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The Manickam-Miklós-Singhi Parameter of Graphs and Degree Sequences

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Abstract

Let G be a simple graph. Consider all weightings of the vertices of G with real numbers whose total sum is nonnegative. How many edges of G have endpoints with a nonnegative sum? We consider the minimum number of such edges over all such weightings as a graph parameter. Computing this parameter has been shown to be NP-hard but we give a polynomial algorithm to compute the minimum of this parameter over realizations of a given degree sequence. We also completely determine the minimum and maximum value of this parameter for regular graphs. [1]

1 Introduction

Suppose there are n real numbers $x_1, \dots, x_n$ with nonnegative sum. How many subsums $x_{i_1} + \cdots + x_{i_k}$ of size k are there which are also nonnegative? The Manickam-Mikls-Singhi (MMS) Conjecture is that if n is at least $4k$, then there are at least $\binom{n-1}{k-1}$ nonnegative subsums. The conjecture was proven for $n \geq \min\{33k^2, 2k^3\}$ by Alon, Huang, and Sudakov [1] and for $n \geq 10^{46}k$ by Pokrovskiy [11].

For a function $w : V \to \mathbb{R}$ and a set $X \subseteq V$, let $w(X) = \sum_{x \in X} w(x)$. For a hypergraph $H = (V, E)$ let $\nu(H)$, $\tau(H)$, $\nu^*(H)$ and $\tau^*(H)$ denote the matching number, the cover number, the fractional matching number and the fractional cover number of H. It is well known that $\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H)$. For $E' \subseteq E$ let $H - E' = (V, E \setminus E')$ be the hypergraph we get from H after deleting the edges in E'. Moreover, for a k-uniform hypergraph H on n vertices, let

$$\text{mms}(H) = \min_{w:V\to\mathbb{R}\,;\,w(V)\geq 0} \big(|\{e \in E \;;\; w(e) \geq 0\}|\big), \text{ and}$$

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[1]This research was done in 2016, during a Research course at Budapest Semesters in Mathematics.

$$\mu(H) = \min_{E' \subseteq E} \big(|E'| \ ; \ \nu^*(H - E') = \tau^*(H - E') < \frac{n}{k}\big).$$

The definition of the hypergraph parameter $\mathrm{mms}(H)$ was introduced by D. Miklós [10], inspired by [1] and [7]. The following theorem was proved in [1] for complete uniform hypergraphs. We repeat their proof with a slight modification for this more general statement.

Theorem 1.1. *For any k-uniform hypergraph H,* $\mathrm{mms}(H) = \mu(H)$.

Proof. First take a weighting w for which $w(V) \geq 0$ and $|\{e \in E \ ; \ w(e) \geq 0\}| = \mathrm{mms}(H)$. Let $E' = \{e \in E \ ; \ w(e) \geq 0\}$, so $\mathrm{mms}(H) = |E'|$. After dividing each weight by $2k \cdot \max_{v \in V}(|w(v)|)$, we may assume that $w(v) < 1/k$ for all $v \in V$. There is an $\varepsilon > 0$ such that for $w'(v) = w(v) + \varepsilon$ we have $w'(v) \leq 1/k$ for every $v \in V$, and also $w'(e) < 0$ whenever $w(e) < 0$. Clearly $w'(V) > w(V) \geq 0$. Let $f(v) = 1/k - w'(v)$ for each $v \in V$. Now f is a fractional cover of $H - E'$ with size $f(V) = n/k - w'(V) < n/k$, proving $\mu(H) \leq \mathrm{mms}(H)$.

Let $E' \subseteq E$ be a subset with $\mu(H) = |E'|$ and $\tau^*(H - E') < \frac{n}{k}$. Let f denote a fractional cover of $H - E'$ with $f(V) = n/k - \delta < n/k$. That is, for each edge $e \in E \setminus E'$ we have $f(e) \geq 1$. For a vertex $v \in V$, define $w(v) = \frac{1}{k} - \frac{\delta}{n} - f(v)$. On one hand $w(V) = n/k - \delta - f(V) = 0$, on the other hand, for any $e \in E$ if $w(e) = 1 - k\delta/n - f(e) \geq 0$, then $f(e) \leq 1 - k\delta/n < 1$, so $e \in E'$ as f is a fractional cover for $H - E'$. □

Let $\delta(H)$ denote the minimum degree in H. It is obvious that $\mathrm{mms}(H) = \mu(H) \leq \delta(H)$. Huang and Sudakov in [7] defined that a k-uniform hypergraph H *has the MMS property* if $\mathrm{mms}(H) = \delta(H)$. Using this concept the MMS conjecture says that if $n \geq 4k$, then the complete k-uniform hypergraph on n vertices has the MMS property.

In this paper, only simple graphs (2-uniform hypergraphs) are considered. Let $G = (V, E)$ be a graph. A subgraph F is called a *perfect 2-matching* if it is spanning and its every component is either a K_2 or an odd cycle. (*Remark: in the literature its weighted version is usually called a perfect 2-matching when we give weight one to the edges of cycles and weight two for the other edges.*) It is well known that G has a perfect 2-matching if and only if $\nu^*(G) = n/2$. For $S \subseteq V$, let $\Gamma_G(S)$ denote the set of vertices in $V \setminus S$ having at least one neighbor in S. In 1953, Tutte characterized the graphs having a perfect 2-matching.

