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Csaba Király\* and Zoltán Szigeti\*\*

## Abstract

The fundamental result of Edmonds [5] started the area of packing arborescences and the great number of recent results shows increasing interest of this subject. Two types of matroid constraints were added to the problem in [2, 3, 9], here we show that both constraints can be added simultaneously. This way we provide a solution to a common generalization of the reachability-based packing of arborescences problem of the first author [14] and the matroid intersection problem of Edmonds [4].

## 1 Introduction

This paper considers problems on arborescence packings in rooted digraphs where a **rooted digraph** is a digraph  $D = (V + s, A)$  with a designated root vertex  $s$ . Throughout this paper a **packing** in a digraph means arc-disjoint subgraphs. Different types of matroid constraints will be added simultaneously to the arborescence packing problem in such a way that the problem obtained contains the matroid intersection problem. The solution provided to this problem in this paper applies ideas from the proof of the matroid intersection theorem of Edmonds [4].

An  **$s$ -arborescence** is a directed tree on a vertex-set containing the **root** vertex  $s$  in which each vertex has in-degree 1 except  $s$ . An  $s$ -arborescence in a rooted digraph  $D = (V + s, A)$  is **spanning** if its vertex set is  $V + s$ . For definitions from matroid theory, we refer to the next section.

Edmonds [5] solved the packing problem of  $k$  spanning  $s$ -arborescences in a rooted digraph. It is well-known that this problem can be formulated as a matroid intersection problem. Indeed, if the first matroid is the  $k$ -sum of the graphic matroid of the underlying undirected graph of  $D$  and the second matroid is the direct sum of the uniform matroids  $U_{|\partial(v)|, k}$  on the set  $\partial(v)$  of arcs entering  $v$  for  $v \in V$ , then the set of the arc sets of the union of  $k$  arc-disjoint spanning  $s$ -arborescences of  $D$  is the set of common bases of these two matroids.

Frank [9] (and later Bernáth and T. Király [2]) observed that one can go further, namely in the above construction the uniform matroids can be replaced by arbitrary matroids. It is mentioned in [9] that this way one may get a solution to the problem of matroid-restricted packing of  $k$  spanning  $s$ -arborescences where a packing of  $s$ -arborescences  $T_1, \dots, T_k$  in a rooted digraph  $D = (V + s, A)$  is said to be **matroid-restricted** if, given a matroid  $\mathcal{M}_v$  on  $\partial(v)$  for every  $v \in V$ ,

$$\{A(T_i) \cap \partial(v) : T_i \text{ contains } v\} \text{ is independent in } \mathcal{M}_v \text{ for every } v \in V. \quad (1)$$

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If  $\mathcal{M}$  is the direct sum of the matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ , then a matroid-restricted packing is called an  $\mathcal{M}$ -restricted packing.

Another way of adding a matroid constraint to the problem of packing arborescences, was proposed by Durand de Gevigney, Nguyen, Szigeti in [3]. A packing of  $s$ -arborescences  $T_1, \dots, T_t$  in a rooted digraph  $D = (V + s, A)$  is said to be **matroid-based** if, given a matroid  $\mathcal{M}$  on the set  $\partial(V)$  of arcs leaving  $s$ ,

$$\{\partial(V) \cap A(T_i[s, v]) : T_i \text{ contains } v\} \text{ is a base of } \mathcal{M} \text{ for every } v \in V, \quad (2)$$

where  $\mathbf{T}[s, v]$  denotes the unique path from  $s$  to  $v$  in an  $s$ -arborescence  $T$ . Durand de Gevigney, Nguyen, Szigeti [3] solved the problem of matroid-based packing of  $s$ -arborescences. Bérczi and Frank proposed later a more natural problem of matroid-based packing of *spanning*  $s$ -arborescences (see in [1]). Recently, a superset of the authors of this paper in [8] proved that this problem is NP-complete.

We propose in this paper to solve the problem of matroid-based matroid-restricted packing of  $s$ -arborescences where both of the above matroid constraints are added. Note that the proposed problem contains the problems of matroid-based packing of  $s$ -arborescences and matroid-restricted packing of spanning  $s$ -arborescences. It is not surprising that it also contains the problem of matroid intersection. Indeed, if  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are two matroids on  $S$ , then the problem of matroid-based matroid-restricted packing of  $s$ -arborescences for the instance of digraph, with two vertices  $s$  and  $v$  and parallel arcs  $sv$  each corresponding to an element of  $S$ , and matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , reduces to the matroid intersection problem.

Observe that, by the above mentioned negative result of [8], the problem of matroid-based matroid-restricted packing of spanning  $s$ -arborescences is NP-complete, however, we will solve this problem for special cases where the first matroid is restricted to several fundamental classes. Observe that, by [2, Corollary 3.2] and by the matroid intersection algorithm of Edmonds, the problem of matroid-based matroid-restricted packing of spanning  $s$ -one-arborescences can be solved in polynomial time where an  **$s$ -one-arborescence** is an  $s$ -arborescence with only one arc leaving its root  $s$ .

In fact, we will propose an even more general problem. To be able to do this, we mention another direction in which the problem of packing spanning arborescences was generalized. Kamiyama, Katoh, Takizawa [13] solved the packing problem of  $k$  reachability  $s$ -one-arborescences where an  $s$ -one-arborescence with a root arc  $e$  in a rooted digraph  $D = (V + s, A)$  is said to be a **reachability**  $s$ -one arborescence if its vertex set is the set of vertices reachable from  $s$  by a directed path of  $D$  with first arc  $e$ .

