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Acyclic orientations with degree constraints

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Abstract

In this note we study the complexity of some generalizations of the notion of st -numbering. Suppose that given some functions f and g , we want to order the vertices of a graph such that every vertex v is preceded by at least $f(v)$ of its neighbors and succeeded by at least $g(v)$ of its neighbors. We prove that this problem is solvable in polynomial time if $fg \equiv 0$, but it becomes NP-complete for $f \equiv g \equiv 2$. This answers a question of the first author posed in 2009.

1 Introduction

In this paper $G = (V, E)$ always denotes an undirected connected graph, parallel edges are allowed but loops are not. We use $n = |V|$, $d(v)$ for the degree of a vertex $v \in V$ and $d(v, Y)$ for the number of edges going from v to Y . For a function $f : V \rightarrow \mathbb{N}$, we use the notation $f(X) = \sum_{x \in X} f(x)$ for any $X \subseteq V$.

In a digraph, the indegree (number of incoming arcs) of a vertex v (or of a set $X \subseteq V$) is denoted by $\varrho(v)$ (or $\varrho(X)$, resp.) and the outdegree is denoted by $\delta(v)$ (or $\delta(X)$, resp.).

The first named author proposed to study the complexity of the following problem in 2009 in Category “Orientations” of EgresOpen [1].

Problem 1 ([1]). *Given $G = (V, E)$ and $s, t \in V$ and positive integers k, ℓ , decide whether G has an acyclic orientation where for every vertex $v \in V \setminus \{s, t\}$, there are k pairwise arc-disjoint directed paths from s to v , and ℓ pairwise arc-disjoint directed paths from v to t .*

In this paper we settle the complexity of this problem by show that it is NP-complete, already for $k = \ell = 2$.

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A more general problem that plays a central role in this note, is the following. We are given $G = (V, E)$ and two functions $f : V \rightarrow \mathbb{N}$ and $g : V \rightarrow \mathbb{N}$ with $f(v) + g(v) \leq d(v)$ for each $v \in V$. An orientation of G is called (f, g) -bounded if $f(v) \leq \varrho(v) \leq d(v) - g(v)$ for each $v \in V$.

Problem 2 (Degree constrained acyclic orientation problem). *Given G, f, g , decide whether G has an acyclic (f, g) -bounded orientation.*

Note that $f(v)$ is a lower bound for the indegree of v while $g(v)$ is a lower bound for the outdegree of v as we are dealing with orientations, so $\delta(v) = d(v) - \varrho(v)$.

Claim 1.1. *Problem 1 is equivalent to the degree constrained acyclic orientation problem if $f(v) = k$ and $g(v) = \ell$ for all $v \in V \setminus \{s, t\}$; and $f(s) = g(t) = 0$, $g(s) = f(t) = d(s)$.*

Proof. If the prescribed pairwise arc-disjoint directed paths exist, then obviously for every $v \in V \setminus \{s, t\}$ we have $\varrho(v) \geq k$ and $\delta(v) \geq \ell$.

Suppose we have an orientation \vec{G} where all $v \in V \setminus \{s, t\}$ have $\varrho(v) \geq k$ and $\delta(v) \geq \ell$. This orientation defines a (not necessarily unique) topological order, $V = \{v_1, \dots, v_n\}$, such that for every directed edge $v_i v_j$ we have $i < j$ (we may suppose $v_1 = s$ and $v_n = t$). By symmetry and by Menger's theorem, it is enough to prove that for any $X \subseteq V$ if $s \notin X$, then $\varrho(X) \geq k$. Let v_i be the first vertex of X . As $s \notin X$ we have $i > 1$. Clearly $\varrho(X) \geq \varrho(v_i) \geq k$. \square

2 Polynomially solvable cases

First we examine the special case of Problem 2 when $g(v) = 0$ for all v , i.e., only lower bounds on the indegrees are given. For a set $X \subseteq V$, we call a vertex $x \in X$ a *potential source* if $f(x) \leq d(x, V \setminus X)$.