Theorem 1.2 (Tutte [12])**.** *A graph G has a perfect 2-matching if and only if every independent set S of vertices satisfy $|\Gamma_G(S)| \geq |S|$.*

Putting Theorems 1.1 and 1.2 together gives the following corollary, first observed by a previous research group at Budapest Semesters in Mathematics [10]:

Theorem 1.3. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

1. $\mathrm{mms}(G) = \mu(G) \leq k$*;*
2. *There exists a set $S \subseteq V$ and a set E' of k edges such that in the graph $G - E'$, S is independent with fewer than $|S|$ neighbors;*

3. *There exists a set E' of k edges such that $G - E'$ has no perfect 2-matching, i.e., E' blocks (covers) every perfect 2-matching.*

Corollary 1.4. Let $G = (V, E)$ be a graph. Then $\mu(G) = 0$ if and only if there exists an independent set $S \subseteq V$ such that $|\Gamma_G(S)| < |S|$.

Corollary 1.5. A graph G has the MMS property (i.e., $\mu(G) = \delta(G)$) if no fewer edges than $\delta(G)$ can block every perfect 2-matching.

For a graph $G = (V, E)$ we define some notation. The degree of a vertex v is denoted by $d_G(v)$. Let $S, T \subseteq V$ be two disjoint subsets of the vertices. Let $i_G(S)$ denote the number of edges having both end-vertices in S, and let $d_G(S, T)$ denote the number of edges having one end-vertex in S and the other end-vertex in T. Moreover, we use the following unusual notation. Let $E_G(S; V \setminus T)$ denote the set of edges having either both end-vertices in S or one end-vertex in S and the other end-vertex in $(V \setminus T) \setminus S$. For simplicity, this latter set will be denoted by $V - S - T$. Thus $|E_G(S; V \setminus T)| = i_G(S) + d_G(S, V - S - T)$.

Corollary 1.6. Let G be a graph. Then

$$\mu(G) = \min |E_G(S; V \setminus T)|,$$

where S and T range over all disjoint subsets of V such that $|S| > |T|$.

Proof. Suppose that E' is a set of $\mu(G)$ edges so that some S is independent in $G - E'$ with fewer than $|S|$ neighbors. Let T be the neighborhood of S in $G - E'$; then $E_G(S; V \setminus T) \subseteq E'$. Conversely, for disjoint S and T with $|S| > |T|$, the subgraph $G - E_G(S; V \setminus T)$ has S independent with $|T| < |S|$ neighbors. □

If S and T are disjoint subsets of V with $|S| > |T|$ and $\mu(G) = |E_G(S; V \setminus T)|$, then we say that the pair (S, T) *realizes* $\mu(G)$.

As bipartite graphs have no odd cycles, we also get:

Corollary 1.7. Let G be a bipartite graph. Then $\mu(G)$ is the minimum number of edges that can block every perfect matching.

So $\mu(G)$ is a nice graph parameter. One can ask whether it is computable or not.

Theorem 1.8 (Dourado et al. [4]). *For a bipartite graph G, checking whether $\mu(G) < \delta(G)$ is NP-complete.*

A recent trend in graph theory is the following: given a graph parameter—perhaps one which is NP-hard to compute—what values does that parameter take over all graphs with the same degree sequence? Dvořák and Mohar in [5] and later Bessy and Rautenbach in [3] investigated the possible values of the chromatic number and clique number over a given degree sequence, obtaining nice bounds relating them. Hence, we will investigate the possible values of $\mu(G)$ for all G realizing a given degree sequence.

Given a degree sequence $\mathbf{d}$, let $\underline{\mu}(\mathbf{d})$ denote the minimum value of μ over all graphs with degree sequence $\mathbf{d}$. One of our main results is the following:

Theorem 1.9. *Given a degree sequence* $\mathbf{d}$*, there is a polynomial algorithm to determine* $\underline{\mu}(\mathbf{d})$*.*

We also compute the maximum and minimum values of μ over all regular degree sequences.

2 Degree sequences

Given a graph, its *degree sequence* is the list of degrees of the vertices of the graph.

Definition 1. A *graphical degree sequence* is a sequence of nonnegative integers $(d_1, d_2, \ldots, d_n)$ with even sum such that there is a simple graph G on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ in which the degree of of vertex v_i is d_i for $1 \le i \le n$.

Definition 2. For a graphical degree sequence $\mathbf{d}$, we define

$$\underline{\mu}(\mathbf{d}) = \min\{\mu(G) : G \text{ realizes } \mathbf{d}\},$$
$$\overline{\mu}(\mathbf{d}) = \max\{\mu(G) : G \text{ realizes } \mathbf{d}\}.$$

We refer to these as *lower* μ and *upper* μ of a degree sequence, respectively.

A fundamental tool of realizations of a degree sequence is the *swap*: given an alternating 4-cycle of edges and non-edges in a graph, swapping the edges and non-edges gives a new realization of the same degree sequence. Given two graphs G and G' which realize the same degree sequence, there always exists a sequence of swaps which may be applied to G to obtain G' (see, for example, [2, pp. 153-154]). We thus investigate the effect of a swap on μ, making use of Corollary 1.6.

