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Attila Joó ⋆

**Abstract**

In the finite case an $r$-vertex-flame is a finite directed graph $F$ with $r \in V(F)$ in which for every vertex $v \neq r$ the indegree of $v$ is equal to $\kappa_F(r,v)$ (the local connectivity from $r$ to $v$ in $F$). G. Calvillo Vives proved that if $D$ is a finite directed graph with $r \in V(D)$, then there is a spanning subdigraph $F$ of $D$ such that $F$ is an $r$-vertex-flame and $\kappa_F(r,v) = \kappa_D(r,v)$ holds for every $v$. Our goal is to find the "right" infinite generalization of this theorem. We extend the definition of flame to the infinite case by demanding for every $v$ an internally disjoint system of $r \to v$ paths which uses all the ingoing edges of $v$. Instead of just preserving the local connectivities from $r$ as cardinals we want to preserve in $F$ for every $v$ an Aharoni-Berger cut (see subsection 1.2) of $D$ from $r$ to $v$. The main result of this paper is to accomplish this for countable digraphs.

## 1 Introduction

### 1.1 Notation

We use some standard notation form set theory. We use the $\aleph$-operation, we write $\kappa$ for cardinals and $\alpha, \beta, \gamma$ for ordinals. The smallest limit ordinal (i.e. the set of the natural numbers) is denoted by $\omega$. For a family of sets $\mathcal{X}$, the union of the elements of $\mathcal{X}$ is denoted by $\bigcup \boldsymbol{\mathcal{X}}$. The operation $\bigcap \boldsymbol{\mathcal{X}}$ is defined similarly but only for $\mathcal{X} \neq \varnothing$. For ordered pairs we write simply $\boldsymbol{uv}$ instead of the usual $\langle u, v \rangle$ and $X - x$ and $X + x$ are abbreviations of $X \setminus \{x\}$ and $X \cup \{x\}$.

The digraphs in this paper are simple and they may have arbitrary size until we say explicitly otherwise. They always have a unique vertex $r$ which has no ingoing edges and we denote the set of the other vertices by $V$. Thus formally the digraphs have the form $\boldsymbol{D = (V + r, A)}$ where $r \notin V$ and

$$A \subseteq (V + r) \times (V + r) \setminus (\{vv : v \in (V + r)\} \cup \{vr : v \in V\}).$$

If we handle an edge set $E \subseteq A$ as a digraph, then we always think of the corresponding spanning subdigraph $(V + r, E)$ of $D$. The set of the ingoing (outgoing) edges of a

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vertex set $X$ with respect to the edge set (spanning subdigraph) $E \subseteq A$ is denoted by $\mathsf{in}_E(X)$ $(\mathsf{out}_E(X))$. (When we apply set functions to singletons we usually omit the extra brackets.) For the in-neighbours (out-neighbours) of a vertex $v$ we write $N_E^-(v)$ $(N_E^+(v))$. If the edge $e$ is from $u$ to $v$, then $\mathsf{tail}(e) = u$ and $\mathsf{head}(e) = v$. The entrance of $X$ in $E$ is $\{\mathsf{head}(e) : e \in \mathsf{in}_E(X)\}$ and we denote it by $\mathsf{ent}_E(X)$. We write $\mathsf{int}_E(X)$ for $X \setminus \mathsf{ent}_E(X)$.

Paths are always assumed to be finite and directed, repetition of vertices is not allowed. For a path $P$, let $\mathsf{start}(P)$ $(\mathsf{end}(P))$ be the first (last) vertex of $P$. We call $P$ a **trivial path** if $V(P) = \{\mathsf{start}(P), \mathsf{end}(P)\}$ (i.e. it has at most two vertices). We say that $P$ is an $X \to Y$ path if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$. For a path-system (set of paths) $\mathcal{P}$ let $V_{\mathbf{start}}(\mathcal{P}) = \{\mathsf{start}(P) : P \in \mathcal{P}\}$ and $V_{\mathbf{end}}(\mathcal{P}) = \{\mathsf{end}(P) : P \in \mathcal{P}\}$. We denote the set of the first (last) edges of the paths in $\mathcal{P}$ by $A_{\mathbf{first}}(\mathcal{P})$ $(A_{\mathbf{last}}(\mathcal{P}))$. Finally, $V(\mathcal{P})$ $(A(\mathcal{P}))$ stands for the union of the vertices (edges) of the paths in $\mathcal{P}$. A path-system $\mathcal{P}$ is called **$v$-disjoint** if for any $P \neq Q \in \mathcal{P}$ we have $V(P) \cap V(Q) = \{v\}$. A system $\mathcal{P}$ of $u \to v$ paths is internally disjoint if for any $P \neq Q \in \mathcal{P}$ we have $V(P) \cap V(Q) = \{u, v\}$. For $v \in V$ let $\kappa_A(r, v)$ be the largest cardinal $\kappa$ for which there is an internally disjoint system $\mathcal{P}$ of $r \to v$ paths with $|\mathcal{P}| = \kappa$. Note that this maximum exists even in the infinite case and by the weak form of infinite Menger's theorem we have

$$\kappa_{A-rv}(r, v) = \min\{|U| : U \subseteq V \setminus \{r, v\},\ U \text{ covers all the nontrivial } r \to v \text{ paths}\}.$$

## 1.2 The Aharoni-Berger cuts

P. Erdős observed during his school years that Menger's minimax theorem remains true in infinite digraphs (by saying cardinals instead of numbers) and he conjectured the "right" infinite generalization of the theorem. He recognized that cardinality is usually a too rough measure when we generalize a minimax theorem to infinite. The nature of cardinal arithmetic makes cardinalities less meaningful than natural numbers with respect to these problems. To obtain deep and important statements, the infinite generalization needs to reflect the combinatorial structure instead of just the quantities. In the case of Menger's theorem this infinite generalization was known as the Erdős-Menger conjecture which was the most important unsolved problem in the theory of infinite graphs for five decades according to the experts in the field. Let us shortly recall this problem.

A direct consequence of Menger's original theorem is that if $D$ is a finite digraph with $S, T \subseteq V(D)$ and $\mathcal{P}$ is a largest system of (pairwise) disjoint $S \to T$ paths and $C$ is a smallest vertex-cut between the vertex sets $S$ and $T$ (i.e. vertex set which covers all the directed paths from $S$ to $T$), then $C$ consists of one vertex from each path in $\mathcal{P}$. The Erdős-Menger conjecture states that in the infinite case we still always have a path-system $\mathcal{P}$ and a cut $C$ satisfying the conditions above.

R. Aharoni, Nash-Williams and S. Shelah proved in 1984 (see [4]) the infinite generalization of the Marriage theorem which was a milestone in the proof of the Erdős-Menger conjecture. In 1987 Aharoni settled affirmatively the countable case of the conjecture (see [2]). Finally R. Aharoni and E. Berger proved the Erdős-Menger conjecture in its full generality in 2009 (see [3]).

We call an internally disjoint system $\mathcal{P}$ of $r \to v$ paths orthogonal to the vertex set $C$ and write $\boldsymbol{\mathcal{P} \bot C}$ if one can obtain $C$ by picking one internal vertex from each path in $\mathcal{P}$. Let us denote by $\boldsymbol{\mathcal{AB}_A^{\mathrm{path}}(v)}$ the **Aharoni-Berger path-systems** from $r$ to $v$ in $A$ i.e. the set of those systems $\mathcal{P}$ of internally disjoint $r \to v$ paths in $A$ for which there is a $C \bot \mathcal{P}$ such that $C$ covers all the $r \to v$ paths in $A - rv$. The celebrated result [3] of R. Aharoni and E. Berger guarantees that $\mathcal{AB}_A^{\mathrm{path}}(v) \neq \varnothing$ (apply it with $S := N_{A-rv}^{+}(r)$ and $T := N_{A-rv}^{-}(v)$ after the deletion of $r$ and $v$). Similarly $\boldsymbol{\mathcal{AB}_A^{\mathrm{cut}}(v)}$ is the set of the **Aharoni-Berger $\boldsymbol{rv}$-cuts** i.e. the set of those $C$ vertex sets which covers all the $r \to v$ paths in $A - rv$ and $C \bot \mathcal{P}$ for a suitable $\mathcal{P} \in \mathcal{AB}_A^{\mathrm{path}}(v)$.