Condition 1. *For every non-empty set $X \subseteq V$, there exists an $x \in X$ which is a potential source for X .*

Theorem 2.1 (Folklore). *There exists an acyclic orientation of G where $\varrho(v) \geq f(v)$ for every $v \in V$ if and only if Condition 1 is satisfied. Moreover, such an orientation can be produced (and simultaneously Condition 1 can be checked) by a greedy algorithm.*

Proof. If the orientation exists, then for each X take x as the first vertex of it (by a topological order). Clearly $f(x) \leq \varrho(x) \leq d(x, V \setminus X)$.

Suppose Condition 1 is satisfied. By applying it to $X = V$, we obtain a vertex v_1 with $f(v_1) = 0$. Let $G' := G - v_1$ and for any $v \in V \setminus \{v_1\}$ let $f'(v) := \max(f(v) - d(v_1, v), 0)$. Clearly Condition 1 is satisfied for the new G' and f' , thus, by induction, there is an acyclic orientation of G' where the indegrees are lower-bounded by f' . The required orientation of G is given by orienting the edges incident with v_1 from v_1 .

If, at some point of this recursive process, we do not have any vertex v such that the current $f(v)$ is 0, then the set X of the current vertices violates Condition 1.

Otherwise we get the orientation required, thus by the first part of the proof Condition 1 is satisfied. \square

We can go a little bit further.

Theorem 2.2. *If $G = (V, E)$ and $f(v)g(v) = 0$ for every $v \in V$ (i.e., every vertex has either only a lower bound or only an upper bound on the indegree), then we can decide in polynomial time whether there is an (f, g) -bounded acyclic orientation of G .*

Proof. Let $A = \{v \in V \mid f(v) = 0, g(v) > 0\}$, $B = \{v \in V \mid f(v) = 0, g(v) = 0\}$, $C = \{v \in V \mid f(v) > 0, g(v) = 0\}$, by the condition of the theorem these three sets partition V . Call an acyclic orientation ABC if there is a topological order where the vertices of A come first, then the vertices of B , and finally the vertices of C . It is easy to observe that if an (f, g) -bounded acyclic orientation of G exists, then there is another one which is ABC. Let $G' = G[A]$, and $f'(a) = 0$, $g'(a) = \max(g(a) - d(a, V \setminus A), 0)$ for $a \in A$. Furthermore let $G'' = G[C]$ and $f''(c) = \max(f(c) - d(c, V \setminus C), 0)$, $g''(c) = 0$ for $c \in C$. By Theorem 2.1, we can check in polynomial time whether there is an (f', g') -bounded acyclic orientation of G' and whether there is an (f'', g'') -bounded acyclic orientation of G'' . \square

We call a vertex v *strict* if $f(v) + g(v) = d(v)$. The next special case we study is, when every vertex is strict.

Theorem 2.3. *There exists an acyclic orientation of G where $\varrho(v) = f(v)$ for every $v \in V$ if and only if $f(V) = |E|$ and Condition 1 is satisfied. Moreover, such an orientation can be produced (and simultaneously Condition 1 can be checked) by a greedy algorithm.*

Proof. Observe that if $f(V) = |E|$, then $\varrho(v) \geq f(v)$ for every v if and only if $\varrho(v) = f(v) = d(v) - g(v)$ for every v . \square

We have one more sporadic example where the problem is known to be in P, see Theorem 3.5 in the last subsection.

3 NP-complete cases

Theorem 3.1. *The degree constrained acyclic orientation problem (Problem 2) is NP-complete.*

We prove a much stronger theorem about a very restricted version of Problem 2.

Theorem 3.2. *The degree constrained acyclic orientation problem is NP-complete even if every vertex v is ϱ -lower-bounded by 0 or 1 (i.e., $f(v) \leq 1$ and $g(v) = 0$) except one vertex y , which is strict (i.e., $f(y) + g(y) = d(y)$).*

Proof. As the problem is obviously in class NP, it is enough to show its hardness. We reduce the well-known NP-complete problem VERTEX-COVER [3] to this restricted version. In the problem VERTEX-COVER, a graph $G = (V, E)$ and an integer $k \leq |V|$ is given and we have to decide whether there is a set $T \subseteq V$ with $|T| = k$ such that every edge has at least one endvertex in T .

