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Independent and maximal branching packing in infinite matroid-rooted digraphs

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Abstract

We prove a common generalization of the maximal independent arborescence packing theorem of Cs. Király [13] (which itself is a common generalization of the reachability based arborescence packing result [12] and a matroid based arborescence packing result [5]) and two of our earlier works about packing branchings in infinite digraphs, namely [11] and [10].

1 Introduction

Edmonds' branching packing theorem [6] has been generalized in several different directions. The most up to date survey in 2016 about these results that we know is in [7]. Branching packing problems are mostly investigated in finite digraphs but it turned out that in some cases one can relax the finiteness condition of the digraph to some restriction of the forward-infinite or backward-infinite directed paths (see [11] and [10]). The main result of this paper (Theorem 3.7) is to give such an infinite generalization of [13] which itself is a common generalization of the reachability based arborescence packing result [12] and a matroid-based arborescence packing result [5]. We replace the finiteness of D by some restriction of the behaviour of either its forward-infinite (Condition 3.5) or its backward-infinite paths (Condition 3.6). We also show by examples that some obvious further weakenings of our conditions are not possible.

2 Notations

We use some basic set theoretic notation. For the power set of X, we write $\mathcal{P}(X)$. Intersection has higher priority than union. The variables $\alpha, \beta, \gamma, \xi$ always stand for ordinals. We denote the smallest infinite cardinal (i.e. the set of the natural numbers) by ω. If κ is a cardinal, then κ^+ is its successor cardinal. The restriction of a function F to the subset X of its domain is denoted by $F|_X$, and $F[X]$ stands for the image of $F|_X$. We use the abbreviation $B - x + y$ for the set $(B \setminus \{x\}) \cup \{y\}$.

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2.1 Digraphs

The digraphs $D = (V, A)$ in this paper may have multiple edges but does not have loops. If $e \in A$, then $\boldsymbol{D - e}$ is an abbreviation of $(V, A \setminus \{e\})$. For $X \subseteq V$, we denote by $\boldsymbol{D[X]}$ the subdigraph induced by X. If the edge e goes from u to v, then $\mathbf{tail}(\boldsymbol{e}) = u$ and $\mathbf{head}(\boldsymbol{e}) = v$. The set of the ingoing and outgoing edges of $X \subseteq V$ are denoted by $\mathbf{in}_{\boldsymbol{D}}(\boldsymbol{X})$ and $\mathbf{out}_{\boldsymbol{D}}(\boldsymbol{X})$, respectively. For a singleton $\{v\}$, we write $\mathsf{in}_D(v)$ instead of $\mathsf{in}_D(\{v\})$ and we use this kind of abbreviation in connection with singletons in the case of the other set-functions as well.

The paths in this paper are directed, repetition of vertices is not allowed, and they are finite unless we say explicitly otherwise. We may define paths by the corresponding vertex sequence if parallel edges do not appear there. This sequence determines on ordering $<_P$ on $V(P)$. We denote by $\mathbf{start}(\boldsymbol{P})$ the $<_P$-smallest and by $\mathbf{end}(\boldsymbol{P})$ the $<_P$-largest vertex of a path P. For $u <_P v$, the subdigraph of P induced by the elements of the interval $[u, v]$ is denoted by $\boldsymbol{P[u, v]}$ and called the segment of P from u to v. The initial segments of P are the segments in the form $P[\mathsf{start}(P), v]$. We define terminal segments similarly. If an initial segment of P is identical to a terminal segment of Q, then we may join them to a walk and simplify that to a path that we call the **concatenation** of P and Q. We say that **path $\boldsymbol{P}$ goes from $\boldsymbol{X}$ to $\boldsymbol{Y}$** (or shortly P is a $\boldsymbol{X \to Y}$ path) if $\mathsf{start}(P) \in X$ and $\mathsf{end}(P) \in Y$. Path P goes **strictly** from X to Y if exactly the first vertex of P is in X and exactly the last is in Y (**strict** $X \to Y$ path). Let $\mathbf{to}_{\boldsymbol{D}}(\boldsymbol{X})$ be the set of those vertices from which X is reachable by a directed path in D. A path may consist of a single vertex in which case it is a **trivial path**. For a system $\mathcal{P}$ of paths, let $\boldsymbol{A(\mathcal{P})} = \bigcup_{P \in \mathcal{P}} A(P)$ and we denote by $\boldsymbol{A_{last}(\mathcal{P})}$ the set of the last edges of the (not forward-infinite) paths in $\mathcal{P}$.

A digraph D is called a **branching** if it is a directed forest in which every vertex is reachable by a unique path from $X := \{v \in V(D) : |\mathsf{in}_D(v)| = 0\}$. This X is the **root set** of the branching.

2.2 Infinite matroids

There were several attempts to extend the notion of matroid by allowing infinite ground sets but keeping the concept of duality. Finally in [4] the authors achieved this goal which made possible the intensive development of the field. In this paper we need to use just some very basic facts about infinite matroids. Most of these are well-known for finite matroids and have the same proof in the infinite case thus readers with knowledge only about the finite matroids have no disadvantage. In this subsection, we give the notations and the facts that we will use in connection with matroids.

The pair $\mathcal{M} = (S, \mathcal{I})$ is a matroid if $\mathcal{I} \subseteq \mathcal{P}(S)$ and it satisfies the following axioms.

1. $\varnothing \in \mathcal{I}$,
2. $I \subseteq I' \in \mathcal{I}$ implies $I \in \mathcal{I}$,
3. if B is a $\subseteq$-maximal element of $\mathcal{I}$ and $I \in \mathcal{I}$ is not maximal, then is an $i \in B \setminus I$ such that $(I \cup \{i\}) \in \mathcal{I}$,

4. if $I \in \mathcal{I}$ and $I \subseteq X \subseteq S$, then the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a $\subseteq$-maximal element.

The elements of $\mathcal{I}$ are called **independent** sets the other subsets of S are **dependent**. The $\subseteq$-maximal independent sets (they exist by using axiom 4 with $I := \varnothing$ and $X := S$) are the **bases** of the matroid. In notation, sometimes we will not distinguish the matroid from its ground set unless it would lead misunderstanding.

Fact 2.1. If B_1 and B_2 are bases of the same matroid and $B_1 \setminus B_2$ is finite, then $|B_1 \setminus B_2| = |B_2 \setminus B_1|$.

It implies that if there is a finite base, then all the bases are finite and have the same size $\boldsymbol{r(\mathcal{M})}$ which is called the **rank** of $\mathcal{M}$. ZFC alone is not able to decide if the bases of a matroid have necessarily the same cardinality. The Generalized Continuum Hypothesis decides the question affirmatively (as shown by D. A. Higgs in [8]) but it is false under some other set theoretic assumptions (proved by N. Bowler and S. Geschke in [3]). Hence if there is no finite base, then the rank is simply ∞. For $S' \subseteq S$, the pair $(S', \mathcal{I} \cap \mathcal{P}(S'))$ is a matroid, it is the submatroid of $\mathcal{M}$ that we get by **restriction** to S'. For $S' \subseteq S$, we denote by $\boldsymbol{r(S')}$ the rank of the submatroid (corresponding to) S'. A $\subseteq$-minimal dependent set is called a **circuit**.

Fact 2.2. A set $S' \subseteq S$ is dependent if and only if it contains a circuit (which is not straightforward for an infinite S').

Fact 2.3. The relation $\{\langle x, y\rangle \in S \times S : \exists C \text{ circuit with } x, y \in C\}$ is transitive.

By adding the diagonals, we may extend the relation above to an equivalence relation. The equivalence classes are called the **components** of the matroid. A matroid is called **finitary** if all of its circuits are finite. In these matroids, an infinite set is independent if and only if all of its finite subsets are independent, in fact this property characterize the finitary matroids.

Fact 2.4. Fix a base B of (the submatroid corresponding to) $S' \subseteq S$. Then the subsets I of $S \setminus S'$, for which $(I \cup B) \in \mathcal{I}$, forms a matroid on $S \setminus S'$ and it does not depend on the choice of B. It is the submatroid that we get by **contracting** S'.

For $S', S'' \subseteq S$, we may restrict first $\mathcal{M}$ to $S' \cup S''$ and in the resulting matroid contract S''. In this case, we denote the resulting submatroid by $\boldsymbol{S'/S''}$.

If $\mathcal{M}_\xi = (S_\xi, \mathcal{I}_\xi)$ $(\xi < \kappa)$ are matroids with pairwise disjoint ground sets, then the **direct sum** $\mathcal{M} := \boldsymbol{\bigoplus_{\xi<\kappa} \mathcal{M}_\xi}$ of the matroids $\{\mathcal{M}_\xi\}_{\xi<\kappa}$ is the matroid on $\bigcup_{\xi<\kappa} S_\xi$ where $I \in \mathcal{I}_\mathcal{M}$ if and only if $(I \cap S_\xi) \in \mathcal{I}_\xi$ for all $\xi < \kappa$. Every matroid is the direct sum of its components.

$i \in S$ is called a **loop** if $\{i\}$ is a circuit. We denote by $\boldsymbol{\mathsf{span}(S')}$ the union of S' and the loops of S/S'.

Fact 2.5. span is a closure operator.

Fact 2.6 (weak circuit elimination)**.** If C_1, C_2 are circuits with $i \in C_1 \cap C_2$, then $C_1 \cap C_2 - i$ contains a circuit.

Corollary 2.7. *If $i \in \mathsf{span}(I) \setminus I$ for some independent set I, then there is a unique circuit $C \subseteq I \cup \{i\}$. Necessarily $i \in C$ since I is independent.*

For $i \in \mathsf{span}(I)$, let us define $\boldsymbol{C(i, I)} = \begin{cases} \text{the singleton } \{i\} & \text{if } i \in I, \\ \text{the unique circuit } C \text{ above} & \text{if } i \notin I. \end{cases}$

Fact 2.8. If B is a base and $i \in S \setminus B$, then for any $j \in C(i, B)$ the set $B - j + i$ is a base again.

Corollary 2.9. *If I is independent and $i \in I \cap \mathsf{span}(J)$ for some $J \subseteq S$, then there is some $j \in J$ ($j = i$ is allowed) such that $I - i + j$ is independent. Furthermore i and j are in the same component of the matroid and if I was a base, then Fact 2.1 ensures that $I - i + j$ is a base as well.*

One can find a detailed survey about the theory of infinite matroids in the Habilitation thesis of N. Bowler [2].

2.3 Matroid-rooted digraphs

We call a triple $\mathfrak{R} = (D_\mathfrak{R}, \mathcal{M}_\mathfrak{R}, \pi_\mathfrak{R})$ a **matroid-rooted digraph** if $D_\mathfrak{R} = (V, A)$ is a digraph, $\mathcal{M}_\mathfrak{R} = (S, \mathcal{I})$ is a matroid and $\pi_\mathfrak{R} : S \to \mathcal{P}(V) \setminus \{\varnothing\}$. We will omit the subscripts whenever they are clear from the context. For an $I \in \mathcal{I}$ and $T \subseteq V$ an **(I, T)-linkage** is a system of edge-disjoint paths $\{P_i\}_{i \in I}$ indexed by the elements of I such that P_i goes from $\pi(i)$ to T. In a **strict** linkage, P_i goes strictly from $\pi(i)$ to T. We say that I is **T-linkable** if such a linkage exists. A **branching packing $\mathcal{B}$** with respect to $\mathfrak{R}$ is a system of edge-disjoint branchings $\mathcal{B} = \{\mathcal{B}_i\}_{i \in S}$ in D where the root set of $\mathcal{B}_i$ is $\pi(i)$. A branching packing is **trivial** if none of the branchings in it have any edges. For $X \subseteq V$, let $\boldsymbol{\mathcal{S}(X)} = \{i \in S : \pi(i) \cap X \neq \varnothing\}$. The matroid-rooted digraph is called **independent** if $\mathcal{S}(v) \in \mathcal{I}$ for all $v \in V$. A branching packing is called independent if the matroid-rooted digraph $\mathfrak{B} := (D, \mathcal{M}, \pi_\mathfrak{B})$ is independent where $\pi_\mathfrak{B}(i) = V(\mathcal{B}_i)$. Let us denote $\mathsf{span}\left(\mathcal{S}(\mathsf{to}_D(X))\right)$ by $\boldsymbol{\mathcal{N}(X)}$ (the **need of** X). Clearly $\mathcal{N}(X) = \mathsf{span}\left(\bigcup_{v \in X} \mathcal{N}(v)\right)$. The branching packing $\mathcal{B}$ is **maximal** if for all $v \in V$ the set $\mathcal{S}_\mathfrak{B}(v) := \{i \in S : \pi_\mathfrak{B}(i) \cap X \neq \varnothing\}$ spans $\mathcal{N}(v)$. Hence a branching packing $\mathcal{B}$ is independent and maximal if and only if $S_\mathfrak{B}(v)$ is a base of $\mathcal{N}(v)$ for all $v \in V$.

3 Preparations

3.1 The linkage condition

For the existence of a maximal independent branching packing, the independence of $\mathfrak{R}$ is obviously necessary since $\mathcal{S}(v) \subseteq \mathcal{S}_\mathfrak{B}(v)$ holds for any branching packing $\mathcal{B}$. The maximality criteria leads to the following necessary condition.

Condition 3.1 (linkage condition)**.** For all $v \in V$, there exists a (B, v)-linkage in D where B is a base of $\mathcal{N}(v)$.

If we suppose that $\mathcal{M}$ and D are finite, then independence and the linkage condition are enough to ensure the existence of a maximal, independent branching packing as shown by Cs. Király in [13]. In fact, instead of Condition 3.1 he used the condition "$r(\mathcal{S}(X)) + |\mathsf{in}_D(X)| \geq r(\mathcal{N}(X))$ holds for all nonempty $X \subseteq V$". Simple examples show that the literal infinite generalization of this inequality with cardinals fails to be sufficient in the infinite case. In fact it does not even imply our Condition 3.1 although they are equivalent in the finite case.

We need a formally stronger (but in fact equivalent) version of Condition 3.1 which is more similar with the condition of Cs. Király.

Condition 3.2. For all nonempty $X \subseteq V$, there exists a (B, X)-linkage in D where B is a base of $\mathcal{N}(X)$.

A linkage above is called a **linkage for $\boldsymbol{X}$** if it is strict and B contains a base of $\mathcal{S}(X)$. Clearly, one can always ensure these extra regularity conditions by taking the appropriate segments of the paths and replace some of them with trivial paths. Sometimes we will not want to deal with these trivial paths. Throwing them away, the indices of the remaining paths form a base of $\mathcal{N}(X)/\mathcal{S}(X)$. A **reduced linkage for $\boldsymbol{X}$** is a strict (B, X)-linkage where B is a base of $\mathcal{N}(X)/\mathcal{S}(X)$.

Observation 3.3. Condition 3.2 is equivalent to demanding the existence of a (reduced) linkage for all nonempty $X \subseteq V$.