The first author [14] provided a common generalization of the problems of matroid-based packing of  $s$ -arborescences and packing of  $k$  reachability  $s$ -one-arborescences, namely the problem of reachability-based packing of  $s$ -arborescences, where a packing of  $s$ -arborescences  $T_1, \dots, T_t$  in a rooted digraph  $D = (V + s, A)$  is said to be **reachability-based** if, given a matroid  $\mathcal{M}$  on  $\partial(V)$  with rank function  $r$ ,

$$\{\partial(V) \cap A(T_i[s, v]) : T_i \text{ contains } v\} \text{ is independent in } \mathcal{M} \text{ of size } r(\partial_s(P(v))) \text{ for all } v \in V, \quad (3)$$

where  $P(v)$  denotes the set of vertices in  $V$  from which  $v$  is reachable by a directed path in  $D$  and  $\partial_s(X)$  denotes the set of arcs from  $s$  to  $X$ .

In this paper, we will solve the problem of reachability-based matroid-restricted packing of  $s$ -arborescences. We will show that, by applying the proof method of the matroid intersection theorem, the problem of reachability-based matroid-restricted packing of  $s$ -arborescences can be reduced to the problem of reachability-based packing of  $s$ -arborescences.

## 2 Definitions

Let  $D = (V + s, A)$  be a rooted digraph. For  $X \subseteq V$ , let  $\overline{X} = V + s - X$ . For  $Z \subseteq \overline{X}$ ,  $\partial_Z(\mathbf{X})$  denotes the set of arcs from  $Z$  to  $X$ . If  $Z = \overline{X}$ , then  $Z$  is omitted from the index. By consequence,  $|\partial(X)|$  is the in-degree of the set  $X$ . Let  $\mathbf{P}(\mathbf{X})$  denote the set of vertices in  $V$  from which  $X$  can be reached by a directed path. Note that, by definition,  $P(X)$  contains  $X$  and does not contain the vertex  $s$ .

We need some basic terminologies from matroid theory, we refer to [10] for more details. A function  $b : 2^\Omega \rightarrow \mathbb{Z}$  is called **submodular** if for all  $X, Y \subseteq \Omega$ ,

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$

A function  $p : 2^\Omega \rightarrow \mathbb{Z}$  is called **supermodular** if  $-p$  is submodular. By the results of Iwata, Fleischer and Fujishige [12] and independently by Schrijver [15], a submodular function can be minimized in polynomial time.

For a set function  $r : 2^S \rightarrow \mathbb{Z}_+$ ,  $\mathcal{M} = (\mathbf{S}, r)$  is called a **matroid** if  $r$  is 0 on the  $\emptyset$ , monotone non-decreasing, subcardinal ( $r(\mathbf{Q}) \leq |\mathbf{Q}|$ ) and submodular. The members of  $\mathcal{I} = \{\mathbf{Q} \subseteq \mathbf{S} : r(\mathbf{Q}) = |\mathbf{Q}|\}$  are called **independent** sets of the matroid and  $r$  is called the **rank function** of the matroid. It is well known that a matroid can also be defined by its independent sets. Let  $\mathbf{Q} \subseteq \mathbf{S}$ . The maximal independent sets in  $\mathbf{Q}$  are called **bases** of  $\mathbf{Q}$ . Note that all bases of  $\mathbf{Q}$  are of the same size, namely  $r(\mathbf{Q})$ . The bases of  $\mathbf{S}$  are called the bases of  $\mathcal{M}$ . We say that an element  $\mathbf{s}$  of  $\mathbf{Q}$  is a **bridge** of  $\mathbf{Q}$  if  $r(\mathbf{Q} - \mathbf{s}) = r(\mathbf{Q}) - 1$ . We define  $\text{Span}_{\mathcal{M}}(\mathbf{Q}) = \{\mathbf{s} \in \mathbf{S} : r(\mathbf{Q} \cup \{\mathbf{s}\}) = r(\mathbf{Q})\}$ . Note that  $\text{Span}_{\mathcal{M}}$  is monotone.

As examples, let us mention the following matroids:

1. **graphic matroid**:  $\mathcal{I} =$  edge sets of forests in a graph;
2. **transversal matroid**:  $\mathcal{I} =$  subsets of  $S$  that can be covered by a matching in a bipartite graph  $G = (S, T; E)$ ;
3. **uniform matroids**  $U_{n,k}$ :  $\mathcal{I} = \{\mathbf{Q} \subseteq \mathbf{S} : |\mathbf{Q}| \leq k\}$  where  $|\mathbf{S}| = n$ ;
4. **free matroid**:  $U_{n,n}$ .

Note that uniform matroids form a special class of transversal matroids where  $G$  is the complete bipartite graph  $K_{n,k}$ .