Given $G = (V, E)$ and k , we need to construct $G' = (V', E')$ and functions f, g . Let $V' = V \cup E \cup \{y\}$ (where $y \notin V \cup E$). Let $m = |E|$, $M = m + 1$ and $E' = \{ve \mid v \in V, e \in E, v \text{ is incident to } e\} \cup \{ye \mid e \in E\} \cup M\{yv \mid v \in V\}$, where the last edge-set is meant to have M parallel edges between y and any vertex $v \in V$. The construction of G' is finished, its degree-function is denoted by d' .

Let $f(v) = 0$ for $v \in V$, $f(e) = 1$ for $e \in E$ and $f(y) = m + kM$. Let $g(v) = 0$ for $v \in V$, $g(e) = 0$ for $e \in E$ and $g(y) = (n - k)M = d'(y) - f(y)$. We need to prove that G has a cover T of size k if and only if G' has an acyclic orientation satisfying these degree bounds. First suppose T is such a cover and let $E_2 \subseteq E$ denote the edges with both endvertices in T . As T is a cover, every edge in $E_1 = E \setminus E_2$ connects T to $V \setminus T$. Define an order on V' as follows. We start by putting vertices of T (in any order), then elements of E_1 , then y , and finally vertices of $V \setminus T$ (in any order). We still need to place each $uv \in E_2$, we put such an edge between u and v (thus it will precede exactly one of its endvertices). This order defines an acyclic orientation of G' (edges oriented from earlier vertex to the later one). The indegree of any $e \in E$ is exactly one and $\varrho(y) = m + kM$, thus this acyclic orientation is indeed (f, g) -bounded.

For the other direction, suppose there exists an acyclic (f, g) -bounded orientation of G' , and fix any topological order. First we claim that y is preceded by every $e \in E$ and by exactly k elements of V (call this latter subset T). If y is preceded by at most $k - 1$ elements of V , then $\varrho(y) \leq (k - 1)M + m < f(y) = kM + m$. If y is preceded by at least $k + 1$ elements of V , then $\varrho(y) \geq (k + 1)M > d'(y) - g(y) = kM + m$. Accordingly, exactly k vertices of V precedes y , and as its indegree is exactly $kM + m$, necessarily every $e \in E$ must also precede it.

It remains to prove that T is a cover in G . Suppose this is not the case, there is an $e = uv \in E$ with $u \notin T$ and $v \notin T$. As e precedes y and y precedes both u and v , $\varrho(e) = 0 < f(e) = 1$, a contradiction. \square

The graph G' used in the proof is not simple. However, one can split every edge e of G' with a new vertex w_e and define $f(w_e) = g(w_e) = 1$. This gives the following corollary.

Corollary 3.3. *The degree constrained acyclic orientation problem for simple graphs is NP-complete even if for every vertex v either $f(v) + g(v) = d(v)$ (i.e., v is strict) or $f(v) \leq 1$ and $g(v) = 0$ (v is lower-bounded by 0 or 1).*

Theorem 3.4. *Problem 1 is NP-complete.*

Proof. We reduce Problem 2 to Problem 1. Given G, f, g , we construct an instance of Problem 1 as follows. First we fix $k = \ell$ as the maximum degree in G . We add two new vertices, s and t . Then for each vertex v we add $k - f(v)$ parallel edges between s and v , and we also add $k - g(v)$ parallel edges between t and v . Let $G' = (V', E')$ denote the resulting graph (where $V' = V \cup \{s, t\}$).

By Claim 1.1, it is enough to show that the following are equivalent.

- G' has an acyclic orientation with source s and sink t where for every $v \in V$, we have $\varrho'(v) \geq k$ and $\delta'(v) \geq k$.
- G has an acyclic (f, g) -bounded orientation.

First suppose we have the above acyclic orientation of G' . After deleting s and t , we get an acyclic orientation of G where, for every $v \in V$, we have $\varrho(v) = \varrho'(v) - k + f(v) \geq f(v)$, and $\delta(v) = \delta'(v) - k + g(v) \geq g(v)$.

Next suppose we have an acyclic orientation of G where for every $v \in V$ we have $f(v) \leq \varrho(v) \leq d(v) - g(v)$. To orient G' , we keep the orientation of the original edges and orient the edges of form sv from s to v , and the edges of form vt from v to t ; resulting in an acyclic orientation of G' . As $\varrho'(v) = \varrho(v) + k - f(v)$, we get $\varrho'(v) \geq k$. We also have $\delta'(v) = \delta(v) + k - g(v) \geq k$. \square

3.1 Problem 1 for small k values

We proved that Problem 1 is NP-complete if k and ℓ are part of the input. What can we say about its status if k and ℓ are fixed small numbers?

