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Viktória E. Kaszanitzky, Bernd Schulze, and Shin-ichi Tanigawa

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Abstract

In [8] Hendrickson proved that $(d+1)$-connectivity and redundant rigidity are necessary conditions for a generic (non-complete) bar-joint framework to be globally rigid in $\mathbb{R}^d$. Jackson and Jordán [9] confirmed that these conditions are also sufficient in $\mathbb{R}^2$, giving a combinatorial characterization of graphs whose generic realizations in $\mathbb{R}^d$ are globally rigid. In this paper, we establish analogues of these results for infinite periodic frameworks under fixed lattice representations. Our combinatorial characterization of globally rigid generic periodic frameworks in $\mathbb{R}^2$ in particular implies toroidal and cylindrical counterparts of the theorem by Jackson and Jordán.

1 Introduction

A bar-joint framework (or simply framework) in $\mathbb{R}^d$ is a pair (G,p), where $G = (V,E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map. We think of a framework as a straight line realization of G in $\mathbb{R}^d$ in which the length of an edge $uv \in E$ is given by the Euclidean distance between the points $p(u)$ and $p(v)$. A well-studied problem in discrete geometry is to determine the rigidity of frameworks. A framework (G,p) is called *(locally) rigid* if, loosely speaking, it cannot be deformed continuously into another non-congruent framework while maintaining the lengths of all edges. It is well-known that a generic framework (G,p) is rigid in $\mathbb{R}^d$ if and only if *every* generic realization of G in $\mathbb{R}^d$ is rigid [1, 6]. In view of this fact, a graph G is said to be

*Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom, and Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Magyar tudósok krt 2., Budapest, 1117, Hungary E-mail: viktoria@cs.elte.hu Supported by EPSRC First Grant EP/M013642/1 and by the Hungarian Scientific Research Fund (OTKA, grant number K109240).

**Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom E-mail: b.schulze@lancaster.ac.uk Supported by EPSRC First Grant EP/M013642/1.

***Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan, and Centrum Wiskunde & Informatica (CWI), Postbus 94079, 1090 GB Amsterdam, The Netherlands E-mail: tanigawa@kurims.kyoto-u.ac.jp Supported by JSPS Postdoctoral Fellowships for Research Abroad, JSPS Grant-in-Aid for Scientific Research(A)(25240004), and JSPS Grant-in-Aid for Young Scientists (B) (No. 15K15942).

rigid in $\mathbb{R}^d$ if some/any generic realization of G is rigid in $\mathbb{R}^d$. A classical theorem by Laman [16] says that G is rigid in $\mathbb{R}^2$ if and only if G contains a spanning subgraph H such that $|F| \leq 2|V(F)| - 3$ for every nonempty $F \subseteq E(H)$, where $V(F)$ denotes the set of vertices incident to F.

Another central property in rigidity theory is global rigidity. A framework (G, p) is called *globally rigid*, if every framework (G, q) in $\mathbb{R}^d$ with the same edge lengths as (G, p) has the same distances between all pairs of vertices as (G, p). Although deciding the global rigidity of a given framework is a difficult problem in general, the problem becomes tractable if we restrict attention to *generic* frameworks [7], i.e. frameworks with the property that the coordinates of all points $p(v), v \in V$, are algebraically independent over $\mathbb{Q}$.

In 1992 Hendrickson [8] established the following necessary condition for a generic framework in $\mathbb{R}^d$ to be globally rigid.

Theorem 1.1 (Hendrickson [8]). *If (G, p) is a generic globally rigid framework in $\mathbb{R}^d$, then G is a complete graph with at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$, where G is called redundantly rigid if $G - e$ is rigid for every $e \in E(G)$.*

Although the converse direction is false for $d \geq 3$ as pointed out by Connelly, it turned out to be true for $d \leq 2$.

Theorem 1.2 (Jackson and Jordán [9]). *Let (G, p) be a generic framework in $\mathbb{R}^2$. Then (G, p) is globally rigid if and only if G is a complete graph with at most 3 vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

Based on the theory of stress matrices by Connelly [2, 3], Gortler, Healy, and Thurston [7] gave an algebraic characterization of the global rigidity of generic frameworks. This in particular implies that all generic realizations of a given graph share the same global rigidity properties in $\mathbb{R}^d$, as in the case of local rigidity. However the problem of extending Theorem 1.2 to higher dimension remains unsolved.

Largely motivated by practical applications in crystallography, materials science and engineering, as well as by mathematical applications in areas such as sphere packings, the rigidity analysis of infinite periodic frameworks has seen an increased interest in recent years. A particularly relevant to our work is an extension of Laman's theorem to periodic frameworks with fixed-lattice representations by Ross [21]. In her rigidity model, a periodic framework can deform continuously under a fixed periodicity constraint (i.e., each orbit of points is fixed).

In this paper, we shall initiate the global rigidity counterpart of the rigidity theory of periodic frameworks. The global rigidity of periodic frameworks is considered at the same level as Ross's rigidity model [21, 22], and we shall extend Theorem 1.1 and Theorem 1.2 to periodic frameworks. Analogous to Theorem 1.1 there are two types of necessary conditions, a graph connectivity condition (Lemma 3.1) and a redundant rigidity condition (Lemma 3.7). Our main result (Theorem 4.2) is that these necessary conditions are also sufficient in $\mathbb{R}^2$, thus giving a first combinatorial characterization of the global rigidity of generic periodic frameworks. The proof of this result is inspired by the work in [10, 23]. In particular, it does not require the notion of stress matrices [2, 3]. Note also that our proof does not rely on periodic global rigidity in $\mathbb{R}^2$ being a generic property, meaning that all generic realizations of a periodic graph in the plane share the same global rigidity properties.

As for the rigidity of periodic frameworks, an extension of Laman's theorem was established in a more general setting by Malestein and Theran [17], where the underlying lattice may deform during a motion of a framework (but it is still subject to being periodic). Extending our result to this general setting would be an important challenging open problem.

Important corollaries of our main theorem are toroidal and cylindrical counterparts of Theorem 1.2. Here we only give a statement for cylindrical frameworks, but the statement for toroidal frameworks can be derived in a similar fashion (see Theorem 6.1). Consider a straight-line drawing of a graph G on a flat cylinder $\mathcal{C}$. We regard it as a bar-joint framework on $\mathcal{C}$. Using the metric inherited from its representation as $\mathbb{R}^2/L$ for a fixed one-dimensional lattice L, the local/global rigidity is defined. Ross's theorem [21] for periodic frameworks implies that a generic framework on $\mathcal{C}$ is rigid if and only if the underlying graph contains a spanning subgraph H such that $|F| \leq 2|V(F)| - 2$ for every $F \subseteq E(H)$ and $|F| \leq 2|V(F)| - 3$ for every nonempty *contractible* $F \subseteq E(H)$, where F is said to be contractible if every cycle in F is contractible on $\mathcal{C}$.

Theorem 1.3. *A generic framework (G, p) with $|V(G)| \geq 3$ on a flat cylinder $\mathcal{C}$ is globally rigid if and only if it is redundantly rigid on $\mathcal{C}$, 2-connected, and has no contractible subgraph H with $|V(H)| \geq 3$ and $|B(H)| = 2$, where $B(H)$ denotes the set of vertices in H incident to some edge not in H.*

The paper is organized as follows. In Section 2, we define the concept of global rigidity for periodic frameworks with a fixed lattice representation, and then establish Hendrickson-type necessary conditions for a generic periodic framework to be globally rigid in $\mathbb{R}^d$ in Section 3. In Section 4 we then show that for $d = 2$ the necessary conditions established in Section 3 are also sufficient for generic global rigidity (Theorem 4.2). Section 5 is devoted to the proofs of the combinatorial lemmas stated in Section 4.

2 Preliminaries

2.1 Periodic graphs

A simple graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is called *k-periodic* if $\mathrm{Aut}(\tilde{G})$ contains a subgroup Γ isomorphic to $\mathbb{Z}^k$ such that the quotient $\tilde{G}/\Gamma = (\tilde{V}/\Gamma, \tilde{E}/\Gamma)$ is finite. Γ is called a *periodicity* of $\tilde{G}$. A k-periodic graph can be represented succinctly by assigning an orientation and an element of Γ to each edge of $\tilde{G}/\Gamma$. To see this, choose a representative from each vertex orbit, and denote these representative vertices by $v_1, \dots, v_n$, where n denotes the number of vertex orbits. Then each edge orbit between Γv_i and Γv_j can be written as $\{\{\gamma v_i, \beta_{v_i v_j} \gamma v_j\} : \gamma \in \Gamma\}$ for a unique $\beta_{v_i v_j} \in \Gamma$. Hence, by directing the edge orbit from Γv_i to Γv_j in $\tilde{G}/\Gamma$ and then assigning the label $\beta_{v_i v_j}$, we have encoded enough information to recover $\tilde{G}$. The resulting directed (multi-)graph G with the group-labeling $\psi : E(G) \to \Gamma$ described above is called the *quotient Γ-labeled graph* of $\tilde{G}$, and is denoted by the pair (G, ψ). (See Figure 1 for two examples.) Although $V(G)$ denotes the set of vertex orbits, it is often convenient to identify $V(G)$ with the set $\{v_1, \ldots, v_n\}$ of the representative vertices, and throughout the paper we will follow this convention by assuming that a fixed representative vertex has been chosen from each vertex orbit.

Let Γ be a group isomorphic to $\mathbb{Z}^k$. In general, a pair (G, ψ) of a directed (multi-)graph G and a map $\psi : E(G) \to \Gamma$ is called a *Γ-labeled graph*. Throughout this paper we will assume that G is loopless. Although G is directed, its orientation is used only for the reference of the group-label, and we are free to change the orientation of each edge by imposing the property on that if an edge has a label γ in one direction, then it has γ^{-1} in the other direction. More precisely, two edges e_1, e_2 are regarded as *identical* if they are parallel with the same direction and the same label, or with the opposite direction and the opposite labels. Throughout the paper, all Γ-labeled graphs are assumed to be *semi-simple*, that is, no identical two edges exist (although parallel edges may exist).

For a given Γ-labeled graph (G, ψ), one can construct a k-periodic graph $\tilde{G}$ by setting $V(\tilde{G}) = \{\gamma v_i : v_i \in V(G), \gamma \in \Gamma\}$ and $E(\tilde{G}) = \{\{\gamma v_i, \psi(v_i v_j)\gamma v_j\} : (v_i, v_j) \in E(G), \gamma \in \Gamma\}$. This $\tilde{G}$ is called the *covering* of (G, ψ).

We define a *walk* as an alternating sequence $v_1, e_1, v_2 \ldots, e_k, v_{k+1}$ of vertices and edges such that v_i and v_{i+1} are the endvertices of e_i. For a closed walk $C = v_1, e_1, v_2 \ldots, e_k, v_1$ in (G, ψ), let $\psi(C) = \prod_{i=1}^{k} \psi(e_i)^{\mathrm{sign}(e_i)}$, where $\mathrm{sign}(e_i) = 1$ if e_i has forward direction in C, and $\mathrm{sign}(e_i) = -1$ otherwise. For a subgraph H of G define Γ_H as the subgroup of Γ generated by the elements $\psi(C)$, where C ranges over all closed walks in H. The *rank* of H is defined to be the rank of Γ_H. Note that the rank of G may be less than the rank of Γ, in which case the covering graph $\tilde{G}$ contains an infinite number of connected components (see Figure 1(a)).

We say that an edge set F is *balanced* if the subgraph induced by F has rank zero, i.e., $\psi(C) = \mathrm{id}$ for every closed walk C in F.

One useful tool to compute the rank of a subgraph is the *switching* operation. A *switching* at $v \in V(G)$ by $\gamma \in \Gamma$ changes ψ to ψ' defined by $\psi'(e) = \gamma\psi(e)$ if e is directed from v, $\psi'(e) = \gamma^{-1}\psi(e)$ if e is directed to v, and $\psi'(e) = \psi(e)$ otherwise.

2.2 Periodic frameworks

In the context of graph rigidity, a pair (G, p) of a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$ is called a *(bar-joint) framework* in $\mathbb{R}^d$. A periodic framework is a special type of infinite framework defined as follows.

Let $\tilde{G}$ be a k-periodic graph with periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be a nonsingular homomorphism with $k \leq d$, where L is said to be nonsingular if $L(\Gamma)$ has rank k. A pair $(\tilde{G}, \tilde{p})$ of $\tilde{G}$ and $\tilde{p} : \tilde{V} \to \mathbb{R}^d$ is said to be an *L-periodic framework* in $\mathbb{R}^d$ if

$$\tilde{p}(v) + L(\gamma) = \tilde{p}(\gamma v) \qquad \text{for all } \gamma \in \Gamma \text{ and all } v \in \tilde{V}. \tag{2.1}$$

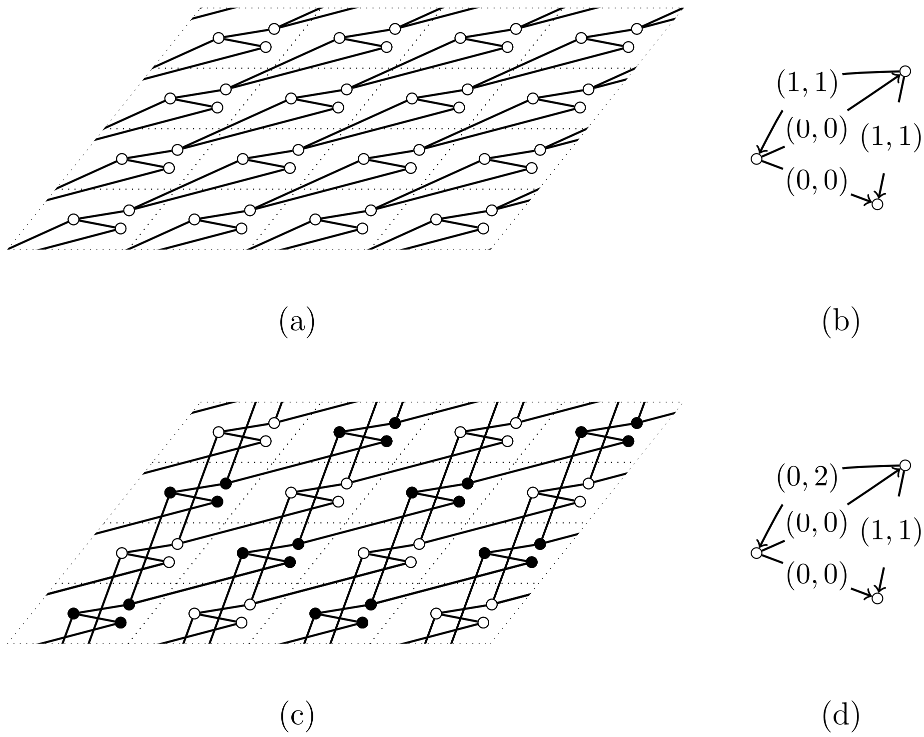

Figure 1: The framework in (a) has infinitely many connected components, and its quotient $\mathbb{Z}^2$-labeled graph shown in (b) is of rank one. The framework in (c) has two connected components, and its quotient $\mathbb{Z}^2$-labeled graph shown in (d) is of rank two. The vertices of the two components are depicted with two distinct colors in (c). We remark that neither of these frameworks is L-periodically globally rigid, by Lemmas 3.1 and 3.7, respectively.

We also say that a pair $(\tilde{G}, \tilde{p})$ is *k-periodic* in $\mathbb{R}^d$ if it is L-periodic for some nonsingular homomorphism $L : \Gamma \to \mathbb{R}^d$. Note that the rank k of the periodicity may be smaller than d. For instance, if $k = 1$ and $d = 2$, then $(\tilde{G}, \tilde{p})$ is an infinite strip framework in the plane. Any connected component of the framework in Figure 1(a), for example, is such a strip framework (with $\Gamma = \langle (1, 1) \rangle$).

An L-periodic framework $(\tilde{G}, \tilde{p})$ is *generic* if the set of coordinates is algebraically independent over the rationals modulo the ideal generated by the equations (2.1). For simplicity of description, throughout the paper, we shall assume that L is a rational-valued function[1], i.e., $L : \Gamma \to \mathbb{Q}^d$.

Let $(\tilde{G}, \tilde{p})$ be an L-periodic framework and let (G, ψ) be the quotient Γ-labeled graph of $\tilde{G}$. Following the convention that $V(G)$ is identified with the set $\{v_1, \dots, v_n\}$ of representative vertices, one can define the *quotient Γ-labeled framework* as the triple

[1]This assumption is not essential. Let $\gamma_1, \dots, \gamma_k$ be a set of generators of Γ and let H be the set of coordinates of the vectors $L(\gamma_i)$ for $1 \leq i \leq k$. Then one can apply the subsequent arguments even if L is not rational by replacing $\mathbb{Q}$ with the field extension $\mathbb{Q}(H)$. If L is not rational-valued, then we say that $(\tilde{G}, \tilde{p})$ is *generic* if the set of coordinates is algebraically independent over $\mathbb{Q}(H)$ modulo the ideal generated by the equations (2.1).