We will need the following operations on matroids. Let  $\mathcal{M} = (\mathbf{S}, r)$  be a matroid. For  $\mathbf{Q} \subseteq \mathbf{S}$ ,  $\mathcal{M}|_{\mathbf{Q}}$  is the matroid with rank function  $r|_{\mathbf{Q}}$  obtained from  $\mathcal{M}$  by restriction on  $\mathbf{Q}$ . For  $\mathbf{s} \in \mathbf{S}$ ,  $\mathcal{M} - \mathbf{s}$  is the matroid obtained from  $\mathcal{M}$  by **deletion** of  $\mathbf{s}$ , that is, a matroid on  $\mathbf{S} - \mathbf{s}$  with rank function  $r|_{\mathbf{S} - \mathbf{s}}$ , while  $\mathcal{M}/\mathbf{s}$  is the matroid obtained from  $\mathcal{M}$  by **contraction** of  $\mathbf{s}$ , that is, a matroid on  $\mathbf{S} - \mathbf{s}$  with a rank function  $r_{\mathcal{M}/\mathbf{s}}(\mathbf{Q}) = r(\mathbf{Q} \cup \mathbf{s}) - 1$ . The  **$k$ -sum** of the matroid  $\mathcal{M}$  is the matroid whose independent sets are those sets that can be partitioned into  $k$  independent sets of  $\mathcal{M}$ . For matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$  on disjoint sets  $\mathbf{S}_1$  and  $\mathbf{S}_2$  with rank functions  $r_1$  and  $r_2$ , their **direct sum**  $\mathcal{M}_1 \oplus \mathcal{M}_2$  is the matroid on  $\mathbf{S}_1 \cup \mathbf{S}_2$  with rank function  $r_{\oplus}(\mathbf{Q}) = r_1(\mathbf{Q} \cap \mathbf{S}_1) + r_2(\mathbf{Q} \cap \mathbf{S}_2)$  for all  $\mathbf{Q} \subseteq \mathbf{S}_1 \cup \mathbf{S}_2$ . Note that  $\mathbf{s}$  is a bridge in  $\mathcal{M}$  if and only if  $\mathcal{M} \simeq (\mathcal{M} - \mathbf{s}) \oplus U_{1,1}$ .

## 3 Results

The first result on packing arborescences is due to Edmonds [5].

**Theorem 3.1** ([5]). *Let  $D = (V + s, A)$  be a rooted digraph and  $k$  a positive integer. There exists a packing of  $k$  spanning  $s$ -arborescences in  $D$  if and only if*

$$|\partial(X)| \geq k \text{ for all } \emptyset \neq X \subseteq V. \quad (4)$$

Edmonds [4] proved a much more general result on the intersection of two arbitrary matroids.

**Theorem 3.2** ([4]). *Let  $\mathcal{M}_1 = (\mathbf{S}, r_1)$  and  $\mathcal{M}_2 = (\mathbf{S}, r_2)$  be two matroids and  $k$  a positive integer. There exists a common independent set of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  of size  $k$  if and only if*

$$r_1(\mathbf{X}) + r_2(\mathbf{S} - \mathbf{X}) \geq k \text{ for all } \mathbf{X} \subseteq \mathbf{S}. \quad (5)$$

For matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$  on the same set  $\mathbf{S}$ , one can find in polynomial time a maximum cardinality common independent set by the matroid intersection algorithm of Edmonds [4].

Theorem 3.1 was generalized in many directions. First, we mention the following result that can be proved by Theorem 3.2.

**Theorem 3.3** ([2, 9]). *Let  $D = (V + s, A)$  be a rooted digraph,  $k$  a positive integer and  $\mathcal{M}_2 = (A, r_2)$  a matroid which is the direct sum of the matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ . There exists an  $\mathcal{M}_2$ -restricted packing of spanning  $s$ -arborescences in  $D$  if and only if*

$$r_2(\partial(X)) \geq k \text{ for all } \emptyset \neq X \subseteq V. \quad (6)$$

Durand de Gevigney, Nguyen and Szigeti [3] proved the following extension of Theorem 3.1.

**Theorem 3.4** ([3]). *Let  $D = (V + s, A)$  be a rooted digraph and  $\mathcal{M}_1 = (\partial(V), r_1)$  a matroid. There exists an  $\mathcal{M}_1$ -based packing of  $s$ -arborescences in  $D$  if and only if*

$$r_1(\partial_s(X)) + |\partial_{V-X}(X)| \geq r_1(\partial(V)) \text{ for all } \emptyset \neq X \subseteq V. \quad (7)$$

In [14], the first author generalized Theorem 3.4 as follows.

**Theorem 3.5** ([14]). *Let  $D = (V + s, A)$  be a rooted digraph and  $\mathcal{M}_1 = (\partial(V), r_1)$  a matroid. There exists an  $\mathcal{M}_1$ -reachability-based packing of  $s$ -arborescences in  $D$  if and only if*

$$r_1(\partial_s(X)) + |\partial_{V-X}(X)| \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V. \quad (8)$$

In this paper, we prove the following result that is a common generalization of all the results previously mentioned in this paper.

**Theorem 3.6.** *Let  $D = (V + s, A)$  be a rooted digraph,  $\mathcal{M}_1 = (\partial(V), r_1)$  and  $\mathcal{M}_2 = (A, r_2)$  two matroids such that  $\mathcal{M}_2$  is the direct sum of the matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ . There exists an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences in  $D$  if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s(X). \quad (9)$$

When we require  $\mathcal{M}_1$ -based packings, (9) can be simplified as follows.

**Corollary 3.7.** *Let  $D = (V + s, A)$  be a rooted digraph,  $\mathcal{M}_1 = (\partial(V), r_1)$  and  $\mathcal{M}_2 = (A, r_2)$  two matroids such that  $\mathcal{M}_2$  is the direct sum of the matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ . There exists an  $\mathcal{M}_1$ -based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences in  $D$  if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(V)) \text{ for all } \emptyset \neq X \subseteq V \text{ and } F \subseteq \partial_s(X). \quad (10)$$

It is proved in [8] that the problem of matroid-based packing of *spanning* arborescences is NP-complete, however, (7) is a necessary and sufficient condition for the case of several fundamental matroid classes, as follows.