Lemma 2.1. *Suppose* G *and* G' *have the same degree sequence and can be obtained from each other via a single swap. Then*

$$|\mu(G) - \mu(G')| \le 1.$$

Proof. Suppose that $V(G) = V(G') = V$ and that G is changed to G' by a single swap. Let S and T be disjoint subsets of V. We compare $E_G(S; V \setminus T)$ and $E_{G'}(S; V \setminus T)$. Since G' has exactly two edges that are not also edges of G, we observe

$$|E_{G'}(S; V \setminus T)| \le |E_G(S; V \setminus T)| + 2$$

If equality holds, then both of the swapped edges in G' are in $E_{G'}(S; V \setminus T)$, while neither of the swapped edges in G are in $E_G(S; V \setminus T)$. Say the swapped edges in G' are $su, s'u'$ where $s, s' \in S$ and $u, u' \notin T$. Then either su' or ss' is one of the swapped edges in G, and is in $E_G(S; V \setminus T)$. Hence equality cannot hold.

If (S, T) realizes $\mu(G)$, we conclude that

$$\mu(G') \le |E_{G'}(S; V \setminus T)| \le |E_G(S; V \setminus T)| + 1 = \mu(G) + 1.$$

By symmetry, $\mu(G) \le \mu(G') + 1$ as well. □

As any two realizations of a given degree sequence are related by a sequence of swaps, the possible values of μ over a degree sequence form an interval. Thus, complete information is given by the minimum and maximum possible values of μ over a degree sequence.

Theorem 2.2. *Suppose* $\mathbf{d}$ *is a graphical degree sequence,* k *is an integer and* $\underline{\mu}(\mathbf{d}) \leq k \leq \overline{\mu}(\mathbf{d})$. *Then there is a graph* G *realizing* $\mathbf{d}$ *such that* $\mu(G) = k$.

Proof. Let G and G' be realizations of $\mathbf{d}$ with $\mu(G) = \underline{\mu}(\mathbf{d})$ and $\mu(G') = \overline{\mu}(\mathbf{d})$. There is a sequence of graphs $G = G_1, G_2, \dots, G_m = G'$ such that adjacent graphs are related by a swap. By Lemma 2.1, $\mu(G_i)$ and $\mu(G_{i+1})$ differ by at most one. Hence k must be equal to $\mu(G_i)$ for some i. □

3 Computation

We now show that $\underline{\mu}(\mathbf{d})$ is computable in polynomial time. This relies on the characterization of μ given in Corollary 1.6. The fundamental strategy for computing $\underline{\mu}(\mathbf{d})$ is the following. For any fixed disjoint pair (S, T) of subsets of $V = \{v_1, \dots, v_n\}$ with $|S| > |T|$, first compute the minimum of $|E_G(S; V \setminus T)|$ over all graphs G on V realizing $\mathbf{d}$. Then compute the minimum of these minimum values over all pairs (S, T).

Of course, there are too many pairs (S, T) of disjoint subsets. The following lemmas help to reduce the number of pairs (S, T) we need to check.

Lemma 3.1. *Suppose that* G *is a graph on vertex set* V*, and* V *is colored by red and blue. Let* v_1, v_2 *be vertices in* V *such that* $d_G(v_1) \geq d_G(v_2)$. *Then there exists a graph* G' *on* V *such that* $d_G(v) = d_{G'}(v)$ *for all* $v \in V$*, and such that* v_1 *has at least as many red neighbors in* G' *as* v_2 *had in* G.

Proof. Assume that v_1 has strictly fewer red neighbors than v_2 in G (if this is not the case, then $G' = G$ is good). We can find a red vertex v_{red} that is a neighbor of v_2 but not of v_1. Since $d_G(v_1) \geq d_G(v_2)$, v_1 has strictly more blue neighbors than v_2, so there is a blue vertex v_{blue} which is a neighbor of v_1 but not of v_2. Swap edges v_2v_{red} and v_1v_{blue} so that v_1 has one more and v_2 has one less red neighbor. To form G', repeat this process until v_1 has at least as many red neighbors as v_2 had in G. □

Lemma 3.2. *Let* G *be a graph with vertex set* V*, and* S *and* T *be disjoint subsets of* V. *Then there exists another graph* G' *on* V *and disjoint subsets* S' *and* T' *of* V *with the same sizes as* S *and* T *such that* $d_G(v) = d_{G'}(v)$ *for all* $v \in V$*, the vertices of* S' *are of lowest degree in* G'*, the vertices of* T' *are of highest degree in* G'*, and*

$$|E_{G'}(S'; V \setminus T')| \leq |E_G(S; V \setminus T)|.$$

Proof. Consider all graphs G' on V such that $d_G(v) = d_{G'}(v)$ for all $v \in V$, and all disjoint subsets S' and T' of V satisfying $|S'| = |S|$, $|T'| = |T|$, and

$$|E_{G'}(S'; V \setminus T')| \leq |E_G(S; V \setminus T)|.$$

Choose the G', S', and T' that minimize the difference $\gamma := \sum_{s\in S'} d_{G'}(s) - \sum_{t\in T'} d_{G'}(t)$. Define $U' := V - S' - T'$. It is claimed that S' consists of vertices of lowest degree and T' consists of vertices of highest degree. If not, one of three cases must hold.