(G, ψ, p) with $p : V(G) \ni v_i \mapsto \tilde{p}(v_i) \in \mathbb{R}^d$. In general, a Γ-*labeled framework* is defined to be a triple (G, ψ, p) of a finite Γ-labeled graph (G, ψ) and a map $p : V(G) \to \mathbb{R}^d$. The *covering* of (G, ψ, p) is a k-periodic framework $(\tilde{G}, \tilde{p})$, where $\tilde{G}$ is the covering of G and $\tilde{p}$ is uniquely determined from p by (2.1).

We say that a Γ-labeled framework (G, ψ, p) is *generic* if the set of coordinates in p is algebraically independent over the rationals. Note that an L-periodic framework $(\tilde{G}, \tilde{p})$ is generic if and only if the quotient (G, ψ, p) of $(\tilde{G}, \tilde{p})$ is generic.

2.3 Rigidity and global rigidity

Let $G = (V, E)$ be a graph. Two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are said to be *equivalent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \qquad \text{for all } uv \in E.$$

They are *congruent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \qquad \text{for all } u, v \in V.$$

A framework (G, p) is called *globally rigid* if every framework (G, q) in $\mathbb{R}^d$ which is equivalent to (G, p) is also congruent to (G, p).

We may define the corresponding periodicity-constrained concept as follows. An L-periodic framework $(\tilde{G}, \tilde{p})$ in $\mathbb{R}^d$ is *L-periodically globally rigid* if every L-periodic framework in $\mathbb{R}^d$ which is equivalent to $(\tilde{G}, \tilde{p})$ is also congruent to $(\tilde{G}, \tilde{p})$. Note that if the rank of the periodicity is equal to zero, then L-periodic global rigidity coincides with the global rigidity of finite frameworks.

A key notion to analyze the L-periodic global rigidity is the L-periodic *rigidity*. By the periodicity constraint (2.1), the space of all L-periodic frameworks in $\mathbb{R}^d$ for a given graph $\tilde{G}$ can be identified with Euclidean space $\mathbb{R}^{dn}$ so that the topology is defined. A framework $(\tilde{G}, \tilde{p})$ is called *L-periodically rigid* if there is an open neighborhood N of $\tilde{p}$ in which every L-periodic framework $(\tilde{G}, \tilde{q})$ which is equivalent to $(\tilde{G}, \tilde{p})$ is also congruent to $(\tilde{G}, \tilde{p})$.

2.4 Characterizing L-periodic rigidity

A key tool to analyze the local or the global rigidity of finite frameworks is the length-squared function and its Jacobian, called the rigidity matrix. One can follow the same strategy to analyze local or global periodic rigidity.

For a Γ-labeled graph (G, ψ) and $L : \Gamma \to \mathbb{R}^d$, define $f_{G,L} : \mathbb{R}^{d|V(G)|} \to \mathbb{R}^{|E(G)|}$ by

$$f_{G,L}(p) = \left(\ldots, \|p(v_i) - (p(v_j) + L(\psi(v_i v_j)))\|^2, \ldots\right) \qquad (p \in \mathbb{R}^{d|V(G)|}).$$

For a finite set V, the *complete* Γ-*labeled graph* $K(V, \Gamma)$ on V is defined to be the graph on V with the edge set $\{(u, \gamma v) : u, v \in V, \gamma \in \Gamma\}$. We simply denote $f_{K(V,\Gamma),L}$ by $f_{V,L}$.

By (2.1) we have the following fundamental fact.

Proposition 2.1. *Let $(\tilde{G}, \tilde{p})$ be an L-periodic framework and let $(G = (V, E), \psi, p)$ be a quotient Γ-labeled framework of $(\tilde{G}, \tilde{p})$. Then $(\tilde{G}, \tilde{p})$ is L-periodically globally (resp. locally) rigid if and only if for every $q \in \mathbb{R}^{d|V|}$ (resp. for every q in an open neighborhood of p in $\mathbb{R}^{d|V|}$), $f_{G,L}(p) = f_{G,L}(q)$ implies $f_{V,L}(p) = f_{V,L}(q)$.*

In view of this proposition, we say that a Γ-labeled framework (G, ψ, p) is *L-periodically globally (or, locally) rigid* if for every $q \in \mathbb{R}^{d|V|}$ (resp. for every q in an open neighborhood of p in $\mathbb{R}^{d|V|}$), $f_{G,L}(p) = f_{G,L}(q)$ implies $f_{V,L}(p) = f_{V,L}(q)$, and we may focus on characterizing the L-periodic global (or, local) rigidity of Γ-labeled frameworks.

For $X \subset \mathbb{R}^d$, an isometry $h : \mathbb{R}^d \to \mathbb{R}^d$ of $\mathbb{R}^d$ is said to be *X-invariant* if $h(x)-h(0) = x$ for every $x \in X$. The following is the first fundamental fact for analyzing periodic rigidity.

Proposition 2.2. *Let V be a finite set, $p, q : V \to \mathbb{R}^d$ two maps, and let $L : \Gamma \to \mathbb{R}^d$ be nonsingular. Suppose that $p(V)$ affinely spans $\mathbb{R}^d$. Then $f_{V,L}(p) = f_{V,L}(q)$ holds if and only if q can be written as $q = h \circ p$ for some $L(\Gamma)$-invariant isometry h of $\mathbb{R}^d$.*

Proof. To see the necessity, assume that $f_{V,L}(p) = f_{V,L}(q)$. Then $\|p(u) - p(w)\| = \|q(u) - q(w)\|$ for every pair of elements $u, w \in V$. Since $p(V)$ affinely spans the whole space, this implies that there is a unique isometry $h : \mathbb{R}^d \to \mathbb{R}^d$ such that $q(u) = h(p(u))$ for every $u \in V$. In other words, there is an orthogonal matrix S such that $q(u) - q(v) = S(p(u) - p(v))$ for every $u, v \in V$. By $f_{V,L}(p) = f_{V,L}(q)$, we have $\|p(v) - (p(u) + L(\gamma))\| = \|q(v) - (q(u) + L(\gamma))\|$ for every $u, v \in V$ and every $\gamma \in \Gamma$, which implies $\langle (I-S)(p(u)-p(v)), L(\gamma)\rangle = 0$, and hence $\langle p(u)-p(v), (I-S^\top)L(\gamma)\rangle = 0$. Since $p(V)$ affinely spans the whole space, $(I - S^\top)L(\gamma) = 0$ for every γ. In other words, S fixes each element in $L(\Gamma)$ as required.

Conversely, suppose that q is represented as $q(v) = Sp(v) + t$ for some $t \in \mathbb{R}^d$ and some orthogonal matrix S that fixes each element in $L(\Gamma)$. Then we have $\|q(v) - (q(u)+L(\gamma))\| = \|S(p(v)-(p(u)+L(\gamma)))\| = \|p(v)-(p(u)+L(\gamma))\|$ for every $u, v \in V$ and every $\gamma \in \Gamma$, implying $f_{V,L}(q) = f_{V,L}(p)$.

The following algebraic characterization is the periodic version of one of the fundamental facts in rigidity theory, and is known even in the periodic case if $k = d$ [22].

Proposition 2.3. *Let (G, ψ, p) be a generic Γ-labeled framework in $\mathbb{R}^d$ with $|V(G)| \geq d + 1$ and rank k periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be nonsingular. Then (G, ψ, p) is L-periodically rigid if and only if*

$$\operatorname{rank} df_{G,L}|_p = d|V(G)| - d - \binom{d-k}{2},$$

where $df_{G,L}|_p$ denotes the Jacobian of $f_{G,L}$ at p.

Proof. Since the rank of Γ is k, the set of $L(\Gamma)$-invariant isometries forms a $(d + \binom{d-k}{2})$-dimensional manifold. Hence $\operatorname{rank} df_{V,L}|_p = d|V(G)| - d - \binom{d-k}{2}$. By the standard argument using the inverse function theorem, it follows that (G, ψ, p) is L-periodically rigid if and only if $\operatorname{rank} df_{V,L}|_p = \operatorname{rank} df_{G,L}|_p$, implying the statement.

For $d = 2$ Ross [21] gave a combinatorial characterization of the rank of $df_{G,L}|_p$ for generic (G, ψ, p), which implies the following. (Her statement is only for $k = 2$, but the proof can easily be adapted to the case when $k = 1$.)

Theorem 2.4 (Ross [21]). *Let (G, ψ, p) be a generic Γ-labeled framework in $\mathbb{R}^2$ with rank $k \geq 1$ periodicity Γ and let $L : \Gamma \to \mathbb{R}^2$ be nonsingular. Then (G, ψ, p) is L-periodically rigid if and only if (G, ψ) contains a spanning subgraph (H, ψ_H) satisfying the following count conditions:*

- $|E(H)| = 2|V(G)| - 2$;
- $|F| \leq 2|V(F)| - 3$ *for every nonempty balanced* $F \subseteq E(H)$;
- $|F| \leq 2|V(F)| - 2$ *for every nonempty* $F \subseteq E(H)$;

where $V(F)$ denotes the set of vertices incident to F.

Note that if $k = 0$, then an L-periodic framework is simply a finite framework, and generic rigid frameworks in $\mathbb{R}^2$ are characterized by the celebrated Laman theorem [16].

3 Necessary Conditions

In this section we provide necessary conditions for L-periodic global rigidity. As in the finite case, there are two types of conditions, a connectivity condition and a redundant rigidity condition. These two conditions are stated in Lemma 3.1 and Lemma 3.7, respectively.

3.1 Necessary connectivity conditions

Let (G, ψ) be a Γ-labeled graph with Γ having rank k. For a subgraph H of G, $B(H)$ is defined to be the set of vertices in H incident to some edge in $E(G) \setminus E(H)$, and we denote $I(H) = V(H) \setminus B(H)$. A subgraph H is said to be an (s,t)-*block* if the rank of H is s, $|B(H)| = t$, and $I(H) \neq \emptyset$.

Lemma 3.1. *Let (G, ψ, p) be a generic Γ-labeled framework in $\mathbb{R}^d$ with $|V(G)| \geq 2$, rank k periodicity Γ, and nonsingular $L : \Gamma \to \mathbb{R}^d$. If (G, ψ, p) is L-periodically globally rigid, then G contains no (s,t)-block H with $(s+1)t \leq d$ such that $V(H) \neq V(G)$ or $s < k$.*

Proof. We first remark that the following holds for any Γ-labeled graph (H, ψ_H) and any $v \in V(H)$:

$$\begin{aligned}&\text{If a hyperplane } \mathcal{H} \text{ of } \mathbb{R}^d \text{ contains } \{p(v) + L(\gamma) : \gamma \in \Gamma_H\}, \text{ then}\\ &f_{H,L}(g \circ p) = f_{H,L}(p) \text{ holds for the reflection } g : \mathbb{R}^d \to \mathbb{R}^d \text{ with respect to } \mathcal{H}.\end{aligned} \tag{3.1}$$

This follows from Proposition 2.2 by noting that the reflection g is $L(\Gamma_H)$-invariant.

Suppose that G has an (s,t)-block H satisfying the property of the statement. If $t = 0$ and $V(H) \neq V(G)$ (i.e., G is disconnected), then by translating $p(V(H))$ we obtain q with $f_{G,L}(p) = f_{G,L}(q)$ and $f_{V,L}(p) \neq f_{V,L}(q)$, contradicting the global rigidity of (G, ψ, p). Thus we have

$$t > 0 \text{ or } V(H) = V(G). \tag{3.2}$$

Define B' by $B' = B(H)$ if $t > 0$ and otherwise $B' = \{x\}$ by picking any vertex $x \in V(G)$, and consider the set of points $P = \{p(v) + L(\gamma) : v \in B', \gamma \in \Gamma_H\}$. We have the following:

$$\text{The affine span aff}P \text{ of } P \text{ is a proper subspace of } \mathbb{R}^d. \tag{3.3}$$

Indeed, if $t > 0$, then $B' = B(H)$ and aff$P$ has dimension $(s+1)t - 1$, which is less than d by the lemma assumption. On the other hand, if $t = 0$, then $B' = \{x\}$ and aff$P$ has dimension s, which is less than d by (3.2) and the lemma assumption. Thus (3.3) follows.

By (3.3) we can take a hyperplane $\mathcal{H}$ that contains aff$P$. Since p is generic, such a hyperplane can be taken such that $\mathcal{H}$ contains no point in $\{p(v) : v \in V(G) \setminus B'\}$. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection of $\mathbb{R}^d$ with respect to $\mathcal{H}$, and we define $q : V \to \mathbb{R}^d$ by $q(v) = g(p(v))$ if $v \in V(H)$, and $q(v) = p(v)$ otherwise.

We first show $f_{G,L}(p) = f_{G,L}(q)$. Take any edge $e = uv$ in G. If $e \notin E(H)$, then $p(u) = q(u)$ and $p(v) = q(v)$ (in particular, g is identity on $p(B(H))$), implying $\|p(u) - (p(v) + L(\psi(e)))\| = \|q(u) - (q(v) + L(\psi(e)))\|$. Otherwise, (3.1) implies $\|p(u) - (p(v) + L(\psi(e)))\| = \|q(u) - (q(v) + L(\psi(e)))\|$. Thus $f_{G,L}(p) = f_{G,L}(q)$ follows.

To derive a contradiction we shall show $f_{V,L}(p) \neq f_{V,L}(q)$ by splitting the proof into two cases depending on whether $V(H) = V(G)$ or not.

Suppose that $V(G) \neq V(H)$. By the definition of an (s,t)-block, we can take a $v \in I(H)$. Then $f_{V,L}(p) \neq f_{V,L}(q)$ holds since $\|p(v) - p(u)\| \neq \|q(v) - q(u)\|$ for any $u \in V(G) \setminus V(H)$.

Suppose that $V(H) = V(G)$. Let $x \in B'$ and $u \in I(H) \setminus B'$. (As x was chosen arbitrary from $V(G)$, we may suppose $I(H) \setminus B' \neq \emptyset$.) By the lemma assumption, $s < k$ holds, and hence we can take $\gamma^* \in \Gamma$ that is not spanned by Γ_H. As γ^* is not spanned by Γ_H, we could take the above hyperplane $\mathcal{H}$ such that $p(x) - L(\gamma^*)$ is outside of $\mathcal{H}$. Note that $p(u) \neq q(u)$ as $p(u) \notin \mathcal{H}$ (by $u \in I(H) \setminus B'$). Hence the set of points equidistant from $p(v)$ and $q(v)$ is $\mathcal{H}$, which in turn implies that the set of points equidistant from $p(v) + L(\gamma^*)$ and $q(v) + L(\gamma^*)$ is $\mathcal{H} + L(\gamma^*)$. Therefore, as $p(x) \notin \mathcal{H} + L(\gamma^*)$ but $p(x) \in \mathcal{H}$, $\|p(x) - (p(u) + L(\gamma^*))\| \neq \|p(x) - (q(u) + L(\gamma^*))\| = \|q(x) - (q(u) + L(\gamma^*))\|$. This implies $f_{V,L}(p) \neq f_{V,L}(q)$.

Consider, for example, the framework shown in Figure 2(a) and its quotient $\mathbb{Z}^2$-labeled graph (G, ψ) shown in (b). The subgraph H of (G, ψ) induced by the dashed edges is a $(1,1)$-block with $V(H) \neq V(G)$. Thus, by Lemma 3.1, the framework in (a) is not globally L-periodically rigid. Here aff(P) is one of the thin black lines in (a) connecting the copies of the black vertices, and g is the reflection in aff(P). The framework $(\tilde{G}, \tilde{q})$ is obtained from $(\tilde{G}, \tilde{p})$ by reflecting each connected component of the

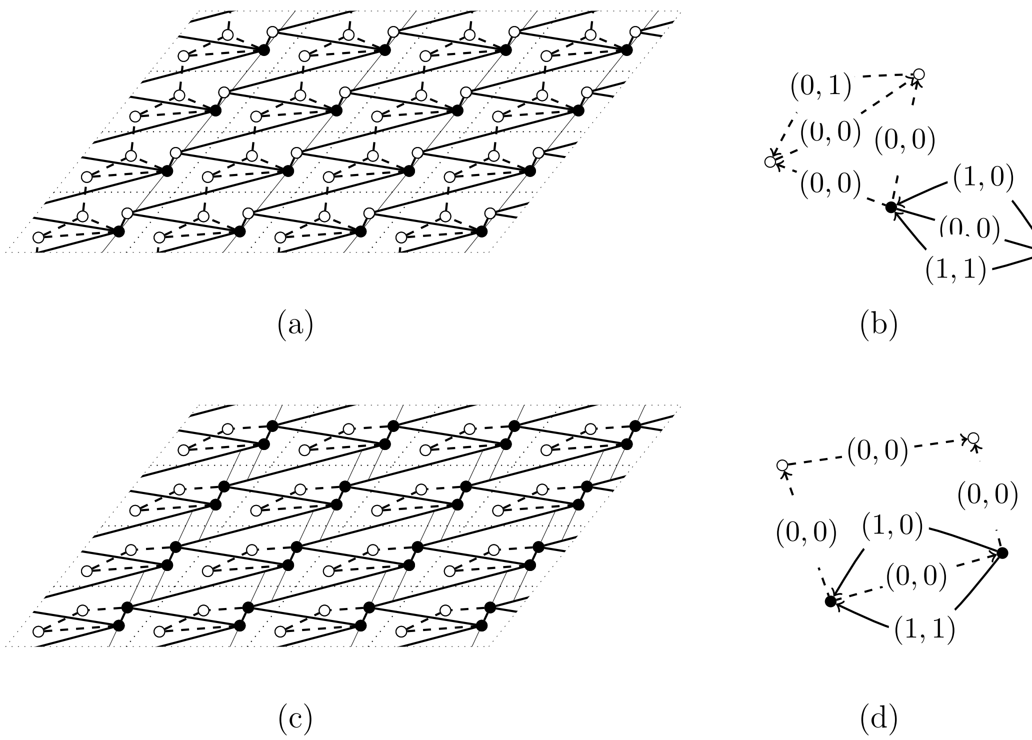

Figure 2: Examples of frameworks which are not globally L-periodically rigid, illustrating the proof of Lemma 3.1 for a rank 2 group $\Gamma = \mathbb{Z}^2$.

framework with dashed edges in the corresponding parallel copy of $\text{aff}(P)$ containing the black vertices of the component.