## 1.3 The large flames of Calvillo Vives

Let $D = (V + r, A)$ be a fixed finite digraph. An $L \subseteq A$ is called large (with respect to $A$) if it preserves the local connectivities from $r$ i.e. $\kappa_A(r, v) = \kappa_L(r, v)$ holds for every $v \in V$. For a large $L$ clearly $|L| \geq \sum_{v \in V} \kappa_A(r, v)$ because necessarily $|\mathsf{in}_L(v)| \geq \kappa_A(r, v)$ for each $v \in V$. An $F \subseteq A$ is called a flame if $\kappa_F(r, v) = |\mathsf{in}_F(v)|$ for every $v \in V$. Note that for a large $L$ the number $|L|$ reaches the lower bound above if and only if $L$ is a flame.

**Theorem 1** (G. Calvillo Vives). *In every finite digraph $D = (V + r, A)$ there exists a large flame.*

*Remark* 2. G. Calvillo Vives formulated originally his theorem by using edge-connectivity instead of vertex-connectivity in the definition of flame and largeness but from his original theorem one can derive easily the vertex-version above (see subsection 3.4). As far we know the theorem appeared only in his PhD thesis which is not available in electronic form.

## 1.4 The "right" infinite generalization via Aharoni-Berger cuts

The equations which define a large flame $F^*$, namely $\kappa_A(r, v) = \kappa_{F^*}(r, v) = |\mathsf{in}_{F^*}(v)|$ ($v \in V$), make sense in the infinite case as well. Even so, cardinality is a too rough measure to consider an infinite generalization based on it satisfactory. A natural strengthening of the condition $\kappa_F(r, v) = |\mathsf{in}_F(v)|$ allows us to extend the notion of flame beyond finite digraphs. An $F \subseteq A$ is a **flame** in $D = (V + r, A)$ if for every vertex $v$ there is a system $\mathcal{P}$ of internally disjoint $r \to v$ paths in $F$ such that $A_{\mathrm{last}}(\mathcal{P}) = \mathsf{in}_F(v)$. Note that $F$ is a flame if and only if for every $v \in V$ we have $(N_F^{-}(v) - r) \in \mathcal{AB}_F^{\mathrm{cut}}(v)$.

Generalizing largeness is less straightforward. Let us call an $L \subseteq A$ $\boldsymbol{v}$**-large** if

- $rv \in L$ if $rv \in A$,
- $\mathcal{AB}_L^{\mathrm{cut}}(v) \cap \mathcal{AB}_A^{\mathrm{cut}}(v) \neq \varnothing$.

*Remark* 3. If $\kappa_A(r, v)$ is finite, then $v$-largeness of $L$ is equivalent with $\kappa_L(r, v) = \kappa_A(r, v)$ but in the general case $v$-largeness is stronger than this cardinal equation.

An $L \subseteq A$ is defined to be **large** if it is $v$-large for every $v \in V$. The main result of this paper is the following.

**Theorem 4.** *In every countable digraph $D = (V + r, A)$ there exists a large flame $F^*$.*

### 1.5 The arising difficulties compared to the finite case

In the original proof of Calvillo Vives (more precisely in the vertex-analogue of his proof) he has a flame $F$ and he picks a system $\mathcal{P}$ of internally disjoint $r \rightarrow v$ paths of size $\kappa_A(r, v)$ in $A$ such that $A(\mathcal{P}) \cap (A \setminus F)$ is $\subseteq$-minimal among these path-systems. One can show that this choice guarantees that $F \cup A(\mathcal{P})$ is a flame (which is $v$-large). Applying this $|V|$-times we obtain the desired large flame. The obvious difficulty with this approach in the infinite case (using Aharoni-Berger path-systems instead of just cardinality-wise maximal ones) is that the existence of such a $\subseteq$-minimal choice of a path-system is highly unclear. The other difficulty with successive extension of flames in general is that in the infinite case the union of a $\subseteq$-increasing sequence of flames may fail to be a flame and thus we may have problem in the limit steps of the construction. Let us describe two more proofs for the finite case.

One can show that if $F \subseteq A$ is a flame which is not $v$-large for some $v \in V$, then there is an $e \in A \setminus F$ for which $F + e$ is a flame (see Claim 26). In the finite case we may build up $F^*$ from an arbitrary flame, say $\varnothing$, applying this fact repeatedly. (In fact, we need here only $\kappa_A(r, v) < \aleph_0$ for $v \in V$ instead of the finiteness of $D$.)

It is not too hard to check that in the finite case if $\mathcal{P}$ is a system of internally disjoint $r \rightarrow v$ paths with $|\mathcal{P}| = \kappa_A(r, v)$, then the deletion of all the ingoing edges of $v$ except $A_{\text{last}}(\mathcal{P})$ keeps largeness. Repeating this $|V|$-times leads us to a proof of the finite theorem. It turns out that the analogue operation is working in the infinite case even with our stronger structural definition of largeness (see Corollary 25) but the intersection of a $\subseteq$-decreasing sequence of large sets may fail to be a large.

# 2 The proof of the main result

### 2.1 A brief remainder about augmenting paths

The augmenting path technique is well-known from the standard proof of the maxflow-mincut theorem (see for example [7] p. 122). It also has some more general versions involving polymatroidal flows (see [8]). We need here only the basic form that one can use for the augmentation of internally disjoint paths-systems.

Let $\mathcal{P}$ be an internally disjoint system of $s \rightarrow t$ paths in $A \not\ni st$. Let $A_{\mathcal{P}}$ be the auxiliary digraph that we obtain from $A$ by changing the direction of the edges in $A(\mathcal{P})$. We say that these are the backward edges and the unchanged edges are the forward edges of $A_{\mathcal{P}}$. An augmenting path $Q$ is a path in $A_{\mathcal{P}}$ with $\mathsf{start}(Q) = s$ such that for any forward edge $e \in A(Q)$ with $\mathsf{head}(e) \in V(\mathcal{P}) - t$ there is a next edge in $Q$ which is a backward edge.

**Proposition 5.** *If $Q$ is an augmenting path with first edge $e$ and last edge $f$ and $\mathsf{end}(Q) = t$, then there exists an internally disjoint system $\mathcal{R}$ of $s \rightarrow t$ paths with*

*$A_{first}(\mathcal{R}) = A_{first}(\mathcal{P}) + e$ and $A_{last}(\mathcal{R}) = A_{last}(\mathcal{P}) + f$. If there is no augmenting path terminating in $t$, then for*

$$B := \{v \in V : \text{ there is no augmenting path } Q \text{ with } \mathsf{end}(Q) = v\}$$

*we have $\mathcal{P} \perp \mathsf{ent}_A(B)$.*

Sometimes we need to use augmenting paths in slightly different situations. For example when $\mathcal{P}$ is a system of $t$-disjoint $S \to t$ paths. In this case an augmenting path is supposed to start in $S \setminus V_{\text{start}}(\mathcal{P})$ and $V_{\text{start}}(\mathcal{P})$ will be extended by $\mathsf{start}(Q)$ instead of $A_{\text{first}}(\mathcal{P})$ by $e$. Furthermore, if augmentation is not possible, then there is a set $B \ni t$ such that one can obtain $\mathsf{ent}_A(B)$ by choosing exactly one vertex other than $t$ from some paths in $\mathcal{P}$. (From Proposition 5 one can derive this version by picking a new vertex $s$ and draw the new edges $\{sv : v \in S\}$)

## 2.2 Chain of bubbles

A set $B \subseteq V$ is a ***v*-bubble** in $E$ if $v \in B$ and there exists a system $\mathcal{P} = \{P_u : u \in \mathsf{ent}_E(B)\}$ of $v$-disjoint $\mathsf{ent}_E(B) \to v$ paths in $E$ with $\mathsf{start}(P_u) = u$. Let us denote the set of the $v$-bubbles in $E$ by $\boldsymbol{\mathsf{bubb}_E(v)}$. Clearly $\{v\} \in \mathsf{bubb}_E(v)$.