Case 1. There exists $v_1 \in S'$ and $v_2 \in T'$ such that $d(v_2) < d(v_1)$.
Color $U' \cup S'$ red. Apply the swaps described in Lemma 3.1 if necessary so that v_1 has at least as many red neighbors in the new graph as v_2 had in G'. Then exchange v_1 and v_2 so that S' now contains v_2 in place of v_1 and T' contains v_1 in place of v_2.
Case 2. There exists $v_1 \in U'$ and $v_2 \in T'$ such that $d(v_2) < d(v_1)$.
Color S' red. As in the previous case, apply the swaps of Lemma 3.1 so that v_1 has at least as many red neighbors in the new graph as v_2 had in G'. Then exchange v_1 and v_2 so that U' contains v_2 in place of v_1 and T' contains v_1 in place of v_2.
Case 3. There exists $v_1 \in S'$ and $v_2 \in U'$ such that $d(v_2) < d(v_1)$.
Color U' red. Apply the swaps of Lemma 3.1 so that v_1 has at least as many red neighbors in the new graph as v_2 had in G'. Then exchange v_1 and v_2 so that S' contains v_2 in place of v_1 and U' contains v_1 in place of v_2.

In every case, Lemma 3.1 shows that the quantity $|E_{G'}(S'; V\setminus T')|$ has not increased, while γ has decreased, which is a contradiction. Thus, it must be that S' and T' contain the vertices of lowest and highest degree, respectively. □

Lemma 3.3. *If G is a graph on n vertices, then there exist S, T which are disjoint subsets of $V(G)$ such that (S,T) realizes $\mu(G)$, and $|S| \le (n+1)/2$ and $|T| = |S| - 1$.*

Proof. Take any pair (S,T) realizing $\mu(G)$. If $|S| \le (n+1)/2$, then we are done, as vertices outside of $S \cup T$ may be added to T (if necessary) to attain $|T| = |S| - 1$. If $|S| > (n+1)/2$, then $|T|$ is at most $n - (n+1)/2 = (n-1)/2$. In this case, deleting $|S| - (|T|+1)$ vertices from S provides the desired (S,T). (In both cases $|E_G(S; V\setminus T)|$ does not increase.) □

The last ingredient in our algorithm for computing $\underline{\mu}(\mathbf{d})$ is a polynomial-time algorithm computing the minimum cost b-factor.

Definition 3. Given a graph G on n vertices and a nonnegative integer weight $b(v)$ for each vertex v of G, a subgraph F is called a *b-factor* if $d_F(v) = b(v)$ for every vertex $v \in V$.

The weighted problem is the following: given also a nonnegative integer cost c for each edge of G, what is the minimum total cost of a b-factor?

By the gadget of Tutte [13] it is easy to reduce this problem to finding a minimum cost perfect matching in a graph having $O(n^2)$ vertices. This later problem is solvable in polynomial time by Edmonds [6].

Theorem 1.9. *$\underline{\mu}(\mathbf{d})$ can be computed in polynomial time.*

Proof. We give an algorithm to compute $\underline{\mu}(\mathbf{d})$. We may assume that $d_1 \ge d_2 \ge \cdots \ge d_n$.

Given **d**, check first whether it is graphical (for example using the linear time algorithm of [8]). Let K be the complete graph on $V = \{v_1 \dots, v_n\}$ and let $b(v_i) = d_i$ for all i. A subgraph of K is a b-factor if and only if it realizes **d**.

For all $k = 1, \dots, \lfloor (n+1)/2 \rfloor$, execute the following process:

Let $S = \{v_{n-k+1}, \dots, v_n\}$ and $T = \{v_1, \dots, v_{k-1}\}$. Define a cost $c(uv)$ of each edge uv of K: $c(uv) = 0$ unless $u \in S$ and $v \notin T$, in which case $c(uv) = 1$. For a given b-factor, G, the cost of G is exactly $|E_G(S; V \setminus T)|$. Then calculate a minimum-cost b-factor. Call its cost $\mathrm{OPT}(k)$.

Finally output $\min_{1 \le k \le \lfloor (n+1)/2 \rfloor}(\mathrm{OPT}(k))$. By Lemmas 3.2 and 3.3, this is exactly $\underline{\mu}(\mathbf{d})$. □

4 Existence of perfect 2-matchings

What is surprising about $\underline{\mu}$ is that minimizing over exponentially many realizations of a given degree sequence is possible. In contrast, it is unknown what the computational complexity of $\overline{\mu}$ is.

Nonetheless, the relationship between perfect 2-matchings and perfect matchings lets us make some headway in checking whether $\overline{\mu}$ is positive, that is, if there exists a realization of a degree sequence with a perfect 2-matching. Since even cycles have a perfect matching, the only obstruction to having a perfect matching in a graph with positive μ can be the existence of odd cycles in every perfect 2-matching. These may be addressed by the following lemma:

Lemma 4.1. *Let G be a graph with a perfect 2-matching. Then there is another graph G' with the same degree sequence as G and a perfect 2-matching with at most one odd cycle.*

Proof. Let F be a perfect 2-matching in G with the minimum number of odd cycles. If F has more than one odd cycle, we show how to construct another realization of the degree sequence of G with a perfect 2-matching with fewer odd cycles. Say C_1 and C_2 are distinct odd cycles in F. Our strategy is to either show that $G[V(C_1) \cup V(C_2)]$ has a perfect matching or make a single swap in G to achieve the same conclusion.

Let $u_1v_1 \in E(C_1)$ and $u_2v_2 \in E(C_2)$. If both u_1u_2 and v_1v_2 are also edges in G, we could replace u_1v_1 and u_2v_2 in F with u_1u_2 and v_1v_2 getting an even cycle that has a perfect matching. But if neither u_1u_2 and v_1v_2 are edges in G, then we could swap u_1v_1 and u_2v_2 to u_1u_2 and v_1v_2 to form G', now $G'[V(C_1) \cup V(C_2)]$ has a perfect matching.