Figure 2(c) shows another example of a framework which is not globally L-periodically rigid by Lemma 3.1. Consider the corresponding quotient $\mathbb{Z}^2$-labeled graph (G, ψ) shown in Figure 2(d). The subgraph H induced by the dashed edges is a $(0, 2)$-block of G with $V(H) = V(G)$ but $0 = s < k = 2$. Here $\text{aff}(P)$ is one of the lines in (c) indicated by thin black line segments connecting pairs of black vertices, and g is again the reflection in $\text{aff}(P)$. The framework $(\tilde{G}, \tilde{q})$ is obtained from $(\tilde{G}, \tilde{p})$ as described in the previous case.

Sometimes Lemma 3.1 can be strengthened by decomposing graphs. Consider for example the case $d = 2$. Suppose that (G_1, ψ_1, p_1) is redundantly L-periodically rigid but contains a $(1, 1)$-block H. Then by Lemma 3.1 (G, ψ, p) is not L-periodically globally rigid. Now consider attaching a new L-periodically globally rigid framework (G_2, ψ_2, p_2) at a vertex in $I(H)$ with $|V(G_1) \cap V(G_2)| = 1$. Then the resulting framework is clearly not L-periodically globally rigid but H is no longer a $(1, 1)$-block and the resulting graph may satisfy the cut condition (and may also be redundantly L-periodically rigid). In general, if a framework has a cut vertex, then we should look at each 2-connected component individually based on the following fact.

Lemma 3.2. *Let (G, ψ, p) be a Γ-labeled framework with rank d periodicity Γ, and suppose that it can be decomposed into two frameworks (G_i, ψ_i, p_i) $(i = 1, 2)$ with*

$|V(G_1) \cap V(G_2)| = 1$. Then (G, ψ, p) is L-periodically globally rigid if and only if each (G_i, ψ_i, p_i) is L-periodically globally rigid.

Proof. Note that, if the underlying periodicity group Γ has rank d, then every $L(\Gamma)$-invariant isometry is a translation. Hence the claim follows from Proposition 2.2.

A similar statement to Lemma 3.2 holds if we assume that the intersection of the two frameworks forms an L-periodically globally rigid subframework. Extending it to a more general gluing scenario (and sharpening the necessary condition for global periodic rigidity) is left as an open problem.

3.2 The necessity of redundant L-periodic rigidity

Let X be a smooth manifold and $f : X \to \mathbb{R}^m$ be a smooth map. Then $x \in X$ is said to be a *regular point* of f if the Jacobian $df|_x$ has maximum rank, and is a *critical point* of f otherwise. Also $f(x)$ is said to be a *regular value* of f if, for all $y \in f^{-1}(f(x))$, y is a regular point of f. Otherwise $f(x)$ is called a *critical value* of f.

We use the following lemmas.

Lemma 3.3. *(See, e.g., [10]) Let $f : \mathbb{R}^d \to \mathbb{R}^k$ be a polynomial map with rational coefficients and p be a generic point in $\mathbb{R}^d$. If $df|_p$ is row-independent, then $f(p)$ is generic in $\mathbb{R}^k$.*

Lemma 3.4. *(See, e.g., [7]) Let $f : \mathbb{R}^d \to \mathbb{R}^k$ be a polynomial map with rational coefficients and p be a generic point in $\mathbb{R}^d$. Then $f(p)$ is a regular value of f.*

For a vector p in $\mathbb{R}^d$, let $\mathbb{Q}(p)$ be the field generated by the entries of p and the rationals. For a field F and an extension K, let $td[K : F]$ denote the transcendence degree of the extension. For a field K, let $\overline{K}$ be the algebraic closure of K. We also need the following lemma which will be used in the proof of Lemma 4.5. (See, e.g., [11, Proposition 13] for the proof.)

Lemma 3.5. *Let $f : \mathbb{R}^d \to \mathbb{R}^k$ be a polynomial map with rational coefficients and p be a generic point in $\mathbb{R}^d$. Suppose that $df|_p$ is nonsingular. Then for every $q \in f^{-1}(p)$ we have $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$.*

We now return to our discussion of L-periodically globally rigid frameworks. Let Γ be a group isomorphic to $\mathbb{Z}^k$, $t = \max\{d-k, 1\}$, (G, ψ) be a Γ-labeled graph with $|V| \geq t$, and $L : \Gamma \to \mathbb{R}^d$ be nonsingular. We pick any t vertices $v_1, \dots, v_t$, and define the augmented function of $f_{G,L}$ by $\hat{f}_{G,L} := (f_{G,L}, g)$, where $g : \mathbb{R}^{d|V|} \to \mathbb{R}^{d+\binom{d-k}{2}}$ is a rational polynomial map given by

$$g(p) = (p_1(v_1), \dots, p_d(v_1), p_1(v_2), \dots, p_{d-1}(v_2), \dots, p_1(v_t), \dots, p_{d-t+1}(v_t)) \qquad (p \in \mathbb{R}^{d|V|})$$

with $p_i(v_j)$ denoting the i-th coordinate of $p(v_j)$. Augmenting $f_{G,L}$ by appending g corresponds to "pinning down" some coordinates to eliminate trivial continuous motions. The following claim is implicit in the proof of Proposition 2.3.

Proposition 3.6. *Let (G, ψ, p) be a Γ-labeled framework in $\mathbb{R}^d$ with rank k periodicity and $L : \Gamma \to \mathbb{R}^d$. Suppose that p is generic and $|V(G)| \geq \max\{d-k, 1\}$. Then*

$$\operatorname{rank} d\hat{f}_{G,L}|_p = \operatorname{rank} df_{G,L}|_p + d + \binom{d-k}{2}.$$

We say that (G, ψ, p) is *redundantly L-periodically rigid* if $(G-e, \psi, p)$ is L-periodically rigid for every $e \in E(G)$. (For simplicity, we slightly abuse notation here and denote the restriction of ψ to $E(G) - e$ also by ψ.)

Lemma 3.7. *Let $(G = (V, E), \psi, p)$ be a generic Γ-labeled framework in $\mathbb{R}^d$ with rank k periodicity Γ and nonsingular $L : \Gamma \to \mathbb{R}^d$. Suppose also that $|V| \geq d+1$ if $k \geq 1$ and $|V| \geq d+2$ if $k = 0$. If (G, ψ, p) is L-periodically globally rigid, then (G, ψ, p) is redundantly L-periodically rigid.*

Proof. The proof idea is from [13]. Suppose for a contradiction that $(G-e, \psi, p)$ is not L-periodically rigid for some $e \in E$. Since (G, ψ, p) is L-periodically globally rigid, it is L-periodically rigid. Hence by Proposition 2.3 and Proposition 3.6 we have $\operatorname{rank} d\hat{f}_{G-e,L}|_p = d|V| - 1$. Since p is generic, Lemma 3.3 implies that

$$td[\mathbb{Q}(\hat{f}_{G-e,L}(p)) : \mathbb{Q}] \geq d|V| - 1. \tag{3.4}$$

$\hat{f}_{G-e,L}(p)$ is a regular value of $\hat{f}_{G-e,L}$ by Lemma 3.4. Hence its preimage is a 1-dimensional smooth manifold (see, e.g., [20]). Since this manifold is bounded and closed, it is compact, and it consists of a disjoint union of cycles by the classification of 1-dimensional manifolds. Let $\mathcal{O}$ be the component that contains p.

Consider $f_{e,L} : \mathbb{R}^{d|V|} \to \mathbb{R}$ which returns $\|p(u) - (p(v) + L(\psi(e))\|^2$ for the edge $e = uv$. Since $\hat{f}_{G-e,L}(p)$ is a regular value, we have $\operatorname{rank} df_{e,L}|_p = \operatorname{rank} d\hat{f}_{G,L}|_p - \operatorname{rank} d\hat{f}_{G-e,L}|_p = 1$ (see, e.g., [12, Lemma 3.4] for the proof of the first equation). Hence $df_{e,L}|_p$ is nonzero, and the intermediate value theorem implies that there is a $q \in \mathcal{O}$ with $f_{e,L}(q) = f_{e,L}(p)$ and $q \neq p$. We can assign an orientation to $\mathcal{O}$ and we may assume that q is chosen as close to p as possible in the forward direction.

$f_{e,L}(p) = f_{e,L}(q)$ implies that $\hat{f}_{G,L}(p) = \hat{f}_{G,L}(q)$. This implies $\hat{f}_{V,L}(p) = \hat{f}_{V,L}(q)$ since (G, p) is L-periodically globally rigid. By Proposition 2.2, q can be written as $q = h \circ p$ for some $L(\Gamma)$-invariant isometry h. Since $p(v_1) = q(v_1)$, there is an orthogonal matrix S such that $q(u) = Sp(u) + (I - S)p(v_1)$ for every u and S fixes each element in $L(\Gamma)$.

Take a path $\gamma : [0,1] \ni t \mapsto p_t \in \mathcal{O}$ with $\gamma(0) = p$ and $\gamma(1) = q$, and define a path $\gamma' : [0,1] \ni t \mapsto h \circ p_t \in \mathbb{R}^{d|V|}$. Since h fixes each element in $L(\Gamma)$, we have $f_{G,L}(p) = f_{G,L}(p_t) = f_{G,L}(h \circ p_t)$ for every $t \in [0,1]$. In other words, γ' is a path in $\mathcal{O}$.

If γ and γ' cover $\mathcal{O}$ then we can assume that f_e increases as we pass through p in the forward direction. Then f_e has to increase as we pass through q. Thus there are values t_1, t_2 with $0 < t_1 < t_2 < 1$ and $f_e(p_{t_2}) < f_e(p_1) = f_e(q) = f_e(p) = f_e(p_0) < f_e(p_{t_1})$. Using the intermediate value theorem, we then get a contradiction, because there exists a point p' with $f_e(p') = f_e(p)$ between p_0 and p_1.

If γ and γ' do not cover $\mathcal{O}$, then there exists a $t \in [0,1]$ such that $\gamma(t) = \gamma'(t)$. At this t, we have $p_t(u) = h(p_t(u))$ for every $u \in V$. In other words, $p_t(V)$ is contained in the invariant subspace H of h, which is a proper affine subspace of $\mathbb{R}^d$ as $p \neq q$. Let $d'(< d)$ be the affine dimension of H. Since H contains $L(\Gamma)$ whose basis is rational, H is determined by $(d'+1)d$ parameters, at most $(d'+1-k)d$ of which are independent over $\mathbb{Q}$. Thus we get $td[\mathbb{Q}(p_t):\mathbb{Q}] \leq (d'+1-k)d + (|V|-(d'+1))d'$. On the other hand, since $\mathbb{Q}(\hat{f}_{G-e,L}(p)) = \mathbb{Q}(\hat{f}_{G-e,L}(p_t)) \subseteq \mathbb{Q}(p_t)$, we also have $td[\mathbb{Q}(p_t):\mathbb{Q}] \geq td[\mathbb{Q}(\hat{f}_{G-e,L}(p)):\mathbb{Q}] \geq d|V|-1$ by (3.4). Thus $|V| \leq 1 + d' + \frac{1-kd}{d-d'} \leq d + \frac{1-kd}{d-d'}$. The last term is at most d if $k \geq 1$ and at most $d+1$ if $k=0$, which contradicts the assumption of the statement.

4 Characterizing Periodic Global Rigidity

In this section we characterize periodic global rigidity in the plane based on the necessary conditions given in Section 3. We need to introduce one more term to describe the main theorem combinatorially. Given a Γ-labeled graph (G,ψ), Proposition 2.3 implies that (G,ψ,p) is L-periodically rigid for some generic p if and only if (G,ψ,p) is L-periodically rigid for every generic p. Moreover, Theorem 2.4 says that the choice of L is not important as long as L is nonsingular. In view of these facts, we say that (G,ψ) is *periodically rigid* in $\mathbb{R}^d$ if (G,ψ,p) is L-periodically rigid for some (any) generic $p: V(G) \to \mathbb{R}^d$ and for some (any) nonsingular $L:\Gamma \to \mathbb{R}^d$.

We are now ready to state our main theorems.

Theorem 4.1. *Let (G,ψ,p) be a generic Γ-labeled framework in $\mathbb{R}^2$ with rank one periodicity Γ and $|V(G)| \geq 3$, and let $L:\Gamma \to \mathbb{R}^2$ be nonsingular. Then (G,ψ,p) is L-periodically globally rigid if and only if (G,ψ) is redundantly periodically rigid in $\mathbb{R}^2$, 2-connected, and has no $(0,2)$-block.*

Theorem 4.2. *Let (G,ψ,p) be a generic Γ-labeled framework in $\mathbb{R}^2$ with rank two periodicity Γ and $|V(G)| \geq 3$, and let $L:\Gamma \to \mathbb{R}^2$ be nonsingular. Then (G,ψ,p) is L-periodically globally rigid if and only if each 2-connected component (G',ψ') of (G,ψ) is redundantly periodically rigid in $\mathbb{R}^2$, has no $(0,2)$-block, and has rank two.*

Combining Theorem 4.1 and Theorem 4.2 with Proposition 2.1, we have the main theorem in this paper.

Theorem 4.3. *A generic L-periodic framework $(\tilde{G},\tilde{p})$ with at least three vertex orbits is L-periodically globally rigid in the plane if and only if its quotient Γ-labeled graph (G,ψ) satisfies the combinatorial condition in Theorem 1.2, Theorem 4.1 or Theorem 4.2 depending on the rank of the periodicity.*

For smaller frameworks we have the following.

Lemma 4.4. *Let (G,ψ,p) be a generic Γ-labeled framework in $\mathbb{R}^2$ with rank k periodicity Γ, and let $L:\Gamma \to \mathbb{R}^2$ be nonsingular. If $|V(G)|=1$, then (G,ψ,p) is L-periodically globally rigid. If $|V(G)|=2$ and $k \geq 1$, then (G,ψ,p) is L-periodically globally rigid if and only if the rank of Γ_G is k.*

Proof. Let $V(G) = \{u, v\}$, and suppose that every edge is oriented to v.

If $k = 1$, then there exist two edges from u to v with distinct labels. By switching, we may assume that these labels are id and γ. Given $q(v)$, there are only two possible positions for $q(u)$ if (G, ψ, p) is equivalent to (G, ψ, p) . In both cases, the resulting framework is congruent to (G, ψ, p), and hence (G, ψ, p) is L-periodically globally rigid.

If $k = 2$, then there exists a third edge with label γ' not spanned by γ. The given distance between $q(u)$ and $\gamma' q(v)$ then uniquely determines the position of $q(u)$. Thus, (G, ψ, q) is again congruent to (G, ψ, p), and hence (G, ψ, p) is L-periodically globally rigid.

We remark here that there is an efficient algorithm to check whether (G, ψ) satisfies the combinatorial conditions of the main theorem. After finding the 2-connected components of G the method given in [15] can be used here as well to check their redundant periodic rigidity. Their rank can also be checked easily by finding an equivalent gain function by switchings (see, e.g., [15]). Finally, for each pair of vertices we can check whether they are the boundary of a (0,2)-block.

The proofs of Theorem 4.1 and Theorem 4.2 are almost identical and consist of two parts, an algebraic part and a combinatorial part. The algebraic part is solved in Lemma 4.5, and the combinatorial part is solved in Lemmas 4.6 and 4.7.

Let (G, ψ) be a Γ-labeled graph and let v be a vertex of G. We say that v is *nondegenerate* if for every neighbor u of v, the set $\{L(\psi(e)) : e = vu \in E(G)\}$ is affinely independent. Suppose that every edge incident to v is directed from v. For each pair of nonparallel edges $e_1 = vu$ and $e_2 = vw$ in (G, ψ), let $e_1 \cdot e_2$ be the edge from u to v with label $\psi(vu)^{-1}\psi(vw)$. We define (G_v, ψ_v) to be the Γ-labeled graph obtained from (G, ψ) by removing v and inserting $e_1 \cdot e_2$ for every pair of nonparallel edges e_1, e_2 incident to v (unless an edge identical to $e_1 \cdot e_2$ is already present in (G, ψ)). The following is the periodic generalization of an observation given in [10, 23].