**Lemma 6.** *Let $\alpha$ be an ordinal number and let $\langle B_\beta : \beta < \alpha \rangle$ be a sequence where $B_\beta \in \mathsf{bubb}_E(v_\beta)$. Let us denote $\bigcup_{\gamma<\beta} B_\gamma$ by $\boldsymbol{B_{<\beta}}$. If for each $\beta < \alpha$ either $v_\beta = v_0$ or $v_\beta \in \mathsf{int}_E(B_{<\beta})$, then $B_{<\alpha} \in \mathsf{bubb}_E(v_0)$.*

*Proof:* By transfinite recursion we construct a sequence $\langle \mathcal{P}_\beta : \beta \leq \alpha \rangle$ such that for all $\beta \leq \alpha$

1. $\mathcal{P}_\beta = \{P_v^\beta : v \in \mathsf{ent}_E(B_{<\beta})\}$ is a path-system which witnesses $B_{<\beta} \in \mathsf{bubb}_E(v_\beta)$,
2. for $\gamma < \beta$, every $P \in \mathcal{P}_\beta$ has a terminal segment which is in $\mathcal{P}_\gamma$.

Let $\mathcal{P}_0$ be an arbitrary path-system which witnesses $B_0 \in \mathsf{bubb}_E(v_0)$. If $\beta$ is a limit ordinal and $v \in \mathsf{ent}_E(B_{<\beta})$, then there is a smallest $\gamma(v) < \beta$ for which $v \in \mathsf{ent}_E(B_{<\gamma(v)})$. It is routine to check that letting $P_v^\beta := P_v^{\gamma(v)}$ satisfies the conditions.

Assume now that $\beta = \gamma + 1$. Fix a path-system $\mathcal{Q} = \{Q_v : v \in \mathsf{ent}_E(B_\gamma)\}$ which shows $B_\gamma \in \mathsf{bubb}_E(v_\gamma)$. For $v \in \mathsf{ent}_E(B_{<\gamma+1}) \cap \mathsf{ent}_E(B_{<\gamma})$ let $P_v^{\gamma+1} := P_v^\gamma$. If $v \in \mathsf{ent}_E(B_{<\gamma+1}) \setminus \mathsf{ent}_E(B_{<\gamma})$, then $v \in \mathsf{ent}_E(B_\gamma)$. By the assumption about $v_\gamma$, all the paths in $\mathcal{Q}$ meet $B_{<\gamma}$. Furthermore, if two paths in $\mathcal{Q}$ have the same vertex as first intersection with $B_{<\gamma}$, then it must be $v_0 = v_\gamma$. Consider the initial segment $Q'_v$ of $Q_v$ up to the first common vertex $u$ with $B_{<\gamma}$. Join $Q'_v$ and $P_u^\gamma$ to obtain $P_v^{\gamma+1}$. ■

**Corollary 7.** $\mathsf{bubb}_E(v)$ *is closed under arbitrary large union.*

A $B \in \mathsf{bubb}_E(v)$ is defined to be big in $E$ if $\mathsf{in}_E(B) \cap \mathsf{in}_E(v) \subseteq \{rv\}$. We write $\boldsymbol{\mathsf{bubb}^*_E(v)}$ for the big $v$-bubbles in $E$. For $C \in \mathcal{AB}_E^{\text{cut}}(v)$ let us denote the set of vertices that are unreachable from $r$ in $E - rv$ after the deletion of the vertices $C$ by $\boldsymbol{B_{C,E}}$.

**Proposition 8.** *For every* $C \in \mathcal{AB}_E^{cut}(v)$ *we have* $C = \mathsf{ent}_{E-rv}(B_{C,E})$

*Proof:* Since $B_{C,E} \supseteq C \supseteq \mathsf{ent}_{E-rv}(B_{C,E})$ the only not entirely trivial part of the statement is that $C$ may not have an element $u$ in $\mathsf{int}_{E-rv}(B_{C,E})$. If such an $u$ exists, then by picking a $\mathcal{P} \in \mathcal{AB}_E^{\mathrm{path}}(v)$ with $\mathcal{P} \perp C$ the path $P \in \mathcal{P}$ with $u \in V(P)$ would meet $C$ again at $\mathsf{ent}_{E-rv}(B_{C,E})$ contradicting $\mathcal{P} \perp C$. ■

**Proposition 9.** *For* $C \in \mathcal{AB}_E^{cut}(v)$ *we have* $B_{C,E} \in \mathsf{bubb}_E^*(v)$.

*Proof:* Pick a $\mathcal{P} \in \mathcal{AB}_E^{\mathrm{path}}(v)$ with $\mathcal{P} \perp C$, then the terminal segments of the paths in $\mathcal{P}$ from $C$ (adding the single vertex path $v$ to $\mathcal{P}$ if $rv \in E$) witnesses $B_{C,E} \in \mathsf{bubb}_E(v)$ (applying Proposition 8). Since $C = \mathsf{ent}_{E-rv}(B_{C,E})$ covers the $r \to v$ paths in $E - rv$ we must have $\mathsf{in}_E(B) \cap \mathsf{in}_E(v) \subseteq \{rv\}$. ■

Note that the $B$ what we obtained in Proposition 5 is a big $t$-bubble.

**Proposition 10.** $\mathsf{bubb}_E^*(v)$ *is an upward and cofinal subset of* $(\mathsf{bubb}_E(v), \subseteq)$.

*Proof:* The 'upward' part is trivial from the definition. To show it is cofinal let $B \in \mathsf{bubb}_E(v)$ be arbitrary. Pick a $C \in \mathcal{AB}_E^{\mathrm{cut}}(v)$. By Proposition 9 we have $B_{C,E} \in \mathsf{bubb}_E^*(v)$. By Corollary 7 and upwardness we obtain $B \cup B_{C,E} \in \mathsf{bubb}_E^*(v)$. ■

**Corollary 11.** *The largest element* $\bigcup \mathsf{bubb}_E(v) =: \boldsymbol{B_{v,E}}$ *of the poset* $(\mathsf{bubb}_E(v), \subseteq)$ *is in* $\mathsf{bubb}_E^*(v)$.

## 2.3 Intermezzo: The complete lattice of Aharoni-Berger cuts

For $C_1, C_2 \in \mathcal{AB}_E^{\mathrm{cut}}(v)$ let $\boldsymbol{C_1 \leq_E C_2}$ if $C_2$ covers all the $r \to C_1$ paths in $E - rv$.

**Theorem 12.** $(\mathcal{AB}_E^{cut}(v), \leq_E)$ *is a complete lattice.*

*Proof.* A set $I \subseteq V$ is called **fillable** in $E$ if there is a system $\mathcal{Q} = \{Q_u : u \in \mathsf{ent}_E(I)\}$ of $r$-disjoint $r \to \mathsf{ent}_E(I)$ paths with $\mathsf{end}(Q_u) = u$. The set of the fillable sets in $E$ containing a given $v \in V$ is denoted by $\boldsymbol{\mathsf{fill}_E(v)}$. Note that $V \in \mathsf{fill}_E(v)$.

*Observation* 13. One can obtain $\mathsf{fill}_E(v)$ by changing the direction of the edges in $E$, interchange the role of $r$ and $v$ and take the complements of the $r$-bubbles of the resulting system.