**Theorem 3.8** ([8]). *Let  $D = (V + s, A)$  be a rooted digraph and  $\mathcal{M}_1 = (\partial(V), r_1)$  a matroid of rank 2 or a graphic matroid or a transversal matroid. There exists an  $\mathcal{M}_1$ -based packing of spanning  $s$ -arborescences in  $D$  if and only if (7) holds.*

Observe that the arc set  $A'$  of an  $\mathcal{M}_1$ -based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences is independent in  $\mathcal{M}_2$  hence restricting  $\mathcal{M}_2$  to  $A'$  we get the free matroid. Moreover, as an  $\mathcal{M}_1$ -based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences is obviously  $\mathcal{M}_1$ -based, (7) also holds for  $(V + s, A')$ . Hence we get the following corollary from Corollary 3.7 and Theorem 3.8.

**Corollary 3.9.** *Let  $D = (V + s, A)$  be a rooted digraph,  $\mathcal{M}_1 = (\partial(V), r_1)$  a matroid of rank 2, a graphic matroid, or a transversal matroid, and  $\mathcal{M}_2 = (A, r_2)$  a matroid that is the direct sum of matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ . There exists an  $\mathcal{M}_1$ -based  $\mathcal{M}_2$ -restricted packing of spanning  $s$ -arborescences in  $D$  if and only if (10) holds.*

## 4 Preliminaries

Before proving Theorem 3.6, we provide some lemmas that will be useful later. Let  $D, \mathcal{M}_1$  and  $\mathcal{M}_2$  be as in Theorem 3.6. For  $X \subseteq V$  and  $F \subseteq \partial_s(X)$ , let

$$\begin{aligned} \mathbf{b}(\mathbf{X}, \mathbf{F}) &:= r_1(F) + r_2(\partial(X) - F), \\ \mathbf{p}(\mathbf{X}) &:= r_1(\partial_s(P(X))). \end{aligned}$$

The submodularity of  $b$  was proved in [2]. However, we need a bit more hence we give the full proof of the following lemma.

**Lemma 4.1.** *Let  $X, X' \subseteq V$ ,  $F \subseteq \partial_s(X)$  and  $F' \subseteq \partial_s(X')$ . Then*

$$b(X, F) + b(X', F') \geq b(X \cap X', F \cap F') + b(X \cup X', F \cup F'). \quad (11)$$

Moreover, if  $e \in (\partial(X) - F) - (\partial(X') - F')$ , then

$$r_1(F) + r_1(F') + r_2(\partial(X) - (F \cup e)) + r_2((\partial(X') - F') \cup e) \geq b(X \cap X', F \cap F') + b(X \cup X', F \cup F'). \quad (12)$$

*Proof.* First note that

$$(\partial_{\overline{X}}(x) - F) \cap (\partial_{\overline{X'}}(x) - F') \supseteq \partial_{\overline{X \cup X'}}(x) - (F \cup F') \quad \text{for every } x \in X \cup X', \text{ and} \quad (13)$$

$$(\partial_{\overline{X}}(x) - F) \cup (\partial_{\overline{X'}}(x) - F') \supseteq \partial_{\overline{X \cap X'}}(x) - (F \cap F') \quad \text{for every } x \in X \cap X'. \quad (14)$$

By  $\mathcal{M}_2 = \bigoplus_{v \in V} \mathcal{M}_v$ , the monotonicity and the submodularity of  $r_2$ , (13) and (14), we get

$$\begin{aligned} & r_2(\partial(X) - F) + r_2(\partial(X') - F') \\ = & \sum_{x \in X} r_x(\partial_{\overline{X}}(x) - F) + \sum_{x \in X'} r_x(\partial_{\overline{X'}}(x) - F') \\ = & \sum_{x \in X - X'} r_x(\partial_{\overline{X}}(x) - F) + \sum_{x \in X' - X} r_x(\partial_{\overline{X'}}(x) - F') \\ & + \sum_{x \in X \cap X'} (r_x(\partial_{\overline{X}}(x) - F) + r_x(\partial_{\overline{X'}}(x) - F')) \\ \geq & \sum_{x \in (X - X') \cup (X' - X) \cup (X \cap X')} r_x(\partial_{\overline{X \cup X'}}(x) - (F \cup F')) + \sum_{x \in X \cap X'} r_x(\partial_{\overline{X \cap X'}}(x) - (F \cap F')) \\ = & r_2(\partial(X \cup X') - (F \cup F')) + r_2(\partial(X \cap X') - (F \cap F')). \end{aligned}$$

We get (11) by the above inequality and by the submodularity of  $r_1$ .

Note that, by  $e = uv \in (\partial(X) - F) - (\partial(X') - F')$ ,

$$(\partial_{\overline{X}}(v) - (F \cup e)) \cap ((\partial_{\overline{X'}}(v) - F') \cup e) = (\partial_{\overline{X}}(v) - F) \cap (\partial_{\overline{X'}}(v) - F'), \quad (15)$$

$$(\partial_{\overline{X}}(v) - (F \cup e)) \cup ((\partial_{\overline{X'}}(v) - F') \cup e) = (\partial_{\overline{X}}(v) - F) \cup (\partial_{\overline{X'}}(v) - F'). \quad (16)$$

By (15), (16) and the previous proof, (12) follows.  $\square$

Although  $p(X)$  is not supermodular in general, we will prove the supermodular inequality for specific pairs in the next lemma, following an idea from [14].