Lemma 4.5. *Let (G, ψ, p) be a generic Γ-labeled framework with $|V(G)| \geq d + 1$ and $L : \Gamma \to \mathbb{R}^d$ be nonsingular. Suppose that G has a nondegenerate vertex v of degree $d + 1$ for which*

- *$(G - v, \psi)$ is L-periodically rigid in $\mathbb{R}^d$, and*
- *(G_v, ψ_v, p') is L-periodically globally rigid in $\mathbb{R}^d$ with notation $p' = p|_{V(G)-v}$.*

Then (G, ψ, p) is L-periodically globally rigid in $\mathbb{R}^d$.

Proof. Pin the framework (G, ψ, p) (as done in Section 3.2) and take any $q \in \hat{f}_{G,L}^{-1}(\hat{f}_{G,L}(p))$. Since $|V(G)| \geq d + 1$, we may assume that v is not "pinned" (i.e., v is different from the vertices selected when augmenting $f_{G,L}$ to $\hat{f}_{G,L}$). Our goal is to show that $p = q$.

Let p' and q' be the restrictions of p and q to $V(G) - v$, respectively. Since $(G - v, \psi, p')$ is L-periodically rigid, $d\hat{f}_{G-v,L}|_{p'}$ is nonsingular by Proposition 2.3 and Proposition 3.6. Hence it follows from Lemma 3.5 that $\overline{\mathbb{Q}(p')} = \overline{\mathbb{Q}(q')}$. This in turn implies that q' is generic.

We may assume that all the edges incident to v are directed from v. Let $e_0 = vv_0, e_1 = vv_1, \dots, e_d = vv_d$ denote the edges incident to v, where $v_i = v_j$ may hold. By switching we may assume $\psi(vv_0) = \mathrm{id}$. For each $1 \le i \le d$, let

$$\begin{aligned} \gamma_i &= \psi(vv_i), \\ x_i &= p(v_i) + L(\gamma_i) - p(v_0), \\ y_i &= q(v_i) + L(\gamma_i) - q(v_0), \end{aligned}$$

and let P and Q be the $d \times d$-matrices whose i-th row is x_i and y_i, respectively. Note that since v is nondegenerate and p', q' are generic, $x_1, \dots, x_d$ and $y_1, \dots, y_d$ are, respectively, linearly independent, and hence P and Q are both nonsingular.

Let $x_v = p(v) - p(v_0)$ and $y_v = q(v) - q(v_0)$. We then have $\|x_v\| = \|y_v\|$ since G has the edge vv_0 with $\psi(vv_0) = \mathrm{id}$. Due to the existence of the edge e_i we also have

$$\begin{aligned} 0 &= \langle p(v_i) + L(\gamma_i) - p(v), p(v_i) + L(\gamma_i) - p(v)\rangle - \langle q(v_i) + L(\gamma_i) - q(v), q(v_i) + L(\gamma_i) - q(v)\rangle \\ &= \langle x_i - x_v, x_i - x_v\rangle - \langle y_i - y_v, y_i - y_v\rangle \\ &= (\|x_i\|^2 - \|y_i\|^2) - 2\langle x_i, x_v\rangle + 2\langle y_i, y_v\rangle, \end{aligned}$$

where we used $\|x_v\| = \|y_v\|$. Denoting by δ the d-dimensional vector whose i-th coordinate is equal to $\|x_i\|^2 - \|y_i\|^2$, the above d equations can be summarized as

$$0 = \delta - 2P^T x_v + 2Q^T y_v$$

which is equivalent to

$$y_v = (Q^T)^{-1} P^T x_v - \frac{1}{2}(Q^T)^{-1}\delta.$$

By putting this into $\|x_v\|^2 = \|y_v\|^2$, we obtain

$$x_v^T(I_d - PQ^{-1}(PQ^{-1})^T)x_v - (\delta^T Q^{-1}(Q^{-1})^T P^T)x_v + \frac{1}{4}\delta^T Q^{-1}(Q^{-1})^T\delta = 0, \qquad (4.1)$$

where I_d denotes the $d \times d$ identity matrix.

Note that each entry of P is contained in $\overline{\mathbb{Q}(p')}$, and each entry of Q is contained in $\overline{\mathbb{Q}(q')}$. Since $\overline{\mathbb{Q}(p')} = \overline{\mathbb{Q}(q')}$, this implies that each entry of PQ^{-1} is contained in $\overline{\mathbb{Q}(p')}$. On the other hand, since p is generic, the set of coordinates of $p(v)$ (and hence those of x_v) is algebraically independent over $\overline{\mathbb{Q}(p')}$. Therefore, by regarding the left-hand side of (4.1) as a polynomial in x_v, the polynomial must be identically zero. In particular, we get

$$I_d - PQ^{-1}(PQ^{-1})^T = 0.$$

Thus, PQ^{-1} is orthogonal. In other words, there is some orthogonal matrix S such that $P = SQ$, and we get $\|p(v_i) + L(\gamma_i) - p(v_0)\| = \|x_i\| = \|Sy_i\| = \|y_i\| = \|q(v_i) + L(\gamma_i) - q(v_0)\|$ for every $1 \le i \le d$. Therefore, $q' \in f_{G_v,L}^{-1}(f_{G_v,L}(p'))$. Since (G_v, ψ_v, p) is L-periodically globally rigid, this in turn implies that $f_{V-v,L}(p') = f_{V-v,L}(q')$. Thus we have $p' = q'$.

Since v is nondegenerate and p is generic, $\{p(v_i) + L(\gamma_i) : 0 \le i \le d\}$ affinely spans $\mathbb{R}^d$. Hence there is a unique extension of $p' : V - v \to \mathbb{R}^d$ to $r : V \to \mathbb{R}^d$ such that $f_{G,L}(r) = f_{G,L}(p)$. Thus we obtain $p = q$.

The combinatorial part consists of the following two lemmas whose proof will be given in the next sections separately.

Lemma 4.6. *Let (G, ψ) be a Γ-labeled graph with rank $k \geq 1$ periodicity Γ. Suppose that (G, ψ) is 2-connected, redundantly periodically rigid in $\mathbb{R}^2$, and has no $(0, 2)$-block. Then at least one of the following holds:*

(i) There exists $e \in E(G)$ such that the Γ-labeled graph $(G - e, \psi)$ is 2-connected, redundantly periodically rigid, and has no $(0, 2)$-block.

(ii) G has a vertex of degree three.

Lemma 4.7. *Let (G, ψ) be a Γ-labeled graph with rank $k \geq 1$ periodicity Γ and $|V(G)| \geq 4$. Suppose that (G, ψ) is 2-connected, redundantly periodically rigid in $\mathbb{R}^2$, and has no $(0, 2)$-block. Then the minimum degree of G is at least three and the following hold for every vertex v of degree three.*

- *v is nondegenerate.*
- *$(G - v, \psi)$ is periodically rigid in $\mathbb{R}^2$.*
- *(G_v, ψ_v) is 2-connected, redundantly periodically rigid in $\mathbb{R}^2$, and has no $(0, 2)$-block.*

Assuming the correctness of Lemma 4.6 and Lemma 4.7 we are now ready to prove our main theorems. **Proof.** [Proof of Theorem 4.1] To see the necessity, first observe the following:

$$\begin{aligned}&(G, \psi) \text{ has no } (s, t)\text{-block } H \text{ with } (s + 1)t \leq 2 = d \text{ such that } V(H) \neq V(G)\\&\text{or } s < 1 = k \text{ if and only if } (G, \psi) \text{ is 2-connected and has no } (0, 2)\text{-block.}\end{aligned} \tag{4.2}$$

Indeed, if (G, ψ) is not 2-connected, then it has an (s, t)-block H with $s \leq k = 1$, $t \leq 1$, and $V(H) \neq V(G)$. Conversely, suppose that (G, ψ) has an (s, t)-block H with $(s + 1)t \leq 2$ such that $V(H) \neq V(G)$ or $s < k = 1$. If H is not a $(0, 2)$-block, then $s \geq 1$, $t \leq 1$ and $V(H) \neq V(G)$ hold. Hence G is not 2-connected. Thus by (4.2) the necessity of the conditions follows from Lemmas 3.1 and 3.7.

The proof of the sufficiency is done by induction on the lexicographic order of the list $(V(G), E(G))$. Suppose that $|V(G)| = 3$. By the 2-connectivity, G contains a triangle and the minimum degree in G is at least three. Let G' be an inclusionwise minimal spanning subgraph that is 2-connected and redundantly periodically rigid. Then it consists of five edges, two parallel classes and one simple edge. Since G' has a vertex of degree three, we can use Lemmas 4.5 and 4.4 to deduce that (G', ψ) (and hence (G, ψ)) is L-periodically globally rigid.

Assume that $|V(G)| \geq 4$. Suppose that G has a vertex of degree three. By Lemma 4.7, (G_v, ψ_v) is 2-connected, redundantly periodically rigid, and has no $(0, 2)$-block. Hence, by induction, (G_v, ψ_v, p) is L-periodically globally rigid. Therefore it follows from Lemma 4.7 and Lemma 4.5 that (G, ψ, p) is L-periodically globally rigid.

Thus we may assume that G has no vertex of degree three. Then by Lemma 4.6 there is an edge e such that $(G-e,\psi)$ is 2-connected, redundantly periodically rigid, and has no $(0,2)$-block. By induction, $(G-e,\psi,p)$ (and hence (G,ψ,p)) is L-periodically globally rigid.

Proof. [Proof of Theorem 4.2] By Lemma 3.2 we may assume that G is 2-connected. Then the necessity of the conditions again follows from Lemmas 3.1 and 3.7.

The proof of the sufficiency is similar to that of Theorem 4.1, and it is done by induction on the lexicographic order of the list $(V(G),E(G))$. The case when $|V(G)|=3$ is exactly the same as that for Theorem 4.1. Hence we assume $|V(G)|\geq 4$.

Suppose that G has a vertex of degree three. By Lemma 4.7, (G_v,ψ_v) is 2-connected, redundantly periodically rigid, and has no $(0,2)$-block. Also, by the definition of (G_v,ψ_v), we have $\Gamma_{G_v}=\Gamma_G$. Hence, by induction, (G_v,ψ_v,p) is L-periodically globally rigid. Therefore it follows from Lemma 4.7 and Lemma 4.5 that (G,ψ,p) is L-periodically globally rigid.

Thus we may assume that G has no vertex of degree three. Then by Lemma 4.6 there is an edge e such that $(G-e,\psi)$ is 2-connected, redundantly periodically rigid, and has no $(0,2)$-block. If the rank of Γ_{G-e} is equal to two, then we can apply the induction hypothesis, meaning that $(G-e,\psi,p)$ (and hence (G,ψ,p)) is L-periodically globally rigid. Therefore, we assume that the rank of Γ_{G-e} is smaller than two.

Since $(G-e,\psi)$ is periodically rigid, Theorem 2.4 implies that the rank of Γ_{G-e} is nonzero. Thus the rank of Γ_{G-e} is one. By switching, we may suppose that every label in $E(G-e)$ is in Γ_{G-e} and consider the restriction $L':\Gamma_{G-e}\to\mathbb{R}^2$ of L. By Theorem 4.1, $(G-e,\psi,p)$ is L'-periodically globally rigid. To show that (G,ψ,p) is L-periodically globally rigid, take any $q:V(G)\to\mathbb{R}^2$ such that $f_{G,L}(p)=f_{G,L}(q)$. Since $(G-e,\psi,p)$ is L'-periodically globally rigid, Proposition 2.2 implies that q can be written as $q=h\circ p$ for some $L(\Gamma_{G-e})$-invariant isometry h. Since the rank of Γ_{G-e} is one and the dimension of the ambient space is two, h is either the identity or the reflection along the line whose direction is in the span of $L(\Gamma_{G-e})$.

Let i and j be the endvertices of e and let $\gamma_e=\psi(e)$. Using the orthogonal matrix S representing the reflection along the span of $L(\Gamma_{G-e})$, we have $q(i)-q(j)=S(p(i)-p(j))$. By looking at the length constraint for e, $\|p(i)-(p(j)+L(\gamma_e))\|=\|q(i)-(q(j)+L(\gamma_e))\|$, which implies $\langle (I-S)(p_i-p_j),L(\gamma_e)\rangle=0$, or equivalently, $\langle p_i-p_j,(I-S^\top)L(\gamma_e)\rangle=0$. Note that S is determined by $L(\Gamma_{G-e})$, and independent from p_i-p_j. Hence, the generic assumption for p implies $(I-S^\top)L(\gamma_e)=0$. Since $\gamma_e\notin\Gamma_{G-e}$, this implies $I=S$. Thus we get $f_{V,L}(p)=f_{V,L}(q)$, and (G,ψ,p) is L-periodically globally rigid.

5 Proof of the Combinatorial Part

Due to Theorem 2.4, we may work in a purely combinatorial world in order to prove Lemma 4.6 and Lemma 4.7.

Since all the graphs we treat in the following discussions are Γ-labeled graphs, we shall use the following convention. Throughout the section, we omit the labeling function ψ to denote a Γ-labeled graph (G,ψ). The underlying graph of each Γ-

labeled graph is directed, but the direction is used only to refer to the group labeling. So the underlying graph is treated as an undirected graph if we are interested in its graph-theoretical properties, such as the connectivity, vertex degree, and so on.

Let $G = (V, E)$ be a Γ-labeled graph. For disjoint sets $X, Y \subseteq V$, $d_G(X, Y)$ denotes the number of edges between X and Y, and let $d_G(X) := d_G(X, V \setminus X)$ and $d_G(v) = d_G(\{v\})$ for $v \in V$. For $X \subseteq V$, $i_G(X)$ denotes the number of edges induced by X. For an edge set F, let $G[F] = (V(F), F)$.

Given two Γ-labeled graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. Note that $G_1 \cup G_2$ and $G_1 \cap G_2$ are Γ-labeled graphs whose labeling are inherited from those of G_1 and G_2. Note also that, when taking the union or the intersection, the labels are taken into account and two edges are recognized as the same edge if and only if they are identical. Hence $G_1 \cup G_2$ may contain parallel edges even if G_1 and G_2 are simple.

Several terms defined for edge sets will be used for graphs G by implicitly referring to $E(G)$. If there is no confusion, terms for graphs will be conversely used for edge sets E by referring to the graph $(V(E), E)$.

We use the following terminology from matroid theory. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are *related* if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. If $\mathcal{M}$ has components $E_1, E_2, \ldots, E_t$ and $\mathcal{M}_i$ is the matroid restriction of $\mathcal{M}$ onto E_i then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_t$ is the direct sum of the $\mathcal{M}_i$'s.

5.1 Count matroids

Let V be a finite set and Γ be a group isomorphic to $\mathbb{Z}^k$. We say that an edge set E is *independent* if $|F| \leq 2|V(F)| - 3$ for every nonempty balanced $F \subseteq E$ and $|F| \leq 2|V(F)| - 2$ for every nonempty $F \subseteq E$. Proposition 2.3 and Theorem 2.4 imply that the family of all independent edge sets forms the family of independent sets of a matroid on $K(V, \Gamma)$, which is denoted by $\mathcal{R}_2(V, \Gamma)$ or simply by $\mathcal{R}_2(V)$. (This can also be checked in a purely combinatorial fashion, see, e.g., [15].) Let r_2 be the rank function and cl_2 be the closure operator of $\mathcal{R}_2(V)$.

We say that G is *M-connected* if the restriction of $\mathcal{R}_2(V(G))$ to $E(G)$ is connected.

By Theorem 2.4, G is periodically rigid in $\mathbb{R}^2$ if and only if $r_2(G) = 2|V(G)| - 2$, where $r_2(G) := r_2(E(G))$. Note that if G is balanced, then the restriction of $\mathcal{R}_2(V)$ to $E(G)$ is isomorphic to the 2-dimensional generic rigidity matroid. Hence we say that G is *rigid* if G is balanced and $r_2(G) = 2|V(G)| - 3$.

We can use known properties of the 2-dimensional generic rigidity matroids for balanced sets. The following statements can be found in [9] (with a slightly different terminology).

Lemma 5.1. *If a Γ-labeled graph G is balanced and M-connected, then G is rigid.*

Lemma 5.2. *Let G_1 and G_2 be two rigid graphs. Suppose that $|V(G_1) \cap V(G_2)| \geq 2$ and $G_1 \cup G_2$ is balanced. Then $G_1 \cup G_2$ is rigid.*

The following simple property was first observed in [21].

Lemma 5.3 ([21]). *Let v be a vertex of degree two in G with $|V(G)| \geq 2$. Then $r_2(G - v) = r_2(G) - 2$.*

Lemma 5.3 enables us to show an analogue of Lemma 5.2.

Lemma 5.4. *Let G_1 and G_2 be Γ-labeled graphs.*

(i) If G_1 is periodically rigid, G_2 is rigid and $|V(G_1) \cap V(G_2)| \geq 2$, then $G_1 \cup G_2$ is periodically rigid.