Proposition 3.4. *Condition 3.1 and 3.2 are equivalent.*

Proof: Condition 3.1 is just the restriction of Condition 3.2 to the singleton sets $X = \{v\}$ $(v \in V)$. We give a proof sketch for the nontrivial direction. Well-order X and pick a linkage $\{P_i\}_{i \in G_0}$ for the smallest element x_0 of X. Take a linkage $\{P'_i\}_{i \in B}$ for the following element x_1. Let $B' = \{i \in B : i \notin \mathsf{span}(G_0)\}$. We claim that for $j \in B'$ the path P'_j may not have a common edge (or even common vertex) with any path in $\{P_i\}_{i \in G_0}$. Indeed, if it has, then $j \in \mathcal{N}(x_0)$ and therefore $j \in \mathsf{span}(G_0)$ (since G_0 spans $\mathcal{N}(x_0)$), contradicting the choice $j \in B'$. But then $G_1 := G_0 \cup B'$ spans $\mathcal{N}(\{x_0, x_1\})$ and the path-system $\{P_i\}_{i \in G_0} \cup \{P'_i\}_{i \in B'}$ is edge-disjoint. One can finish the proof by transfinite recursion taking union of the path-systems at limit steps and trim the final system to be independent at the end. ●

3.2 The statement of the main result and feasible extensions

We propose the following two possible relaxations of the finiteness of D and $\mathcal{M}$.

Condition 3.5. The matroid $\mathcal{M}$ has finite rank, and for any forward-infinite path P the set $\mathcal{S}(V(P))$ spans $\mathcal{N}(V(P))$.

Condition 3.6. The matroid $\mathcal{M}$ has at most countably many components, all of which has finite rank. Furthermore, for any backward-infinite path P the set $\mathcal{S}(V(P))$ spans $\mathcal{N}(V(P))$.

Now we state our main result.

Theorem 3.7. *If the matroid-rooted digraph $\mathfrak{R} = (D, \mathcal{M}, \pi)$ satisfies independence, the linkage condition, and either Condition 3.5 or Condition 3.6, then there is an independent, maximal branching packing for $\mathfrak{R}$.*

Instead of dealing with the branchings directly, we introduce the notion of feasible extension of an $\mathfrak{R}$. Let $i_0 \in S$ and $e_0 \in A$ such that $e_0 \in \mathsf{out}_D(\pi(i_0))$ and $\mathcal{S}(\mathsf{head}(e_0)) \cup \{i_0\}$ is independent. The matroid-rooted digraph obtained by $\boldsymbol{(i_0, e_0)}$**-extension** from $\mathfrak{R} = (D, \pi, \mathcal{M})$ is $\mathfrak{R}_1 := (D - e_0, \mathcal{M}, \pi_1)$ where

$$\pi_1(i) = \begin{cases} \pi(i) & \text{if } i \neq i_0 \\ \pi(i) \cup \{\mathsf{head}(e_0)\} & \text{if } i = i_0. \end{cases}$$

This extension is an imitation of giving edge e_0 to branching $\mathcal{B}_{i_0}$. A matroid-rooted digraph $\mathfrak{R}'$ is an **extension** of $\mathfrak{R}$ if there is a transfinite sequence (build-sequence) of matroid-rooted digraphs $\langle \mathfrak{R}_\xi : \xi \leq \alpha \rangle$ (where $\mathfrak{R}_\xi = (D_\xi, \mathcal{M}, \pi_\xi)$ and $D_\xi = (V, A_\xi)$) with the following properties.

1. $\mathfrak{R}_0 = \mathfrak{R},\ \mathfrak{R}_\alpha = \mathfrak{R}'$,
2. $\mathfrak{R}_{\beta+1}$ is an (i_β, e_β)-extension of $\mathfrak{R}_\beta$ for some i_β, e_β,
3. for a limit β we have $\pi_\beta(i) = \bigcup_{\gamma<\beta} \pi_\gamma(i)$ and $D_\beta := (V, \bigcap_{\gamma<\beta} A_\gamma)$.

For an $\mathfrak{R}'$ extension of $\mathfrak{R}$, the sequence above is not necessarily unique but the **order** $|\alpha|$ of the extension is $(|\alpha| = |A(D_{\mathfrak{R}}) \setminus A(D_{\mathfrak{R}'})|)$. Let $S' \subseteq S$. If $\pi_{\mathfrak{R}}(i) = \pi_{\mathfrak{R}'}(i)$ whenever $i \notin S'$, then we say that $\mathfrak{R}'$ is an $\boldsymbol{S'}$**-extension** of $\mathfrak{R}$. We define the limit of a transfinite sequence of consecutive extensions in the same way as we defined the limit of transfinite sequence of (i, e)-extensions at the limit steps. It is routine to check that for any $v \in V$ and any $\mathfrak{R}'$ extension of $\mathfrak{R}$, we have $\mathcal{N}_{\mathfrak{R}'}(v) \subseteq \mathcal{N}(v)$. We call $\mathfrak{R}'$ a **feasible extension** (with respect to $\mathfrak{R}$) if it satisfies the following condition.

Condition 3.8. $\mathfrak{R}'$ is independent and satisfies the linkage condition; furthermore, $\mathcal{N}_{\mathfrak{R}'}(v) = \mathcal{N}(v)$ for all $v \in V$.

In longer terms: $\mathfrak{R}'$ is independent, and for all $v \in V$ there is a (B, v)-linkage where B is a base of $\mathcal{N}(v)$. It is easy to see that finding a branching packing $\{\mathcal{B}_i\}_{i \in S}$ for $\mathfrak{R}$ is equivalent to finding a feasible extension $\mathfrak{R}'$ of $\mathfrak{R}$ such that $\mathcal{S}_{\mathfrak{R}'}(v)$ is a base of $\mathcal{N}(v)$ for all $v \in V$. Here $A(\mathcal{B}_i)$ will consist of those edges e for which we had an (i, e)-extension in some fixed build-sequence of the extension $\mathfrak{R}'$.

Our plan is to construct a build-sequence of such an $\mathfrak{R}'$ extension. Any extension of an infeasible extension of $\mathfrak{R}$ is an infeasible extension of $\mathfrak{R}$, thus every member of the build-sequence needs to be feasible. On the one hand, a feasible extension of a feasible extension of $\mathfrak{R}$ is clearly a feasible extension of $\mathfrak{R}$. On the other hand, the limit of feasible extensions is not necessary feasible, therefore it is not enough to to ensure the existence of one single feasible (i, e)-extension. (In the finite case of course it is enough since after at most $|A|$-many (i, e)-extensions we are done. Furthermore, in this case, for any independent $\mathfrak{R}$ that satisfies the linkage condition there exists a feasible (i, e)-extension unless the trivial branching packing is already maximal.)

3.3 Counterexamples

As we have already mentioned, independence and the linkage condition are not enough to ensure the existence of an independent maximal branching packing. We show this fact by an example (Figure 1) where we do not even have a feasible (i, e)-extension although for any vertex v the set $\mathcal{S}(v)$ is not a base of $\mathcal{N}(v)$. Let $V = \{u_n\}_{n<\omega} \cup \{v_n\}_{n<\omega}$. The edges are u_1v_0, v_1u_0 furthermore, for $n < \omega$

$$u_nu_{n+1},\ v_nv_{n+1},\ u_{2n+3}u_{2n+1},\ v_{2n+3}v_{2n+1}.$$

Finally take the free matroid on $\{0, 1\}$, let $\pi(0) = \{u_{2n}\}_{n<\omega}$, and let $\pi(1) = \{v_{2n}\}_{n<\omega}$. It is routine to check (by using Figure 1) that linkage condition holds, i.e. every vertex is simultaneously reachable by edge-disjoint paths from the sets $\pi(0)$ and $\pi(1)$. To justify that there is no feasible (i, e)-extension, we give for any $e \in \mathsf{out}_D(\pi(0))$ a vertex set X_e such that for the $(0, e)$-extension $\mathfrak{R}_1$ we have

$$\mathcal{N}_{\mathfrak{R}_1}(X_e) = \{0\} \subsetneq \{0, 1\} = \mathcal{N}(X_e),$$

which shows the infeasibility. We also do the same for any $e \in \mathsf{out}_D(\pi(1))$. For $n < \omega$, let $X_{u_nu_{n+1}} = \{u_k\}_{n<k<\omega}$ and let $X_{v_nv_{n+1}} = \{v_k\}_{n<k<\omega}$.

In the example above, $\pi(0)$ and $\pi(1)$ are infinite. One can show that if we have a free matroid of arbitrary size and the set $\pi(i)$ is finite for some i, then there exists an edge e for which the (i, e)-extension is feasible. Even so, it does not help to construct an independent, maximal branching packing. Indeed, we give an other counterexample with the same matroid where we have $\pi(0) = \{u\}$ and $\pi(1) = \{v\}$. Pick a 2-edge-connected digraph D that contains vertices u, v such that there is no edge-disjoint back and forth paths between u and v. (Such a digraph exists, even with arbitrary large finite edge-connectivity as we have shown in [9].) From the 2-edge-connectivity it follows that every vertex can be reached simultaneously from u and v by edge-disjoint paths, thus the linkage condition holds. On the other hand, a maximal branching packing should contain back and forth paths between u and v which do not exist in D.

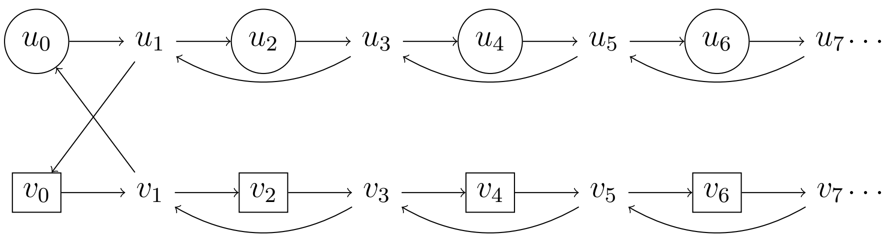

Figure 1: An independent matroid-rooted digraph that satisfies the linkage condition but has no feasible (i, e)-extension although $\mathcal{S}(v)$ is not a base of $\mathcal{N}(v)$ for any v. $\mathcal{M}$ is the free matroid on $\{0, 1\}$. Elements of $\pi(0)$ are circled and elements of $\pi(1)$ are in a rectangle in the figure.

In the examples above, the structure of the matroid was as simple as possible but the one-way infinite paths do not satisfy any of Condition 3.5 or Condition 3.6. Let us give another counterexample (Figure 2) in which, beyond the independence and the linkage condition, there is no infinite path at all (not even undirected) and the matroid is just a little bit more complicated than what Condition 3.6 allows.

Let $V = \{u_n\}_{n<\omega} \cup \{v_n\}_{n<\omega} \cup \{w\}$ and let $A = \{u_n v_n\}_{n<\omega} \cup \{v_n w\}_{n<\omega}$. The matroid will be a countable subset of the vectorspace $\mathbb{R}^\omega$ with the linear independence. We define

$$\mathcal{S}(u_n) := \{(\underbrace{0,\dots,0}_{n},1,0,\dots)\},\ \mathcal{S}(v_n) := \{(\underbrace{0,\dots,0}_{n},1,1,0,\dots),(\underbrace{0,\dots,0}_{n},1,-1,0,\dots)\},$$

$$\mathcal{S}(w) := \varnothing.$$

The resulting matroid-rooted digraph is clearly independent. The unique elements of the sets $\mathcal{S}(u_n)$ form a base of $\mathcal{N}(w)$, and paths u_n, v_n, w $(n < \omega)$ form a linkage for w. Considering the other vertices, $\mathcal{S}(u_n)$ and $\mathcal{S}(v_n)$ already span $\mathcal{N}(u_n)$ and $\mathcal{N}(v_n)$ respectively, thus the linkage condition holds. On the one hand, a hypothetical independent and maximal branching packing may not use any edge of the from $u_n v_n$ otherwise it would violate independence at v_n. On the other hand we claim that one cannot obtain a base for $\mathcal{N}(w)$ by taking at most one element from each $\mathcal{S}(v_n)$. Indeed, a nontrivial linear combination of such vectors must have a nonzero component other than the 0th which ensures that they cannot span $(1, 0, \dots)$. Hence there is no independent and maximal branching packing.

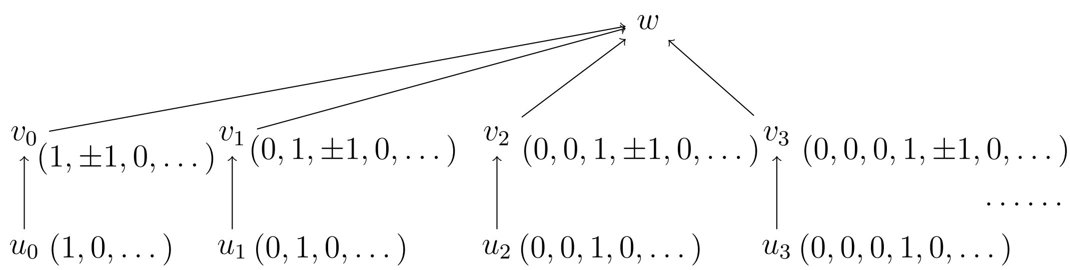

Figure 2: An independent matroid-rooted digraph $\mathfrak{R} = (D, \mathcal{M}, \pi)$ that satisfies the linkage condition. Furthermore D does not contain even undirected infinite paths and $\mathcal{M}$ is countable and finitary but there is no independent, maximal branching packing. For every vertex v, we listed the elements of $\mathcal{S}(v)$ next to v.

There is an asymmetry in the matroid restriction part of Condition 3.5 and Condition 3.6. In our last example, we show that one cannot replace the "$\mathcal{M}$ have finite rank " part of Condition 3.5 by the condition that $\mathcal{M}$ has countably many components all of which has a finite rank. Let be $V = \{t\} \cup \{(m, n) \in \omega \times \omega : m \leq n\}$. The set A consists of the following edges (see Figure 3). For all $m, n < \omega$, for which it makes sense

1. infinitely many parallel edges from $(m, n + 1)$ to (m, n),

2. edge from (m, n) to $(m + 1, n)$,

3. edge from $(2m+2, n)$ to $(2m, n)$,
4. edge from (m, m) to t,
5. edge from t to $(2m+1, n)$ (*not in the figure!*).

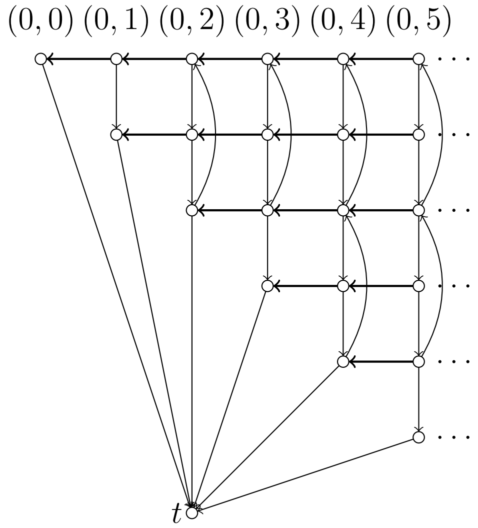

Figure 3: An illustration that in Condition 3.5 one cannot replace the restriction of the matroid by the weaker restriction of Condition 3.6. The outgoing edges of t (a single edge to each vertex in an odd row) are not on the figure. The thick horizontal edges stand for infinitely many parallel edges.