**Lemma 4.2.** *Let  $X$  and  $X'$  be two subsets of  $V$  and  $v \in X \cap X'$  such that  $X' \subseteq P(v)$ . Then*

$$p(X) + p(X') \leq p(X \cap X') + p(X \cup X'). \quad (17)$$

*Proof.* Since the reachability is transitive and  $v \in X \cap X'$ , we get  $P(X') \subseteq P(X \cap X')$  and hence  $\partial_s(P(X')) \subseteq \partial_s(P(X \cap X'))$ . Furthermore,  $P(X) \subseteq P(X \cup X')$  is obvious hence  $\partial_s(P(X)) \subseteq \partial_s(P(X \cup X'))$ . Thus, by the monotonicity of the rank function  $r_1$ , we get (17).  $\square$

## 5 The proof

Observe that the existence of an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences and that of  $s$ -one-arborescences are equivalent as an  $s$ -arborescence can be split into multiple  $s$ -one-arborescences. Hence, in the following proof, we will use  $s$ -one-arborescences.

**Necessity:** Let  $\{T_1, \dots, T_t\}$  be an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -one-arborescences in  $D$ . As each  $T_i$  is an  $s$ -one-arborescence,  $\partial(V) \cap A(T_i) = \partial(V) \cap A(T_i[s, v])$  for every  $v \in V(T_i)$ . For every vertex  $v \in V$ , let  $\mathbf{B}_v = \{e_i = \partial(V) \cap A(T_i), v \in V(T_i)\}$ . Let now  $X \subseteq V, F \subseteq \partial_s(X)$  and  $\mathbf{B} = \bigcup_{v \in X} \mathbf{B}_v$ . Since  $\text{Span}_{\mathcal{M}_1}$  is monotone, by (3) and definition of  $P(X)$ , we have  $\text{Span}_{\mathcal{M}_1}(\mathbf{B}) \supseteq \bigcup_{v \in X} \text{Span}_{\mathcal{M}_1}(\mathbf{B}_v) \supseteq \bigcup_{v \in X} \partial_s(P(v)) = \partial_s(P(X))$ . Then, since  $r_1$  is monotone, we have the following inequality (\*)  $r_1(\mathbf{B}) \geq r_1(\partial_s(P(X)))$ . For each  $e_i \in \mathbf{B} - F$ , there exists a vertex  $v \in X$  such that  $e_i \in \mathbf{B}_v$  and then since  $T_i$  is an  $s$ -arborescence and  $v \in V(T_i) \cap X$ , there exists  $a_i \in A(T_i) \cap (\partial(X) - F)$ . Since  $r_2$  is monotone,  $\{a_i : e_i \in \mathbf{B} - F\}$  is independent in  $\mathcal{M}_2$ , these arborescences are edge-disjoint,  $r_1$  is subcardinal and by (\*), we have  $r_2(\partial(X) - F) \geq r_2(\{a_i : e_i \in \mathbf{B} - F\}) = |\{a_i : e_i \in \mathbf{B} - F\}| = |\mathbf{B} - F| \geq |\mathbf{B}| - |F| \geq r_1(\mathbf{B}) - r_1(F) \geq r_1(\partial_s(P(X))) - r_1(F)$  that is, (9) is satisfied.

**Sufficiency:** We suppose that the theorem is not true. Let us take a counterexample  $(D, \mathcal{M}_1, \mathcal{M}_2)$  ((9) is satisfied but the desired packing of  $s$ -one-arborescences does not exist) that first minimizes the number of arcs in  $D$  and then the number of non-bridge edges in  $\mathcal{M}_2$ .

We say that a pair consisting of  $X \subseteq V$  and  $F \subseteq \partial_s(X)$  is **tight** if  $b(X, F) = p(X)$  and is **critical for an edge**  $e$  if  $(X, F)$  is tight and  $e$  is a bridge in  $\mathcal{M}_2|_{\partial(X)-F}$ .

**Case 1.** First suppose that there exists an edge  $e$  for which no critical pair exists. Then the following hold.

$$r_1(F) + r_2(\partial^{D-e}(X) - F) \geq r_1(\partial_s^{D-e}(P_{D-e}(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s^{D-e}(X), \quad (18)$$

$$r_1(\partial_s^{D-e}(P_{D-e}(w))) = r_1(\partial_s(P(w))) \text{ for every } w \in V. \quad (19)$$

*Proof.* First, suppose to the contrary that there exist  $X \subseteq V$  and  $F \subseteq \partial_r^{D-e}(X)$  that violates (18). Then, by (9), the subcardinality of  $r_2$ ,  $(X, F)$  violates (18) and the monotonicity of  $r_1$ , we have  $r_1(\partial_s(P(X))) \leq r_1(F) + r_2(\partial(X) - F) \leq r_1(F) + r_2(\partial^{D-e}(X) - F) + 1 \leq$

$r_1(\partial_s^{D-e}(P_{D-e}(X))) \leq r_1(\partial_s(P(X)))$ , so equality holds everywhere. Hence  $(X, F)$  is tight in  $D$  and  $e$  is a bridge in  $\mathcal{M}_2|_{\partial(X)-F}$ . Therefore,  $(X, F)$  is critical for  $e$ , a contradiction.