Thus, we suppose towards contradiction that for all $u_1v_1 \in E(C_1)$ and $u_2v_2 \in E(C_2)$, exactly one of u_1u_2 and v_1v_2 is an edge in G. Similarly, we may suppose exactly one of u_1v_2 and u_2v_1 is an edge in G.

Suppose without loss of generality that uv is an edge in C_1 such that both u and v are connected to $w_1 \in V(C_2)$. Let the vertices of C_2 be (in order) $w_1, \dots, w_\ell$. Then both of w_1's neighbors in C_2 (i.e., w_2 and w_ℓ) must have no edges to any of u and v by our assumption. Similarly, w_3 and $w_{\ell-1}$ must be connected to both u and v, and so on. This allows us to 2-color C_2 according to whether a vertex has neither or

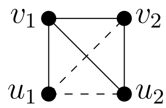

Figure 1: The situation for all $u_1v_1 \in E(C_1)$ and $u_2v_2 \in E(C_2)$, up to symmetry.

both vertices u and v as neighbors, a contradiction since C_2 is odd. So either F does not have the minimal number of odd cycles, or we may swap to a new graph with a perfect 2-matching with fewer odd cycles, as desired. □

For n even, this reduces the existence of a perfect 2-matching to the existence of a perfect matching:

Theorem 4.2. *Let* $\mathbf{d}$ *be a graphical degree sequence of even length. Then* $\mathbf{d}$ *has a realization with a perfect 2-matching if and only if* $\mathbf{d}$ *has a realization with a perfect matching.*

Proof. One direction is clear, since a perfect matching is also a perfect 2-matching. Conversely, if $\mathbf{d}$ has a realization with a perfect 2-matching, Lemma 4.1 shows that there exists a realization with a perfect 2-matching with at most one odd cycle. Since $\mathbf{d}$ is of even length, the number of odd cycles in a perfect 2-matching must be even, so our perfect 2-matching in this realization has no odd cycles and hence it is a perfect matching. □

The k-factor theorem of Kundu [9] gives (as a special case) conditions for a degree sequence to have a realization with a perfect matching. In particular, deciding whether a degree sequence has a realization with a perfect matching can be decided in polynomial time.

Theorem 4.3 (Kundu [9])**.** *Let* $\mathbf{d}$ *be a graphical degree sequence. A realization* G *of* $\mathbf{d}$ *with a perfect matching exists if and only if the sequence* $\mathbf{d} - \mathbf{1^n} = (d_1 - 1, \dots, d_n - 1)$ *is also graphical.*

Corollary 4.4. Let $\mathbf{d}$ be a graphical degree sequence with even length. $\overline{\mu}(\mathbf{d}) > 0$ if and only if $\mathbf{d} - \mathbf{1^n}$ is also graphical.

5 Regular degree sequences

In this section we completely determine the values $\underline{\mu}(\mathbf{d})$ and $\overline{\mu}(\mathbf{d})$ for all regular degree sequences.

5.1 The minimal value of μ

Before computing exact values of $\underline{\mu}(\mathbf{d})$, we find bounds on the value of μ. Since deleting all edges incident to a given vertex in G destroys all perfect 2-matchings, we immediately have an upper bound:

Lemma 5.1. *For any graph G, $\mu(G) \le \delta(G)$.*

For regular graphs, the following lower bound also exists.

Lemma 5.2. *For any k-regular graph G, $\mu(G) \ge k/2$.*

Proof. Let (S, T) realize $\mu(G)$. After deleting the μ edges of $E_G(S; V \setminus T)$, the number of edges leaving S is at least $k|S| - 2\mu$. However, the number of edges leaving T is at most $k|T| \le k(|S| - 1)$. Thus, $k|S| - 2\mu(G) \le k|S| - k$, so $\mu(G) \ge k/2$. □

We will show the upper bound and the lower bound given by Lemmas 5.1 and 5.2 are tight for large enough n (Theorem 5.5). However, if (S, T) realizes $\mu(G)$ and we know the sizes of the sets S and T, we can bound $\mu(G)$ more narrowly. The following lemma (together with Lemma 3.3) gives conditional bounds on the sizes of S and T.

Lemma 5.3. *Suppose that G has minimum degree $\delta(G) \ge 2$. Suppose that $\mu(G) \le \delta(G) - 1$, and S and T are disjoint sets of vertices of G such that (S, T) realizes $\mu(G)$. Then $|S| \ge \delta(G) - 1$, and if $|S| = \delta(G) - 1$ then $\mu(G) = \delta(G) - 1$.*

Proof. Let us write $s = |S|$ and $\delta = \delta(G)$. After deleting the $\mu(G)$ edges of $E_G(S; V \setminus T)$, a vertex in S can have at most $s - 1$ neighbors. Hence, each vertex of S is incident to at least $\delta - s + 1$ edges to be deleted in G. Thus,

$$\delta - 1 \ge \mu(G) \ge \frac{s}{2}(\delta - s + 1). \tag{1}$$

This reduces to the quadratic $s^2 - (\delta + 1)s + 2(\delta - 1) \ge 0$, which is zero at $s = 2$ or $s = \delta - 1$. If $s = 1$, then $\delta - s + 1 = \delta$ edges were deleted contradicting $\mu(G) < \delta$. If $s = 2$ and $s < \delta - 1$, then there are $2(\delta - 1) > \delta$ deleted edges, a contradiction again. Hence $s \ge \delta - 1$, as desired.