Observe that after the deletion of t just finitely many vertices are reachable from any vertex, which shows that there is no forward-infinite path in $D := (V, A)$. Let $\mathcal{M}$ be the free matroid on ω and let $\pi(n) = \{(0, n)\}$. It is easy to check (using Figure 3) that $\mathcal{N}(v) = \omega$ for all $v \in V$ and the linkage condition holds. We have to show that there are no edge-disjoint spanning branchings with the prescribed root sets. Suppose to the contrary that there is and fix one, say $\mathcal{B} = \{\mathcal{B}_n\}_{n<\omega}$. The only possibility for $\mathcal{B}_0$ to reach t is to use the edge $((0, 0), t)$. Suppose that we already know for some $0 < N$ that $\mathcal{B}_n$ contains the path $(0, n), (1, n), \ldots, (n, n), t$ whenever $n < N$. By using just the remaining edges, t is no longer reachable from columns $0, \ldots, N-1$. It easy to check (using Figure 3) that for $\mathcal{B}_N$ the path $(0, N), (1, N), \ldots, (N, N), t$ is the only possible option to reach t. On the other hand after the deletion of the edges of these paths for all n the vertices $\{(0, n) : 1 \leq n < \omega\}$ are no longer reachable from $\{(0, 0), t\}$. This prevents $\mathcal{B}_0$ from being a spanning branching rooted at $(0, 0)$ which is a contradiction.

4 Duality and the characterisation of the infeasible $(\boldsymbol{i},\boldsymbol{e})$-extensions

Assume that the linkage condition and independence hold for $\mathfrak{R}$ and let us focus first just on a single (i_0, e_0)-extension $\mathfrak{R}_1$ of $\mathfrak{R}$. We cannot ruin independence in this extension, as it is built into the definition of the (i,e)-extension. If for some nonempty $X \subseteq V$ any linkage for X necessarily uses all the ingoing edges of X, then we call X **tight** (with respect to $\mathfrak{R}$). If X is tight and $i_0 \in \mathsf{span}(\mathcal{S}(X))$, then X is called $\boldsymbol{i_0}$**-dangerous**. We claim that if e_0 is an ingoing edge of an i_0-dangerous set X, then the (i_0, e_0)-extension is infeasible. On the one hand, $i_0 \in \mathsf{span}(\mathcal{S}(X))$ implies that $\mathsf{span}(\mathcal{S}_{\mathfrak{R}_1}(X))) = \mathsf{span}(\mathcal{S}(X))$ and hence $\mathcal{N}(X)/\mathcal{S}_{\mathfrak{R}_1}(X) = \mathcal{N}(X)/\mathcal{S}(X)$. On the other hand, by the tightness of X (with respect to $\mathfrak{R}$) any (B,X)-linkage where B is a base of $\mathcal{N}(X)/\mathcal{S}(X) = \mathcal{N}(X)/\mathcal{S}_{\mathfrak{R}_1}(X)$ uses all the ingoing edges of X including e_0 thus there is no more a desired linkage for X with respect to $\mathfrak{R}_1$. It will turn out that surprisingly this is the only possible reason for the infeasibility of an (i_0, e_0)-extension. In the finite case, one can justify this easily in the following way. We use without proof that if $\mathcal{M}$ has finite rank, then the consequence

$$\mathfrak{R}' \text{ is independent and } r(\mathcal{S}_{\mathfrak{R}'}(X)) + \left|\mathsf{in}_{D_{\mathfrak{R}'}}(X)\right| \geq r(\mathcal{N}(X)) \text{ for all nonempty } X \subseteq V$$

of Condition 3.8 is actually equivalent with it. Furthermore, tightness of X is equivalent with the fact equality holds for X in the inequality above. (Of course in the finite case we do not need to know this equivalence or anything about our Condition 3.8 at all. One can simply define tightness based on the inequality.)

If the (i_0, e_0)-extension is infeasible in the finite case and X^* is a violating set with respect to the resulting $\mathfrak{R}_1$, then the extension necessarily reduces the number of ingoing edges of X^* (i.e. $|\mathsf{in}_D(X^*)| = |\mathsf{in}_{D-e_0}(X^*)| + 1$ and hence $e_0 \in \mathsf{in}_D(X^*)$) but does not increase the rank of the submatroid corresponding to X^* (i.e. $r(\mathcal{S}_{\mathfrak{R}_1}(X^*)) = r(\mathcal{S}(X^*)))$ thus $i_0 \in \mathsf{span}(\mathcal{S}(X^*))$, furthermore there must be originally equality for X^*. Summarizing these we obtain that e_0 is an ingoing edge of the i_0-dangerous set X^*.

As we mentioned, the same characterisation of infeasible extensions remains true in the general case, although we need to use more complex arguments to prove it. The rest of the section contains this proof and the corresponding preparations.

A set $X \subseteq V$ is called $\boldsymbol{t}$**-good** for some $t \in V$ if there is a system of edge-disjoint paths $\{P_b\}_{b\in B} \cup \{P_e\}_{e\in \mathsf{in}_D(X)}$ in $D[X]$ such that B is a base of $\mathcal{S}(X)$ and $\{P_b\}_{b\in B}$ is a (B,t)-linkage and P_e goes from $\mathsf{head}(e)$ to t.

Definition 4.1 (complementarity conditions)**.** The **complementarity conditions** for an (I,t)-linkage $\{P_i\}_{i\in I}$ and a vertex set $X \ni t$ are the following.

1. $I_{in} := I \cap \mathcal{S}(X)$ is a base of $\mathcal{S}(X)$,
2. paths $\{P_i\}_{i\in I_{in}}$ lie in $D[X]$,

3. for $i \in I \setminus \mathcal{S}(X) =: I_{out}$ we have $|A(P_i) \cap \mathsf{in}_D(X)| = 1$,
4. $\bigcup_{i \in I_{out}} A(P_i) \supseteq \mathsf{in}_D(X)$.

For $i \in I_{out}$, let us denote by e_i the first edge of P_i that enters X. Note that if the complementarity conditions hold for $\mathcal{P}$ and X, then X is t-good, as shown by the paths

$$\{P_i\}_{i \in I_{in}} \cup \{P_i[\mathsf{head}(e_i), t]\}_{i \in I_{out}}.$$

One can replace the conditions 2,3,4 by the single condition

$$A_{last}(\{P_i[\mathsf{start}(P_i), \mathsf{head}(e_i)]\}_{i \in I_{out}}) = \mathsf{in}_D(X).$$

Conjecture 4.2. *We always have some (I, t)-linkage $\mathcal{P}$ and an $X \ni t$ such that $\mathcal{P}$ and X satisfy the complementarity conditions.*

Note that for the free matroid this conjecture is a reformulation of the famous Infinite Menger theorem [1] of Aharoni and Berger. On the other hand, in the finite case much more general versions are true (see for example [14]).

Claim 4.3. *There exists a $\subseteq$-largest t-good set.*

Proof: First of all we always have a smallest t-good set, namely $\{t\}$.

Proposition 4.4. *For any $\subseteq$-increasing, nonempty chain $\langle X_\beta : \beta < \alpha \rangle$ of t-good sets, $\bigcup_{\beta < \alpha} X_\beta$ is t-good.*

Proof: Note that the definition of t-goodness is equivalent if we demand a generator system G (a set that contains a base) instead of a base B of $\mathcal{S}(X)$. We define for all $\beta \leq \alpha$ a path-system $\mathcal{P}_\beta$ that shows the t-goodness of X_β. Let $\mathcal{P}_0$ be an arbitrary system that witnesses the t-goodness of X_0. If some $\mathcal{P}_\beta = \{P_g\}_{g \in G_\beta} \cup \{P_e : e \in \mathsf{in}_D(X_\beta)\}$ has been defined, then we obtain $\mathcal{P}_{\beta+1}$ in the following way. Let $\mathcal{P} = \{P'_g\}_{g \in G'} \cup \{P'_e : e \in \mathsf{in}_D(X_{\beta+1})\}$ be an arbitrary linkage that shows the t-goodness of $X_{\beta+1}$. Throw away the elements of G' that are spanned by G_β and take the union of G_β and the reminder of G' to obtain $G_{\beta+1}$. For $g \in G_\beta$, we keep the path P_g unchanged. Observe that for $g \in G_{\beta+1} \setminus G_\beta$, the path $P'_g \in \mathcal{P}$ may not start inside X_β, because then G_β would span g since G_β is a generator for $\mathcal{S}(X_\beta)$. For $g \in G_{\beta+1} \setminus G_\beta$, let e_g be the first edge of P'_g that enters X_β. We obtain P_g as a concatenation of paths $P'_g[\mathsf{start}(P'_g), \mathsf{head}(e_g)]$ and $P_{e_g} \in \mathcal{P}_\beta$. We do the same terminal segment replacement process with all the paths $\{P'_e : e \in \mathsf{in}_D(X_{\beta+1})\}$ as well for getting $\{P_e : e \in \mathsf{in}_D(X_{\beta+1})\}$. Note that the resulting system $\mathcal{P}_{\beta+1}$ is really edge-disjoint.

Let $\beta \leq \alpha$ be a limit ordinal. Observe that $G_\beta := \bigcup_{\gamma < \beta} G_\gamma$ is a generator system for $\mathcal{S}(X_\beta)$. Indeed, if $i \in \mathcal{S}(X_\beta)$, then $i \in \mathcal{S}(X_\gamma)$ for some $\gamma < \beta$ hence $i \in \mathsf{span}(G_\gamma) \subseteq \mathsf{span}(G_\beta)$. If $e \in \mathsf{in}_D(X_\beta)$, then $e \in \mathsf{in}_D(X_\gamma)$ for some $\gamma < \beta$ thus P_e has already been defined, as well as the paths $\{P_b\}_{b \in G_\beta}$. Furthermore, the path-system $\mathcal{P}_\beta := \{P_e\}_{e \in \mathsf{in}_D(X_\gamma)} \cup \{P_b\}_{b \in G_\beta}$ is obviously edge-disjoint since any two elements of it are already members of $\mathcal{P}_\gamma$ for some $\gamma < \beta$. ●

Proposition 4.5. *If X and Y are t-good sets, then $X \cup Y$ is t-good as well.*

Proof: Let $\mathcal{P} = \{P_b\}_{b \in B_X} \cup \{P_e\}_{e \in \mathsf{in}_D(X)}$ and $\mathcal{Q} = \{Q_b\}_{b \in B_Y} \cup \{Q_e\}_{e \in \mathsf{in}_D(Y)}$ be path-systems that show the t-goodness of X and Y respectively. Note that all the common edges of the two path-systems are in $A(D[X \cap Y])$. Let us define $B'_Y = \{b \in B_Y : b \notin \mathsf{span}(B_X)\}$. For

$$s \in B'_Y \cup [(\mathsf{in}_D(Y) \cap \mathsf{in}_D(X \cup Y)) \setminus \mathsf{in}_D(X)],$$

let R_s be the path that we obtain by taking the initial segment of Q_s up to the first vertex in X and concatenate it with P_e where e is the last edge of this terminal segment. The path-system

$$\{P_s : s \in B_X \cup (\mathsf{in}_D(X) \cap \mathsf{in}_D(X \cup Y))\} \cup \{R_s : s \in B'_Y \cup (\mathsf{in}_D(Y) \cap \mathsf{in}_D(X \cup Y)) \setminus \mathsf{in}_D(X)\}$$

shows that $X \cup Y$ is t-good. ●

Proposition 4.4 and 4.5 imply that the union of arbitrary many t-good sets is t-good thus the union of all of them (it is not an empty union because $\{t\}$ is in it) as well. ●●

Our main tool to characterize the infeasible (i, e)-extensions is the following theorem.

Theorem 4.6. *If the (I, t)-linkage $\mathcal{P} = \{P_i\}_{i \in I}$ does not satisfy the complementarity conditions with the largest t-good set T, then there is a t-linkable I' for which $\mathsf{span}(I') \supsetneq \mathsf{span}(I)$.*

Proof: Assume that $\mathcal{P}$ and T do not satisfy the complementarity conditions and for $i \in I \setminus \mathcal{S}(T) =: I_{out}$ the first edge of P_i that enters T is e_i. First we show that we may suppose without loss of generality that there is a path-system $\{P_i\}_{i \in B} \cup \{P_e\}_{e \in \mathsf{in}_D(T)}$ such that

1. $\{P_i\}_{i \in B} \cup \{P_e\}_{e \in \mathsf{in}_D(T)}$ shows the t-goodness of T,
2. $B \subseteq I$ and $\{P_i\}_{i \in B} \subseteq \mathcal{P}$,
3. for $i \in I_{out}$ we have $A(P_i) \cap \mathsf{in}_D(T) = \{e_i\}$, and $P_i[\mathsf{head}(e_i), t] = P_{e_i}$.

Indeed, otherwise let J be a maximal $I/\mathcal{S}(T)$-independent subset of I_{out} and for $j \in J$ take the segments $\{P_j[\mathsf{start}(P_j), \mathsf{head}(e_j)]\}_{j \in J}$ from $\mathcal{P}$ and extend it to an $(J \cup B, t)$-linkage $\mathcal{Q}$ by using the t-goodness of T. Clearly $I \subseteq \mathsf{span}(J \cup B)$. We may assume that $\mathsf{span}(I) = \mathsf{span}(J \cup B)$, otherwise $I' := J \cup B$ would be a suitable choice for the theorem itself. We check that $A(\mathcal{Q}) \cap \mathsf{in}_D(T) = \{e_j\}_{j \in J} \subsetneq \mathsf{in}_D(T)$ by applying the fact that $\mathcal{P}$ and T do not satisfy the complementarity conditions. Assume that the first complementarity condition fails for $\mathcal{P}$ and T. We know $\mathcal{S}(X) \subseteq \mathsf{span}(I)$ because of $\mathsf{span}(I) = \mathsf{span}(J \cup B)$. Thus $\{e_j\}_{j \in J} = \mathsf{in}_D(T)$ would mean that complementarity conditions hold for $\mathcal{P}$ and T which is not the case. Finally the edges $\{e_j\}_{j \in J} \setminus \mathsf{in}_D(T)$ are unused by $\mathcal{Q}$, hence $\mathcal{Q}$ and T do no satisfy the complementarity conditions either. If $\mathcal{P}$ and T satisfy the first complementarity condition, then by using the alternative formulation of complementarity conditions 2,3,4 (see at the end of Definition 4.1) we obtain that

$$\mathsf{in}_D(X) \setminus A_{last}(\{P_i[\mathsf{start}(P_i), \mathsf{head}(e_i)]\}_{i\in I_{out}}) \neq \varnothing.$$

These edges will be unused by $\mathcal{Q}$.