Now, suppose to the contrary that there exists  $w \in V$  that violates (19). Let  $F' = \partial_s^{D-e}(P_{D-e}(w))$ ,  $F = \partial_s(P(w))$  and  $X' = P_{D-e}(w)$ . By the definition of  $X'$  and  $F'$ ,  $\partial(X') - F' \subseteq \{e\}$ . Then, by  $w$  violates (19),  $F' \subseteq F$ ,  $P(w) \subseteq P(X')$ , the monotonicity of  $r_1$ , (9) and the subcardinality of  $r_2$ , we have  $r_1(F') + 1 \leq r_1(F) = r_1(\partial_s(P(w))) \leq r_1(\partial_s(P(X'))) \leq r_1(F') + r_2(\partial(X') - F') \leq r_1(F') + 1$ . Thus equality holds everywhere. Hence  $(X', F')$  is tight, and  $r_2(\partial(X') - F') = r_2(e) = 1$ , that is,  $e$  is a bridge in  $\mathcal{M}_2|_{\partial(X')-F'}$ . Therefore,  $(X', F')$  is critical for  $e$ , a contradiction.  $\square$

By (18),  $(D - e, \mathcal{M}_1 - e, \mathcal{M}_2 - e)$  satisfies the condition of the theorem. By  $|A(D - e)| < |A(D)|$ , it is not a counterexample, so there exists an  $(\mathcal{M}_1 - e)$ -reachability-based  $(\mathcal{M}_2 - e)$ -restricted packing  $T_1, \dots, T_t$  of  $s$ -one-arborescences in  $D - e$ , that is, for every  $v \in V$ ,  $\{A(T_i) \cap \partial^{D-e}(v) : v \in V(T_i)\}$  is independent in  $\mathcal{M}_v - e$  (and hence in  $\mathcal{M}_v$ ) and  $\{A(T_i) \cap \partial^{D-e}(V) : v \in V(T_i)\}$  is independent in  $\mathcal{M}_1 - e$  (and hence in  $\mathcal{M}_1$ ) of size  $r_1(\partial_r^{D-e}(P_{D-e}(v)))$  that is, by (19), of size  $r_1(\partial_s(P(v)))$ . Then  $T_1, \dots, T_t$  is an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -one-arborescences in  $D$ , and the proof is complete in this case.

**Case 2.** Suppose now that there exists a non-bridge edge  $e = uv$  in  $\mathcal{M}_2$ . Since we are not in Case 1, there exists a critical pair  $(X, F)$  for  $e$  such that  $X$  is minimal.

**Claim 5.1.**  $X \subseteq P(v)$ .

*Proof.* Let  $(X', F') = (P(v), \partial_s(P(v)))$ . By  $\partial(X') - F' = \emptyset$  and  $\partial_s(P(X')) = F'$ , we get  $r_2(\partial(X') - F') = r_2(\emptyset) = 0 = r_1(\partial_s(P(X'))) - r_1(F')$ , that is  $(X', F')$  is tight. By the tightness of  $(X, F)$  and  $(X', F')$ , Lemma 4.2, (9) and (11), we have  $b(X, F) + b(X', F') = p(X) + p(X') \leq p(X \cap X') + p(X \cup X') \leq b(X \cap X', F \cap F') + b(X \cup X', F \cup F') \leq b(X, F) + b(X', F')$ . Hence equality holds everywhere, in particular,  $(X \cap X', F \cap F')$  is tight. Note that, by  $X' = P(v)$  and  $uv \in \partial(X) - F$ ,  $e \in Y := \partial(X \cap X') - (F \cap F')$ . Suppose that  $e$  is not a bridge in  $\mathcal{M}_2|_Y$ . Then there exists an  $\mathcal{M}_2$ -base  $B'$  of  $Y$  not containing  $e$ . Since no edge exists from  $X - X'$  to  $X \cap X'$ ,  $B' \subseteq \partial(X) - F$  so there exists an  $\mathcal{M}_2$ -base  $B$  of  $\partial(X) - F$  containing  $B'$ . Since  $B'$  was an  $\mathcal{M}_2$ -base of  $Y$ ,  $e \notin B$ . Thus  $e$  is not a bridge in  $\mathcal{M}_2|_{\partial(X)-F}$ , which is a contradiction. So  $e$  is a bridge in  $\mathcal{M}_2|_{\partial(X \cap X')-(F \cap F')}$ , thus  $(X \cap X', F \cap F')$  is a critical pair for  $e$ . It follows, by the minimality of  $X$ , that  $X \subseteq X' = P(v)$ .  $\square$

Let  $\mathcal{M}'_2 = (\mathcal{M}_2/e) \oplus e$  (with rank function  $r'_2$ ), that is,  $\mathcal{M}'_2$  is obtained from  $\mathcal{M}_2$  by contracting  $e$  and adding back  $e$  as a bridge. Note that  $\mathcal{M}'_2$  will still be a direct sum of its submatroids on  $\partial(w)$  for  $w \in V$  as  $\mathcal{M}'_2 = \bigoplus_{w \in V-v} \mathcal{M}_w \oplus \mathcal{M}'_v$  where  $\mathcal{M}'_v = (\mathcal{M}_v/e) \oplus e$ . We show now that (9) with respect to  $\mathcal{M}'_2$  holds, that is,

$$b'(X', F') := r_1(F') + r'_2(\partial(X') - F') \geq r_1(\partial_s(P(X'))) \text{ for all } X' \subseteq V \text{ and } F' \subseteq \partial_s(X'). \quad (20)$$