The first case is well known, it was solved in [2] where st -numbering was introduced.

Theorem 3.5 ([2]). *If $k = \ell$ is fixed to one, then we can answer Problem 1 in polynomial time, namely there is a required orientation if and only if $G + st$ is biconnected.*

On the other hand, we can prove the following.

Theorem 3.6. *Problem 1 is NP-complete for $k = \ell = 2$.*

Proof. The problem is obviously in NP. We will reduce the problem NOT-ALL-EQUAL 3SAT [3] to it. In the problem NOT-ALL-EQUAL 3SAT there are variables x_1, \dots, x_n and clauses C_1, \dots, C_m , each clause contains exactly three literals (a literal is a variable or a negated variable), and we need to decide whether there is a truth assignment to the variables such that every clause has at least one TRUE and at least one FALSE literal in it. Given a NOT-ALL-EQUAL 3SAT instance, we first construct the multigraph G as follows.

$V := \{s = a_0, a_1, a_2, \dots, a_{4n+2m}, a_{4n+2m+1} = t, y_1, \dots, y_n, z_1, \dots, z_n, C_1, \dots, C_m\} \cup \{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\}$.

First we add edges to make the “skeleton”. We add two parallel edges between a_{2i} and a_{2i+1} for $i = 0, \dots, 2n + m$. We also add one edge between a_{2i-1} and a_{2i} for $i = 1, \dots, 2n + m$. Then we connect a_{2i-1} to y_i and y_i to a_{2i} , and we also connect $a_{2n+2m+2i-1}$ to z_i and z_i to $a_{2n+2m+2i}$ for $i = 1, \dots, n$. To finish the skeleton we connect $a_{2n+2j-1}$ to C_j and C_j to a_{2n+2j} for $j = 1, \dots, m$.

We are left to connect the literals to the skeleton. For $i = 1, \dots, n$, we connect both x_i and \bar{x}_i to y_i and z_i . Then for $j = 1, \dots, m$, we connect C_j to the three literals it contains. Finally we add two parallel edges between s and any literal, and also between t and every literal.

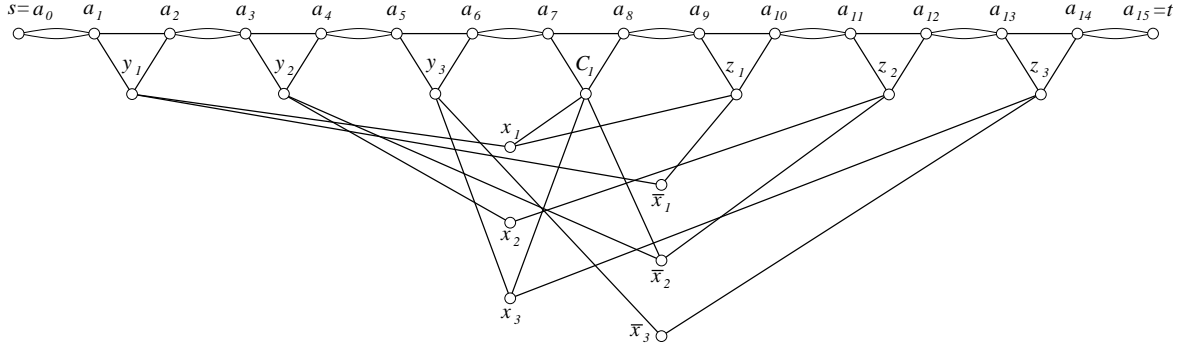


Figure 1: The construction for $n = 3$, $m = 1$, $C_1 = x_1 \vee \bar{x}_2 \vee x_3$ (except the parallel edges between the literals and s and t added at the end).

The construction of G is finished, see Figure 1. Note that the vertices a_1, \dots, a_{4n+2m} have degree four as well as vertices y_1, \dots, y_n and z_1, \dots, z_n , while vertices C_1, \dots, C_m have degree five.