Now we turn to the proof of the theorem. Let us denote $\{i \in S : I + i \in \mathcal{I}\} = I \cup (S \setminus \mathsf{span}(I))$ by $I^\star$. We build an auxiliary digraph by extending D. Pick the new vertices $\{u_i\}_{i\in I^\star}$, $\{w_i\}_{i\in S}$ and s and draw the following additional edges

1. $\{su_i : i \in I^\star\}$,
2. $\{u_iw_i : i \in I^\star\}$,
3. $\{w_iv : i \in S \wedge v \in \pi(i)\}$.

We denote the resulting digraph by $\boldsymbol{D_0^+} = (V^+, A_0)$. For $i \in I$, we extend the path P_i with the new initial vertices s, u_i, w_i to obtain the $s \to t$ path P_i^+ in D_0^+. Let $\boldsymbol{\mathcal{P}^+} = \{P_i^+\}_{i\in I}$. Finally, change the direction of the edges in $A(\mathcal{P}^+)$ to obtain $\boldsymbol{D_0^*}$. We call these redirected edges **backward edges** and the others **forward edges**. Let $\boldsymbol{U_0^+}$ be the set of vertices of D_0^* that are **u**nreachable from s and let $\boldsymbol{U_0} = U_0^+ \cap V$.

Assume first that $t \notin U_0$ i.e. there is an $s \to t$ path P^+ in D_0^+. Let its first edge be su_{i_0}. Note that $i_0 \in I^\star \setminus I$. Use the standard **augmentation path** technique to obtain a system of edge-disjoit $s \to t$ paths $\{Q_i^+\}_{i\in I+i_0}$ in D_0^+ where the first edge of Q_i^+ is su_i. More precisely, do the following. First of all, keep unchanged the paths $P_i^+ \in \mathcal{P}^+$ that have no common edges with P^+. After that, take the symmetric difference of $A(P^+)$ and the united edge sets of the, say k many, elements of $\mathcal{P}^+$ from which P^+ uses some backward edges. From the resulting edge set build $k+1$ edge disjoint $s \to t$ paths by the greedy method. Finally $\{Q_i^+\}_{i\in I+i_0}$ consists of the paths we kept unchanged and these $k+1$ new paths. By cutting off the three initial vertices of the paths Q_i^+, we obtain a system of edge-disjoint paths $\mathcal{Q} = \{Q_i\}_{i\in I\cup\{i_0\}}$ in D such that for any $i \in I \cup \{i_0\}$, path Q_i goes from $\pi(i)$ to t, i.e. we get an $(I + i_0, t)$-linkage. In this case, $I' := I + i$ is appropriate.

Suppose that $t \in U_0$. Clearly the paths in $\mathcal{P}^+$ use all the edges in $\mathsf{in}_{D_0^+}(U_0^+)$ and none of the edges in $\mathsf{out}_{D_0^+}(U_0^+)$. Therefore the same holds for $\mathcal{P}$ with respect to D and U_0. We claim that $T \subseteq U_0$. Assume, to the contrary, that we have a strict $s \to T$ path P^+ in D_0^* with last edge f. It follows from our additional assumptions about $\mathcal{P}$ (the first paragraph of this proof) that it does not use any edge from $\mathsf{out}_D(T)$, thus f cannot be a backward edge. If $f \in \mathsf{in}_D(T) \setminus A(\mathcal{P})$, then path $P_f \in \{P_e\}_{e\in\mathsf{in}_D(T)}$ would show the reachability of t from s in D_0^* contradicting $t \in U_0$. Finally, suppose that $f = w_iv$ for some $v \in T$. Then $i \in \mathcal{S}(T)$ and therefore $i \in \mathsf{span}(B)$. For $j \in S \setminus I^\star$, the vertex w_j has no ingoing edges in D_0^* hence we know that $i \in I^\star$. Thus $I + i$ is independent hence $B+i$ as well. It follows that necessarily $i \in B$. But then the unique ingoing edge of w_i in D_0^* comes from $\mathsf{start}(P_i) \in T$ and it contradicts the strictness of the $s \to T$ path P^+.

Let $\boldsymbol{I_{in,0}} = I \cap \mathcal{S}(U_0)$ and let $I_{out,0} = I \setminus \mathcal{S}(U_0)$. We claim that for $i \in I_{in,0}$ we have $\mathsf{start}(P_i) \in U_0$. Indeed, otherwise the backward edge $\mathsf{start}(P_i)w_i$ and any forward edge w_iv with $v \in \pi(i) \cap U_0$ would lead to a contradiction with the definition of U_0^+. Since $A(\mathcal{P}) \cap \mathsf{out}_D(U_0) = \varnothing$, we obtain that the path-system $\{P_i\}_{i \in I_{in,0}}$ lies in $D[U_0]$. Then clearly the paths $\{P_i\}_{i\in I_{out,0}}$ have to use all the edges in $\mathsf{in}_D(U_0)$ because $\mathsf{in}_D(U_0) \subseteq A(\mathcal{P})$. It follows that each of them uses exactly one such an edge, thus $\mathcal{P}$ and U_0 satisfy all but possibly the first complementary conditions.

Let us define $\boldsymbol{F_0} := \mathcal{S}(U_0) \setminus \mathsf{span}(I_{in,0})$. Observe that $F_0 \neq \varnothing$, otherwise the first complementarity condition would hold for $\mathcal{P}$ and U_0 and hence U_0 would be a t-good set with $U_0 \supsetneq T$ (clearly $U_0 \neq T$, since $\mathcal{P}$ and T do not satisfy the complementarity conditions by assumption) which contradicts the maximality of T.

We know that $\mathcal{S}(U_0) \subseteq \mathsf{span}(I)$, since for $i \in S \setminus \mathsf{span}(I)$ the path s, u_i, w_i, v where $v \in \pi(i)$ shows that $\pi(i) \cap U_0 = \varnothing$ i.e. $i \notin \mathcal{S}(U_0)$. Fix a well-ordering of $I_{out,0}$. For $i \in F_0$, let $s_0(i)$ be the smallest element of $I_{out,0} \cap C(i, I)$. Extend D_0^+ with the new edges $\{u_{s_0(i)}w_i : i \in F_0\}$ to obtain $D_1^+ = (V^+, A_1)$. We get D_1^* by changing the direction of edges in $A(\mathcal{P})$ in D_1^+. Assume first that there is some $s \to t$ path P^+ in D_1^*. For the first edge su_{i_0} of P^+, we have $i_0 \in I^\star \setminus I$. Consider $S_{rep} := \{i \in F_0 : u_{s_0(i)}w_i \in A(P^+)\}$ and take the smallest element $s_0(i)$ of $s_0[S_{rep}]$. The set $(I + i_0) - s_0(i) + i$ is independent, spans $I + i_0$, and the remaining elements of $s_0[S_{rep}]$ have the same fundamental circuit on it as on I. We can do recursively in increasing order the other replacements thus $I' := (I + i_0) + S_{rep} - s_0[S_{rep}]$ is independent and spans $I + i_0$. Applying P^+ in the augmentation path method results in a desired (I', t)-linkage.

Assume that such a P^+ does not exist. Let $U_1^+ \ni t$ be the set of the vertices of D_1^* that are not reachable from s and let $U_1 = U_1^+ \cap V$. Because of the new edges, $U_1^+ \subseteq U_0^+$ holds. Observe that the vertices $\{w_i\}_{i\in\mathcal{S}(T)}$ did not get any new ingoing edge (B ensures $\mathcal{S}(T) \cap F_0 = \varnothing$) hence $T \subseteq U_1$ follows in the same way as we proved $T \subseteq U_0$. Let us define $I_{in,1} = I \cap \mathcal{S}(U_1)$ and $I_{out,1} = I \setminus \mathcal{S}(U_1)$. The complementarity conditions hold for $\mathcal{P}$ and U_1 except the first which may not, and $\mathcal{S}(U_1) \subseteq \mathsf{span}(I)$ holds. The proof of these facts are the same as for U_0. Note that the new edges ensure that $F_0 \cap \mathcal{S}(U_1) = \varnothing$ hence for $F_1 := \mathcal{S}(U_1) \setminus \mathsf{span}(I_{in,1})$ we have $F_0 \cap F_1 = \varnothing$. Let us extend the well-ordering of $I_{out,0}$ to a well ordering of $I_{out,1}$ in such a way that $I_{out,1} \setminus I_{out,0}$ is a terminal segment in it. This choice ensures that for an edge $u_{s_0(i)}w_i$ the element $s_0(i)$ is the smallest in $I_{out,1} \cap C(i, I)$, not just in $I_{out,0} \cap C(i, I)$. For $i \in F_1$, let $s_1(i)$ be the smallest element of $I_{out,1} \cap C(i, I)$. We obtain D_2^+ from D_1^+ by adding the new edges $\{u_{s_1(i)}w_i : i \in F_1\}$.

We define the corresponding notions $D_2^*, U_2^+, U_2, I_{in,2}, I_{out,2}, F_2$ and continue the process recursively. Suppose, to the contrary that we do not find a desired I'. Let us define $D_\omega^+ = (V^+, \bigcup_{n<\omega} A_n)$ and the corresponding notions as earlier. Note that $U_\omega = \bigcap_{n<\omega} U_n \supseteq T$ and it satisfies all but the first complementarity conditions (thus $F_\omega \neq \varnothing$) with $\mathcal{P}$. Obviously $I_{in,\omega} \subseteq \bigcap_{n<\omega} I_{in,n}$ but in fact $I_{in,\omega} = \bigcap_{n<\omega} I_{in,n}$ holds. Indeed, $i \in \bigcap_{n<\omega} I_{in,n}$ implies that P_i lies in $\bigcap_{n<\omega} U_n = U_\omega$ and $\mathsf{start}(P_i) \in \pi(i)$ shows $i \in I_{in,\omega}$. Let $i \in F_\omega = \mathcal{S}(U_\omega) \setminus \mathsf{span}(I_{in,\omega})$ be arbitrary. Then $i \notin F_n$ for all $n < \omega$ otherwise by the new edges we would have $i \notin \mathcal{S}(U_{n+1}) \supseteq \mathcal{S}(U_\omega)$. On the other hand, $i \in \mathcal{S}(U_\omega) \subseteq \mathcal{S}(U_n)$ and by putting these together we obtain $i \in \mathsf{span}(I_{in,n})$ for all $n < \omega$. We can not have $i \in I_{in,n}$ for all $n < \omega$ since then $i \in I_{in,\omega}$ would follows. Suppose that $i \notin I_{in,n}$ if $n > n_0$. Thus for any $n > n_0$ we have $(C(i, I) - i) \subseteq I_{in,n}$

but then

$$(C(i,I) - i) \subseteq \bigcap_{n_0 < n} I_{in,n} = \bigcap_{n<\omega} I_{in,n} = I_{in,\omega}$$

witnesses $i \in \mathsf{span}(I_{in,\omega})$, hence $i \notin F_\omega$ and thus $F_\omega = \varnothing$ which is a contradiction. ■

It worth mentioning the following two consequences of the Theorem above.

Corollary 4.7. *Let $\mathcal{P}$ be a (I,t)-linkage and assume that there exists a t-linkable I' with $\mathsf{span}(I') \supsetneq \mathsf{span}(I)$. Then one can choose such an I' and a (I',t)-linkage $\mathcal{Q}$ such that either $A_{last}(\mathcal{Q}) \subseteq A_{last}(\mathcal{P})$ or the following hold:*

1. $|I' \setminus I| = |I \setminus I'| + 1 < \omega$,
2. $A_{last}(\mathcal{P}) \subseteq A_{last}(\mathcal{Q})$,
3. $|A_{last}(\mathcal{Q}) \setminus A_{last}(\mathcal{P})| = 1$,
4. $|\mathcal{Q} \setminus \mathcal{P}| = |\mathcal{P} \setminus \mathcal{Q}| + 1 < \omega$.

Proof: Follow the proof of Theorem 4.6 without dealing with T hence without the modification described in the first paragraph of that proof. Then we either obtain a desired (I',t)-linkage that satisfies conditions 1-4 or a t-good set U_n that satisfies the complementarity conditions with $\mathcal{P}$. We may assume that the second possibility happens. Pick an arbitrary (J,t)-linkage $\mathcal{R} = \{R_i\}_{i\in J}$ with $\mathsf{span}(J) \supsetneq \mathsf{span}(I)$. Let e_i be the first edge of R_i that enters U_n, and let K be a maximal $J/\mathcal{S}(U_n)$-independent subset of $J \setminus \mathcal{S}(U_n)$. Keep the segments $R_i[\mathsf{start}(R_i), \mathsf{head}(e_i)]$ for $i \in K$. Concatenate $R_i[\mathsf{start}(R_i), \mathsf{head}(e_i)]$ with $P_j[\mathsf{head}(e_i), t]$, where P_j is the unique element of $\mathcal{P}$ for which $A(P_j) \cap \mathsf{in}_D(U_n) = \{e_i\}$, to obtain Q_i, and let $Q_i = P_i$ for $i \in I \cap \mathcal{S}(X)$. It is routine to check that $(I \cap \mathcal{S}(X)) \cup K =: I'$ and $\mathcal{Q} := \{Q_i\}_{i\in I'}$ are appropriate. ●

Corollary 4.8. *Suppose that I is t-linkable. Then there is no t-linkable $I' \supsetneq I$ if and only if there is a vertex set $X \ni t$ such that $\mathcal{S}(X) \subseteq \mathsf{span}(I)$ and for $I_{out} := I \setminus \mathcal{S}(X)$ any strict (I_{out}, X)-linkage $\mathcal{Q}$ we have $A_{last}(\mathcal{Q}) = \mathsf{in}_D(X)$.*

Proof: Apply Theorem 4.6 with the free matroid on $I \cup (S \setminus \mathsf{span}(I))$. ●

Now we are able to prove the characterization of the infeasible (i,e)-extensions.