**Proposition 14.** $\mathsf{fill}_E(v)$ *is closed under arbitrary large intersection.*

*Proof:* It follows directly from Corollary 7 via Observation 13. ■

Let $\boldsymbol{\mathsf{fill}_E^*(v)}$ consists of those $I \in \mathsf{fill}_E(v)$ for which $\mathsf{in}_E(I) \cap \mathsf{in}_E(v) \subseteq \{rv\}$. Clearly $\mathsf{fill}_E^*(v)$ is still closed under arbitrary large intersection. Let us define $\mathsf{bubb}_E^*(v) \cap \mathsf{fill}_E^*(v) =: \boldsymbol{\mathcal{AB}_E^{\mathrm{set}}(v)}$.

*Observation* 15.

1. $X \in \mathcal{AB}_E^{\mathrm{set}}(v)$ if and only if $\mathsf{ent}_{E-rv}(X) \in \mathcal{AB}_E^{\mathrm{cut}}(v)$,

2. $B_{C,E} \in \mathcal{AB}_E^{\text{set}}(v)$ for every $C \in \mathcal{AB}_E^{\text{cut}}(v)$ (see Proposition 8),

3. $X \subseteq B_{\mathsf{ent}_{E-rv}(X),E}$ for every $X \in \mathcal{AB}_E^{\text{set}}(v)$.

**Lemma 16.** *For $I \in \mathsf{fill}_E^*(v)$ and $I \supseteq B \in \mathsf{bubb}_E^*(v)$, there exists an $X \in \mathcal{AB}_E^{set}(v)$ with $B \subseteq X \subseteq I$.*

*Proof:* Pick in $E-rv$ a (totally) disjoint system $\mathcal{P}$ of $\mathsf{ent}_{E-rv}(I) \to \mathsf{ent}_{E-rv}(B)$ paths and a vertex set $C$ which covers all the $\mathsf{ent}_{E-rv}(I) \to \mathsf{ent}_{E-rv}(B)$ paths in $E-rv$ in such a way that $C$ consists of one vertex from each path in $\mathcal{P}$. (It is possible by the original form of the infinite Menger's theorem in [3].) Since any $r \to v$ path in $E-rv$ has an $\mathsf{ent}_{E-rv}(I) \to \mathsf{ent}_{E-rv}(B)$ segment $C$ covers all the $r \to v$ paths. By extending the paths in $\mathcal{P}$ to $r \to v$ paths by using the facts $I \in \mathsf{fill}_E^*(v)$ and $B \in \mathsf{bubb}_E^*(v)$ we obtain a $\mathcal{P}'$ with $\mathcal{P}' \perp C$ thus $C \in \mathcal{AB}_E^{\text{cut}}(v)$. It is routine to check that $X := B_{C,E}$ is appropriate. ■

**Corollary 17.** *$B_{v,E}$ is the $\subseteq$-largest element of $\mathcal{AB}_E^{set}(v)$ ($B_{v,E}$ is defined in Corollary 11)*

*Proof:* Clearly $B_{v,E} \supseteq \bigcup \mathcal{AB}_E^{\text{set}}(v)$ since $\mathcal{AB}_E^{\text{set}}(v) \subseteq \mathsf{bubb}_E(v)$ thus the nontrivial part of the statement is $B_{v,E} \in \mathcal{AB}_E^{\text{set}}(v)$ . We already know $B_{v,E} \in \mathsf{bubb}_E^*(v)$ (see Corollary 11). By Applying Lemma 16 with $B := B_{v,E}$ and $I := V$, there is an $X \in \mathcal{AB}_E^{\text{set}}(v)$ with $B_{v,E} \subseteq X \subseteq V$. Since $\mathcal{AB}_E^{\text{set}}(v) \subseteq \mathsf{bubb}_E(v)$, necessarily $B_{v,E} = X$. ■

**Proposition 18.** *$\mathsf{ent}_{E-rv}(B_{v,E})$ is the largest element of $\mathcal{AB}_E^{cut}(v)$.*

*Proof:* By Corollary 17 and Observation 15/1 we have $\mathsf{ent}_{E-rv}(B_{v,E}) \in \mathcal{AB}_E^{\text{cut}}(v)$. Let $C \in \mathcal{AB}_E^{\text{cut}}(v)$ be arbitrary. Then $B_{C,E} \in \mathcal{AB}_E^{\text{set}}(v)$ by Observation 15/2 which implies by Corollary 17 that $B_{C,E} \subseteq B_{v,E}$ which shows $C \leq_E \mathsf{ent}_{E-rv}(B_{v,E})$. ■

**Claim 19.** *The poset $(\mathcal{AB}_E^{set}(v), \subseteq)$ is a complete lattice.*

*Proof:* It is enough to show that for a nonempty $\mathcal{X} \subseteq \mathcal{AB}_E^{\text{set}}(v)$ the sets

$$s(\mathcal{X}) := \bigcap \left\{ X \in \mathcal{AB}_E^{\text{set}}(v) : X \supseteq \bigcup \mathcal{X} \right\}$$
$$i(\mathcal{X}) := \bigcup \left\{ X \in \mathcal{AB}_E^{\text{set}}(v) : X \subseteq \bigcap \mathcal{X} \right\}.$$

are in $\mathcal{AB}_E^{\text{set}}(v)$. We know that $\bigcup \mathcal{X} \in \mathsf{bubb}_E^*(v)$ and $s(\mathcal{X}) \in \mathsf{fill}_E^*(v)$ because $\mathsf{bubb}_E^*(v)$ ($\mathsf{fill}_E^*(v)$) is closed under union (intersection). By applying Lemma 16 there exists an $X^* \in \mathcal{AB}_E^{\text{set}}(v)$ with $\bigcup \mathcal{X} \subseteq X^* \subseteq s(\mathcal{X})$. Since $\bigcup \mathcal{X} \subseteq X^* \in \mathcal{AB}_E^{\text{set}}(v)$ we have $X^* \in \left\{ X \in \mathcal{AB}_E^{\text{set}}(v) : X \supseteq \bigcup \mathcal{X} \right\}$ and hence $X^* \supseteq s(\mathcal{X})$. Therefore $s(\mathcal{X}) = X^* \in \mathcal{AB}_E^{\text{set}}(v)$. The proof for the infimum is similar. ■

**Proposition 20.** *The function $f : \mathcal{AB}_E^{cut}(v) \to \mathcal{AB}_E^{set}(v)$ defined by the formula $f(C) := B_{C,E}$ (see Observation 15/2) embeds $(\mathcal{AB}_E^{cut}(v), \leq_E)$ to $(\mathcal{AB}_E^{set}(v), \subseteq)$.*

*Proof:* It is injective since we calculated $f^{-1}$ in Proposition 8. Assume that $B_{C_1,E} \subseteq B_{C_2,E}$. Then an $r \to C_1 \subseteq B_{C_1,E}$ path $P$ in $E-rv$ necessarily meets $B_{C_2,E}$ and thus with $\mathsf{ent}_{E-rv}(B_{C_2,E}) = C_2$ as well.