(ii) If G_1 and G_2 are periodically rigid and $|V(G_1) \cap V(G_2)| \geq 2$, then $G_1 \cup G_2$ is periodically rigid.

Proof. Let $G_i = (V_i, E_i)$ for $i = 1, 2$. To see the first claim, suppose that G_1 is periodically rigid and G_2 is rigid. Since G_2 is rigid it has a spanning balanced set, and by switching we may suppose that $\psi(e) = \mathrm{id}$ for every edge e in the spanning balanced set. Thus we have $K^0(V_2) \subseteq \mathrm{cl}_2(E_2)$, where $K^0(V_2)$ denotes the set of edges on V_2 whose labels are the identity. Since $|V_1 \cap V_2| \geq 2$, there are two distinct vertices u and v in $V_1 \cap V_2$. Let F be the edge set of the complete bipartite subgraph of $K^0(V_2)$ whose one partite set is $\{u, v\}$ and whose other partite set is $V_2 \setminus V_1$. By Lemma 5.3, we have $r_2(E_1 \cup F) = r_2(E_1) + 2|V_2 \setminus V_1|$. Since $F \subseteq K^0(V_2) \subseteq \mathrm{cl}_2(E_2)$, we get

$$r_2(E_1 \cup E_2) \geq r_2(E_1 \cup F) = 2|V_1| - 2 + 2|V_2 \setminus V_1| = 2|V_1 \cup V_2| - 2.$$

Thus $G_1 \cup G_2$ is periodically rigid.

We can use the same argument to prove the second statement. In this case we have $K^0(V_2) \subset K(V_2, \Gamma) \subseteq \mathrm{cl}_2(E_2)$ and we can again use Lemma 5.3 to deduce that $r_2(E_1 \cup E_2) = 2|V_1 \cup V_2| - 2$, which completes the proof.

5.2 *M*-connectivity and ear decomposition

Jackson and Jordán [9] used ear decompositions of connected rigidity matroids as a key tool in their proof of Theorem 1.2. Let $C_1, C_2, \ldots, C_t$ be a non-empty sequence of circuits of the matroid $\mathcal{M}$. For $1 \leq j \leq t$, we denote $D_j = C_1 \cup C_2 \cup \cdots \cup C_j$ and define the *lobe* $\tilde{C}_j$ of C_j by $\tilde{C}_j = C_j \setminus D_{j-1}$. We say that $C_1, C_2, \ldots, C_t$ is a *partial ear decomposition* of $\mathcal{M}$ if for every $2 \leq i \leq t$ the following properties hold:

(E1) $C_i \cap D_{i-1} \neq \emptyset$,

(E2) $C_i \setminus D_{i-1} \neq \emptyset$,

(E3) no circuit C' satisfying (E1) and (E2) has $C' \setminus D_{i-1}$ properly contained in $C_i \setminus D_{i-1}$.

An *ear decomposition* of $\mathcal{M}$ is a partial ear decomposition with $D_t = E$. We need the following facts about ear decompositions.

Lemma 5.5 ([5,9]). *Let $\mathcal{M}$ be a matroid. Then the following hold.*

(a) $\mathcal{M}$ is connected if and only if $\mathcal{M}$ has an ear decomposition.

(b) If $\mathcal{M}$ is connected then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$.

(c) If $C_1, C_2, \ldots, C_t$ is an ear decomposition of $\mathcal{M}$ then $r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1$ for $2 \leq i \leq t$.

For an M-connected graph $G = (V, E)$, *an ear decomposition of E* means an ear decomposition of the restriction of $\mathcal{R}_2(V)$ to E.

When the matroid $\mathcal{M}$ is specialized to the generic 2-dimensional rigidity matroid, several properties of ear-decompositions were proved in [9, 14]. We will use the following (with some of the terminology adjusted to fit our context).

Lemma 5.6 ([9]). *Let G be balanced M-connected, and let $C_1, \ldots, C_t$ be an ear decomposition of $E(G)$ with $t \geq 2$. Let $Y = V(C_t) \setminus \bigcup_{i=1}^{t-1} V(C_i)$. Then the following hold.*

(a) *Either $Y = \emptyset$ and $|\tilde{C}_t| = 1$, or $Y \neq \emptyset$ and every edge $e \in \tilde{C}_t$ is incident to Y.*

(b) $|\tilde{C}_t| = 2|Y| + 1$.

We say that G is *minimally M-connected* if G is M-connected and for every $e \in E(G)$, $G - e$ is not M-connected. In [14, Lemma 3.4.3], Jordán pointed out that Lemma 5.6 immediately implies the existence of a degree three vertex in a minimally M-connected graph. His proof actually gives a slightly stronger statement as follows.

Lemma 5.7. *Let G be balanced and minimally M-connected, and let $C_1, \ldots, C_t$ be an ear decomposition of $E(G)$. Let $Y = V(C_t) - V(\bigcup_{i=1}^{t-1} C_i)$. If $t \geq 2$, then Y contains a vertex which has degree three in G. Moreover, if $|Y| \geq 2$, then Y contains at least two vertices of degree three in G.*

Proof. Since G is minimally M-connected, $|\tilde{C}_t| > 1$. Lemma 5.6(a) says that, if $|\tilde{C}_t| > 1$, then $Y \neq \emptyset$ and every edge in $\tilde{C}_t$ is incident to Y. Moreover, Lemma 5.6(b) says that

$$|\tilde{C}_t| = 2|Y| + 1. \tag{5.1}$$

Hence if $|Y| = 1$ then the vertex in Y has degree three.

If $|Y| \geq 2$, then the edge set of G induced by Y is a proper subset of $\tilde{C}_t$, and hence $i_G(Y) \leq 2|Y| - 3$. Combining this with (5.1), we have that the total degree of the vertices of Y is $d_G(Y, V(G) \setminus Y) + 2i_G(Y) = |\tilde{C}_t| + i_G(Y) = 4|Y| - 2$. Thus Y contains at least two vertices that have degree three in G.

It is well known that every circuit in the generic rigidity matroid (i.e., every balanced circuit) is 2-connected and 3-edge-connected. The next lemma shows that even if G is unbalanced or M-connected, we can guarantee the same connectivity.

Lemma 5.8. *If G is M-connected, then G is 2-connected and 3-edge-connected.*

Proof. The statement is known if G is balanced (see, e.g., [9]). Hence we assume that G is unbalanced.

It suffices to consider the case when G is an unbalanced circuit. Suppose that G can be decomposed into two edge-disjoint subgraphs G_1 and G_2 with $|V(G_1)\cap V(G_2)|\leq 1$ and $|V(G_i)|\geq 2$ for $i=1,2$. Then $2|V(G)|-1=|E(G)|=|E(G_1)|+|E(G_2)|\leq 2|V(G_1)|-2+2|V(G_2)|-2\leq 2|V(G)|-2$, which is a contradiction. Thus G is 2-connected.

The same counting argument works for showing that G is 3-edge-connected. Suppose that there are two vertex-disjoint induced subgraphs G_1 and G_2 connected by at most two edges in G with $V(G_1)\cup V(G_2)=V(G)$. Then we have $2|V(G)|-1=|E(G)|\leq|E(G_1)|+|E(G_2)|+2\leq 2|V(G_1)|-2+2|V(G_2)|-2+2=2|V(G)|-2$, which is a contradiction.

5.3 Redundant rigidity and M-connectivity

Based on the ear decomposition, in this section we shall reveal a connection between redundant rigidity and M-connectivity. In particular, we show that, under a certain connectivity condition, these properties are equivalent. This is an analogue of the fact that M-connectivity is equivalent to redundant rigidity under (near) 3-connectivity in the plane [9, Theorem 3.2].

In Lemma 5.10 we first give the unbalanced version of Lemma 5.1. For the proof we need the following technical lemma.

Lemma 5.9. *Let E_1 and E_2 be balanced edge sets in a Γ-labeled graph G. Suppose that $E_1\cup E_2$ is unbalanced. Then the graph $(V(E_1)\cap V(E_2), E_1\cap E_2)$ is disconnected.*

Proof. This is a special case of [15, Lemma 2.4]. Since the proof is easy, we include it for completeness. Suppose that $(V(E_1)\cap V(E_2), E_1\cap E_2)$ is connected. Then one can take a spanning tree T in $(V(E_1\cup E_2), E_1\cup E_2)$ such that $T\cap E_i$ is a spanning tree in $(V(E_i), E_i)$ for $i=1,2$. By switching, we may assume that $\psi(e)=\mathrm{id}$ for every $e\in T$. Since E_i is balanced, we get $\psi(e)=\mathrm{id}$ for every $e\in E_1\cup E_2$, contradicting that $E_1\cup E_2$ is unbalanced.

Lemma 5.10. *If G is unbalanced and M-connected, then G is periodically rigid.*

Proof. Since G is M-connected, $E(G)$ has an ear decomposition $C_1,\dots,C_t$. It follows from $|D_i\cap C_{i+1}|\geq 1$ that there are at least two vertices in $V(D_i)\cap V(C_{i+1})$. We will prove by induction on i that

$$r_2(D_i)=2|V(D_i)|-3 \text{ if } D_i \text{ is balanced, and } r_2(D_i)=2|V(D_i)|-2 \text{ otherwise.} \tag{5.2}$$

This follows from the definition of the underlying matroid if $i=1$. Hence, we assume $i\geq 2$.

If D_{i-1} is unbalanced, then by induction D_{i-1} is periodically rigid. By Lemma 5.4, $D_{i-1}\cup C_i$ is periodically rigid, implying (5.2).

If D_{i-1} is balanced, then by induction it is rigid. If C_i is periodically rigid, then by Lemma 5.4(i), $D_{i-1}\cup C_i$ is periodically rigid. If C_i is rigid and $D_{i-1}\cup C_i$ is balanced, then by Lemma 5.2, $D_{i-1}\cup C_i$ is rigid.

For (5.2) it remains to show $r_2(D_i) = 2|V(D_i)| - 2$ if C_i and D_{i-1} are balanced and $D_{i-1} \cup C_i$ is unbalanced. Let $s := 2|V(C_i \cap D_{i-1})| - 3 - |C_i \cap D_{i-1}|$. Since $C_i \cap D_{i-1}$ is balanced and independent, we have $s \geq 0$. If $s = 0$ then $C_i \cap D_{i-1}$ is rigid as $C_i \cap D_{i-1}$ is balanced. Hence by Lemma 5.8, $C_i \cap D_{i-1}$ is connected. However, since $(V(C_i) \cap V(D_{i-1}), C_i \cap D_{i-1})$ is disconnected by Lemma 5.9, we have $V(C_i) \cap V(D_{i-1}) \neq V(C_i \cap D_{i-1})$. This in particular implies

$$s + 2(|V(C_i) \cap V(D_{i-1})| - |V(C_i \cap D_{i-1})|) \geq 1. \tag{5.3}$$

Since D_{i-1} is rigid and C_i is a balanced circuit, Lemma 5.5(c) implies

$$\begin{aligned} r_2(D_i) &= r_2(D_{i-1}) + |C_i \setminus D_{i-1}| - 1 \\ &= 2|V(D_{i-1})| - 3 + |C_i| - |C_i \cap D_{i-1}| - 1 \\ &= 2|V(D_{i-1})| - 3 + 2|V(C_i)| - 2 - 2|V(C_i \cap D_{i-1})| + 3 + s - 1 \\ &= 2(|V(D_{i-1})| + |V(C_i)| - |V(C_i \cap D_{i-1})|) - 3 + s \\ &= 2|V(D_i)| - 3 + 2(|V(C_i) \cap V(D_{i-1})| - |V(C_i \cap D_{i-1})|) + s \\ &\geq 2|V(D_i)| - 2 \end{aligned}$$

where the last inequality follows from (5.3).

Since $E(G) = D_t$ and G is unbalanced, we conclude that G is periodically rigid by (5.2).

The following is an analogue of [9, Theorem 3.2]. Using the lemmas collected so far, its proof is now a direct adaptation of that of [9, Theorem 3.2].

Theorem 5.11. *Let G be unbalanced and suppose that it has no $(0, 2)$-block. Then G is 2-connected and redundantly periodically rigid if and only if G is M-connected.*

Proof. Suppose first that G is M-connected. By Lemma 5.8 G is 2-connected. Also G is periodically rigid by Lemma 5.10. Since every edge is included in a circuit, we have $r_2(G - e) = r_2(G)$ for every edge $e \in E(G)$, meaning that $G - e$ is periodically rigid. Thus G is redundantly periodically rigid.

Conversely, suppose that G is 2-connected and redundantly periodically rigid. Suppose further for a contradiction that G is not M-connected. Then $E(G)$ can be decomposed into M-connected components $E_1, \ldots, E_t$ with $t \geq 2$. By Lemmas 5.10 and 5.1, we have $r_2(E_i) = 2|V(E_i)| - 3$ if E_i is balanced and $r_2(E_i) = 2|V(E_i)| - 2$ otherwise. Let a_i be the number of vertices in $G[E_i]$ which are shared by other $G[E_j]$. With this notation, we have

$$\begin{aligned} &2|V(G)| - 2 \geq r_2(E(G)) = \sum_{i=1}^{t} r_2(E_i) \\ &= \sum_{E_i:\text{balanced}} (2|V(E_i)| - 3) + \sum_{E_i:\text{unbalanced}} (2|V(E_i)| - 2) \\ &= \sum_{i=1}^{t} (2|V(E_i)| - a_i) + \sum_{E_i:\text{balanced}} (a_i - 3) + \sum_{E_i:\text{unbalanced}} (a_i - 2). \end{aligned} \tag{5.4}$$

Since G is redundantly periodically rigid, every M-connected component contains a circuit. Hence, for each balanced E_i, we have $|V(E_i)| \geq 4$ (since the smallest balanced circuit is K_4). This means that, if $a_i \leq 2$ for a balanced E_i, then $G[E_i]$ is a $(0,2)$-block in G. Hence we have $a_i \geq 3$ for each balanced E_i. On the other hand, for each unbalanced E_i, we have $a_i \geq 2$ since G is 2-connected. Therefore the last two terms in (5.4) are nonnegative. Observe finally that $\sum_{i=1}^{k}(2|V(E_i)| - a_i) \geq 2|V(G)|$, since each vertex of G contained in a unique E_i is counted twice while the rest of the vertices are counted at least twice. Therefore (5.4) is at least $2|V(G)|$, which is a contradiction.

5.4 Proof of Lemma 4.7

We first solve the following main case of the proof of Lemma 4.7.

Lemma 5.12. *Let G be a 2-connected and redundantly periodically rigid graph that has no $(0,2)$-block. Then so is G_v.*

Proof. The 2-connectivity of G_v follows from the 2-connectivity of G. Let E_v be the set of edges incident with v in G, and assume that those edges are directed from v. Let F be the set of edges $e_1 \cdot e_2$ over all pairs of nonparallel edges e_1, e_2 in E_v. Recall that G_v is obtained from G by removing v and adding F, where the label of $e_1 \cdot e_2$ is defined to be $\psi(e_1)^{-1}\psi(e_2)$. Thus, we have that

$$E_v \cup F \text{ is a circuit.} \tag{5.5}$$

We first show that G_v has no $(0,2)$-block. Suppose to the contrary that G_v contains a $(0,2)$-block H. Take a maximal such H. We split the proof into four cases depending on how H intersects with F as follows. (Case 1) If $|E(H) \cap F| = 0$, then H is also a $(0,2)$-block of G, a contradiction. (Case 2) If $|E(H) \cap F| = 1$, then $B(H)$ is exactly the endvertices of the edge e in $E(H) \cap F$. Hence $H \cap G$ would be a $(0,2)$-block in G with $V(H \cap G) = V(H)$ and $B(H \cap G) = B(H)$. This is a contradiction. (Case 3) If $F \subseteq E(H)$, then $(H + E_v) \cap G$ is a $(0,2)$-block in G by (5.5), again a contradiction. (Case 4) The remaining case is when $|N_G(v)| = 3$ and $|E(H) \cap F| = 2$. Recall that F forms a balanced triangle if $|N_G(v)| = 3$. Hence $H + F$ would be balanced and hence be a $(0,2)$-block in G_v, which contradicts the maximality of H. This completes the proof that G_v has no $(0,2)$-block.

To see that G_v is redundantly periodically rigid, take any edge $e \in E(G_v)$. Our goal is to show that $G_v - e$ is periodically rigid. We take an edge f from E_v. Suppose first that there is a circuit C in $G + F$ with $f \in C$ and $e \notin C$. Since $C \subseteq E(G) + F - e$, we have $\mathrm{cl}_2(E(G) + F - e) = \mathrm{cl}_2(E(G) + F - e - f)$. On the other hand, since G is redundantly periodically rigid, $\mathrm{cl}_2(E(G) + F) = \mathrm{cl}_2(E(G) + F - e)$. Thus, $\mathrm{cl}_2(E(G) + F) = \mathrm{cl}_2(E(G) + F - e - f)$, and we get $r_2(G + F - e - f) = r_2(G + F) = 2|V(G)| - 2$. Since v has degree two in $G + F - e - f$, we get $r_2(G_v - e) = 2|V(G)| - 2 - 2 = |V(G_v)| - 2$. Thus, $G_v - e$ is periodically rigid.