Lemma 4.9. *The (i_0, e_0)-extension is infeasible if and only if e_0 enters some i_0-dangerous set.*

Proof: We have already checked the "if" so now we prove the remaining direction. Assume that vertex t witnesses the failure of the linkage condition in the (i_0, e_0)-extension $\mathfrak{R}_1$ of $\mathfrak{R}$. We claim that the largest t-good set X with respect to $\mathfrak{R}_1$ is a desired i_0-dangerous set (with respect to $\mathfrak{R}$). Let $\mathcal{P} = \{P_b\}_{b\in B_{out}}$ be an arbitrary reduced linkage for X with respect to $\mathfrak{R}$, i.e. a strict (B_{out}, X)-linkage where B_{out} is a base of $\mathcal{N}(t)/\mathcal{S}(X)$. Note that B_{out} contains a base B'_{out} of $\mathcal{N}(t)/\mathcal{S}_{\mathfrak{R}_1}(X)$. Let $\mathcal{P}' = \{P_b\}_{b\in B'_{out}}$. Clearly $e_0 \in A(\mathcal{P}')$, otherwise from $\mathcal{P}'$ one can get a (B,t)-linkage with respect to $\mathfrak{R}_1$ where B is a base of $\mathcal{N}(t)$ by using the t-goodness of X. Suppose that $e_0 \in A(P_{b_1})$. Then we are able to construct a strict $(B - b_1, t)$-linkage $\mathcal{L}$ with respect to $\mathfrak{R}_1$ from $\mathcal{P}' \setminus \{P_{b_1}\}$ via t-goodness as above. By Fact 2.1, an augmentation of this linkage in the sense of Theorem 4.6 would lead to a linkage for t with respect to $\mathfrak{R}_1$, which is impossible, therefore, by Theorem 4.6, linkage $\mathcal{L}$ satisfies the complementarity conditions with X. Hence $\mathcal{P}' \setminus \{P_{b_1}\}$ needs to use all the edges in $\mathsf{in}_{D-e_0}(X)$. The only way for this to be true is if e_0 is the last edge of P_{b_1}, $A_{last}(\mathcal{P}') = \mathsf{in}_D(X)$ and $\mathcal{P}' = \mathcal{P}$. Thus $e_0 \in \mathsf{in}_D(X)$ and X is tight with respect to $\mathfrak{R}$. Furthermore, $\mathcal{P}' = \mathcal{P}$ implies $\mathcal{S}(X) = \mathcal{S}_{\mathfrak{R}_1}(X)$, hence $i_0 \in \mathsf{span}(\mathcal{S}(X))$. Therefore, X is i_0-dangerous. ●

5 New matroid-rooted digraphs from tight sets

In finite combinatorics, it is a common proof technique to subdivide the problem into smaller sub-problems by using an appropriate notion of tightness and then solve the smaller sub-problems by induction from which one can obtain a solution for the original problem. Unfortunately, in infinite combinatorics usually the resulting sub-problems are no longer "smaller" in any sense that would make possible such an induction. Even though do not lead to such an immediate success, the investigation of them could be fruitful, as happened in this topic.

Through this chapter we have some fixed matroid-rooted digraph $\mathfrak{R}$ that satisfies independence and the linkage condition.

Claim 5.1. *If X is a tight set and $\{P_b\}_{b\in B_0}$ is a linkage for Z where $\varnothing \neq Z \subseteq X$, then*

1. *the set $B^* := B_0 \cap \mathcal{S}(X)$ is a base of $\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$,*
2. *for $b \in B^*$, path P_b lies in $D[X]$,*
3. *for $b \in B_0 \setminus B^*$ we have $|A(P_b) \cap \mathsf{in}_D(X)| = 1$,*
4. *all the edges $\{e \in \mathsf{in}_D(X) : \mathsf{head}(e) \in \mathsf{to}_D(Z)\}$ are used by the paths $\{P_b\}_{b\in(B_0\setminus B^*)}$.*

Proof: Pick a linkage $\{Q_b\}_{b\in B_1}$ for X and let $B_0 \subseteq B \subseteq B_0 \cup B_1$ a base of $\mathcal{N}(X)$. Note that if $A(P_{b_0}) \cap A(Q_{b_1}) \neq \varnothing$ for some $b_0 \in B_0$ and $b_1 \in B_1 \setminus B_0$, then $b_1 \in \mathcal{N}(Z)$ and therefore $b_1 \in \mathsf{span}(B_0)$. For $b \in B \setminus B_0$, let $P_i = Q_i$, then $\mathcal{P}' := \{P_b\}_{b\in B}$ is a (B, X)-linkage. If $B^* \subseteq B_0$ is not a base of

$$\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X)) \subseteq \mathsf{span}(B_0),$$

then we may pick some $i \in (\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))) \setminus B^*$ for which $B^* + i$ is independent and some

$$j \in C(i, B_0) \setminus B^* = C(i, B) \setminus B^*.$$

$B - j + i$ is a base of $\mathcal{N}(X)$ and $i \in \mathsf{span}(X)$ implies that for a suitable $k \in \mathcal{S}(X)$ the set $B - j + k$ as well (by Corollary 2.9 with $I := B - j + i$ and $J := \mathcal{S}(X)$). Note that $\mathsf{start}(P_j) \notin X$ because $j \in B_0 \setminus B^*$ and therefore $A(Q_i) \cap \mathsf{in}_D(X) \neq \varnothing$. But then we may replace P_j by a trivial path P_k (consisting of a single vertex from $\pi(k) \cap X$) in $\mathcal{P}'$ and the new linkage does not use the edges $A(P_j) \cap \mathsf{in}_D(X) \neq \varnothing$ which contradicts the tightness of X.

If for some $b \in B^*$ path P_b is not entirely in X, then it uses some element of $\mathsf{in}_D(X)$. Replace P_b by a trivial path consisting of an element of $\pi(b) \cap X$ to get a contradiction as above.

Assume that for some $b \in B_0$ path P_b uses more than one ingoing edge of X, then we may replace it in $\mathcal{Q}$ by its own initial segment up to the head of its first edge in $\mathsf{in}_D(X)$ and get contradiction.

If the linkage $\mathcal{P} = \{P_b\}_{b \in B_0}$ does not use all the edges $\{e \in \mathsf{in}_D(X) : \mathsf{head}(e) \in \mathsf{to}_D(Z)\}$, then the linkage $\mathcal{P}' = \{P_b\}_{b \in B}$ does not use these edges as well (since for $b \in B \setminus B_0$ their heads may not even be reachable from $\pi(b)$) which contradicts the tightness of X.●

Corollary 5.2. *Under Condition 3.5 (Condition 3.6), for a tight X, a forward-infinite (backward-infinite) path P of $D[X]$ may not be reachable in D from outside X (equivalently from $\{\mathsf{head}(e) : e \in \mathsf{in}_D(X)\}$).*

Proof: Since

$$\mathcal{N}(V(P)) = \mathsf{span}(\mathcal{S}(V(P))) \subseteq \mathcal{N}(V(P)) \cap \mathsf{span}(\mathcal{S}(X)),$$

by applying the first statement of Claim 5.1 with $Z := V(P)$ we obtain $B^* = B_0$ thus $B_0 \setminus B^* = \varnothing$. Hence the Corollary follows from the fourth statement of Claim 5.1. ●

For a tight X, let $\mathfrak{R}[X]$ be the matroid-rooted digraph with $D_{\mathfrak{R}[X]} = D[X]$, $\mathcal{M}_{\mathfrak{R}[X]} = \mathcal{S}(X) \bigoplus \{i_e : e \in \mathsf{in}_D(X)\}$ where i_e are some new elements, distinct from the elements of S, and we consider $\{i_e : e \in \mathsf{in}_D(X)\}$ as a free matroid. Finally, let $\pi_{\mathfrak{R}[X]}(i) = \pi(i) \cap X$ for $i \in \mathcal{S}(X)$ and let $\pi_{\mathfrak{R}[X]}(i_e) = \{\mathsf{head}(e)\}$ for $e \in \mathsf{in}_D(X)$.

Observation 5.3. For $U \cup \{i\} \subseteq \mathcal{S}(X)$, we have

$$i \in \mathsf{span}(\mathcal{S}(U)) \iff i \in \mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(\mathcal{S}_{\mathfrak{R}[X]}(U)).$$

Applying Claim 5.1 we prove some basic facts related to $\mathfrak{R}[X]$.

Proposition 5.4.

1. *$\mathfrak{R}[X]$ satisfies the linkage condition and independence,*
2. *$\mathsf{span}(\mathcal{N}(Z) \cap \mathcal{S}(X)) = \mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$ $(Z \subseteq X)$,*
3. *$\mathcal{N}_{\mathfrak{R}[X]}(Z) = \mathcal{N}(Z) \cap \mathcal{S}(X) \cup \{i_e : e \in \mathsf{in}_D(X) \wedge \mathsf{head}(e) \in \mathsf{to}_D(Z)\}$ $(Z \subseteq X)$.*

Proof: Let $v \in X$ be arbitrary and pick a linkage $\{P_b\}_{b \in B}$ for v. Take the terminal segments of paths P_b from their first vertex in X. Claim 5.1 and the definition of $\mathfrak{R}[X]$ ensure that the result is a linkage for v with respect to $\mathfrak{R}[X]$. The independence preserving part follows from the fact that the circuits of $\mathcal{M}_{\mathfrak{R}[X]}$ are exactly those circuits of $\mathcal{M}$ that lie in $\mathcal{S}(X)$ and for $Z \subseteq X$ we have $\mathcal{S}(Z) = \mathcal{S}_{\mathfrak{R}[X]}(Z) \cap \mathcal{S}(X)$. Thus if we have a $\mathcal{M}_{\mathfrak{R}[X]}$-circuit $C \subseteq \mathcal{S}_{\mathfrak{R}[X]}(v)$ for some $v \in X$, then $C \subseteq \mathcal{S}(v)$ and C would be an $\mathcal{M}$-circuit as well.

Assume that $i \in \mathsf{span}(\mathcal{N}(Z) \cap \mathcal{S}(X))$. Then by monotonicity $i \in \mathsf{span}(\mathcal{S}(X))$ and $i \in \mathsf{span}(\mathcal{N}(Z)) = \mathcal{N}(Z)$, i.e. $i \in \mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$. Suppose now $i \in \mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$. By the first statement of Claim 5.1, we know that there is a base $B^* \subseteq \mathcal{S}(X) \cap \mathcal{N}(Z)$ of $\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$, hence

$$i \in \mathsf{span}(B^*) \subseteq \mathsf{span}(\mathcal{N}(Z) \cap \mathcal{S}(X)).$$

At the third statement of this Proposition, the inclusion "$\subseteq$" is straightforward. The linkage $\{P_b\}_{b \in B^*}$ from the first two statements of Claim 5.1 ensures $\mathcal{N}_{\mathfrak{R}[X]}(Z) \supseteq \mathcal{N}(Z) \cap \mathcal{S}(X)$ and the last two statements of Claim 5.1 show

$$\mathcal{N}_{\mathfrak{R}[X]}(Z) \supseteq \{i_e : e \in \mathsf{in}_D(X) \wedge \mathsf{head}(e) \in \mathsf{to}_D(Z)\}. \bullet$$

Proposition 5.5. *If X is a tight set and $Z \subseteq X$, then Z is tight with respect to $\mathfrak{R}$ if and only if Z is tight with respect to $\mathfrak{R}[X]$.*

Proof: Suppose that $Z \neq \varnothing$ is not tight with respect to $\mathfrak{R}$ and let $\mathcal{P} = \{P_b\}_{b \in B_0}$ be a linkage for Z such that for some $f \in \mathsf{in}_D(Z)$ we have $f \notin A(\mathcal{P})$. Let q_b be the first vertex of P_b in X. We show that the paths $\{P_b[q_b, \mathsf{end}(P_b)]\}_{b \in B_0}$ witnesses that Z is not tight with respect to $\mathfrak{R}[X]$. By Claim 5.1 we know that these are really paths in $D[X]$. Let $B^* = \{b \in B_0 : \mathsf{start}(P_b) = q_b\}$. According to Claim 5.1 B^* is a base of $\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$. By the forth statement of Claim 5.1, the set of the last edges of paths $\{P_b[\mathsf{start}(P_b), q_b]\}_{b \in B_0 \setminus B^*}$ is $\{e \in \mathsf{in}_D(X) : \mathsf{head}(e) \in \mathsf{to}_D(Z)\} =: A_0$. For $e \in A_0$, let $P_{i_e} = P_b[q_b, \mathsf{end}(P_b)]$ where e is the last edge of $P_b[\mathsf{start}(P_b), q_b]$. The third statement of Proposition 5.4 ensures that the linkage

$$\{P_b\}_{b \in B^*} \cup \{P_{i_e}\}_{e \in A_0}$$

corresponds to a base of $\mathcal{N}_{\mathfrak{R}[X]}(Z)$. This linkage clearly does not use $f \in \mathsf{in}_D(Z)$ but we are not done since we need to show $f \in \mathsf{in}_{D[X]}(Z)$. Suppose, to the contrary that $f \notin \mathsf{in}_{D[X]}(Z)$, then necessarily $f \in \mathsf{in}_D(X) \cap \mathsf{in}_D(Z)$. But then by the last statement of Claim 5.1 we obtain $f \in A(\mathcal{P})$ contradicting to the choice of f. This completes the proof of the "only if" part of the statement.

The proof of the other direction is very similar hence we give just a sketch. Take a linkage for Z which witnesses the untightness of Z with respect to $\mathfrak{R}[X]$. Then give a backward continuation for its paths in the form P_{i_e} by using an arbitrary linkage for Z with respect to $\mathfrak{R}$. The resulting linkage for Z with respect to $\mathfrak{R}$ shows the untightness of Z with respect to $\mathfrak{R}$. $\bullet$

Observation 5.3 leads to the following consequence of the Proposition above.

Corollary 5.6. *If X is a tight set, $Z \subseteq X$ and $i \in S$, then Z is i-dangerous with respect to $\mathfrak{R}$ if and only if Z is i-dangerous with respect to $\mathfrak{R}[X]$.*

Claim 5.7. *If X and Y are tight sets with $X \cap Y \neq \varnothing$, then $X \cap Y$ is tight as well. Furthermore if X and Y are i-dangerous and $i \in \mathcal{N}(X \cap Y)$, then $X \cap Y$ is i-dangerous.*

Proof: Let $\{P_b\}_{b \in B_0}$ be a linkage for $X \cap Y =: Z$. If some edge enters Z, then it enters X or enters Y thus by applying the last statement of Claim 5.1 to X with Z and then to Y with Z we obtain that the paths $\{P_b\}_{b \in B_0}$ use all the edges in $\mathsf{in}_D(Z)$.

Let us turn to the dangerousness part of the Claim. By the first statements of Claim 5.1, $B_X^* := B_0 \cap \mathcal{S}(X)$ is a base of $\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(X))$ and $B_Y^* := B_0 \cap \mathcal{S}(Y)$ is a base of $\mathcal{N}(Z) \cap \mathsf{span}(\mathcal{S}(Y))$. Clearly both B_X^* and B_Y^* need to contain a base of $\mathcal{S}(Z)$. This two bases of $\mathcal{S}(Z)$ must be the same since $B_X^* \cup B_Y^*$ is independent. Therefore on the one hand $B_X^* \cap B_Y^*$ contains a base of $\mathcal{S}(Z)$. On the other hand, by the second statement of Claim 5.1 for $b \in B_X^* \cap B_Y^*$ we have $\mathsf{start}(P_b) \in X \cap Y = Z$ and hence $b \in \mathcal{S}(Z)$ thus $B_X^* \cap B_Y^* \subseteq \mathcal{S}(Z)$. It follows that $B_X^* \cap B_Y^*$ is a base of $\mathcal{S}(Z)$.