Suppose now that $C_1 \leq_E C_2$ and let $u \in B_{C_1,E}$ be arbitrary. Then any $r \to u$ path $Q$ in $E - rv$ has an initial segment of form $r \to C_1$ but then by $C_1 \leq_E C_2$ path $Q$ meets $C_2$ as well which means $u \in B_{C_2,E}$. ■

Let $\mathcal{C} \subseteq \mathcal{AB}_E^{\text{cut}}(v)$ be nonempty and let $\mathcal{X}_\mathcal{C} := \{B_{C,E} : C \in \mathcal{C}\}$. We define $B := \bigcup \mathcal{X}_\mathcal{C} \in \mathsf{bubb}_E^*(v)$ and $\mathcal{Y}_B := \{B_{C,E} : C \in \mathcal{AB}_E^{\text{cut}}(v),\ B_{C,E} \supseteq B\}$ and finally $I := \bigcap \mathcal{Y}_B \in \mathsf{fill}_E^*(v)$. We obtain an $X \in \mathcal{AB}_E^{\text{set}}(v)$ with $B \subseteq X \subseteq I$ by applying Lemma 16. Then by Observation 15/1 we have $C^* := \mathsf{ent}_{E-rv}(X) \in \mathcal{AB}_E^{\text{cut}}(v)$. We claim that the desired supremum of $\mathcal{C}$ is $C^*$. By considering the embedding $f$ from Proposition 20 a $C_0 \in \mathcal{AB}_E^{\text{cut}}(v)$ is an upper bound of $\mathcal{C}$ iff $B_{C_0,E} \supseteq B$ thus $\mathcal{Y}_B$ consists exactly of the $f$-images of the upper bounds of $\mathcal{C}$. On the one hand, by $B_{C^*,E} \supseteq X \supseteq B$ (see Observation 15/3) we have $B_{C^*,E} \in \mathcal{Y}_B$. On the other hand, for $B_{C,E} \in \mathcal{Y}_B$

$$C^* = \mathsf{ent}_{E-rv}(X) \subseteq X \subseteq I \subseteq B_{C,E},$$

which implies that $C^* \leq_E \mathsf{ent}_{E-rv}(B_{C,E}) = C$ (see Proposition 8). The proof of the existence of the infimum of $\mathcal{C}$ is similar.

□

## 2.4 A characterisation of largeness

**Lemma 21.** *$L \subseteq A$ is large if and only if $L \supseteq \mathsf{out}_A(r)$ and for every $v \in V$ : $N_{A-rv}^-(v) \subseteq B_{v,L}$. Furthermore for a large $L$ the $\leq_L$-largest element of $\mathcal{AB}_L^{cut}(v)$ (namely $\mathsf{ent}_{L-rv}(B_{v,L})$) is in $\mathcal{AB}_A^{cut}(v)$.*

*Proof:* A large $L$ contains the edges $\mathsf{out}_A(r)$ by definition and for any $C \in \mathcal{AB}_L^{\text{cut}}(v) \cap \mathcal{AB}_A^{\text{cut}}(v)$ we have

$$N_{A-rv}^-(v) \subseteq B_{C,L} \subseteq B_{v,L}.$$

For the "if" direction let $v \in V$ be arbitrary. By Proposition 18 $\mathsf{ent}_{L-rv}(B_{v,L})$ is the largest element of $\mathcal{AB}_L^{\text{cut}}(v)$. Pick a $\mathcal{P} \in \mathcal{AB}_L^{\text{path}}(v)$ with $\mathcal{P} \perp \mathsf{ent}_{L-rv}(B_{v,L})$.

To ensure $\mathsf{ent}_{L-rv}(B_{v,L}) \in \mathcal{AB}_A^{\text{cut}}(v)$ it is enough to prove that

$$\mathsf{ent}_{L-rv}(B_{v,L}) = \mathsf{ent}_{A-rv}(B_{v,L}).$$

Suppose, to the contrary, that $w \in \mathsf{ent}_{A-rv}(B_{v,L}) \setminus \mathsf{ent}_{L-rv}(B_{v,L})$ and it is showed by some $uw \in A \setminus L$. Then either $w = v$ or $w \in \mathsf{int}_L(B_{v,L})$. Since $u \neq r$, we have $u \in N_{A-rv}^-(v) \subseteq B_{w,L}$. By applying Lemma 6 with $B_0 := B_{v,L}$, $v_0 := v$ and $B_1 := B_{w,L}$, $v_1 := w$ in $L$ we may conclude that $B_{v,L} \cup B_{w,L} \in \mathsf{bubb}_L(v)$. Since $u \in B_{w,L} \setminus B_{v,L}$ it contradicts the definition of $B_{v,L}$. ■

## 2.5 Largeness-preserving edge deletions

**Lemma 22.** *Let $E \subseteq H \subseteq A$ where $H \setminus E \subseteq \mathsf{in}_H(v)$ for some $v \in V$. If $N_{H \setminus E}^-(v) \subseteq B_{v,E}$, then for every $w \in V$ we have $B_{w,E} = B_{w,H}$.*

*Proof:* Let $w \in V$ be fixed.

**Proposition 23.** *If* $B_{w,H} \in \mathsf{bubb}_E(w)$*, then* $B_{w,E} = B_{w,H}$*.*

*Proof:* Observe that $B_{w,H} \in \mathsf{bubb}_E(w)$ implies $B_{w,H} \subseteq B_{w,E}$ by the definition of $B_{w,E}$. Assume, for a contradiction, that $B_{w,H} \subsetneq B_{w,E}$. Then necessarily $\mathsf{ent}_E(B_{w,E}) \subsetneq \mathsf{ent}_H(B_{w,E})$ otherwise a path-system which witnesses $B_{w,E} \in \mathsf{bubb}_E(w)$ would show $B_{w,E} \in \mathsf{bubb}_H(w)$ since $E \subseteq H$ which would imply $B_{w,E} \subseteq B_{w,H}$.

It follows that $v \in \mathsf{ent}_H(B_{w,E}) \setminus \mathsf{ent}_E(B_{w,E})$ showed by an edge $uv \in H \setminus E$. But then $v \in \mathsf{int}_E(B_{w,E})$ and therefore by applying Lemma 6 with $B_0 := B_{w,E},\ v_0 := w$ and $B_1 := B_{v,E},\ v_1 := v$ in $E$ we obtain $B_{v,E} \cup B_{w,E} \in \mathsf{bubb}_H(w)$ which contradicts the definition of $B_{w,E}$ because $u \in B_{v,E} \setminus B_{w,E}$. ■

Let $\mathcal{P}$ be a path-system which witnesses $B_{w,H} \in \mathsf{bubb}_H(w)$. We may suppose that $\mathcal{P}$ uses some edge from $H \setminus E$ otherwise $\mathcal{P}$ shows $B_{w,H} \in \mathsf{bubb}_E(w)$ as well (since $\mathsf{ent}_E(B_{w,H}) \subseteq \mathsf{ent}_H(B_{w,H})$) and we are done by Proposition 23. It implies that either $v \in \mathsf{int}_H(B_{w,H})$ or $v = w$.

**Proposition 24.** $B_{v,E} \subseteq B_{w,H}$

*Proof:* Note that $B_{v,E} \in \mathsf{bubb}_H(v)$ because $N^-_{H \setminus E}(v) \subseteq B_{v,E}$ implies

$$\mathsf{ent}_E(B_{v,E}) = \mathsf{ent}_H(B_{v,E}).$$

By applying Lemma 6 with $B_0 := B_{w,H},\ v_0 := w$ and $B_1 := B_{v,E},\ v_1 := v$ in $H$ we obtain

$$B_{v,E} \cup B_{w,H} \in \mathsf{bubb}_H(w).$$

Thus by the definition of $B_{w,H}$ we have $B_{v,E} \cup B_{w,H} \subseteq B_{w,H}$. ■

Let us handle first the simpler case when $v = w$. Let $\mathcal{Q}$ be a path-system which shows $B_{v,E} \in \mathsf{bubb}_E(v)$. Remember that $B_{v,E}$ contains the tails of the edges in $H \setminus E$ by assumption. For every $P \in \mathcal{P}$ take the initial segment of $P$ up to the first vertex $v_P$ of it which is in $B_{v,E}$ and extend it by the unique $Q \in \mathcal{Q}$ with $\mathsf{start}(Q) = v_P$. The resulting path-system witnesses $B_{w,H} \in \mathsf{bubb}_E(w)$.