So we may suppose that every circuit in $G + F$ that contains f also contains e. Then by (5.5), $e \in F$. Suppose that $G_v - e$ is not periodically rigid.

Claim 5.13. *$G_v - e$ contains an M-connected component D' with $D' \cap (F - e) = \emptyset$.*

Proof. We split the proof into two cases depending on the size of $N_G(v)$.

Case 1: $|N_G(v)| = 3$. We denote $N_G(v) = \{v_1, v_2, v_3\}$, and assume that e connects v_1 and v_2. For $i = 1, 2$, let e_i be the edge in F that connects v_i and v_3, and let D_i be the M-connected component in $G_v - e$ that contains e_i. We first show that $D_1 \neq D_2$. To see this, suppose $D_1 = D_2$. Then $e \in \mathrm{cl}_2(D_1)$ since $\{e, e_1, e_2\}$ is balanced. This however implies that $\mathrm{cl}_2(E(G_v)) = \mathrm{cl}_2(E(G_v - e))$ and hence G_v is not periodically rigid as $G_v - e$ is not. This contradiction implies $D_1 \neq D_2$.

Next, to see the statement of the claim, assume that D_1 and D_2 are all the M-connected components in $G_v - e$. By Lemma 5.1 and Lemma 5.10, $r_2(D_i) = 2|V(D_i)| - 3 + \delta(D_i)$, where $\delta(D_i) = 0$ if it is balanced and $\delta(D_i) = 1$ otherwise. Since $G_v - e$ is not periodically rigid, $2|V(G_v)| - 3 \geq r_2(G_v - e) = r_2(D_1) + r_2(D_2) \geq 2(|V(D_1)| + |V(D_2)|) - 6 + \delta(D_1) + \delta(D_2)$, implying $2|V(D_1) \cap V(D_2)| + \delta(D_1) + \delta(D_2) \leq 3$. Since $V(D_1) \cap V(D_2) \neq \emptyset$, we obtain

$$|V(D_1) \cap V(D_2)| = 1 \text{ and } \delta(D_1) + \delta(D_2) \leq 1.$$

The first equation implies that G has no edge between v_1 and v_2 except possibly e. The second inequality implies that at least D_1 or D_2 is balanced. Without loss of generality we assume that D_2 is balanced. If $|V(D_2)| \geq 3$, then $G[D_2]$ or $G[D_2 - e_2]$ is a subgraph of G, and one of them forms a $(0, 2)$-block in G since $\{v_2, v_3\}$ forms a cut, which contradicts that G has no $(0, 2)$-block. On the other hand, if $|V(D_2)| = 2$, then $D_2 = \{e_2\}$ and G has no edge between v_2 and v_3 except possibly e_2. Therefore $E_v \cup (F \cap E(G))$ induces a $(0, 2)$-block in G, which is again a contradiction. This contradiction completes the proof for Case 1.

Case 2: $|N_G(v)| = 2$. Let e_1 be the edge in F different from e. Let D_1 be the M-connected component in $G_v - e$ that contains e_1. If D_1 is unbalanced, then D_1 is periodically rigid by Lemma 5.10, and hence $e \in \mathrm{cl}_2(D_1)$. This however implies that G_v is not periodically rigid, which is a contradiction. Thus D_1 is balanced.

If D_1 is the only M-component in $G_v - e$, then $G[D_1]$ or $G[D_1 - e_1]$ would be a $(0, 2)$-block in G by $|V(D_1)| \geq 3$. Thus $G_v - e$ has an M-component different from D_1.

Let D' be an M-component in $G_v - e$ satisfying the condition of Claim 5.13. Let $g \in D'$ be arbitrary, and let C_1 be a circuit in $G + F$ with $e, g \in C_1$. By (5.5), $E_v \cup F$ is a circuit. Since $e \in F$ and $g \notin E_v + F$, by the circuit elimination, we have a circuit $C_2 \subseteq (C_1 \cup (E_v + F)) - e$ with $g \in C_2$. We claim

$$C_2 \cap (E_v + e) = \emptyset. \tag{5.6}$$

To see this recall that every circuit in $G + F$ containing f also contains e. As $e \notin C_2$, we get $f \notin C_2$. Moreover, since v has degree three in $G + F$, $f \notin C_2$ implies $E_v \cap C_2 = \emptyset$, implying (5.6).

By (5.6), C_2 is a circuit in $G_v - e$. However, according to the construction of C_2, we have $C_2 \cap (E_v + F) \neq \emptyset$, which means that C_2 intersects $F - e$ by (5.6). Since $D' \cap (F - e) = \emptyset$ and $f \in C_2 \cap D'$, C_2 intersects more than one M-connected component in $G_v - e$, which is a contradiction.

Proof. [Proof of Lemma 4.7] By Lemma 5.3, the minimum degree of G is at least three. Let v be a vertex of degree three, and suppose without loss of generality that all the edges incident to v are directed from v. To prove that v is nondegenerate, we have to show that there is no neighbor u of v such that the set $\{L(\psi(e)) : e = vu \in E(G)\}$ is affinely dependent. If v has three distinct neighbors, then this is clearly true. If v has exactly two neighbors, any parallel edges are not identical, and hence the parallel edges form an unbalanced cycle. Thus the statement is again true. Hence by 2-connectivity of G we conclude that v is nondegenerate.

Next we show that $G - v$ is periodically rigid. Take any edge e incident to v. Since G is redundantly periodically rigid, we have $2|V(G)| - 2 = r_2(G) = r_2(G - e)$. Since v has degree two in $G - e$, we get $r_2(G - v) = r_2(G - e) - 2 = 2|V(G - v)| - 2$ by Lemma 5.3, implying that $G - v$ is periodically rigid.

Finally the statement for G_v has already been proved in Lemma 5.12.

5.5 Proof of Lemma 4.6

The proof of Lemma 4.6 is rather involved, and it consists of three major parts. In Lemma 5.14 we first show the existence of a degree three vertex in a minimally M-connected graph. By this result, we may focus on the case when G is not minimally M-connected, i.e., $E' = \{e \in E(G) : G - e \text{ is } M\text{-connected}\}$ is nonempty. By Lemma 5.11, $G - e$ is 2-connected and redundantly periodically rigid for every $e \in E'$ for which $G - e$ has no $(0, 2)$-block. Thus, what remains to show is that there is an edge $e \in E'$ such that $G - e$ has no $(0, 2)$-block. We prove this by contradiction. A key lemma for this will be Lemma 5.20, which shows that, for any $(0, 2)$-block H_e in $G - e$ for $e \in E'$, H_e contains an edge f in E'. The proof is completed by first taking $e \in E'$ such that $|V(H_e)|$ is as small as possible and then showing that $G - f$ has no $(0, 2)$-block for $f \in E(H_e) \cap E'$.

We start with the following lemma, which is an unbalanced version of Lemma 5.7. The proof strategy is similar, but its proof is technically more involved.

Lemma 5.14. *If G is unbalanced and minimally M-connected, then G has a vertex of degree three.*

Proof. Let $C_1, \ldots, C_t$ be an ear decomposition of $E(G)$. The statement trivially follows from the edge count of G if $t = 1$. Hence we assume $t \geq 2$. Let $D_i = \bigcup_{j=1}^{i} C_j$. We will use the notation $V = V(G)$ and $V' = V(D_{t-1})$. Our goal is to prove that the average degree of $V \setminus V'$ (denoted by $d_{avg}(V \setminus V')$) is less than 4, implying that $V \setminus V'$ must contain a vertex of degree three. The proof is split into two cases depending on whether D_{t-1} is balanced or not.

Case 1: Suppose that D_{t-1} is unbalanced. Then it follows from Lemma 5.11 that D_{t-1} is periodically rigid. Using Lemma 5.5(c) we have

$$|C_t \setminus D_{t-1}| - 1 = r_2(D_{t-1} \cup C_t) - r_2(D_{t-1}) = 2|V| - 2 - (2|V'| - 2) = 2|V \setminus V'| \quad (5.7)$$

since $D_{t-1} \cup C_t$ is periodically rigid and D_{t-1} is unbalanced M-connected.

Since D_{t-1} is periodically rigid, $K(V',\Gamma) \subseteq \mathrm{cl}_2(D_{t-1})$. Therefore, since G is minimally M-connected, no edge in $C_t \setminus D_{t-1}$ is induced by V'. This implies $V \setminus V' \neq \emptyset$, and by Lemma 5.8 we further have

$$d(V \setminus V', V') \geq 3. \tag{5.8}$$

Combining (5.7) and (5.8), we get

$$d_{avg}(V \setminus V') = \frac{2|C_t \setminus D_{t-1}| - d(V \setminus V', V')}{|V \setminus V'|} = \frac{4|V \setminus V'| + 2 - d(V \setminus V', V')}{|V \setminus V'|} < 4.$$

Case 2: Suppose that D_{t-1} is balanced. We may assume that C_t is balanced (otherwise, by Lemma 5.5, we can start the ear decomposition with the unbalanced circuit C_t, and hence we are back in Case 1). We prove that $V \setminus V' \neq \emptyset$.

Since D_{t-1} is balanced, we may assume $\psi(e) = \mathrm{id}$ for every $e \in D_{t-1}$. Let $X = \{e \in D_t | \psi(e) \neq \mathrm{id}\}$. Since D_t is unbalanced, we have $\emptyset \neq X \subseteq C_t \setminus D_{t-1}$. However, since C_t is balanced, $G[C_t] - X$ must be disconnected. This in turn implies $|X| \geq 3$ by Lemma 5.8, and hence

$$|C_t \setminus D_{t-1}| \geq 3. \tag{5.9}$$

On the other hand, by Lemma 5.5(c) we have

$$|C_t \setminus D_{t-1}| - 1 = r(D_{t-1} \cup C_t) - r(D_{t-1}) = 2|V| - 2 - (2|V'| - 3) = 2|V \setminus V'| + 1 \tag{5.10}$$

as $D_{t-1} \cup C_t$ is periodically rigid and D_{t-1} is balanced and M-connected. Combining (5.9) and (5.10), we get $V \setminus V' \neq \emptyset$. By Lemma 5.8 we again have (5.8).

Let F be the set of edges in $C_t \setminus D_{t-1}$ induced by V'. If $F \neq \emptyset$, then we have

$$d_{avg}(V \setminus V') = \frac{2|C_t \setminus (D_{t-1} \cup F)| - d(V \setminus V', V')}{|V \setminus V'|} = \frac{4|V \setminus V'| + 2 - d(V \setminus V', V')}{|V \setminus V'|} < 4$$

where the equation follows from (5.10) and the last inequality follows from (5.8).

Hence we suppose $F = \emptyset$ and every edge in $C_t \setminus D_{t-1}$ is incident to $V \setminus V'$. By Lemma 5.9, the graph $(V(C_t) \cap V(D_{t-1}), C_t \cap D_{t-1})$ is disconnected. Since every edge in $C_t \setminus D_{t-1}$ is incident to $V \setminus V'$, the disconnectivity of $(V(C_t) \cap V(D_{t-1}), C_t \cap D_{t-1})$ implies that $V \setminus V'$ is a cut set in $G[C_t]$. Hence Lemma 5.8 implies $|V \setminus V'| \geq 2$. Since C_t is balanced, the number of edges of C_t induced by $V \setminus V'$ is at most $2|V \setminus V'| - 3$, and we obtain

$$|C_t \setminus D_{t-1}| \leq 2|V \setminus V'| - 3 + d(V \setminus V', V').$$

Combining this with (5.10), we get $d(V \setminus V', V') \geq 5$. Applying (5.10) again gives

$$d_{avg}(V \setminus V') = \frac{2|C_t \setminus D_{t-1}| - d(V \setminus V', V')}{|V \setminus V'|} \leq \frac{4|V \setminus V'| - 1}{|V \setminus V'|} < 4.$$

This completes the proof.

The next target is to prove Lemma 5.18 and Lemma 5.19, which will be needed for the proof of Lemma 5.20. Lemmas 5.18 and 5.19 may be considered as periodic versions of the 2-sum lemma and the cleaving lemma given in [9]. For the proofs we first need three technical lemmas.

Lemma 5.15. *Let X and Y be nonempty edge sets with $X \setminus Y \neq \emptyset$, $Y \setminus X \neq \emptyset$, and $|V(X) \cap V(Y)| = 2$ such that $X \cup Y$ is a circuit. Let $l_X = 2|V(X)| - |X|$ and $l_Y = 2|V(Y)| - |Y|$. Then the following hold.*

(i) *If $X \cup Y$ is balanced, then $(l_X, l_Y, |X \cap Y|) = (3, 3, 0)$.*

(ii) *If $X \cup Y$ is unbalanced, then $(l_X, l_Y, |X \cap Y|) = (3, 2, 0)$, $(l_X, l_Y, |X \cap Y|) = (2, 3, 0)$, or $(l_X, l_Y, |X \cap Y|) = (2, 2, 1)$.*

Proof. Let $l_{X\cup Y} = 2|V(X \cup Y)| - |X \cup Y|$. Then by $|V(X) \cap V(Y)| = 2$ we have

$$l_X + l_Y + |X \cap Y| = 4 + l_{X\cup Y}.$$

If $X \cup Y$ is balanced, then $l_{X\cup Y} = 2, l_X \geq 3$ and $l_Y \geq 3$. Hence $(l_X, l_Y, |X \cap Y|) = (3, 3, 0)$. And if $X \cup Y$ is unbalanced, then $l_{X\cup Y} = 1, l_X \geq 2$ and $l_Y \geq 2$. This gives the result.

The following properties for balanced circuits is implicit in [9]. (Although the proof is omitted, a counting argument identical to Lemma 5.17 gives a proof.)

Lemma 5.16. *Let G be a balanced circuit and H be a $(0, 2)$-block in G. Then H does not have an edge on $B(H)$. Also let $G_0 = H$ and $G_1 = G - I(H)$. Then the following hold.*

(i) *$|E(G_i)| = 2|V(G_i)| - 3$ for $i = 0, 1$.*

(ii) *$|F| \leq 2|V(F)| - 4$ for any nonempty F with $F \subsetneq E(G_i)$ and $\{a, b\} \subseteq V(F)$.*

(iii) *$|F| \leq 2|V(F)| - 3$ for any nonempty F with $F \subseteq E(G_i)$.*

The following is an unbalanced counterpart.

Lemma 5.17. *Let G be an unbalanced circuit and H be a $(0, 2)$-block in G. Let $G_0 = H - f_0'$ if H has an edge f_0' on $B(H)$, and let $G_0 = H$ otherwise. Also let $G_1 = G - I(H)$. Then the following hold.*

(i) *$|E(G_i)| = 2|V(G_i)| - 3 + i$ for $i = 0, 1$.*

(ii) *$|F| \leq 2|V(F)| - 4 + \delta(F)$ for any nonempty F with $F \subsetneq E(G_i)$ and $\{a, b\} \subseteq V(F)$.*

(iii) *$|F| \leq 2|V(F)| - 3 + \delta(F)$ for any nonempty F with $F \subseteq E(G_i)$,*

where $\delta(F) = 0$ if F is balanced and $\delta(F) = 1$ otherwise.

Proof. Denote $G_i = (V_i, E_i)$. Since G is an unbalanced circuit, for every nonempty $F \subsetneq E(G)$, we have $|F| \leq 2|V(F)| - 2$, and $|F| \leq 2|V(F)| - 3$ if F is balanced. This implies (iii). We also have $|E_0| \leq 2|V_0| - 3$ and $|E_1| \leq 2|V_1| - 2$. On the other hand, by $E_0 \cap E_1 = \emptyset$, Lemma 5.15 implies $|E_i| \geq 2|V_i| - 3$, and exactly one of G_0 and G_1 satisfies $|E_i| = 2|V_i| - 2$. Thus we get (i).

To see (ii), take $F \subsetneq E_0$ with $a, b \in V(F)$. Suppose for a contradiction that F violates (ii). Then by (i) $|F \cup E_1| = |F| + |E_1| \geq 2|V(F)| - 3 + 2|V(E_1)| - 2 = 2|V(F \cup E_1)| - 1$, which contradicts the fact that G is a circuit. The same counting argument works for $F \subsetneq E_1$ with $\{a, b\} \subseteq V(F)$. Thus (ii) holds.