Assume now that $i \in \mathsf{span}(\mathcal{S}(X)) \cap \mathsf{span}(\mathcal{S}(Y))$ and $i \in \mathcal{N}(Z)$ (hence $i \in \mathcal{N}(X) \cap \mathcal{N}(Y)$). Then $i\ \mathsf{span}(B_X^*) \cap \mathsf{span}(B_Y^*)$. If $i \in B_X^* \cap B_Y^*$, then $i \in \mathcal{S}(Z)$ and we are done. If exactly one element of $\{B_X^*, B_Y^*\}$ contains i, then $B_X^* \cup B_Y^*$ would contain a circuit through i which is impossible. Finally if i has a fundamental circuit on B_X^* and on B_Y^*, then by the independence of $B_X^* \cup B_Y^*$ and the weak circuit elimination (Fact 2.6) these two circuits must be the same and therefore lie in $(B_X^* \cap B_Y^*) + i$. Hence

$$i \in \mathsf{span}(B_X^* \cap B_Y^*) = \mathsf{span}(\mathcal{S}(Z)). \ \bullet$$

Claim 5.8. *If the (i, e)-extension of $\mathfrak{R}[X]$ is feasible where $i \in S$, then the (i, e)-extension of $\mathfrak{R}$ is feasible as well.*

Proof: Suppose, to the contrary, that it is not. Then by Lemma 4.9 e in an ingoing edge of some i-dangerous set Y. Using the fact that e lies in X we have

$$i \in \mathcal{S}(\mathsf{tail}(e)) \subseteq \mathcal{S}(X) \subseteq \mathsf{span}(\mathcal{S}(X))$$

hence X is i-dangerous too. Edge e witnesses that $i \in \mathcal{N}(X \cap Y)$, thus by Claim 5.7 $Z := X \cap Y$ is i-dangerous with respect to $\mathfrak{R}$ thus by Claim 5.6 it is i-dangerous with respect to $\mathfrak{R}[X]$ as well. But then the (i, e)-extension of $\mathfrak{R}[X]$ is infeasible since $e \in \mathsf{in}_{D[X]}(Z)$ and Z is i-dangerous which is a contradiction. $\bullet$

Corollary 5.9. *Let $S' \subseteq S$. Then a feasible S'-extension $\mathfrak{R}[X]^*$ of $\mathfrak{R}[X]$ of order n determines a unique, feasible S'-extension $\mathfrak{R}^*$ of $\mathfrak{R}$ of order n characterized by the property $\mathfrak{R}^*[X] = \mathfrak{R}[X]^*$.*

6 Augmentations at a prescribed vertex

In this section we prove a Lemma, that allows us a kind of local augmentation. The Lemma will imply immediately Theorem 3.7 in the case of countable D and one can derive from it Theorem 3.7 itself as well without too much effort as we will do it in the last section.

Lemma 6.1. *Assume that $\mathfrak{R} = (D, \mathcal{M}, \pi)$ is independent and satisfies the linkage condition and either Condition 3.5 or Condition 3.6. Then for any $v \in V$ and for any $W \subseteq S$ which is the union of finitely many components of $\mathcal{M}$ there is a finite-order, feasible W-extension $\mathfrak{R}^*$ of $\mathfrak{R}$ such that $\mathcal{S}_{\mathfrak{R}^*}(v) \cap W$ is a base of $\mathcal{N}(v) \cap W$.*

Assume, to the contrary, that the Lemma is false and choose an arbitrary counterexample triple $\mathfrak{R} = (D, \mathcal{M}, \pi), v_0, W$. We may assume (by replacing $\mathfrak{R}$ by some feasible, finite-order W-extension of itself) that we are not able to augmenting at v_0 even by one. More precisely for any feasible, finite-order W-extensions $\mathfrak{R}'$ of $\mathfrak{R}$ we have $\mathcal{S}_{\mathfrak{R}'}(v_0) = \mathcal{S}_{\mathfrak{R}}(v_0)$. Similarly we may suppose that $\mathfrak{R}$ minimize the following expression among the feasible, finite-order W-extensions $\mathfrak{R}'$ of $\mathfrak{R}$.

$$\min\{|A(P_{i_0})| : \{P_i\}_{i \in B} \text{ is a reduced linkage for } v_0 \text{ with respect to } \mathfrak{R}' \text{ and } i_0 \in B \cap W\} \tag{1}$$

Let the minimum for $\mathfrak{R}$ be taken on $P_{i_0} \in \{P_i\}_{i \in B}$. Consider the first edge e_0 of P_{i_0}.

Proposition 6.2. *The (i_0, e_0)-extension of $\mathfrak{R}$ is defined but not feasible.*

Proof: Suppose, to the contrary, that it is undefined i.e. $i_0 \in \mathsf{span}(\mathsf{head}(e_0))$. Then by Corollary 2.9 there is some $i_0' \in \mathcal{S}(\mathsf{head}(e_0)) \cap W \subseteq \mathcal{N}(v_0)$ such that $B - i_0 + i_0'$ is a base of $\mathcal{N}(v_0)/\mathcal{S}(v_0)$ (which implies $\mathsf{head}(e_0) \neq v_0$ since $\{P_i\}_{i \in B}$ is a reduced linkage for v_0). But then we may replace i_0 by i_0' and P_{i_0} by $P_{i_0'} := P_{i_0}[\mathsf{head}(e_0), v_0]$ to get a contradiction with the fact that the minimum at (1) for $\mathfrak{R}$ is $|A(P_{i_0})|$. On the other hand, the (i_0, e_0)-extension cannot be feasible since otherwise the resulting extension would have a smaller minimum showed by the linkage that we would obtain from $\{P_i\}_{i \in B}$ by replacing P_{i_0} with $P_{i_0}[\mathsf{head}(e_0), v_0]$. ●

It follows by Lemma 4.9 that e_0 enters some i_0-dangerous set X.

Proposition 6.3. *The set X does not contain v_0.*

Proof: Suppose, seeking for contradiction, that $v_0 \in X$. Then all the paths in $\{P_b\}_{b \in B}$ meet X. Pick a reduced linkage $\{P_b'\}_{b \in B_X}$ for X and note that if some P_b' have a common edge (or just a common vertex) with a path in $\{P_b\}_{b \in B}$, then $b \in \mathsf{span}(B)$. Let $B_X' = \{b \in B_X : b \notin \mathsf{span}(B)\}$. The set $(B \cup B_X') \setminus \{i_0\}$ contains a base of $\mathcal{N}(X)/\mathcal{S}(X)$ since $B \cup B_X'$ clearly does and $i_0 \in \mathsf{span}(\mathcal{S}(X))$ implies that i_0 is a loop in $\mathcal{N}(X)/\mathcal{S}(X)$. The path-system $(\{P_b\}_{b \in B} \setminus \{P_{i_0}\}) \cup \{P_b'\}_{b \in B_X'}$ is edge-disjoint and shows that X is not tight, since the edge $e_0 \in \mathsf{in}_D(X)$ is unused, which is a contradiction. ●

By the first statement of Proposition 5.4, $\mathfrak{R}[X]$ is independent and satisfies the linkage condition.

Claim 6.4. $\mathfrak{R}[X]$ *satisfies Condition 3.5 (Condition 3.6).*

Proof: Under Condition 3.5 the bases of $\mathcal{M}$ are finite and hence a tight set may have just finitely many ingoing edges. Thus the bases of $\mathcal{M}_{\mathfrak{R}[X]}$ are finite as well and therefore $\mathfrak{R}[X]$ satisfies the first part of Condition 3.5.

In the case of Condition 3.6, observe that the independence of $\mathfrak{R}$ implies that there are no loops in $\mathcal{M}$. Thus from a component $\mathcal{C}$ of $\mathcal{M}$ we get at most $r(\mathcal{C} \cap \mathcal{S}(X)) \leq r(\mathcal{C}) < \infty$ components of $\mathcal{M}_{\mathfrak{R}[X]}$. Since under Condition 3.6 the bases of $\mathcal{M}$ are countable, a tight set may have just countably many ingoing edges. Hence the set of the further single-element components $\{\{i_e\} : e \in \mathsf{in}_D(X)\}$ is countable. Thus $\mathfrak{R}[X]$ satisfies the first part of Condition 3.6.

To show the second part of Condition 3.5 (Condition 3.6) for $\mathfrak{R}[X]$ take a forward-infinite (backward-infinite) path P of $D[X]$ and let $i \in \mathcal{N}_{\mathfrak{R}[X]}(V(P))$ be arbitrary. By Corollary 5.2 and by the third statement of Proposition 5.4 (with $Z := V(P)$), we obtain

$$i \in \mathcal{N}_{\mathfrak{R}[X]}(V(P)) = \mathcal{N}(V(P)) \cap \mathcal{S}(X).$$

Since Condition 3.5 (Condition 3.6) holds for $\mathfrak{R}$ and $i \in \mathcal{N}(V(P))$ and Corollary 5.2 ensures

$$\mathcal{S}(V(P)) = \mathcal{S}_{\mathfrak{R}[X]}(V(P)) \subseteq \mathcal{S}(X),$$

we have

$$i \in \mathsf{span}(\mathcal{S}(V(P))) = \mathsf{span}(\mathcal{S}_{\mathfrak{R}[X]}(V(P))).$$

Hence by Observation 5.3 $i \in \mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(\mathcal{S}_{\mathfrak{R}[X]}(V(P)))$. ●

Let v_1 be the last vertex of P_{i_0} in X.

A restricted version of the following Sublemma was needed in [10] (applying the terminology of this paper $\mathcal{M}$ was there the free matroid and $\mathcal{N}(v) = \mathcal{M}$ for any v). The more general circumstances make the precise formalisation of the proof a bit ugly although the new difficulties are just technical. After we finish the proof of the Sublemma (page 25) we continue the proof of Lemma 6.1.

Sublemma 6.5. $\mathfrak{R}[X], v_1, W \cap \mathcal{S}(X) =: W^*$ is a counterexample for Lemma 6.1.

Proof: Suppose, to the contrary, that it is not, and choose a feasible, finite-order W^*-extension $\mathfrak{R}[X]^*$ of $\mathfrak{R}[X]$ such that $\mathcal{S}_{\mathfrak{R}[X]^*}(v_1) \cap W^*$ is a base of $\mathcal{N}_{\mathfrak{R}[X]}(v_1) \cap W^*$. Then Corollary 5.9 gives a feasible, finite-order W^*-extension $\mathfrak{R}^*$ of $\mathfrak{R}$ such that $\mathfrak{R}^*[X] = \mathfrak{R}[X]^*$.

Proposition 6.6. *There is some $i_1 \in \mathcal{S}_{\mathfrak{R}^*}(v_1) \cap W^*$ for which $B' := B - i_0 + i_1$ is a base of $\mathcal{N}(v_0)/\mathcal{S}(v_0)$.*

Proof: By the i_0-dangerousness of X, we have $i_0 \in \mathsf{span}(\mathcal{S}(X))$ and path P_{i_0} shows $i_0 \in \mathcal{N}(v_1)$. Thus by applying Proposition 5.4 with $Z := \{v_1\}$ we obtain

$$\begin{aligned} i_0 \in\ & \mathcal{N}(v_1) \cap \mathsf{span}(\mathcal{S}(X)) \\ &= \mathsf{span}(\mathcal{N}(v_1) \cap \mathcal{S}(X)) \\ &= \mathsf{span}(\mathcal{N}_{\mathfrak{R}[X]}(v_1) \cap \mathcal{S}(X)). \end{aligned}$$

Since $W \ni i_0$ and any circuit through i_0 lies in W, it implies

$$i_0 \in \mathsf{span}(\mathcal{N}_{\mathfrak{R}[X]}(v_1) \cap \mathcal{S}(X) \cap W) = \mathsf{span}(\mathcal{N}_{\mathfrak{R}[X]}(v_1) \cap W^*) = \mathsf{span}(\mathcal{S}_{\mathfrak{R}[X]^*}(v_1) \cap W^*).$$

Finally apply Corollary 2.9 with $i := i_0$, $I := B$, and $J := \mathcal{S}_{\mathfrak{R}^*}(v_1) \cap W^*$ and let i_1 be the resulting j of Corollary 2.9 . ●

Claim 6.7. *There is a base B^* of $\mathcal{N}(v_0)/\mathcal{S}(v_0)$ and a (B^*, v_0)-linkage $\mathcal{P}^*$ with respect to $\mathfrak{R}^*$ such that $i_1 \in B^*$ and $\mathcal{P}^* \ni P^*_{i_1} := P_{i_0}[v_1, v_0]$. Hence (1) is smaller for $\mathfrak{R}^*$ than for $\mathfrak{R}$ (which is a contradiction that proves Sublemma 6.5).*

Let $B' = B - i_0 + i_1$ (see Proposition 6.6). If the paths $\{P_i\}_{i \in B' - i_1}$ have no edge in $A(D) \setminus A(D_{\mathfrak{R}^*}) =: \boldsymbol{A_{lost}}$, then $\{P^*_{i_1}\} \cup \{P_i\}_{i \in B' - i_1}$ shows that a desired linkage exists and we are done. We may assume that it is not the case. Remember that $A_{lost} \subseteq A(D[X])$.

Consider the indices of those paths from $\{P_i\}_{i \in B' - i_1}$ that meet X i.e. $\boldsymbol{B_{ess}} := \{i \in B' - i_1 : V(P_i) \cap X \neq \varnothing\}$ (the **ess**ential paths). Let us define the **non**essential paths $B_{non} := B' \setminus B_{ess}$ as well. For $i \in B_{ess}$ we denote by q_i and z_i the first and the last vertex of P_i in X respectively. Whenever for some $i \in B_{ess}$ the path $P_i[q_i, z_i]$ use an edge $e \in \mathsf{out}_D(X)$ then there is an edge $h_e \in \mathsf{in}_D(X)$ of $P_i[q_i, z_i]$ which is corresponding to the first "come back" to X after e (see Figure 4). For all such an e, we extend $D_{\mathfrak{R}^*}[X]$ with a new edge $g(i, e)$ that goes from $\mathsf{tail}(e)$ to $\mathsf{head}(h_e)$. Furthermore pick a new vertex t and for all $i \in B_{ess}$ draw an edge f_i from z_i to t to obtain $\boldsymbol{H}$. Let $\boldsymbol{B_{in}} = \{i \in B_{ess} : \mathsf{start}(P_i) \in X\}$ and let $\boldsymbol{B_{out}} = B_{ess} \setminus B_{in}$.

