \omega$ we have $f \in \mathsf{out}_{D(A(\mathcal{W}_n)}(V(\mathcal{A}_n))$ whenever $n > n_0$. But then the infinitely many pairwise distinct edges $\{e_n : n_0 < n < \omega\}$ are all smaller than $f$ in our fixed well-ordering on $A$ which contradicts to the fact that the type of the well-ordering is at most $\omega$. ●

The Theorem follows easily from Lemma 2.12. Indeed, pick a maximal wave $(\mathcal{W}_0, X_0)$ with respect to $\mathfrak{D}_0 := \mathfrak{D}$ where $\mathcal{W}_0 = \{P_n\}_{n<\omega}$. Apply Lemma 2.12 with $P_0 \in \mathcal{W}_0$. The resulting arborescence $\mathcal{A}_0$ contains a unique $s \to t$ path $P_0^*$ which is necessarily a forward-continuation of $P_0$ (usage of a new edge from $\mathsf{in}_D(X_0)$ would lead to infeasibility). Then by Lemma 2.12 we have a maximal wave $(\mathcal{W}_1, X_1)$ where $X_1 \subseteq X_0$ with respect to $\mathfrak{D}_1 := \mathfrak{D}_0(A(\mathcal{A}_0))$ such that $\mathcal{W}_1 = \{P_n^1\}_{1 \leq n < \omega}$ where $P_n^1$ is a forward continuation of $P_n$. Then we apply Lemma 2.12 with the $\mathfrak{D}_1$-wave $(\mathcal{W}_1, X_1)$ and $P_1^1 \in \mathcal{W}_1$ and continue the process recursively. By the construction $\bigcup_{n<m} A(P_n^*)$ is $\mathfrak{D}$-feasible for each $m < \omega$. Since the matroids $\mathcal{M}_v$ are assumed to be finitary $\bigcup_{n<\infty} A(P_n^*)$ is $\mathfrak{D}$-feasible as well thus $\mathcal{P} := \{P_n^*\}_{n\in\mathbb{N}}$ is a desired paths-system that satisfies the complementarity conditions with $X_0$.

## 3 Related open problems

We suspect that one can omit the countability condition for $D$ in Theorem 2.1 by analysing the famous infinite Menger theorem [2] of Aharoni and Berger. We also think that it is possible to put matroid constraints on the outgoing edges of each vertex as well but it contains as a special case the Matroid intersection conjecture restricted to finitary matroids which is an important open problem itself. The finitarity of the matroids are used several times in the proof; we do not know yet if one can omit this condition.

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