Lemma 5.18. *Let G be unbalanced and M-connected and let H be a $(0,2)$-block in G with $B(H)=\{a,b\}$. Then there is exactly one edge f_0 in $K(\{a,b\},\Gamma)$ (which may exist in G) such that $H+f_0$ is balanced M-connected.*

Proof. Since H is balanced, H has at most one edge between a and b. We denote it by f_0' if it exists, and let $G_0 = H - f_0'$ if f_0' exists, and $G_0 = H$ otherwise. Since G_0 is balanced and connected, there is exactly one edge f_0 in $K(\{a,b\},\Gamma)$ whose addition to G_0 keeps the balancedness. (This f_0 coincides with f_0' if f_0' exists.) Moreover, Lemma 5.17 implies $G_0 + f_0$ is a circuit if G is an unbalanced circuit. Thus we have shown the following:

If G is an unbalanced circuit, then there is exactly one edge $f_0 \in K(\{a,b\},\Gamma)$ such that $H + f_0$ is a balanced circuit. (5.11)

Next we consider a general M-connected graph G. Take any $e \in E(H) - f_0'$. Since G is M-connected, G has a circuit C_e containing e and some edge $f \in E(G) \setminus E(H)$. Since C_e is 2-connected by Lemma 5.8, it spans a and b. Hence, $H \cap C_e$ is a $(0,2)$-block in $G[C_e]$, and by (5.11) there is exactly one edge $f_{0,e}$ on $\{a,b\}$ such that $H \cap C_e + f_{0,e}$ is a balanced circuit.

Since H is balanced, we actually have $f_{0,e} = f_{0,e'}$ for any $e, e' \in E(H) - f_0'$. We denote the edge by f_0. Then we have shown that, for every $e \in E(H) - f_0$, $H + f_0$ contains a circuit $H \cap C_e + f_0$ that contains e and f_0. This in turn implies the M-connectivity of $H + f_0$.

The edge f_0 that was proved to exist in Lemma 5.26 is called the *cleaving edge* for the $(0,2)$-block H, and $H + f_0$ is called the *cleavage graph* of H.

Lemma 5.19. *Let G be M-connected, H be a $(0,2)$-block in G, and f_0 be the cleaving edge for H. If $H + f_0 - f$ is M-connected for some $f \in E(H - f_0)$, then $G - f$ is M-connected.*

Proof. By the counting formulae given in Lemma 5.16 and Lemma 5.17, it is not difficult to check the following:

Let C_1 be a circuit in $H + f_0 - f$ and C_2 be a circuit in G. If $f_0 \in C_1$ and $C_2 \setminus E(H) \neq \emptyset$, then $(C_1 - f_0) \cup (C_2 - E(H))$ is a circuit. (5.12)

To see the M-connectivity of $G-f$, we take two edges $e_1 \in E(H-f)$ and $e_2 \in E(G-f)\setminus E(H)$. By the M-connectivity of G, G has a circuit C_2 containing e_1 and e_2. Since C_2 is 2-connected, $\{a,b\} \subset V(C_2)$. On the other hand, since $H+f_0-f$ is M-connected, it has a circuit C_1 with $f_0, e_1 \in C_1$. Thus, by (5.12), $C := (C_1 - f_0) \cup (C_2 - E(H))$ is a circuit which is contained in $G - f$.

Since e_1 and e_2 are chosen arbitrarily over $e_1 \in E(H-f)$ and $e_2 \in E(G-f)\setminus E(H)$, the above construction further implies that for every $e' \in E(H-f)$, $G-f$ has a circuit containing e' and e_2, while for every $e' \in E(G-f)\setminus E(H)$, $G-f$ has a circuit containing e' and e_1. In other words, for every $e' \in E(G-f)$, $G-f$ has a circuit $C_{e'}$ intersecting C, which implies the M-connectivity of $G-f$.

We are now ready to prove the following key lemma.

Lemma 5.20. *Let G be an unbalanced M-connected graph that has no $(0,2)$-block and its minimum degree is at least four. Suppose that $G-e$ is M-connected but $G-e$ has a $(0,2)$-block H_e for some $e \in E(G)$. Then $H_e - f_0$ contains an edge f such that $G-f$ is M-connected, where f_0 is the cleaving edge for H_e.*

Moreover, if an endvertex of e is not in $I(H_e)$, then f can be taken such that $H_e - f - f_0$ is 2-connected.

Proof. Since H_e is balanced and G has no $(0,2)$-block, we may suppose that

$$\text{the label of each edge in } H_e \text{ is the identity and } e \text{ has a non-identity label.} \tag{5.13}$$

Let $\{a,b\} = B(H_e)$ and let $G' = H_e + f_0$ be the cleavage graph for H_e. The endvertices of e are denoted by u and v. At least one endvertex of e is contained in $I(H_e)$, since otherwise H_e would be a $(0,2)$-block in G. Hence we may assume $u \in I(H_e)$. As $G', \psi')$ is (balanced) M-connected by Lemma 5.18, we can take an ear decomposition $C_1, \ldots, C_t$ of $E(G')$ such that $f_0 \in C_1$ and $u \in V(C_1)$.

We first solve the case when an endvertex of e is not in $I(H_e)$. We first remark that $t > 1$. Otherwise C_1 contains at least four vertices of degree three in C_1, one of which is also a degree three vertex in G since an endvertex of e is not in $I(H_e)$. This contradicts that the minimum degree of G is at least four. Thus $t > 1$. By Lemma 5.7 and the minimum degree condition for G, we have $|\tilde{C}_t| = 1$. Let f be the edge in $\tilde{C}_t$. Then $H_e + f_0 - f$ is M-connected. By Lemma 5.19, $G - f$ is also M-connected. It remains to show that $H_e - f - f_0$ is 2-connected. Suppose that $H_e - f - f_0$ is not 2-connected. Then it is not (balanced) rigid, since every rigid graph is 2-connected. This in turn implies that $H_e - f + f_0$ is not M-connected, a contradiction. This completes the proof in the case where an endvertex of e is not in $I(H_e)$.

The difficult case is when both endvertices of e are contained in $I(H_e)$. We may assume that $|\tilde{C}_t| > 1$, for otherwise the edge in $\tilde{C}_t$ has the desired property.

Claim 5.21. *One of the following holds:*

(i) *$t = 1$, u, v, a, b are distinct, and they are exactly the vertices of degree three in G';*

(ii) *$t > 1$, $V(C_t) \setminus V(\bigcup_{i=1}^{t-1} C_i) = \{v\}$, and v has degree three in G'.*

Proof. If $t = 1$, then there are at least four vertices of degree three in G' since $E(G')$ is a balanced circuit. Since the minimum degree of G is at least four, we have (i).

Suppose $t > 1$, and suppose also that (ii) does not hold. Then by Lemma 5.7 there is a vertex w in $V(C_t) - V(\bigcup_{i=1}^{t-1} C_i)$ other than v that has degree three in G'. Since $\{u,a,b\} \subseteq V(C_1)$, w is distinct from them. Hence w has degree three even in G, contradicting the minimum degree condition of G.

Claim 5.21 implies
$$d_G(v) = 4. \tag{5.14}$$
Claim 5.21 also implies $v \in I(H_e)$, and we can take an edge f in G' incident to v. We claim that G has a circuit C^* with $e \in C^*$ and $f \notin C^*$. Indeed, since $G - e$ is

M-connected, $G-e$ has a circuit C with $f \in C$, and G has a circuit C' with $e \in C'$ and $f \in C'$. By the circuit elimination, we get $C^* \subseteq C \cup C' - f$ with $e \in C^*$. By (5.14) we have

$$d_{C^*}(v) = 3 \text{ and } d_{G-f}(v) = 3. \tag{5.15}$$

Claim 5.22. *C^* is unbalanced with $\{a,b\} \subseteq V(C^*)$.*

Proof. If $C^* \subseteq E(H_e) + e$, then we would have $r(C^*) = r(C^* - e) + 1$ by (5.13), which contradicts that C^* is a circuit. Hence C^* must contain at least one edge from $E(G) - E(H_e + e)$. Thus the 2-connectivity of C^* implies $\{a,b\} \subseteq V(C^*)$.

Suppose that C^* is balanced. Then every path between u and v in $C^* - e$ passes through a, since the concatenation of e and a path between u and v avoiding a is unbalanced by (5.13). Hence a is a cut vertex in $C^* - e$, contradicting the rigidity of $G[C^*]$. (Note that u and v are distinct from a by Claim 5.21.)

Claim 5.23. *If $t = 1$, then $V(H_e) \subseteq V(C^*)$.*

Proof. Let $X = V(C^*) \cap V(H_e)$ and $Y = V(H_e) \setminus X$. Also let k be the number of edges between a and b in G. Note that $C_1 = E(H_e) + f_0$ by $t = 1$. (Recall that C_1 is the initial circuit in the ear decomposition $C_1, \ldots, C_t$.)

By Claim 5.22 and $e \in C^*$, $\{u, v, a, b\} \cap Y = \emptyset$. Hence the edge set of G induced by Y is a proper subset of C_1. Hence

$$i_G(Y) \leq 2|Y| - 3 \tag{5.16}$$

if $|Y| \geq 2$. On the other hand, since C^* is an unbalanced circuit by Claim 5.22, we have

$$i_G(X) \geq 2|X| - 3 + k \tag{5.17}$$

by Lemma 5.15. (This can be seen as follows. If $V(C^*) \subseteq V(H_e)$, then C^* contains at most two edges between a and b, and hence $i_G(X) \geq 2|X| - 1 + (k-2) = 2|X| - 3 + k$. Suppose $V(C^*) \setminus V(H_e) \neq \emptyset$. In this case, by Lemma 5.15, C^* contains at most one edge between a and b. If C^* contains an edge between a and b, we have $i_G(X) \geq 2|X| - 2 + (k-1)$ by Lemma 5.15. Otherwise, we have $i_G(X) \geq 2|X| - 3 + k$, again by Lemma 5.15.) Finally we claim

$$i_G(X \cup Y) \leq 2|X \cup Y| - 2 + k. \tag{5.18}$$

To see this, recall that $E(H_e) + f_0$ is a balanced circuit. Also $X \cup Y$ induces $E(H_e) + e$ and the edges on $\{a,b\}$. Thus, if $f_0 \in E(G)$ then we have $i_G(X \cup Y) \leq (2|X \cup Y| - 2) + 1 + (k-1)$, and otherwise we have $i_G(X \cup Y) \leq (2|X \cup Y| - 3) + 1 + k$. Hence (5.18) holds.

Combining (5.16)(5.17)(5.18), we get

$$2i_G(Y) + d(X,Y) = i_G(Y) + i_G(X \cup Y) - i_G(X) \leq 4|Y| - 2 \quad \text{if } |Y| \geq 2,$$

and

$$d(X,Y) = i_G(X \cup Y) - i_G(X) \leq 2|Y| + 1 = 3 \quad \text{if } |Y| = 1.$$

Those imply that, if $Y \neq \emptyset$, then G has a vertex of degree three in Y. By the minimum degree condition, we conclude that $Y = \emptyset$.

In the following two claims we show that $G - f$ is M-connected. For any $e_1, e_2 \in E(G - f)$, we denote $e_1 \sim e_2$ if $G - f$ has a circuit that contains e_1 and e_2. Since $\sim$ is an equivalence relation, for the M-connectivity of $G - f$ it suffices to show $e' \sim e$ for every $e' \in E(G - f)$.

Claim 5.24. *For any $e' \in E(H_e - f - f_0)$, $e' \sim e$. Also $C^* \cup (E(H_e) - f - f_0)$ is periodically rigid.*

Proof. Since C^* is an unbalanced circuit by Claim 5.22, C^* is periodically rigid. If $t = 1$, then Claim 5.23 implies $V(H_e) \subseteq V(C^*)$, and hence $e' \in \mathrm{cl}_2(C^*)$ for every $e' \in E(H_e - f - f_0)$. Thus the claim holds for $t = 1$.

Suppose $t > 1$. Recall that f is incident to v. In view of Claim 5.21, $\tilde{C}_t$ is equal to the set of edges in $G-e$ incident to v. Hence $f \in \tilde{C}_t$, implying $f \notin C_i$ for $1 \leq i \leq t-1$. Also, since C^* is incident to v and $d_{G-f}(v) = 3$, C^* contains all the edges in $G - f$ incident to v. Thus we have

$$C^* \cup (E(H_e) - f - f_0) = C^* \cup \bigcup_{i=1}^{t-1} (C_i - f_0),$$

and the proof of the claim follows by showing the following by induction on i from $i = 1$ to $t - 1$:

$$e' \sim e \text{ for any } e' \in C_i - f_0 \text{ and } C^* \cup \bigcup_{j=1}^{i} (C_j - f_0) \text{ is periodically rigid.} \tag{5.19}$$

To see this, we first note that $f_0 \in \mathrm{cl}_2(C^*)$ since C^* is periodically rigid and $\{a, b\} \subset V(C^*)$. Hence $C^* + f_0$ contains a circuit C_0^* with $f_0 \in C_0^*$.

For $i = 1$, consider any $e' \in C_1 - C^* - f_0$. The circuit elimination implies that $C_1 \cup C_0^* - f_0$ contains a circuit $C_{e'}$ with $e' \in C_{e'}$. Notice that $C_{e'}$ is a circuit in $G-f$ and has a nonempty intersection with C^*. Since $\sim$ is an equivalence relation, this implies $e' \sim e$. Also $C^* \cup C_{e'}$ is periodically rigid by Lemma 5.4. Since $\bigcup_{e' \in C_1 - C^* - f_0} (C^* \cup C_{e'}) = C^* \cup (C_1 - f_0)$, $C^* \cup (C_1 - f_0)$ is periodically rigid.

For $i > 1$, consider any $e' \in C_i - C^* - f_0$. If $f_0 \in C_i$, then we can apply the same argument. (Specifically, the circuit elimination implies that $C_i \cup C_0^* - f_0$ contains a circuit $C_{e'}$ with $e' \in C_{e'}$. Since $C_{e'}$ is a circuit in $G - f$ and has a nonempty intersection with C^*, we get $e' \sim e$.) Suppose $f_0 \notin C_i$. According to the definition of ear decompositions, C_i has a nonempty intersection with $\bigcup_{j=1}^{i-1} C_j$. Take an edge h from the intersection. Note that $h \neq f_0$. Since $\{e', h\} \subseteq C_i$, we get $e' \sim h \sim e$ by induction. Also the periodic rigidity of $C^* \cup \bigcup_{j=1}^{i-1} (C_j - f_0)$ follows from Lemma 5.4 and the induction hypothesis. This completes the proof of (5.19) as well as the proof of the claim.

Claim 5.25. *For any $e' \in E(G - f) \setminus E(H_e - f_0)$, $e' \sim e$.*

Proof. We may assume $e' \notin C^*$. Let $F = C^* \cup (E(H_e) - f - f_0)$. By Claim 5.24, F is periodically rigid. As f is induced by $V(F)$, $f \in \mathrm{cl}_2(F)$, and G has a circuit C'' with

$f \in C'' \subseteq F+f$. On the other hand, G contains a circuit $C_{e'}$ with $e', f \in C_{e'}$ by the M-connectivity of G. Hence, by the circuit elimination, we get a circuit $C'_{e'} \subset C_{e'} \cup C'' - f$ with $e' \in C'_{e'} \subseteq E(G-f)$. Note that every element in F is related to e by Claim 5.24. Since $C'_{e'} \cap F \neq \emptyset$, we conclude that $e' \sim e$.

By Claim 5.24 and Claim 5.25, we conclude that $G - f$ is M-connected.

The following lemma lists properties of $(0,2)$-blocks which we will use frequently in the following. Most of them follow directly from the definition.

Lemma 5.26. *Let G be an unbalanced and M-connected graph having no $(0,2)$-block, and let $e \in E(G)$. Suppose that $G - e$ is unbalanced and M-connected, but $G - e$ has a $(0,2)$-block H_e. Then the following hold.*

(a) *Any edge of $G-e$ induced by $V(H_e)$ is included in H_e unless it is on the boundary $B(H_e)$, i.e., $E_{G-e}(V(H_e)) - E_{G-e}(B(H_e)) \subseteq E_{G-e}(H_e)$.*

(b) *At least one endvertex of e is included in $I(H_e)$.*

(c) *If H_e is an inclusionwise minimal $(0,2)$-block in $G - e$, then the cleavage graph of H_e is 3-connected.*

(d) $|E(G-e) \setminus E(H_e)| \geq 2$.

(e) *$(G-e) - I(H_e)$ is connected.*

Proof. (a) This directly follows from the definition of $B(H_e)$.

(b) If both endvertices of e are not in $I(H_e)$, then H_e would be a $(0,2)$-block in G, a contradiction.

(c) If the cleavage graph of H_e is not 3-connected, then $G - e$ would have a proper subgraph of H_e which is a $(0,2)$-block. This contradicts the minimality of H.

(d) Recall that H_e is balanced. Hence $G - e$ contains at least two edges which are not in H_e, since otherwise G cannot be M-connected.

(e) Let $B(H_e) = \{a, b\}$. Suppose that $(G-e) - I(H_e)$ is disconnected. Since $G-e$ is connected, $(G-e) - I(H_e)$ has two connected components C_a and C_b containing a and b, respectively. In particular, there is no edge between a and b in $G - e$. If any one of the two components is nontrivial, then $G - e$ has a cut vertex. However, since $G - e$ is M-connected, Lemma 5.8 implies that $G - e$ is 2-connected, a contradiction.

Thus we may assume that both components are trivial. Then we have $V(H_e) = V(G)$. By (a), every edge in $E(G-e) - E(H_e)$ lies on $\{a, b\}$. However, since $G - e$ has no edge between a and b, we would have $E(G-e) = E(H_e)$, contradicting (d).