If $s = \delta - 1$, then equality holds in (1), showing that $\mu(G) = \delta - 1$. □

These results allow us to determine the value of lower μ when $\delta(G)$ is large relative to n.

Lemma 5.4. *If G is a graph on n vertices and $\mu(G) < \delta(G)$, then $n \ge 2\delta(G) - 3$. If in addition $n = 2\delta(G) - 3$, then $\mu(G) = \delta(G) - 1$.*

Proof. By Lemma 3.3, we may choose (S, T) realizing $\mu(G)$ such that $|S|$ is at most $(n + 1)/2$. By Lemma 5.3, since (S, T) realizes $\mu(G)$, $|S| \ge \delta(G) - 1$. So $n + 1 \ge 2(\delta(G) - 1)$. If in addition $n = 2\delta(G) - 3$, then $|S| = \delta(G) - 1$, so Lemma 5.3 shows $\mu(G) = \delta(G) - 1$. □

Note that Lemma 5.4 applies to all graphs, not only regular ones. In addition, it proves that the complete graph K_n satisfies $\mu(K_n) = n - 1$ for $n \ge 6$, verifying the Manickam-Mikls-Singhi conjecture for graphs.

We use the notation $\mathbf{k^n}$ for the degree sequence $d_1 = \cdots = d_n = k$. Observe that $\mathbf{k^n}$ is graphical if and only if $k < n$ and kn is even.

Theorem 5.5. *For $n > k$ with n odd and k even,*

$$\underline{\mu}(\mathbf{k^n}) = \begin{cases} k & \text{if } n < 2k-3 \\ k-1 & \text{if } n = 2k-3 \\ k/2 & \text{if } n \geq 2k-1. \end{cases}$$

Proof. Lemma 5.4 establishes the case of $n < 2k-3$, and shows that to prove the case of $n = 2k-3$, it suffices to construct a realization G of the degree sequence $\mathbf{k^{2k-3}}$ with $\mu(G) < k$. Start with the complete bipartite graph $K_{k-1,k-2}$. Let its color classes of size $k-1$ and $k-2$ be denoted by S and T, respectively. Form G by adding a perfect matching to T and a cycle of size $k-1$ to S. The result is a k-regular graph. If the cycle in S is deleted, then S is independent with $|T| < |S|$ neighbors. Since the cycle in S has $k-1$ edges, $\mu(G) \leq k-1$, as desired.

Finally, suppose $n \geq 2k-1$. By Lemma 5.2, it suffices to find a graph G realizing the degree sequence $\mathbf{k^n}$ with $\mu(G) \leq k/2$. Begin with a k-regular bipartite graph with parts of size $(n+1)/2$, which exists since $k \leq (n+1)/2$. G is the graph formed by deleting a vertex and adding a perfect matching to the neighborhood of that vertex. If one deletes the perfect matching added, then what is left is a bipartite graph with parts of size $(n+1)/2$ and $(n-1)/2$, which cannot have a perfect 2-matching. Thus $\mu(G) \leq k/2$, as desired. □

Lemma 5.6. *Let n be even. If G is a k-regular graph on n vertices and $\mu(G) < k$, then $n \geq 3k-2$. If in addition $n \leq 3k-1$, then $\mu(G) = k-1$.*

Proof. By Lemma 3.3, there exists (S,T) which realizes $\mu = \mu(G)$ with $|T| = |S|-1$. Then $S \cup T$ has an odd number of vertices and so is not all of V. Let $U = V - S - T$, and let $u = |U|$, $s = |S|$, $t = |T| = s-1$. Remember that $\mu = |E_G(S; V \setminus T)| = i_G(S) + d_G(S,U)$. By counting the number of half-edges incident to vertices of S, we obtain

$$ks = 2(\mu - d_G(S,U)) + d_G(S,U) + d_G(S,T) = 2\mu - d_G(S,U) + d_G(S,T). \quad (2)$$

Exactly $ku - 2i_G(U) - d_G(S,U)$ edges go from U to T, so by counting the number of half-edges incident to T, we obtain

$$kt = k(s-1) \geq (ku - 2i_G(U) - d_G(S,U)) + d_G(S,T). \quad (3)$$

Since $i_G(U) \leq u(u-1)/2$, subtracting (2) from (3), and simplifying gives

$$2\mu \geq k(u+1) - 2i_G(U) \geq 2k + (u-1)(k-u).$$

As $\mu < k$, we see either $u < 1$ or $u > k$. But U is non-empty, so $u \geq k+1$. Thus, using Lemma 5.3 we get

$$n = u + s + t \geq (k+1) + (k-1) + (k-2) = 3k-2.$$

Since $s + t = 2s - 1 = n - u$, we also have

$$s \leq \frac{n-u+1}{2} \leq \frac{n-k}{2}. \quad (4)$$

If $n \leq 3k-1$, then (4) shows $|S| \leq k-1$, so by Lemma 5.3, $\mu(G) = \delta(G)-1 = k-1$. □

Theorem 5.7. *For $n > k$ with n even,*

$$\underline{\mu}(\mathbf{k^n}) = \begin{cases} k & \text{if } n < 3k-2 \\ k-1 & \text{if } n = 3k-2 \text{ or } 3k-1 \\ \lceil k/2 \rceil & \text{if } n \geq 3k. \end{cases}$$

Proof. The case of $n < 3k-2$ is immediate from Lemma 5.6. To prove the case of $n = 3k-1$ or $3k-2$, all that remains is to demonstrate a realization, G, of the degree sequence $\mathbf{k^n}$ with $\mu(G) < k$:

If $n = 3k-2$, then k must be even as n is even. Hence, by Theorem 5.5, there is a k regular graph on $2k-3$ vertices with a μ value of $k-1$. Let G be the disjoint union of this graph with K_{k+1}.