First we show that if we have an assignment to the variables such that every clause has at least one and at most two TRUE literals in it, then we can make an acyclic orientation of G where for every vertex $v \in V \setminus \{s, t\}$ we have $\varrho(v) \geq 2$ and $\delta(v) \geq 2$. We give the orientation by defining the topological order. First we define the order of the skeleton: $a_0, a_1, y_1, a_2, a_3, y_2, a_4, \dots, y_n, a_{2n}, a_{2n+1}, C_1, a_{2n+2}, a_{2n+3}, \dots, a_{2n+2m-1}, C_m, a_{2n+2m}, a_{2n+2m+1}, z_1, a_{2n+2m+2}, \dots, a_{4n+2m-1}, z_n, a_{4n+2m}, a_{4n+2m+1}$. Note that the vertices a_i are ordered by the indices, and every y_i, z_i, C_j is between its two neighbors. So far each a_i has indegree 2 and outdegree 2 for $1 \leq i \leq 4n + 2m$ and vertices y_i, z_i, C_j have indegree 1 and outdegree 1. Finally we place all the TRUE literals between a_1 and y_1 (in any order), and we place all the FALSE literals between z_n and a_{4n+2m} (in any order). First note that due to the parallel edges added at the end, every literal has at least two incoming and at least two outgoing arcs. As any vertex y_i or z_i is preceded by the TRUE literal, and succeeded by the FALSE literals, every y_i and z_i has indegree and outdegree exactly two. Finally, a clause is preceded by its TRUE literals, we have one or two of them, consequently either its indegree is two and its outdegree is three, or vice versa.

It remains to show that if the required acyclic orientation exists, then we have an assignment to the variables such that each clause gets one or two TRUE values. Take the topological order of a good acyclic orientation. First we claim that the skeleton is in the same order as we defined in the previous part. Suppose for a contradiction that i is the smallest index ($1 \leq i \leq 4n + 2m - 1$) such that a_{i+1} precedes a_i . In this case a_i has at least three incoming arcs, so this case is impossible. Next suppose that w is the first vertex from $y_1, \dots, y_n, C_1, \dots, C_m, z_1, \dots, z_n$ which is not between its two a_i -neighbors. If it is before that place, then its lower-indexed a_i -neighbor has indegree three, if it is after that place, then its higher-indexed a_i -neighbor has outdegree three. Next observe that for all $1 \leq i \leq n$, one of x_i and \bar{x}_i must be before y_i

(consequently before a_{2n}) and the other must be after z_i (consequently after a_{2n+2m}), since otherwise either y_i or z_i does not have the prescribed indegree two. Now we can define the assignment: a literal is TRUE if it precedes a_{2n} . To finish the proof, observe that if C_j has for example three TRUE literals, then its indegree is four, so its outdegree is only one. \square

The graph G used in the above proof is not simple, actually the answer is always NO for a simple graph, as the second vertex cannot have two incoming arcs. However, one can split any edge $e = uv$ of G with a new degree four vertex w_e that is connected to s, u, v, t . Such a w_e must necessarily be between u and v , thus their in- and outdegrees will not be affected. This way we can obtain a graph that is almost simple - only edges adjacent to s or t might have multiplicity 2.

By splitting s into two vertices, s_1 and s_2 , and similarly t into t_1 and t_2 , and dividing the multiple edges among them, we obtain that the following problem is NP-complete.

Problem 3. *Given a simple graph $G = (V, E)$ and $s_1, s_2, t_1, t_2 \in V$, decide whether there is an order of the vertices such that every vertex $v \in V \setminus \{s_1, s_2, t_1, t_2\}$ has at least 2 neighbors preceding it and also at least 2 neighbors succeeding it.*

Some of the questions left open are the following.

Open Problem 1. *Is Problem 1 in P for the special case $k = 2$ and $\ell = 1$?*

Open Problem 2. *Is Problem 2 in P for the special case $f(v) + g(v) \leq 2$ for all $v \in V$?*

Open Problem 3. *Is Problem 2 in P for the special case $f(v) + g(v) \leq 3$ for all $v \in V$?*

Open Problem 4. *Is Problem 2 in P for the special case $f(v)g(v) \leq 1$ for all $v \in V$?*

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