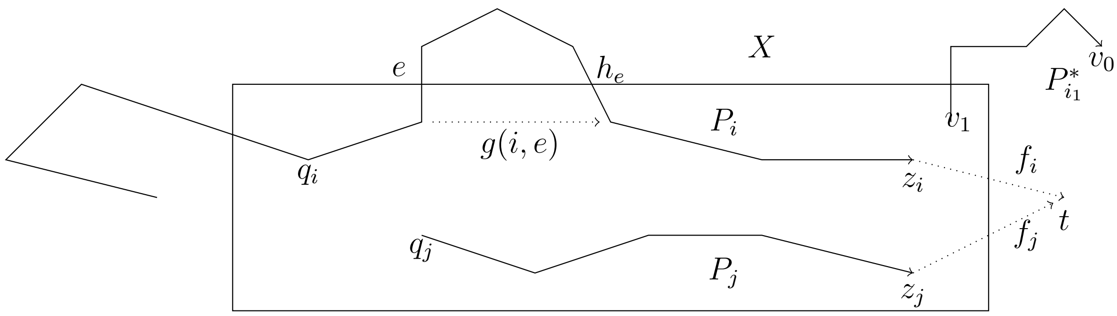

Figure 4: The construction of the digraph H. We have $i \in B_{out}$ and $j \in B_{in}$.

We claim that one can justify Claim 6.7 by proving the following Claim.

Claim 6.8. *There is a $B'_{in} \subseteq \mathcal{S}(X)$ such that the set $(B' \setminus B_{in}) \cup B'_{in} = B_{non} \cup B_{out} \cup B'_{in}$ is a base of $\mathcal{N}(v_0)/\mathcal{S}_{\mathfrak{R}}(v_0)$ and there is a system of edge-disjoint paths $\{Q_i\}_{i \in B_{out} \cup B'_{in}}$ in H such that for $i \in B_{out}$ path Q_i goes from q_i to t and for $i \in B'_{in}$ it goes from $\pi_{\mathfrak{R}[X]^*}(i)$ to t .*

Indeed, for $i \in B_{non}$ let $P^*_i = P_i$ if $i \neq i_1$ and let $P^*_{i_1} = P_{i_0}[v_0, v_1]$. For $i \in B_{out}$, replace first the edges in the form $g(j, e)$ of Q_i with the corresponding path segments $P_j[\mathsf{tail}(e), \mathsf{head}(h_e)]$. Then simplify the resulting walk to a path and delete its last edge, say f_k. Denote the result by $\widetilde{Q}_i$. Concatenate $P_i[\mathsf{start}(P_i), q_i]$ with $\widetilde{Q}_i$ and $P_k[z_k, v_0]$

to obtain P_i^*. In the case $i \in B'_{in}$, we do the same, except we need to concatenate just $\widetilde{Q}_i$ and $P_k[z_k, v_0]$ to get P_i^*. Finally $\{P_i^*\}_{i \in B_{non} \cup B_{out} \cup B'_{in}}$ is a desired linkage.

Let us define a matroid-rooted digraph that makes possible a reformulation of Claim 6.8. For $j \in B_{out}$, let $F(j) = i_e \in S_{\mathcal{M}[X]}$ where e is the unique ingoing edge of q_j in P_j and let F be the identity on $\mathcal{S}(X)$. We define

$$\boldsymbol{\mathcal{M}_{\mathfrak{Q}}} := [\mathcal{S}(X)/(B_{non} \cup B_{out} \cup \mathcal{S}(v_0))] \bigoplus F[B_{out}].$$

Note that $S_{\mathcal{M}_{\mathfrak{Q}}} \subseteq S_{\mathcal{M}_{\mathfrak{R}[X]}}$. Finally let $\boldsymbol{\mathfrak{Q}} := (H, \mathcal{M}_{\mathfrak{Q}}, \pi_{\mathfrak{R}[X]^*}|_{S_{\mathcal{M}_{\mathfrak{Q}}}})$.

Observation 6.9. The $\mathcal{M}_{\mathfrak{Q}}$-independent sets are $\mathcal{M}_{\mathfrak{R}[X]}$-independent and for $S' \subseteq S_{\mathcal{M}_{\mathfrak{R}[X]}}$

$$\mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(S') \cap S_{\mathcal{M}_{\mathfrak{Q}}} \subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(S' \cap S_{\mathcal{M}_{\mathfrak{Q}}}). \tag{2}$$

For any $T \subseteq X$, we have $\mathcal{S}_{\mathfrak{Q}}(T) = \mathcal{S}_{\mathfrak{R}[X]^*}(T) \cap S_{\mathcal{M}_{\mathfrak{Q}}}$ which implies by using (2) with $S' := \mathcal{S}_{\mathfrak{R}[X]^*}(T)$

$$\mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(\mathcal{S}_{\mathfrak{R}[X]^*}(T)) \cap S_{\mathcal{M}_{\mathfrak{Q}}} \subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(\mathcal{S}_{\mathfrak{Q}}(T)). \tag{3}$$

Proposition 6.10. *For $T \subseteq X$, we have $\mathcal{N}_{\mathfrak{R}[X]}(T) \cap S_{\mathcal{M}_{\mathfrak{Q}}} \subseteq \mathcal{N}_{\mathfrak{Q}}(T)$.*

Proof: We know that $\mathcal{N}_{\mathfrak{R}[X]} = \mathcal{N}_{\mathfrak{R}[X]^*}$ since $\mathfrak{R}[X]^*$ is a feasible extension of $\mathfrak{R}[X]$. Obviously $\mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T) \subseteq \mathsf{to}_H(T)$ because $D_{\mathfrak{R}[X]^*}$ is a subdigraph of H. Then by applying (3) of Observation 6.9 with $T := \mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T)$

$$\begin{aligned}\mathcal{N}_{\mathfrak{R}[X]}(T) \cap S_{\mathcal{M}_{\mathfrak{Q}}} &= \mathcal{N}_{\mathfrak{R}[X]^*}(T) \cap S_{\mathcal{M}_{\mathfrak{Q}}} = \mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]^*}}\left[\mathcal{S}_{\mathfrak{R}[X]^*}(\mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T))\right] \cap S_{\mathcal{M}_{\mathfrak{Q}}} \\ &\subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}\left[\mathcal{S}_{\mathfrak{Q}}(\mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T))\right] \subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}\left[\mathcal{S}_{\mathfrak{Q}}(\mathsf{to}_H(T))\right] = \mathcal{N}_{\mathfrak{Q}}(T). \quad \bullet\end{aligned}$$

Clearly $\boldsymbol{B_0} := F[B_{ess}] \subseteq \mathcal{N}_{\mathfrak{Q}}(t)$ is $\mathcal{M}_{\mathfrak{Q}}$-independent. In fact it is a base of $\mathcal{N}_{\mathfrak{Q}}(t)$. Indeed, if there is an $\mathcal{M}_{\mathfrak{Q}}$-independent I with $B_0 \subsetneq I \subseteq \mathcal{N}_{\mathfrak{Q}}(t)$, then we would obtain

$$B' \subsetneq B_{non} \cup B_{out} \cup I \cap \mathcal{S}(X) \subseteq \mathcal{N}(v_0)/\mathcal{S}(v_0)$$

where $B_{non} \cup B_{out} \cup I \cap \mathcal{S}(X)$ is $\mathcal{N}(v_0)/\mathcal{S}(v_0)$-independent which is impossible since B' is a base of $\mathcal{N}(v_0)/\mathcal{S}(v_0)$. Thus an equivalent formulation of Claim 6.8, that we will actually prove, is the following.

Claim 6.11. *There is a $(\widehat{B}, t)$-linkage with respect to $\mathfrak{Q}$ where $\widehat{B}$ is a base of $\mathcal{N}_{\mathfrak{Q}}(t)$.*

Proof: Fix a build sequence of $\mathfrak{R}[X]^*$ from $\mathfrak{R}[X]$ and let be the corresponding sequence of edges $\langle h_m : m < M \rangle$. Note that $\{h_m : m < M\} = A_{lost}$. For $n \leq M$, we denote the extension of H with the edges $\{h_m : n \leq m < M\}$ by H_n and we define $\mathfrak{Q}_n = (H_n, \mathcal{M}_{\mathfrak{Q}}, \pi_{\mathfrak{Q}})$. Note that $H_M = H$ and hence $\mathfrak{Q}_M = \mathfrak{Q}$.

Observation 6.12. If H_n contains an $u \to v$ path and $v \neq t$, then D as well since we can just replace the edges in form $g(j, e)$ by the corresponding paths of D.

Proposition 6.13. *For all $n \leq M$, we have $\mathcal{N}_{\mathfrak{Q}_n}(t) = \mathcal{N}_{\mathfrak{Q}}(t)$.*

Proof: Obviously $\mathcal{N}_{\mathfrak{Q}_n}(t) \supseteq \mathcal{N}_{\mathfrak{Q}}(t) = \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(B_0)$. Suppose, to the contrary that $\mathcal{N}_{\mathfrak{Q}_n}(t) \setminus \mathcal{N}_{\mathfrak{Q}}(t) \neq \varnothing$. Then there is some $i \in \mathcal{N}_{\mathfrak{Q}_n}(t) \setminus \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(B_0)$ such that t (and hence $\{z_i\}_{i \in B_{ess}}$) is reachable from $\pi_{\mathfrak{Q}}(i)$ in H_n. Necessarily $i \in \mathcal{S}(X)$ because $S_{\mathcal{M}_{\mathfrak{Q}}} \setminus \mathcal{S}(X) = F[B_{out}] \subseteq B_0$. Then by Observation 6.12 $\{z_i\}_{i \in B_{ess}}$ is reachable from $\pi_{\mathfrak{Q}}(i) = \pi_{\mathfrak{R}^*}(i) \cap X$ in D. But then from $\pi(i)$ as well, since all the new vertices that get $\pi(i)$ in an extension were originally reachable from $\pi(i)$. It follows that $i \in \mathcal{N}(\{z_i\}_{i \in B_{ess}}) \subseteq \mathcal{N}(v_0)$ because v_0 is reachable in D from any element of $\{z_i\}_{i \in B_{ess}}$. But then $B' \cup \{i\} \subseteq \mathcal{N}(v_0)/\mathcal{S}(v_0)$ would be independent which is a contradiction since B' is a base of $\mathcal{N}(v_0)/\mathcal{S}(v_0)$ and clearly $i \notin B'$ since $i \in S_{\mathcal{M}_{\mathfrak{Q}}}$ and

$$B' \cap S_{\mathcal{M}_{\mathfrak{Q}}} = B_{in} \subseteq B_0 \subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(B_0) \not\ni i. \;\bullet$$

We prove by induction that for all $n \leq M$ there is a (B_n, t)-linkage with respect to $\mathfrak{Q}_n$ where B_n is a base of $\mathcal{N}_{\mathfrak{Q}}(t)$. For $n = M$, we will obtain a desired linkage for Claim 6.11.

Let us start with the case $n = 0$. For $i \in B_{ess}$, consider $P_i[q_i, z_i]$ and for any $e \in \mathsf{out}_D(X) \cap A(P_i[q_i, z_i])$ replace the segment $P_i[\mathsf{tail}(e), \mathsf{head}(h_e)]$ by the single edge $g(i, e)$ to obtain a path $P^0_{F(i)}$ in H_0. The linkage $\mathcal{P}_0 = \{P^0_i\}_{i \in B_0}$ is suitable.

Suppose that there is a (B_n, t)-linkage $\mathcal{P}_n = \{P^n_i\}_{i \in B_n}$ with respect to $\mathfrak{Q}_n$ such that $n < M$ and B_n is a base of $\mathcal{N}_{\mathfrak{Q}}(t)$. We need to give a desired linkage with respect to $\mathfrak{Q}_{n+1}$. We may assume that for some $j_0 \in B_n$ we have $h_n \in A(P^n_{j_0})$ otherwise $\mathcal{P}_n$ would be appropriate. Consider the $(B_n - j_0, t)$-linkage $\mathcal{P}'_n := \{P^n_i\}_{i \in B_n - j_0}$ in $\mathfrak{Q}_{n+1}$. Note that if for some $\mathcal{M}_{\mathfrak{Q}}$-independent $I \subseteq \mathcal{N}_{\mathfrak{Q}_{n+1}}(t) = \mathcal{N}_{\mathfrak{Q}}(t)$ we have

$$\mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(I) \supsetneq \mathsf{span}_{\mathcal{M}_{\mathfrak{Q}}}(B_n - j_0),$$

then by Fact 2.1 I is necessarily a base of $\mathcal{N}_{\mathfrak{Q}}(t)$. Apply Theorem 4.6 with linkage $\mathcal{P}'_n$ in $\mathfrak{Q}_{n+1}$. We may assume that $\mathcal{P}'_n$ and the largest t-good set T^+ of $\mathfrak{Q}_{n+1}$ satisfy the complementarity conditions since otherwise Theorem 4.6 provides us a desired linkage. Let $f_{i(j_0)}$ be the last edge of $P^n_{j_0}$. Clearly $z_{i(j_0)} \in T^+$ otherwise $f_{i(j_0)} \in \mathsf{in}_{H_{n+1}}(T^+) \setminus A(\mathcal{P}'_n)$ contradicting to the complementarity conditions.

We build $\mathcal{P}_{n+1}$ in three steps. First let $\boldsymbol{B^{in}_n} = (B_n - j_0) \cap \mathcal{S}_{\mathfrak{Q}}(T^+)$ and for $i \in B^{in}_n$ let $P^{n+1}_i = P^n_i$. By the first complementarity condition, B^{in}_n is a $\mathcal{M}_{\mathfrak{Q}}$-base of $\mathcal{S}_{\mathfrak{Q}}(T^+)$ and these paths lie inside T^+. Let $T = T^+ - t$ and we define $\boldsymbol{B^{un}_n} = B_n \setminus \mathcal{N}_{\mathfrak{R}[X]}(T)$.

Proposition 6.14. $j_0 \notin B^{un}_n$.

Proof: Since $z_{i(j_0)} \in T$, the path $P^n_{j_0}$ shows by applying Observation 6.12 that T is reachable from $\pi_{\mathfrak{Q}}(j_0) = \pi_{\mathfrak{R}[X]^*}(j_0)$ in D and hence from $\pi_{\mathfrak{R}[X]}(j_0)$ as well thus $j_0 \in \mathcal{N}_{\mathfrak{R}[X]}(T)$. $\bullet$

In the second step we define $P^{n+1}_i := P^n_i$ for $i \in B^{un}_n$. Proposition above ensures that these paths are in H_{n+1}. To construct the third part take a reduced linkage $\mathcal{R} = \{R_i\}_{i \in B_r}$ for T with respect to $\mathfrak{R}[X]^*$. The path-system $\mathcal{R}$ lies in H_{n+1} because $D_{\mathfrak{R}^*}[X]$ is a subdigraph of H_{n+1} . Since $\mathfrak{R}[X]^*$ is a feasible extension of $\mathfrak{R}[X]$, the set B_r is a base of $\mathcal{N}_{\mathfrak{R}[X]}(T)/\mathcal{S}_{\mathfrak{R}[X]^*}(T)$.