Suppose $n = 3k-1$. Begin with two components. One component is the complete graph K_{k+2}. For the other component, start with the complete bipartite graph $K_{k-1,k-2}$. The degree of $k-2$ vertices of one part is $k-1$, and since $k-2$ is odd, it is possible to add a perfect matching to all but one of its vertices, v_1. The degree of the $k-1$ vertices on the other side is $k-2$, so add a cycle, C, of length $k-1$. Choose any vertex from the K_{k+2} component, call it v_2. Let a_1 and a_2 be any neighbors of v_2 in K_{k+2}. Then delete a perfect matching from the remaining $k-1$ vertices of K_{k+2}, so that those $k-1$ vertices now have degree k. Delete the edges a_1v_2 and a_2v_2 and add the edge v_1v_2.

Now the graph is k-regular on $3k-1$ vertices. If the $k-1$ edges of cycle C are deleted, then there is an independent set of size $k-1$ with $k-2$ neighbors. Hence $\mu(G) < k$, as desired.

Finally, suppose $n \geq 3k$. By Lemma 5.2, it suffices to find a graph G realizing the degree sequence $\mathbf{k^n}$ with $\mu(G) \leq \lceil k/2 \rceil$. If k is even, let G' be a graph on $n-(k+1)$ vertices with $\mu(G') = k/2$, and let G be the disjoint union of G' with a complete graph on $k+1$ vertices.

If k is odd and $n \geq 3k+1$ is even, construct G as follows. Let G' be a k-regular bipartite graph with parts of size $(n-k-1)/2$, which exists since $n \geq 3k+1$ implies $(n-k-1)/2 \geq k$. Let H be a graph on $k+2$ vertices with degree sequence $(k, k, \ldots, k, k-1)$. If v is any vertex of G', then $(G'-v) \cup H$ will have $k+1$ vertices of degree $k-1$, while all other vertices have degree k. Construct G by adding a perfect matching to the $k+1$ vertices of degree $k-1$ in $(G'-v) \cup H$. After deleting the $(k+1)/2$ edges in this perfect matching from G, $G'-v$ is a connected component with $\mu = 0$, since it is a bipartite graph with different size color classes. Hence $\mu(G) \leq (k+1)/2$. □

5.2 The maximal value of μ

This section is devoted to proving that $\overline{\mu}(\mathbf{k^n}) = k$ whenever $\mathbf{k^n}$ is graphical (i.e., if $k < n$ and not both k and n are odd), except for some sporadic small cases. We start with the easier case when n is even.

Lemma 5.8. *Suppose n is even and $1 \leq k < n$. Then $\overline{\mu}(\mathbf{k^n}) = k$.*

Proof. It is well known that the edge-set of the complete graph on n vertices decomposes into perfect matchings, i.e., $E(K_n) = M_1 \cup M_2 \cup \cdots \cup M_{n-1}$. Let $G = M_1 \cup \cdots \cup M_k$. Clearly G is k-regular and less than k edges cannot block every perfect matching. □

When n is odd, we compute μ of a certain family of graphs with high symmetry.

Definition 4. For even k, the k-regular *circulant* on n vertices, denoted by $C(k, n)$, is the graph with vertex set $\{1, 2, \dots, n\}$, where i and j are adjacent if $|i - j| \leq k/2$. For $i < j$, here $|i - j|$ denotes $\min(j - i,\ i + n - j)$.

Theorem 5.9. *Let $n \geq 9$ be odd, and $4 \leq k < n$ be even. Then $\mu(C(k, n)) = k$.*

Proof. Let $G = C(k, n)$ and assume towards contradiction that $\mu(G) < k$. By Lemma 3.3, there exists (S, T) which realizes $\mu(G)$ with $s = |S| \leq (n+1)/2$ and $t = |T| = s-1$.

The edges to be deleted to realize $\mu(G)$ are either those internal to S or those that go from S to $\Gamma(S) - T$. The number of these latter edges is at least $|\Gamma_G(S)| - |T|$. Thus,

$$k - 1 \geq \mu(G) \geq i_G(S) + |\Gamma_G(S)| - |T| = i_G(S) + |\Gamma_G(S)| - s + 1. \quad (5)$$

Suppose the vertices of S are $v_1 = v_{s+1} < v_2 < \ldots < v_s$. Let $n_i = v_{i+1} - v_i$ for $i = 1, \dots, s-1$, and $n_s = v_1 + n - v_s$. Thus $\sum n_i = n$.