*Proof.* Assume for a contradiction that there exists  $(X', F')$  that violates (20), that is,  $b'(X', F') \leq p(X') - 1$ . By the definition of contraction,  $r'_2(Y) = r_2(Y)$  if  $e \in Y$  and  $r_2(Y \cup e) - 1$  if  $e \notin Y$ . It follows, by (9) for  $(X', F')$  and the monotonicity of  $r_2$ , that  $p(X') \leq b(X', F') \leq r_1(F') + r_2((\partial(X') - F') \cup e) \leq b'(X', F') + 1$ . By adding the above two inequalities, we get that all these inequalities hold with equalities, so  $v \in X'$ ,  $e \notin \partial(X') - F'$  and  $r_1(F') + r_2((\partial(X') - F') \cup e) = p(X')$ . Since  $(X, F)$  is a critical pair for  $e$ ,  $e \in \partial(X) - F$ ,  $r_1(F) + r_2(\partial(X) - (F \cup e)) + 1 = b(X, F) = p(X)$  and, by Claim 5.1, we have  $X \subseteq P(v)$  hence the condition of Lemma 4.2 is satisfied. By the two equalities above, Lemma 4.2, (9) for the pairs  $(X \cap X', F \cap F')$  and  $(X \cup X', F \cup F')$  and (12), we get a contradiction.  $\square$



By (20),  $(D, \mathcal{M}_1, \mathcal{M}'_2)$  satisfies the condition of the theorem. Note that if  $f$  is a bridge in  $\mathcal{M}_2$ , then it will be a bridge in  $\mathcal{M}'_2$  also. Then the number of non-bridge edges in  $\mathcal{M}'_2$  is one less than in  $\mathcal{M}_2$ , hence  $(D, \mathcal{M}_1, \mathcal{M}'_2)$  is not a counterexample, so there exists an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}'_2$ -restricted packing  $T_1, \dots, T_t$  of  $s$ -one-arborescences in  $D$ , that is, for every  $v \in V$ ,  $Y = \{A(T_i) \cap \partial(v) : v \in V(T_i)\}$  is independent in  $\mathcal{M}'_v$  (and hence, by  $r_v(Y) \leq |Y| = r'_v(Y) \leq r_v(Y)$ , independent in  $\mathcal{M}_v$ ) and  $\{A(T_i) \cap \partial(V) : v \in V(T_i)\}$  is independent in  $\mathcal{M}_1$  of size  $r_1(\partial_s(P(v)))$ . Then, as the independent sets of  $\mathcal{M}'_2$  are also independent in  $\mathcal{M}_2$ ,  $T_1, \dots, T_t$  is an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -one-arborescences in  $D$ , and the proof is complete in this case.

**Case 3.** We may suppose finally that each edge is a bridge in  $\mathcal{M}_2$ , that is,  $\mathcal{M}_2$  is the free matroid. Note that in this case (9) implies (8) and hence we can conclude by Theorem 3.5.  $\square\square$

## 6 Algorithmic aspects

We show in this section how to derive from our proof a polynomial algorithm to find either a reachability-based matroid-restricted packing of  $s$ -one-arborescences or a pair  $(X, F)$  that violates (9).

First we show how to check in polynomial time whether (9) holds. We start with the following observation.

**Lemma 6.1.** *If there exists a pair that violates (9), then there also exists a pair  $(X^*, F^*)$  violating (9) and a vertex  $v$  such that  $v \in X^* \subseteq P(v)$ .*

The proof will be similar to the proof of Claim 5.1.

*Proof.* Let  $(X, F)$  be a pair that violates (9) such that  $X$  is maximal and  $F$  is also maximal with respect to  $X$ . Let  $v \in X$ . If  $X \subseteq P(v)$ , then  $(X^*, F^*) := (X, F)$  is as required. Otherwise, let  $(X', F') = (P(v), \partial_s(P(v)))$ . By  $\partial(X') - F' = \emptyset$  and  $\partial_s(P(X')) = F'$ , we get  $r_2(\partial(X') - F') = r_2(\emptyset) = 0 = r_1(\partial_s(P(X'))) - r_1(F')$ , that is  $(X', F')$  is tight. By (11), as  $(X, F)$  violates (9) and by the tightness  $(X', F')$ , and by Lemma 4.2, we have  $b(X \cap X', F \cap F') + b(X \cup X', F \cup F') \leq b(X, F) + b(X', F') < p(X) + p(X') \leq p(X \cap X') + p(X \cup X')$ . Hence  $(X \cap X', F \cap F')$  or  $(X \cup X', F \cup F')$  is violating (9). By the maximality of  $(X, F)$ , if  $P(v) \not\subseteq X$  or  $\partial_s(P(v)) \not\subseteq F$ , then  $(X \cup X', F \cup F')$  does not violate (9). Thus, in this case,  $(X^*, F^*) = (X \cap X', F \cap F')$  is violating (9), moreover, by the definition of  $X'$ ,  $v \in X^* \subseteq P(v)$  as required. Therefore, we find a violating pair as required except when  $P(v) \subseteq X$  and  $\partial_s(P(v)) \subseteq F$ . However, this cannot hold for all  $v \in X$  as then  $X = \bigcup_{v \in X} P(v) = P(X)$  and  $F = \partial_s(X) = \partial(X)$  hence (9) holds with equality, a contradiction.  $\square$

By Lemma 6.1, (9) holds if and only if for every  $v \in V$ , it holds for all pairs  $(X, F)$  with the addition property that  $v \in X \subseteq P(v)$ . Note that for a fixed vertex  $v$ , for all  $v \in X \subseteq P(v)$ ,  $P(X) = P(v)$ , so the right hand side of (9) is constant.