Proposition 6.15. $B_n \setminus (B_n^{in} \cup B_n^{un}) \subseteq \mathsf{span}_{\mathcal{M}_{\mathfrak{O}}}(B_n^{in} \cup (B_r \cap S_{\mathcal{M}_{\mathfrak{O}}}))$.

Proof: Let $i \in B_n \setminus (B_n^{in} \cup B_n^{un})$ be arbitrary. Then $i \notin \mathsf{span}_{\mathcal{M}_{\mathfrak{O}}}(B_n^{in}) = \mathsf{span}_{\mathcal{M}_{\mathfrak{O}}}(\mathcal{S}_{\mathfrak{O}}(T))$ hence by (3) of Observation 6.9 $i \notin \mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(\mathcal{S}_{\mathfrak{R}[X]^*}(T))$. On the other hand, $i \in \mathcal{N}_{\mathfrak{R}[X]}(T)$ because $i \notin B_n^{un}$. It shows that $i \in \mathcal{N}_{\mathfrak{R}[X]}(T)/\mathcal{S}_{\mathfrak{R}[X]^*}(T)$. Then $i \in \mathsf{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(B_r \cup \mathcal{S}_{\mathfrak{R}[X]^*}(T))$ because the choice of B_r. Hence by Observation 6.9 $i \in \mathsf{span}_{\mathcal{M}_{\mathfrak{O}}}((B_r \cap S_{\mathcal{M}_{\mathfrak{O}}}) \cup \mathcal{S}_{\mathfrak{O}}(T))$ which is enough since $\mathsf{span}_{\mathcal{M}_{\mathfrak{O}}}(B_n^{in}) = \mathcal{S}_{\mathfrak{O}}(T)$. ●

We may take a $B_r' \subseteq B_r \cap S_{\mathcal{M}_{\mathfrak{O}}}$ for which $B_n^{in} \cup B_n^{un} \cup B_r'$ is a maximal $\mathcal{M}_{\mathfrak{O}}$-independent subset of $B_n^{in} \cup B_n^{un} \cup (B_r \cap S_{\mathcal{M}_{\mathfrak{O}}})$. For $i \in B_r'$, concatenate R_i with the terminal segment of the path P_j^n which is corresponding to the last edge of R_i to obtain P_i^{n+1}. These paths also witnesses that $B_r' \subseteq \mathcal{N}_{\mathfrak{O}}(t)$ and therefore $B_n^{in} \cup B_n^{un} \cup B_r'$ is a base of $\mathcal{N}_{\mathfrak{O}}(t)$ since it is independent and spans such a base namely B_n.

We need to check that the paths $\{P_i^{n+1}\}_{i \in B_r'}$ have no common edges with the paths $\{P_i^n\}_{i \in B_n^{in} \cup B_n^{un}}$. The path-system $\{P_i^n\}_{i \in B_n^{in}}$ lies in T^+ and the terminal segments of the paths $\{P_i^{n+1}\}_{i \in B_r'}$ from the first (and only) entering to T^+ are some other elements of $\mathcal{P}_n$ which itself is an edge-disjoint system. Hence the path-system $\{P_i^{n+1}\}_{i \in B_r'} \cup \{P_i^n\}_{i \in B_n^{in}}$ is edge-disjoint. From the definition of B_n^{un} it follows that

$$\mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T) \cap \bigcup_{i \in B_n^{un}} V(P_i^n) = \varnothing.$$

On the other hand,

$$\mathsf{to}_{D_{\mathfrak{R}[X]^*}}(T) \supseteq \bigcup_{i \in B_r} V(P_i^{n+1}) \setminus \{t\},$$

thus the two paths-systems may not even have a common vertex other than t. Now the proof of Claim 6.11 is complete and hence the proof of Claim 6.7 and the proof of Sublemma 6.5 as well. ● ●

We continue the proof of Lemma 6.1. We obtained by Sublemma 6.5 and by Proposition 6.3 that if $\mathfrak{R}_0, v_0, W_0$ is a counterexample triple, then there is a feasible, finite-order W_0-extension $\mathfrak{R}_1$ of $\mathfrak{R}_0$ such that there is a vertex set $X =: X_1 \not\ni v_0$ which is tight wit respect to $\mathfrak{R}_1$ and for a suitable $v_1 \in X_1$ the triple $\mathfrak{R}_1[X], v_1, \mathcal{S}_{\mathfrak{R}_1}(X) \cap W_0 =: W_1$ is a counterexample again. Furthermore we know that there is an $e_1 \in \mathsf{in}_{D_{\mathfrak{R}_1}}(X_1)$ and there is a path, namely $P_{i_1}^*$ see Figure 4, that goes strictly from X_1 to v_0 and starts at v_1. The path P_{i_0} shows that v_1 is reachable outside X_1 in $D_{\mathfrak{R}_1}$. We may apply these observations with the new counterexample triple and iterate the process recursively to get an infinite sequence of counterexample triples $\langle (\mathfrak{R}_n[X_n], W_n, v_n) : n < \omega \rangle$ with $X_0 := V$.

Here $\mathfrak{R}_{n+1}$ is a finite-order, feasible W_n-extension of $\mathfrak{R}_n$ where the extension use edges only from $D[X_n]$, $\langle X_n : n < \omega \rangle$ is a nested sequence of vertex sets such that X_n is tight with respect to $\mathfrak{R}_n$ and $v_n \in X_n$ but $v_n \notin X_{n+1}$. We also have a path P_n in $D_{\mathfrak{R}_n}$ from X_{n+1} to v_n with $\mathsf{start}(P_n) = v_{n+1}$ and some edge $e_{n+1} \in \mathsf{in}_{D_{\mathfrak{R}_{n+1}}[X_n]}(X_{n+1})$.

If $\mathfrak{R}_0$ satisfies Condition 3.6, then we build a backward-infinite path P by concatenating the paths P_n for $n = 1, 2, \ldots$. Then P lies in the $\mathfrak{R}_1$-tight X_1 and $V(P)$ is reachable in $D_{\mathfrak{R}_1}$ from outside X_1 in $D_{\mathfrak{R}_1}$ (showed by P_{i_1}) contradicting to Corollary 5.2. It proves Lemma 6.1 in the case when $\mathfrak{R}_0$ satisfies Condition 3.6.

Suppose now that $\mathfrak{R}_0$ satisfies Condition 3.5. The sequence $\langle \mathcal{N}(X_n) : n < \omega \rangle$ is $\subseteq$-decreasing thus $\langle r(\mathcal{N}(X_n)) : n < \omega \rangle$ is a decreasing sequence of natural numbers therefore by throwing away some initial elements we may assume that $r(\mathcal{N}(X_n))$ does not depend on n. On the other hand, the $(n+1)$-th extension uses edges only from $D_{\mathfrak{R}_n}[X_n]$ thus we have

$$\mathcal{S}_{\mathfrak{R}_n}(X_n) = \mathcal{S}_{\mathfrak{R}_{n+1}}(X_n) \supseteq \mathcal{S}_{\mathfrak{R}_{n+1}}(X_{n+1})$$

and therefore

$$r(\mathcal{S}_{\mathfrak{R}_n}(X_n)) \geq r(\mathcal{S}_{\mathfrak{R}_{n+1}}(X_{n+1})).$$

But then

$$r(\mathcal{N}(X_n)/\mathcal{S}_{\mathfrak{R}_n}(X_n)) = r(\mathcal{N}(X_n)) - r(\mathcal{S}_{\mathfrak{R}_n}(X_n))$$

is an increasing function of n (bounded by $r(\mathcal{M}) < \infty$) hence similarly we may suppose that it is constant, say m_0. Pick a reduced linkage $\mathcal{P}_1$ for X_1 with respect to $\mathfrak{R}_1$. It consists of m_0 paths and these paths use all the elements of $\mathsf{in}_{D_{\mathfrak{R}_1}}(X_1) \ni e_1$ because of the tightness of X_1. Then pick a reduced linkage $\mathcal{Q}$ for X_2 in $\mathfrak{R}_2$. Observe that these paths also use all the elements of $\mathsf{in}_{D_{\mathfrak{R}_1}}(X_1) = \mathsf{in}_{D_{\mathfrak{R}_2}}(X_1)$. Take the set of the terminal segments of the elements of $\mathcal{Q}$ from the first vertex in X_1 and denote it by $\mathcal{Q}'$. From $\mathcal{P}_1$ obtain via concatenation with elements in $\mathcal{Q}'$ a reduced linkage $\mathcal{P}_2$ for X_2 with respect to $\mathfrak{R}_2$. Iterate the process recursively. In a general step we have a reduced linkage $\mathcal{P}_n$ for X_n with respect to $\mathfrak{R}_n$ and we find forward-continuations for the elements of $\mathcal{P}_n$ to obtain a reduced linkage for X_{n+1} with respect to $\mathfrak{R}_{n+1}$. By the tightness of X_{n+1} with respect to $\mathfrak{R}_{n+1}$, necessarily $e_{n+1} \in A(\mathcal{P}_{n+1})$. Eventually we obtain an edge disjoint path-system $\mathcal{P}$ with m_0 members. Since the edges $\{e_n\}_{1 \leq n < \omega} \subseteq A(\mathcal{P})$ are pairwise distinct, there is a $P \in \mathcal{P}$ that contains infinitely many of them. A terminal segment of the forward-infinite path P lies inside X_1 and reachable from outside X_1 in $D_{\mathfrak{R}_1}$ (showed by P itself) which contradicts Corollary 5.2. Now the proof of Lemma 6.1 is complete. ■

7 Careful iteration of local augmentations

Now we are able to prove our main result Theorem 3.7. Suppose first that V is countable and $V = \{v_n\}_{n<\omega}$ and the components of $\mathcal{M}$ are $\{\mathcal{C}_n\}_{n<\omega}$ (if there are just finitely many, then repetition is allowed). Let $\omega \times \omega = \{p_n : n < \omega\}$. We build recursively a sequence $\langle \mathfrak{R}_n : n \leq \omega \rangle$ such that $\mathfrak{R}_0 = \mathfrak{R}$ and if $p_n = \langle m, k \rangle$, then we obtain $\mathfrak{R}_{n+1}$ by applying Lemma 6.1 to $\mathfrak{R}_n = (D_n, \mathcal{M}, \pi_n)$ with v_m and $\mathcal{C}_k$. Finally let $\mathfrak{R}_\omega = (D_\omega, \mathcal{M}, \pi_\omega)$ where $D_\omega = (V, \bigcap_{n<\omega} A(D_n))$ and $\pi_\omega(i) = \bigcup_{n<\omega} \pi_n(i)$. By the construction, for any $v \in V$ and any component $\mathcal{C}$ of $\mathcal{M}$, the set $\mathcal{S}_\omega(v) \cap \mathcal{C}$ is a base of $\mathcal{N}(v) \cap \mathcal{C}$ thus for all $v \in V$ the set $\mathcal{S}_\omega(v)$ is a base of $\mathcal{N}(v)$.

In the general case, we should organize the recursion more warily to ensure that after limit steps Condition 3.8 holds. Let $V = \{v_\xi : \xi < \kappa\}$. To obtain $\mathfrak{R}_{\xi+1}$ from $\mathfrak{R}_\xi$ we consider v_ξ and all the finitely many vertices that lost some ingoing edge since the last limit step. We apply to these vertices v one by one in an arbitrary order Lemma 6.1 with the smallest n for which $\mathcal{S}_\xi(v) \cap \mathcal{C}_n$ is not a base of $\mathcal{N}(v) \cap \mathcal{C}_n$ (if such an n does not exists for some v, then do nothing with that v). Observe that it ensures that after a limit step α a $v \in V$ either keeps all of its ingoing edges or $\mathcal{S}_\alpha(v)$ is a base of $\mathcal{N}(v)$. We need to justify that at limit steps we obtain feasible extensions in the process above.

Proposition 7.1. *Let $\alpha < \kappa$ be a limit ordinal and suppose that $\langle \mathfrak{R}_\beta : \beta < \alpha \rangle$ has been defined as above and this is a chain of feasible extensions of $\mathfrak{R}_0 = \mathfrak{R}$. Then the limit $\mathfrak{R}_\alpha$ of the sequence is also feasible extension of $\mathfrak{R}$.*

Proof: Let $v \in V$ arbitrary and pick a linkage $\{P_b\}_{b \in B}$ for v with respect to $\mathfrak{R}$. If some P_b is not a path in D_α then replace it by the terminal segment Q_b of itself that starts at the head u_b of the last deleted edge of P_b otherwise let $Q_b = P_b$ and let u_b be the first vertex of P_b. Note that the our recursive process guarantees that $b \in \mathsf{span}(\mathcal{S}_\alpha(u_b))$. It is enough to show that there is a transversal for $\{\mathcal{S}_\alpha(u_b)\}_{b \in B}$ which is a base of $\mathcal{N}(v)$. To do so we prove that for any component $\mathcal{C}$ of $\mathcal{M}$ there is a transversal for $\{\mathcal{S}_\alpha(u_b)\}_{b \in B \cap \mathcal{C}}$ which is a base of $\mathcal{N}(v) \cap \mathcal{C}$.

Let $\mathcal{C}$ be fixed and let $B_\mathcal{C} := B \cap \mathcal{C} = \{b_1, \dots, b_{\ell_0}\}$. Pick a base $B'_\mathcal{C} = \{b'_1, \dots, b'_{\ell_0}\}$ of $\mathcal{N}(v) \cap \mathcal{C}$ for which $b'_\ell \in \mathsf{span}(\mathsf{S}_\alpha(u_{b_\ell}))$ holds for all $1 \le \ell \le \ell_0$ and $b'_\ell \in \mathcal{S}_\alpha(u_{b_\ell})$ for as many ℓ as possible. Assume, to the contrary, that $b'_{\ell_1} \notin \mathcal{S}_\alpha(u_{b_{\ell_1}})$ for some $1 \le \ell_1 \le \ell_0$. The fact $b'_{\ell_1} \in \mathsf{span}(\mathcal{S}_\alpha(u_{b_{\ell_1}})) \setminus \mathcal{S}_\alpha(u_{b_{\ell_1}})$ implies that there is a circuit $C \ni b'_{\ell_1}$ such that $(C \setminus \{b'_{\ell_1}\}) \subseteq \mathcal{S}_\alpha(u_{b_{\ell_1}})$. Note that $C \subseteq \mathcal{C}$ because $b'_{\ell_1} \in \mathcal{C}$. Since $(C \setminus \{b'_{\ell_1}\}) \not\subseteq \mathsf{span}(B'_\mathcal{C} - b'_{\ell_1})$ (otherwise $b'_{\ell_1} \in \mathsf{span}(B'_\mathcal{C} - b'_{\ell_1})$), there is some $b''_{\ell_1} \in (C \setminus \{b'_{\ell_1}\})$ for which $B'_\mathcal{C} - b'_{\ell_1} + b''_{\ell_1}$ is still a base of $\mathcal{N}(v) \cap \mathcal{C}$ contradicting to the choice of $B'_\mathcal{C}$. ●

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