Assume now that $w \neq v$ and therefore $v \in \mathsf{int}_H(B_{w,H})$ as we discussed before Proposition 24. Then there is a unique path $P \in \mathcal{P}$ which uses an edge from $H \setminus E$. Let us define $\mathcal{P}' := \mathcal{P} \setminus \{P\}$ and apply the augmenting path method in $E$ with the $w$-disjoint system $\mathcal{P}'$ of $V_{\mathsf{start}}(\mathcal{P}) \to w$ paths. We can suppose that the augmentation is not possible since otherwise the resulting path-system shows $B_{w,H} \in \mathsf{bubb}_E(w)$ and we are done by Proposition 23. Let $B \in \mathsf{bubb}_E(w)$ what the augmentation path method gives us. Then $v \in B$ otherwise the terminal segment of $P$ starting from $v$ would show that we can reach $w$ from $\mathsf{start}(P)$ in the auxiliary digraph of the augmentation path method. We also know that $v \notin \mathsf{ent}_E(B)$ because $\mathsf{ent}_E(B) \subseteq V(\mathcal{P}') - w$ and $v \in V(P)$ holds where $(V(\mathcal{P}') - w) \cap V(P) = \varnothing$. Hence necessarily $v \in \mathsf{int}_E(B)$. By applying Lemma 6 with $B_0 := B,\ v_0 := w$ and $B_1 := B_{v,E},\ v_1 := v$ in $E$ we may conclude $B \cup B_{v,E} \in \mathsf{bubb}_E(w)$, showed by a path-system $\mathcal{R}$. Note that $B \cup B_{v,E}$ contains the tails of the edges in $H \setminus E$ because $B_{v,E}$ does so. Eventually we do the same what we did in the case $v = w$. For every $P \in P$ take the initial segment of $P$

up to the first vertex $v_P$ of it which is in $B \cup B_{v,E}$ and extend it by the unique $R \in \mathcal{R}$ with $\mathsf{start}(R) = v_P$. The resulting path-system witnesses $B_{w,H} \in \mathsf{bubb}_E(w)$ and we are done by Proposition 23 again. ■ ■

**Corollary 25.** *If $L$ is large and $\mathcal{P} \in \mathcal{AB}_L^{path}(v) \cap \mathcal{AB}_A^{path}(v)$ for some $v \in V$, then the deletion of all the ingoing edges of $v$ in $L$ except $A_{last}(\mathcal{P})$ and $rv$ (if $rv \in A$) does not ruin largeness i.e. $L' := L \setminus [\mathsf{in}_L(v) \setminus (A_{last}(\mathcal{P}) + rv)]$ is large.*

*Proof:* Clearly $L' \supseteq \mathsf{out}_A(r)$. By applying Lemma 22 with $E := L'$ and $H := L$ we obtain for every $w \in V$ that $B_{w,L'} = B_{w,L}$ holds. By the largeness of $L$ it implies the largeness of $L'$ via Lemma 21. ■

## 2.6 Large quasi-flames

Let us denote by $\boldsymbol{\mathcal{G}_E(v)}$ the set of those $J \subseteq \mathsf{in}_E(v)$ for which there is a system $\mathcal{P}$ of internally disjoint $r \to v$ paths with $A_{\mathrm{last}}(\mathcal{P}) = J$. Contrary to the finite case $\mathcal{G}_E(v)$ is not necessary a matroid (for the axiomatization of infinite matroids we refer [5]). Matroids representable in this form are called gammoids. Several highly nontrivial sufficient conditions has been found which guarantee $\mathcal{G}_E(v)$ to be a gammoid (see [1], [6]). Using this terminology, $F$ is a flame if and only if $\mathsf{in}_F(v) \in \mathcal{G}_F(v)$ for every $v$. A $Z \subseteq A$ is defined to be a **quasi-flame** if for every $v \in V$ and for every *finite* $J \subseteq \mathsf{in}_Z(v)$ we have $J \in \mathcal{G}_Z(v)$.

**Claim 26.** *Assume that $E \subseteq A$ is not $v$-large. Then there exists an $uw \in A \setminus E$ such that for every $J \in \mathcal{G}_E(w)$ we have $J + uw \in \mathcal{G}_{E+uw}(w)$.*

*Proof:* If $rv \in A \setminus E$, then $rv$ is suitable thus we may suppose that $rv \in E$ if $rv \in A$.

**Proposition 27.** *There is an $uw \in A \setminus E$ such that $u \notin B_{v,E}$ and $w \in \mathsf{int}_{E-rv}(B_{v,E})$.*

*Proof:* Indeed, otherwise $\mathsf{ent}_{E-rv}(B_{v,E}) = \mathsf{ent}_{A-rv}(B_{v,E})$ holds and from the fact $B_{v,E} \in \mathcal{AB}_E^{\mathrm{set}}(v)$ (see Corollary 17) we would be able to conclude that $B_{v,E} \in \mathcal{AB}_A^{\mathrm{set}}(v)$. But then the path-system which shows $C := \mathsf{ent}_{E-rv}(B_{v,E}) \in \mathcal{AB}_E^{\mathrm{cut}}(v)$ (see Observation 15/1) would also show $C \in \mathcal{AB}_A^{\mathrm{cut}}(v)$ contradicting the fact that $E$ is not $v$-large. ■

We show that any edge $uw$ from Proposition 27 is appropriate for Claim 26. Let $\mathcal{P}$ be a system of internally disjoint $r \to C$ paths with $V_{\mathrm{end}}(\mathcal{P}) = C$ (exists by Corollary 17). Reverse the direction of the edges in $E + uw$ and apply the augmenting path method with the reserved $\mathcal{P}$ considering it as an $r$-disjoint system of $C + w \to r$ paths. The augmentation from $w$ must be possible (and necessarily uses $uw$), otherwise it would give us a $\subseteq$-larger $v$-bubble $B$ in $E$ than $B_{v,E}$ (indeed, $u \in B \setminus B_{v,E}$ would hold because the reverse of $uw$).

Let us denote by $\mathcal{Q}$ the path-system what we obtained after the successful augmentation in the original digraph extended by the trivial path $rv$ if $rv \in E$. Let $Q_{uw}$ the unique path in $\mathcal{Q}$ with last edge $uw$. Then $\mathcal{Q} - Q_{uw}$ is a system of $r$-disjoint $r \to \mathsf{ent}_E(B_{v,E})$ paths in $E$ with $V_{\mathrm{end}}(\mathcal{Q} - Q_{uw}) = \mathsf{ent}_E(B_{v,E})$. Let $J \in \mathcal{G}_E(w)$ be arbitrary and let $\mathcal{R}$ be an internally disjoint system of $r \to w$ paths in $E$ with

$A_{\mathrm{last}}(\mathcal{R}) = J$. Let $\mathcal{R}'$ be the set of the terminal segments of the paths in $\mathcal{R}$ starting from the last intersection with $\mathsf{ent}_E(B_{v,E})$. For $R \in \mathcal{R}'$ pick the unique $Q \in \mathcal{Q} - Q_{uw}$ for which $\mathsf{end}(Q) = \mathsf{start}(R)$ and join them to a path and keep $Q_{uw}$ untouched. The resulting path-system shows $J + uw \in \mathcal{G}_{E+uw}(w)$. ■ ■

**Lemma 28.** *There exists a large quasi-flame $Z$.*

*Proof:* We apply transfinite recursion starting with an arbitrary quasi-flame (for example $\varnothing$). In limit steps we take union. In a successor step we extend the current $Z_\alpha$ applying Claim 26 with a suitable $v$ unless $Z_\alpha$ is already large, in which case we stop. Clearly we necessarily stop in less than $|A|^+$ steps. It follows by transfinite induction using Claim 26 that the $Z_\alpha$'s are really quasi-flames. ■

## 2.7 Preserving "gammoids"

**Lemma 29.** *Suppose that $\mathcal{P} \in \mathcal{AB}_E^{path}(v) \cap \mathcal{AB}_A^{path}(v)$ and $rv \notin J \in \mathcal{G}_E(v)$. Then there exists a $\mathcal{P}' \in \mathcal{AB}_E^{path}(v) \cap \mathcal{AB}_A^{path}(v)$ with $A_{last}(\mathcal{P}') \supseteq J$.*