On the one hand, for a fixed set  $X \subseteq V$ ,  $r_1(F) + r_2(\partial(X) - F)$  for all  $F \subseteq \partial_s(X)$  is a submodular function, so by submodular function minimization one can determine in polynomial time, for all  $X \subseteq V$ ,  $q(X) = \min\{r_1(F) + r_2(\partial(X) - F) : F \subseteq \partial_s(X)\}$ . On the other hand, by Lemma 4.1,  $q(X)$  is submodular. Then, using again submodular function minimization, one can check in polynomial time whether for a fixed  $v \in V$ , for all  $v \in X \subseteq P(v)$ ,  $q(X) \geq r_1(\partial_s(P(v)))$ . We may hence conclude that we can check in polynomial time whether (9) holds.

It follows that (8) can also be checked in polynomial time. Then the proof of Theorem 3.5 in [14] provides a polynomial algorithm to find either a Reachability-based packing of  $s$ -one-arborescences or a set that violates (8).

Now we can explain our algorithm. We check first whether (9) holds. As mentioned above, in polynomial time, either we find a set that violates (9) and we stop or we know that (9) holds and we continue. If every edge is a bridge in  $\mathcal{M}_2$  then the problem reduces to the problem of reachability-based packing of  $s$ -one-arborescences and hence we are done by the above remark on the algorithm of [14]. If there exists a non-bridge edge in  $\mathcal{M}_2$ , then let us choose one, say  $e$ . Let us check if  $(D - e, \mathcal{M}_1 - e, \mathcal{M}_2 - e)$  satisfies (18) and (19). (18) is just (9) for the smaller graph, so we can do it. The second one is trivially polynomial to check. If both hold, then recursively we use our algorithm for  $(D - e, \mathcal{M}_1 - e, \mathcal{M}_2 - e)$  and the packing obtained will be a required packing for  $(D, \mathcal{M}_1, \mathcal{M}_2)$ . Otherwise,  $(D, \mathcal{M}_1, \mathcal{M}'_2)$ , where  $\mathcal{M}'_2$  is defined in Case 2 in the proof of Theorem 3.6, satisfies (20) and recursively we use our algorithm for  $(D, \mathcal{M}_1, \mathcal{M}'_2)$  and the packing obtained will be a required packing for  $(D, \mathcal{M}_1, \mathcal{M}_2)$ . Note that during the recursive execution of our algorithm either the number of edges decreases by one or the number of non-bridge edges in  $\mathcal{M}_2$  decreases by one, hence our algorithm stops in polynomial time.

The above argument shows that the following theorem holds.

**Theorem 6.2.** *Let  $D = (V + s, A)$  be a rooted digraph,  $\mathcal{M}_1 = (\partial(V), r_1)$  and  $\mathcal{M}_2 = (A, r_2)$  two matroids such that  $\mathcal{M}_2$  is the direct sum of the matroids  $\mathcal{M}_v = (\partial(v), r_v)$  for  $v \in V$ . There exists a polynomial algorithm to find either an  $\mathcal{M}_1$ -reachability-based  $\mathcal{M}_2$ -restricted packing of  $s$ -arborescences in  $D$  or a pair  $(X, F)$  that violates (9).  $\square$*

## 7 Concluding remarks

### 7.1 An extension for dypergraphs

A **dypergraph** is a directed hypergraph where each oriented hyperedge, called a **dyperedge**, has one head and multiple tails. An  **$s$ -hyperarborescence** is a dypergraph which can be trimmed to an  $s$ -arborescence, that is, each of its dyperedges can be substituted by one arc from one of its tails to its head such that the resulting digraph is an  $s$ -arborescence. [7] showed that all arborescence packing results can be simply generalized to dypergraphs. The idea is to substitute each dyperedge of the input dypergraph by a new vertex such that it is entered by multiple arcs from each of the tails of the dyperedge and it has only one outgoing arc, called a head arc, that has the same head as the dyperedge. By the same construction one can get a generalization of the result presented here, one only needs to add the free matroid on  $\partial(v)$  for each new vertex  $v$  and keep the original matroid  $\mathcal{M}_2$  on the head arcs.

### 7.2 An open question

We conclude this paper with some remarks on the weighted versions of the problems. Suppose that we are given a weight function on the set of arcs of a digraph. The weight of a packing of arborescences is the sum of the weights of the arcs of the arborescences in the packing. It is clear that one can find a packing of  $k$  spanning  $s$ -arborescences of minimum weight (if one exists) with the weighted matroid intersection algorithm [6]. Similarly, a matroid-restricted packing of spanning  $s$ -arborescences of minimum weight (if one exists) can be found with the weighted matroid intersection algorithm. The weighted version of the problem of matroid-based packing of  $s$ -arborescences was solved in [3] by the ellipsoid method [11] and submodular function

minimization [12, 15]. It is an open problem whether there exists a polynomial algorithm to solve the common generalization of these problems, that is to find a matroid-based matroid-restricted packing of  $s$ -arborescences of minimum weight.

Finally, we note that the weighted version of the problem of reachability-based packing of  $s$ -arborescences was solved in [1] by an abstract reformulation of the problem. Obviously, the problem of reachability-based matroid-restricted packing of  $s$ -arborescences of minimum weight also remains open.

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