Between v_i and v_{i+1}, there are precisely $\min\{k, n_i - 1\}$ neighbors of either v_i or v_{i+1}. Hence,

$$|\Gamma_G(S)| = \sum_{i=1}^{s} \min\{k, n_i - 1\}. \quad (6)$$

In addition, if $n_i \leq k/2$, then the vertices v_i and v_{i+1} are adjacent, so

$$i_G(S) \geq |\{i : n_i \leq k/2\}|. \quad (7)$$

Although (7) is a crude estimate, it is enough along with (6) to bound μ for all but a few cases. To this end, define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n_i) = \begin{cases} \min\{k, n_i - 1\} + 1 & \text{if } \ n_i \leq k/2 \\ \min\{k, n_i - 1\} & \text{if } \ n_i \geq k/2 + 1. \end{cases}$$

Now (5), (6), and (7) yield

$$k - 1 \geq \mu(G) \geq \sum_{i=1}^{s} f(n_i) - s + 1. \quad (8)$$

Observe that $f(n_i) \geq 1$ for all i, with equality if and only if $n_i = 1$. We claim that $n_i \leq k$ for all i. For if $n_i > k$ for some i, then $f(n_i) = k$, and so since $f(n_j) \geq 1$ for all other j, (8) implies that

$$k - 1 \geq k + (s - 1) - s + 1 = k,$$

a contradiction. So $n_i \leq k$ for all i, implying that $|\Gamma_G(S)| = n - s$.

We similarly claim that $n_i \leq k/2$ for all but at most one i, yielding $i_G(S) \geq s - 1$. If instead $n_i \geq k/2 + 1$ for at least two different values of i, then again (8) implies

$$k - 1 \geq \sum_{i=1}^{s} f(n_i) - s + 1 \geq 2(k/2 + 1) + (s - 2) - s + 1 = k + 1, \tag{9}$$

a contradiction.

Since $i_G(S) \geq s - 1$, (5) yields

$$k - 1 \geq i_G(S) + |\Gamma_G(S)| - s + 1 \geq (s - 1) + (n - s) - s + 1 = n - s.$$

Equivalently, $n \leq k - 1 + s \leq 2s$ by Lemma 5.3, however, as n is odd, necessarily $n \leq 2s - 1$. As $s \leq (n + 1)/2$, we have $n = 2s - 1$.

If $k = 4$, then $k - 1 \geq \mu(G) \geq i_G(s) \geq s - 1$, yielding $s \leq 4$ and thus $n \leq 7$, contradicting our assumption that $n \geq 9$. So we may suppose $k \geq 6$.

Now it is time to take edges inside S of form $v_i v_{i+2}$ into account. As $\sum_{i=1}^{s}(n_i + n_{i+1}) = 2n = 4s-2$, we have at least two different indices $i \neq j$ such that $n_i+n_{i+1} \leq 3$ and $n_j+n_{j+1} \leq 3$. Consequently $v_i v_{i+2}$ and $v_j v_{j+2}$ are also edges of G, so $i_G(S) \geq s+1$. Now Lemma 5.3 and (5) yield

$$s \geq k - 1 \geq (s + 1) + (n - s) - s + 1 = n - s + 2 = s + 1,$$

a contradiction again. We have now eliminated all cases; $\mu(G) < k$ is impossible. □

Lemma 5.10. *If G is a 4-regular graph on 7 vertices, then $\mu(G) \leq 3$.*

Proof. The complement of G is 2-regular, so it is either a seven-cycle, or the union of a triangle and a 4-cycle. In both cases it is easy to find a set S with $|S| = 4$, connected by at least 3 edges of the complement, so $i_G(S) \leq 3$. □

Lemma 5.11. $\mu(C(4, 7)) = 3$.

Proof. By Lemma 3.3, there exists (S, T) which realizes $\mu(C(4, 7))$ with $s = |S| \leq (n + 1)/2 = 4$ and $t = |T| = s - 1$. By Lemma 5.3, $s \geq k - 1 = 3$, and if $s = 3$, then $\mu(C(4, 7)) = 3$. It is easy to see that if $s = 4$, then $i_{C(4,7)}(S) \geq 3$. □

Theorem 5.12. *For all $n > k$ such that $\mathbf{k^n}$ is graphical, $\overline{\mu}(\mathbf{k^n}) = k$ unless $k = 2$ and n is odd, or $k = 4$ and $n = 5$ or $n = 7$. In these exceptional cases $\overline{\mu}(\mathbf{k^n}) = k - 1$.*

Proof. If n is even, then use Lemma 5.8. From now on we assume that n is odd. If G is a 2-regular graph on an odd number of vertices, then G must contain an odd cycle as a component. Any odd cycle has $\mu = 1$, for deleting one edge leaves an odd path, which contains no cycles and does not have a perfect matching. Since G is a union of cycles, $\mu(G) = 1$. Hence for n odd, $\overline{\mu}(\mathbf{2^n}) = 1$.

When $k = 4$ and $n = 5$, then K_5 is the unique realization of $\mathbf{k^n}$. If we delete the edges of any triangle from K_5, the vertices contained in that triangle become independent with two neighbors. This means that $\mu(K_5) \leq 3$. But we know from Lemma 5.4 that $\mu(K_5) \geq 3$, thus $\mu(K_5) = 3$ and $\overline{\mu}(\mathbf{4^5}) = 3$.

When $k = 4$ and $n = 7$, Lemmas 5.11 and 5.10 together show that $\overline{\mu}(\mathbf{7^4}) = 3$.

Finally, Lemma 5.4 and Theorem 5.9 show that $\mu(\mathbf{k^n}) = k$ in all other cases. □

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