*Proof:* Take a $C \in \mathcal{AB}_E^{\mathrm{cut}}(v) \cap \mathcal{AB}_A^{\mathrm{cut}}(v)$ with $C \perp \mathcal{P}$. Let $\mathcal{P}_0$ consist of the terminal segments of paths in $\mathcal{P}$ from $C$. Take a $\mathcal{Q}$ which witnesses $J \in \mathcal{G}_E(v)$ and let $\mathcal{Q}_0$ be the set of the terminal segments of the paths in $\mathcal{Q}$ from the last intersection with $C$. By applying Pym's theorem (see [9]) we obtain a system $\mathcal{R}$ of $v$-disjoint paths with $V_{\mathrm{start}}(\mathcal{R}) = C$ and $A_{\mathrm{last}}(\mathcal{R}) \supseteq J$. Finally build $\mathcal{P}'$ by joining the initial segments up to $C$ of the paths in $\mathcal{P}$ with the paths in $\mathcal{R}$. ■

**Lemma 30.** *Let $E \subseteq H \subseteq A$ such that $H \setminus E \subseteq \mathsf{in}_H(v)$ for some $v \in V$ and suppose that $N^-_{H\setminus E}(v) \subseteq B_{v,E}$. Let $\mathcal{P}$ be a path-system which shows $B_{v,E} \in \mathsf{bubb}_E(v)$. Then for every $u \neq v$ and for every finite $J \in \mathcal{G}_H(u)$ with $J \supseteq A(\mathcal{P}) \cap \mathsf{in}_E(u)$ we have $J \in \mathcal{G}_E(u)$.*

*Proof:* Suppose, to the contrary, that $J$ is a counterexample. Then by applying Menger's theorem there is a vertex set $B \subseteq V$ which spans the edges in $J$ and $|\mathsf{ent}_E(B) - u| = |J| - 1$. Furthermore, necessarily $v \in \mathsf{ent}_H(B) \setminus \mathsf{ent}_E(B)$ (thus specially $rv \notin E$) and there exists an internally disjoint system $\mathcal{Q}$ of $r \to u$ paths with $\mathcal{Q} \perp (\mathsf{ent}_H(B) - u)$ and $A_{\mathrm{last}}(\mathcal{Q}) = J$ (see Figure 1). Observe that $rv \notin H$ as well since $rv \notin H \setminus E$ by the assumption $N^-_{H\setminus E}(v) \subseteq B_{v,E}$. Note that one can extend the paths in $\mathcal{P}$ backward to obtain an internally disjoint system $\mathcal{R}$ of $r \to v$ paths with $\mathcal{R} \perp \mathsf{ent}_E(B_{v,E})$ (see Corollary 17). By Proposition 18 we have $\mathsf{ent}_E(B_{v,E}) \in \mathcal{AB}_E^{\mathrm{cut}}(v)$ and hence $\mathcal{R} \in \mathcal{AB}_E^{\mathrm{path}}(v)$. Since $N^-_{H\setminus E}(v) \subseteq B_{v,E}$, the vertex set $\mathsf{ent}_E(B_{v,E})$ covers the $r \to v$ paths in $H$ as well which implies that $\mathcal{R} \in \mathcal{AB}_E^{\mathrm{path}}(v) \cap \mathcal{AB}_H^{\mathrm{path}}(v)$ and

$$\kappa_H(r,v) = \kappa_E(r,v) = |\mathcal{R}| = |\mathcal{P}| \leq |\mathsf{ent}_E(B)| \leq |J| < \aleph_0. \tag{1}$$

For every $P \in \mathcal{P}$ consider the longest terminal segment $P'$ of $P$ for which $V(P') \subseteq B$. The vertices $\mathsf{start}(P')$ are distinct elements of $\mathsf{ent}_E(B) - u$ (here we use the fact that if a $P \in \mathcal{P}$ uses an ingoing edge of $u$, then it is in $J$ thus it cannot come from out of $B$). Extend all the paths $P'$ to an $r \to v$ path by using the initial segment

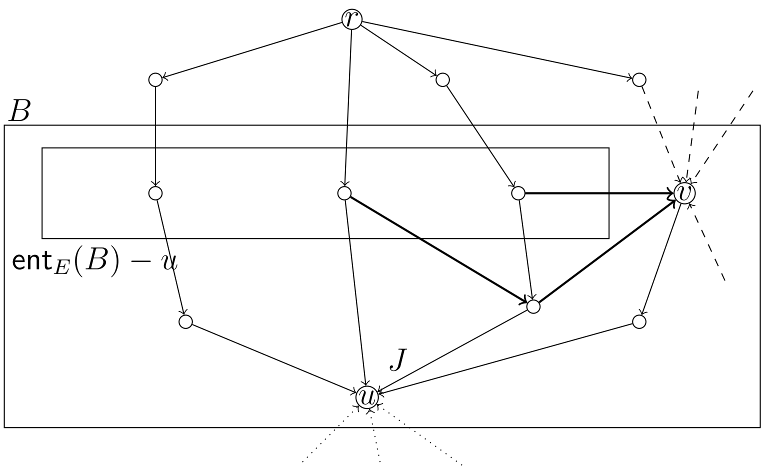


Figure 1: The elements of $H \setminus E$ are dashed, $|Q| = 4$ and $A(\mathcal{Q})$ contains the normal edges and the left dashed edge. The edge set $J$ consists of the normal ingoing edges of $u$. The paths $P'$ are thickened. The dotted edges are not really relevant we just want to emphasize that such edges may exist.

up to $\mathsf{start}(P')$ of the unique $Q \in \mathcal{Q}$ which goes through $\mathsf{start}(P')$ and take the initial segment up to $v$ of the unique $Q \in \mathcal{Q}$ which contains $v$. We have just constructed $|\mathcal{P}| + 1$ internally disjoint $r \to v$ paths in $H$ which contradicts the equation at 1. ■

## 2.8 Proof of the Theorem

Until now we did not use the assumption that $D$ is countable but now we will.

*Proof of Theorem 4:* Let $V = \{v_n\}_{n<\omega}$ and let $Z$ be a large quasi-flame (exists by Lemma 28). We construct a sequence $(E_n)$ by recursion such that for every $n < \omega$

0. $E_0 = \mathsf{out}_A(r)$
1. $E_n \subseteq E_{n+1} \subseteq Z$,
2. $\mathcal{AB}_A^{\text{path}}(v_n) \cap \mathcal{AB}_{E_{n+1}}^{\text{path}}(v_n) \neq \varnothing$,
3. $\mathsf{in}_{E_{n+1}}(v_n) \in \mathcal{G}_{E_{n+1}}(v_n)$,
4. $\mathsf{in}_{E_n}(v_i) = \mathsf{in}_{E_{n+1}}(v_i)$ for $i < n$,
5. $\mathsf{in}_{E_n}(v_m)$ is finite for $m \geq n$.

Suppose that the construction is done and let $F^* := \bigcup_{n=0}^{\infty} E_n$. Largeness of $F^*$ follows immediately from properties 0,2 and properties 1,3 and 4 ensure that $F^*$ is a flame. Thus if we are able to construct the desired $(E_n)$, we are done.

Suppose that $E_i$ is defined for $i \leq n$ and so far the conditions hold. By the largeness of $Z$, we have $\mathcal{AB}_Z^{\text{path}}(v_n) \cap \mathcal{AB}_A^{\text{path}}(v_n) \neq \varnothing$. By property 5, $\mathsf{in}_{E_n}(v_n)$ is finite which implies $\mathsf{in}_{E_n}(v_n) \in \mathcal{G}_Z(v_n)$ since $Z$ is a quasi-flame. We may pick an

$\mathcal{P} \in \mathcal{AB}_Z^{\text{path}}(v_n) \cap \mathcal{AB}_A^{\text{path}}(v_n)$ with $A_{\text{last}}(\mathcal{P}) \supseteq \mathsf{in}_{E_n}(v_n) - rv_n$ (apply Lemma 29 with $J := \mathsf{in}_{E_n}(v_n) - rv_n$ and $E := Z$). If we define $E_{n+1}$ as $E_n \cup A(\mathcal{P})$, then it satisfies the properties 1,2,3,5. Unfortunately it can violate property 4 since the set $A(\mathcal{P})$ may contain for some $i < n$ an ingoing edge of $v_i$ which is not in $\mathsf{in}_{E_n}(v_i) = \mathsf{in}_{E_{i+1}}(v_i)$. It turns out that the deletion of these forbidden edges from $Z$ does not ruin the properties of $Z$ which we need in the construction.