We are now ready to prove Lemma 4.6.

Proof. [Proof of Lemma 4.6] By Theorem 5.11 it suffices to show that, for any M-connected graph G having no $(0,2)$-block and whose minimum degree is at least four, there exists $e \in E(G)$ such that $G - e$ is M-connected and has no $(0,2)$-block. By Lemma 5.14, G is not minimally M-connected. In other words, $E' = \{e \in E : G - e \text{ is M-connected}\}$ is not empty.

Suppose that $G-e'$ has a $(0,2)$-block for every $e' \in E'$. Let $H_{e'}$ be an inclusionwise minimal $(0,2)$-block in $G - e'$ for each $e' \in E'$, and take $e \in E'$ such that $|V(H_e)|$

is as small as possible. Since the minimum degree of G is at least four, Lemma 5.20 implies that H_e contains an edge f such that $G - f$ is M-connected.

Let $B(H_e) = \{a, b\}$ and $B(H_f) = \{x, y\}$. The cleaving edges for H_e and H_f are denoted by f_{ab} and f_{xy}, respectively. By Lemma 5.18 and Lemma 5.26(c), $H_e + f_{ab}$ and $H_f + f_{xy}$ are (balanced) M-connected and 3-connected. Also, by Lemma 5.20, we may suppose that

$$H_e - f - f_{ab} \text{ is 2-connected if an endvertex of } e \text{ is not in } I(H_e). \tag{5.20}$$

We first claim the following technical fact.

Claim 5.27. *If $V(G) = V(H_f)$, then x and y are contained in $I(H_e)$.*

Proof. Suppose $V(G) = V(H_f)$ but x is not contained in $I(H_e)$. Let $F = E(G - f) \setminus E(H_f)$. By $V(G) = V(H_f)$ and $|B(H_f)| = 2$, F is the set of parallel edges on $\{x, y\}(= B(H_f))$.

Since $f \in E(H_e)$ and $H_e + f_{ab}$ is 3-connected, H_e contains a cycle C_a (resp., C_b) that passes through f and avoids a (resp., b). Since x is not contained in $I(H_e)$, C_a or C_b avoids x, which means that $C_a \cap F = \emptyset$ or $C_b \cap F = \emptyset$. Without loss of generality assume $C_a \cap F = \emptyset$. Hence $C_a \subset H_f + f$. Since $f \in C_a \subset H_e \cap (H_f + f)$ and H_e and H_f are balanced, we conclude that $H_f + f$ is balanced. This however implies that $H_f + f$ is a $(0, 2)$-block in G, which is a contradiction.

We have four cases depending on the relative positions among $\{a, b, x, y\}$.

Case 1: $x \in I(H_e)$ and $y \in V(G) \setminus V(H_e)$. In this case $\{a, b\}$ is a cut of $G - e - f$ since x and y belong to different components in $(G - e) - a - b$. By Claim 5.27, $\{x, y\}$ is also a cut of $G - e - f$. (If $\{x, y\}$ is not a cut in $G - e - f$, then it is also not a cut in $G - f$ and hence $V(G) = V(H_f)$ follows. Hence Claim 5.27 implies $x, y \in I(H_e)$, contradicting $y \in V(G) \setminus V(H_e)$.) Now, from the fact that $\{a, b\}$ and $\{x, y\}$ are cuts of $G - e - f$, $G - e - f$ can be decomposed into four subgraphs $G_1, \ldots, G_4$ by taking $G_1 = (H_e - f) \cap (H_f - e)$, $G_2 = (H_f - e) \cap (G - I(H_e))$, $G_3 = (G - I(H_f)) \cap (H_e - f)$ and $G_4 = (G - I(H_f)) \cap (G - I(H_e))$. Then G_1, G_2 and G_3 are balanced, and each G_i has at least two vertices. Thus, $r_2(G - e - f) \leq \sum_{i=1}^{4} r_2(G_i) \leq \sum_{i=1}^{4} 2|V(G_i)| - 11 \leq 2|V(G)| - 3$, which contradicts that $G - e - f$ is periodically rigid as $G - e$ is unbalanced M-connected.

Case 2: $x \in I(H_e)$ and $y \in I(H_e)$. We first show the following:

$$(H_e - f) \cup (H_f - e) = G - e - f. \tag{5.21}$$

By the minimum choice of $V(H_e)$, there must exist $w \in V(H_f) \setminus I(H_e)$. By Lemma 5.26(e), $G - I(H_e)$ is connected and hence there is a path from w to any vertex in $G - I(H_e)$. Since such a path avoids $\{x, y\}$ by $\{x, y\} \subseteq I(H_e)$, we get $V(G) \setminus I(H_e) \subseteq I(H_f)$. (Note that such a path starts from the interior of H_f and remains in the interior, since it avoids the boundary $\{x, y\}$.) Hence every edge induced by $V(G) \setminus I(H_e)$ is included in H_f by Lemma 5.26(a), implying (5.21).

We next prove

$$(H_e - f) \cap (H_f - e) \text{ contains a path between } a \text{ and } b. \tag{5.22}$$

Suppose not. Since $x, y \in I(H_e)$ and $a, b \in V(H_f)$ by (5.21), there are only two possibilities for $H_f - e$: (i) a and b are both cut vertices of $H_f - e$, or (ii) a (or b) is a cut vertex of $H_f - e$ and x, y belong to the same component in $(H_f - e) - a$ (or $(H_f - e) - b$). Now consider the cleavage graph $H_f + f_{xy}$ of H_f. $H_f + f_{xy}$ is obtained from $H_f - e$ by adding f_{xy} and e. However, the above two possibilities imply that $H_f + f_{xy}$ is not 3-connected, which is a contradiction.

We next prove

$$G - e - f \text{ is balanced.} \tag{5.23}$$

To see this suppose $G - e - f$ has an unbalanced cycle C. Suppose that $C \cap (E(H_e) - f) = \emptyset$. Then, by (5.21), $C \subseteq E(H_f) - e$ holds, and hence H_f is unbalanced, which is a contradiction to the balancedness of H_f. Thus, $C \cap (E(H_e) - f) \neq \emptyset$, and the symmetric argument also gives $C \cap (E(H_f) - e) \neq \emptyset$. Therefore, as $B(H_e) = \{a, b\}$, C can be decomposed into two nonempty paths P_1 and P_2 such that P_1 is a path in $H_e - f$ between a and b. By (5.21) P_2 is a path in $H_f - e$ between a and b. Also by (5.22) we have a path P_3 between a and b in $(H_e - f) \cap (H_f - e)$. Since $P_1 \cup P_2$ is unbalanced, $P_1 \cup P_3$ or $P_2 \cup P_3$ is unbalanced. This however contradicts that both H_e and H_f are balanced since $P_1 \cup P_3 \subseteq H_e$ and $P_2 \cup P_3 \subseteq H_f$. Thus we get (5.23).

By (5.23) we can suppose that the label of each edge in $G - e - f$ is identity. Since $f \in E(H_e)$, H_e contains a cycle passing through f. Since H_e is balanced, the label of f is the identity. Thus $G - e$ is balanced, and this contradicts that G is unbalanced and M-connected.

Case 3: $x \in I(H_e)$ and $y \in B(H_e)$. Without loss of generality we assume $b = y$. By Claim 5.27, $\{x, y\}$ is a cut of $G - e - f$. Due to the minimality of H_e, $G - e - f$ can be decomposed into three subgraphs G_1, G_2, G_3 by taking $G_1 = (H_e - f) \cap (H_f - e)$, $G_2 = (H_e - f) \cap (G - I(H_f))$ and $G_3 = (G - I(H_e)) \cap (H_f - e)$.

Observe that either $G_1 \cup G_3 + f_{xy}$ or $G_1 \cup G_3 + f_{xy} + e$ is the cleavage graph of H_f. Since a is a cut vertex in $G_1 \cup G_3$, by the 3-connectivity of the cleavage graph, e connects a vertex in $V(G_1) \setminus \{a\}$ with a vertex in $V(G_3) \setminus \{a\}$. Since $V(G_3) \subseteq V(G) \setminus I(H_e)$, we conclude that an endvertex of e is not in $I(H_e)$. Thus, by (5.20), $H_e - f - f_{ab}$ is 2-connected. However, $G_1 \cup G_2 = H_e - f - f_{ab}$ holds, and x is a cut vertex in $G_1 \cup G_2$. This is a contradiction.

Case 4: $x \notin I(H_e)$ and $y \notin I(H_e)$. We claim that

$$I(H_e) \not\subseteq I(H_f) \text{ and } H_e - f - f_{ab} \not\subseteq H_f. \tag{5.24}$$

Note that $I(H_e) \subseteq I(H_f)$ implies $H_e - f - f_{ab} \subseteq H_f$, by Lemma 5.26(a). Hence it suffices to show $H_e - f - f_{ab} \not\subseteq H_f$. To see this suppose $H_e - f - f_{ab} \subseteq H_f$. Since $f \in E(H_e)$, we can take a cycle C in $H_e - f_{ab}$ that contains f. This cycle is balanced since H_e is balanced. By $H_e - f - f_{ab} \subseteq H_f$, C is contained in $H_f + f$. As C is balanced with $f \in C$, it turns out that $H_f + f$ is balanced with $B(H_f + f) = B(H_f)$, and $H_f + f$ is a $(0, 2)$-block in G, a contradiction. Thus we have (5.24).

We next claim $I(H_e) \cap I(H_f) \neq \emptyset$. Indeed, an endvertex of f is in $I(H_f)$ by Lemma 5.26(b), and this vertex is in $V(H_e)$ by $f \in E(H_e)$. If this vertex belongs to $I(H_e)$, we are done. If this vertex is in $B(H_e)$, then the other endvertex of f is in

$I(H_e)$. This vertex also belongs to $I(H_f)$, since H_e has a path between the endvertices of f avoiding $\{x, y\}$ by the 3-connectivity of $H_e + f_{ab}$ and $x, y \notin I(H_e)$.

Thus we can take $v^* \in I(H_e) \cap I(H_f)$. Suppose that $\{x, y\} \neq \{a, b\}$, say $a \notin \{x, y\}$. Since $H_e + f_{ab}$ is 3-connected, for any $u \in I(H_e)$, H_e has a path from u to v^* that avoids b and f. Since $x, y \notin I(H_e)$, this in turn implies that $G - f$ has a path between $v^* \in I(H_f)$ and $u \in I(H_e)$ that avoids $\{x, y\}$. Thus $I(H_e) \subseteq I(H_f)$, which is a contradiction to (5.24).

Thus we have $\{x, y\} = \{a, b\}$. In other words, $B(H_e) = B(H_f)$. We next prove

$$e \in E(H_f). \tag{5.25}$$

Suppose not. Then we have the following:

$$\text{a vertex } u \in V(G) \text{ belongs to } I(H_f) \text{ if and only if } G - e - f \text{ has a path from } u \text{ to } v^* \text{ that avoids } a \text{ and } b. \tag{5.26}$$

Indeed, if $G - e - f$ has a path from $u \in V(G)$ to v^* avoiding a and b, then $u \in I(H_f)$ by $v^* \in I(H_f)$. On the other hand, if $u \in I(H_f)$, then $H_f - x - y$ has a path between u and v^* since $H_f + f_{xy}$ is 3-connected. This path exists in $G - e - f$ since $e \notin E(H_f)$.

To complete the proof of (5.25), recall $v^* \in I(H_e) \cap I(H_f)$. Then (5.26) implies $I(H_f) \subseteq I(H_e)$. This actually implies $I(H_f) = I(H_e)$ due to the minimality of $V(H_e)$, which however contradicts (5.24). Thus (5.25) holds.

By the 3-connectivity of $H_f + f_{xy}$ and $\{x, y\} = \{a, b\}$, $H_f - e$ has a path between v^* and a that avoids b and a path between v^* and b that avoids a. As $\{a, b\}$ is a cut in $G - e$, those two paths are contained in H_e. Also, since $f \notin H_f$, those paths are actually contained in $(H_e - f) \cap (H_f - e)$. Thus $(H_e - f) \cap (H_f - e)$ contains a path between a and b, and hence $(H_e - f) \cup (H_f - e)$ is balanced (by the same argument as (5.23)). Thus we may assume that the label of each edge in $(H_e - f) \cup (H_f - e)$ is identity.

Recall that $f \in E(H_e)$. Due to the 3-connectivity of $H_e + f_{ab}$, H_e has a cycle passing through f. Since H_e is balanced, the label of f is identity. In the same way, (5.25) implies that the label of e is identity. In total, $H_e \cup H_f$ is balanced. However, by $f \in E(H_e)$, $e \in E(H_f)$, and $B(H_e) = B(H_f)$, $H_e \cup H_f$ turns out to be a $(0, 2)$-block in G. This contradiction completes the proof.

6 Concluding Remarks

6.1 Global rigidity of cylindrical/troidal frameworks

Given $\ell \in \mathbb{R}^2$, we consider the following equivalence relation of $\mathbb{R}^2$:

$$\text{for } a, b \in \mathbb{R}^2, a \sim_1 b \text{ if and only if } a = b + z\ell \text{ for some } z \in \mathbb{Z}.$$

A flat cylinder $\mathcal{C}_\ell = \mathcal{C}$ is obtained by factoring out $\mathbb{R}^2$ with the relation $\sim_1$. Similarly, given a pair of vectors $\ell_1 \ell_2$ of $\mathbb{R}^2$, consider an equivalence relation of $\mathbb{R}^2$ by:

$$\text{for } a, b \in \mathbb{R}^2, a \sim_2 b \text{ if and only if } a = b + z_1\ell_1 + z_2\ell_2 \text{ for some pair } z_1, z_2 \in \mathbb{Z}.$$

A flat torus $\mathcal{T}_{\ell_1,\ell_2} = \mathcal{T}$ is obtained by factoring out $\mathbb{R}^2$ with the relation $\sim_2$.

Given a straight-line drawing of an undirected graph on $\mathcal{C}$ (resp. on $\mathcal{T}$), we regard it as a bar-joint framework. Note that such frameworks on $\mathcal{C}$ (resp. on $\mathcal{T}$) has a one-to-one correspondence with 1-periodic (resp. 2-periodic) frameworks in $\mathbb{R}^2$ through $\sim_1$ (resp. $\sim_2$). Hence our combinatorial characterization of the global rigidity of periodic frameworks immediately implies a characterization of the global rigidity of cylindrical/toroidal frameworks.

The underlying combinations of a drawing on $\mathcal{C}$ (resp. on $\mathcal{T}$) is captured by using a $\mathbb{Z}$-labeled graph (G, ψ) (resp. $\mathbb{Z}^2$-labeled graph), where each label determines the geodesic between two endvertices. A cylindrical framework (resp. a toroidal framework) is defined as a pair (G, ψ, p) of a $\mathbb{Z}$-labeled graph (G, ψ) (resp. $\mathbb{Z}^2$-labeled graph) and $p : V(G) \to \mathcal{C}$ (resp. $p : V(G) \to \mathcal{T}$). Note that a subgraph H is balanced if and only if it is contractible on the surface. Hence Theorem 4.1 can be translated to Theorem 1.3 for cylindrical frameworks. For toroidal frameworks the statement becomes as follows.

Theorem 6.1. *A generic framework (G, p) with $|V(G)| \geq 3$ on $\mathcal{T}$ is globally rigid if and only if each two-connected component is redundantly rigid on $\mathcal{T}$, has no contractible subgraph H with $|V(H)| \geq 3$ and $|B(H)| = 2$, and has rank two.*

6.2 Open problems

There are quite a few remaining questions. As mentioned in the introduction, an important challenging problem is to extend our results to more general settings of global rigidity of periodic frameworks, as was done for local rigidity in [17, 18].

As mentioned above, Theorems 4.1 and 4.2 may be viewed as a characterization of generic globally rigid bar-joint frameworks on a flat cylinder or a flat torus, respectively. A natural open problem is to establish counterparts of Theorem 1.2 in other flat Riemannian manifolds.

A similar question would be about the global rigidity of frameworks on a flat cone. Since a flat cone with cone angle $2\pi/n$ is the quotient of $\mathbb{R}^2$ by an n-fold rotation, the global rigidity of such frameworks can be understood by the global rigidity of frameworks in $\mathbb{R}^2$ with n-fold rotational symmetry (under the given symmetry constraints). The corresponding local rigidity question has been studied in [15, 19] and an extension of Laman's theorem is known. Since the space of trivial motions is only of dimension one in this case (only rotations are trivial), an unbalanced rigidity circuit G satisfies the count $|E(G)| = 2|V(G)|$. Therefore, we cannot guarantee the existence of a vertex of degree three in G. This constitutes the key obstacle in applying our current proof method to this problem.

We may also ask about the global rigidity of finite frameworks with other point group symmetries. However, for the same reason, it also remains open to characterize the symmetry-forced global rigidity of finite bar-joint frameworks in $\mathbb{R}^2$ that are generic modulo reflection or dihedral symmetry. In fact, it is currently not even known whether symmetry-forced global rigidity is a generic property for any point group in dimension 2. Necessary redundant rigidity and connectivity conditions, however, may be obtained in a similar fashion as described in Section 3.

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