**Claim 31.** *The set*

$$Z_n := Z \setminus \bigcup_{i<n} \left(\mathsf{in}_Z(v_i) \setminus \mathsf{in}_{E_{i+1}}(v_i)\right)$$

*is still a large quasi-flame.*

*Proof:* We prove by induction that $Z_i$ is a large quasi-flame for $i = 0, \dots, n$. For $i = 0$, we have $Z_0 = Z$ thus it is true. Suppose that $n > 0$ and we know this for some $i < n$. By properties 2 and 3, there is a $\mathcal{Q} \in \mathcal{AB}_{E_{i+1}}^{\text{path}}(v_i) \cap \mathcal{AB}_A^{\text{path}}(v_i)$ with $A_{\text{last}}(\mathcal{Q}) \supseteq \mathsf{in}_{E_{i+1}}(v_i) - rv_i$ (just apply Lemma 29 with $J := \mathsf{in}_{E_{i+1}}(v_i) - rv_i$ and $E := E_{i+1}$). Since $Z_i$ is large, Corollary 25 ensures that $Z_{i+1} = Z_i \setminus \left(\mathsf{in}_{Z_i}(v_i) \setminus \mathsf{in}_{E_{i+1}}(v_i)\right)$ is large as well.

The path-system $Q$ from the previous paragraph shows $\mathsf{in}_{Z_{i+1}}(v_i) \in \mathcal{G}_{Z_{i+1}}(v_i)$. Let $u \neq v_i$ and let $J \subseteq \mathsf{in}_{Z_{i+1}}(u)$ be finite. Since $Z_i$ is a quasi-flame we know that $J \in \mathcal{G}_{Z_i}(u)$. We want to apply Lemma 30 with $H := Z_i,\ E := Z_{i+1},\ v := v_i,\ J$. By the largeness of $Z_{i+1}$ we may pick a $C \in \mathcal{AB}_{Z_{i+1}}^{\text{cut}}(v_i) \cap \mathcal{AB}_A^{\text{cut}}(v_i)$. Then we have

$$N^-_{Z_i \setminus Z_{i+1}}(v_i) \subseteq N^-_{A - rv_i}(v_i) \subseteq B_{C,A} \subseteq B_{C,Z_{i+1}} \subseteq B_{v_i,Z_{i+1}}.$$

Suppose that $\mathcal{P}$ witnesses $B_{v_i,Z_{i+1}} \in \mathsf{bubb}_{Z_{i+1}}(v_i)$. Since $\left|A(\mathcal{P}) \cap \mathsf{in}_{Z_{i+1}}(u)\right| \leq 1$, we may assume, by extending $J$ by a single new edge if it is necessary, that $J \supseteq A(\mathcal{P}) \cap \mathsf{in}_{Z_{i+1}}(u)$. Finally we may conclude $J \in \mathcal{G}_{Z_{i+1}}$ by Lemma 30. Thus $Z_{i+1}$ is a large quasi-flame and hence by induction $Z_n$ as well. ■

By applying $Z_n$ instead of $Z$ at the construction in the paragraph before Claim 31 leads to an $E_{n+1}$ which satisfies property 4 as well. Let us point out that $F^* = \bigcap_{n=0}^{\infty} Z_n$ holds as well. □

# 3 Open problems

## 3.1 Beyond countability

It is not too hard to see that we can replace in Theorem 4 the countability of $D$ by the weaker assumption that $\kappa_A(r, v) \leq \aleph_0$ for every $v \in V$. We believe that more is true.

**Conjecture 32.** *We may omit the countability of $D$ in Theorem 4.*

## 3.2 Upper and lower bounds

**Question 33.** *Suppose that there is a prescribed flame $F$ and a large set $L \supseteq F$. Is it true that there is a large flame $F^*$ with $F \subseteq F^* \subseteq L$?*

In the finite case it is true. Indeed, first we may simply replace $A$ by $L$ because largeness with respect to $L$ implies largeness with respect to $A$ (since $\kappa_{F^*}(r,v) = \kappa_L(r,v)$ and $\kappa_L(r,v) = \kappa_A(r,v)$ imply $\kappa_{F^*}(r,v) = \kappa_A(r,v)$). Then we build a large quasi-flame $F^*$ by applying the proof of Lemma 28 with the initial (quasi-)flame $F$ instead of $\varnothing$. Note that here we need only the finiteness of the cardinals $\kappa_A(r,v)$ instead of the finiteness of $D$ since it already guarantees that the union of an $\subseteq$-increasing chain of flames is a flame.

## 3.3 Huge flames i.e. preserving all the Aharoni-Berger cuts

Call $v$-huge an $H \subseteq A$ if

- $rv \in H$ if $rv \in A$,
- $\mathcal{AB}_H^{\mathrm{cut}}(v) \supseteq \mathcal{AB}_A^{\mathrm{cut}}(v)$.

If $\kappa_A(v) < \aleph_0$, then $v$-largeness and $v$-hugeness of $H$ means the same, namely $\kappa_H(r,v) = \kappa_A(r,v)$ but it is easy to see that in the general case $v$-hugeness is strictly stronger. An $H \subseteq A$ is defined to be huge if it is $v$-huge for every $v \in V$.

**Question 34.** *Does there always exist a huge flame?*

## 3.4 The edge version of flames and largeness

Let $D = (V + r, A)$ be as usual. We call an $F \subseteq A$ an $r$-edge-flame in $D$ if for every $v \in V$ there is a system $\mathcal{P}$ of *edge-disjoint* $r \to v$ path such that $A_{\mathrm{last}}(\mathcal{P}) = \mathsf{in}_F(v)$. A set $C \subseteq A$ is an Aharoni-Berger edge-cut between $r$ and $v$ if $C$ covers all the $r \to v$ paths in $D$ and there is a system $\mathcal{P}$ of edge-disjoint $r \to v$ paths such that one can obtain $C$ by choosing one edge from each element of $\mathcal{P}$. Let us denote the set of such a cuts by $\overrightarrow{\mathcal{AB}}_A^{\mathrm{cut}}(v)$. A set $L \subseteq A$ is defined to be $v$-edge-large if $\overrightarrow{\mathcal{AB}}_L^{\mathrm{cut}}(v) \cap \overrightarrow{\mathcal{AB}}_A^{\mathrm{cut}}(v) \neq \varnothing$. An $L \subseteq A$ is edge-large if it is $v$-edge-large for every $v \in V$.

**Question 35.** *Is it true that in every (countable) digraph $D = (V + r, A)$ there is an edge-large edge-flame $F^*$?*

It seems that most of the tools we developed is working in the edge-version as well. A major new difficulty is that a $\mathcal{P} \in \overrightarrow{\mathcal{AB}}_A^{\mathrm{path}}(v)$ may have infinitely many ingoing edges at a vertex other than $v$ (not just at most one as in the vertex version) and therefore our quasi-flame method cannot play the role what it played in the vertex version.

If the answer for Question 35 is yes, then one can derive the vertex-version from it by simply splitting the vertices to an edge and multiply the old edges to $|V|^+$ parallel copies (i.e. the same way as one can reduce directed vertex-Menger to the directed edge-Menger). Similarly easy reduction in the other direction seems not possible (if we blow up a vertex to a vertex set, we may loose the edge-flame condition at this vertex).

The edge-version analogues of all of our earlier open questions are also open.

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