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Discrete Decreasing Minimization, Part I, Base-polyhedra with Applications in Network Optimization

András Frank and Kazuo Murota

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Discrete Decreasing Minimization, Part I, Base-polyhedra with Applications in Network Optimization

András Frank⋆ and Kazuo Murota⋆⋆

Abstract

Motivated by resource allocation problems, Borradaile et al. (2017) investigated orientations of an undirected graph in which the sequence of in-degrees of the nodes, when arranged in a decreasing order, is lexicographically minimal in the sense that the largest in-degree is as small as possible, within this, the next largest in-degree is as small as possible, and so on. They called such an orientation egalitarian but we prefer to use the term *decreasingly minimal* (=dec-min) to avoid confusion with another egalitarian-felt orientation where the smallest in-degree is as large as possible, within this, the next smallest in-degree is as large as possible, and so on. Borradaile et al. proved that an orientation is dec-min if and only if there is no dipath from s to t with in-degrees $\varrho(t) \geq \varrho(s) + 2$. They conjectured that an analogous statement holds for strongly connected dec-min orientations, as well. We prove not only this conjecture but its extension to k-edge-connected orientations, as well, even if additional in-degree constraints are imposed on the nodes.

Resource allocation was also the motivation behind an earlier framework by Harvey et al. (2006) who introduced and investigated semi-matchings of bipartite graphs. As a genaralization of their results, we characterize degree-constrained subgraphs of a bipartite graph $G = (S, T; E)$ which have a given number of edges and their degree-sequence in S is decreasingly minimal. We also provide a solution to a discrete version of Megiddo's 'lexicographically' optimal (fractional) network flow problem (1974, 1977).

Furthermore, we exhibit a generalization of a result of Levin and Onn (2016) on 'shifted' matroid optimization, and describe a way of finding a basis of each of k matroids so that the sum of their incidence vectors is decreasingly minimal.

Our main goal is to integrate these cases into a single framework. Namely, we characterize dec-min elements of an M-convex set (which is nothing but the set

⋆MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány P. s. 1/c, Budapest, Hungary, H-1117. e-mail: `frank@cs.elte.hu` . The research was partially supported by the Hungarian Scientific Research Fund - OTKA, No K109240.

⋆⋆Department of Economics and Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan, e-mail: `murota@tmu.ac.jp` . The research was supported by CREST, JST, Grant Number JPMJCR14D2, Japan, and JSPS KAKENHI Grant Number 26280004.

of integral points of an integral base-polyhedron), and prove that the set of dec-min elements is a special M-convex set arising from a matroid base-polyhedron by translation. The topic of our investigations may be interpreted as a discrete counter-part of the work by Fujishige (1980) on the (unique) lexicographically optimal base of a base-polyhedron. On the dual side, as an extension of a result of Borradaile at al. (2018) on density decomposition of networks, we exhibit a canonical chain (and partition) associated with a base-polyhedron. We also show that dec-min elements of an M-convex set are exactly those which minimize the square-sum of components, and describe a new min-max formula for the minimum square-sum.

Our approach gives rise to a strongly polynomial algorithm for computing a dec-min element, as well as the canonical chain. The algorithm relies on a submodular function minimizer oracle in the general case, which can, however, be replaced by more efficient classic flow- and matroid algorithms in the relevant special cases.

In Part II, we offer a broader structural view from discrete convex analysis (DCA). In particular, min-max formulas will be derived as special cases of general DCA results. Furthermore, the relationship between continuous and discrete problems will also be clarified. Finally, Part III of the series describes a strongly polynomial algorithm for the discrete decreasing minimization problem over the intersection of two base-polyhedra, and also over submodular flows.

Keywords: network optimization, resource allocation, lexicographic minimization, submodular optimization, matroids, base-polyhedra, M-convex sets, polynomial algorithms

1 Introduction

1.1 Background problems

There are three independent sources of the topic we study.

1.1.1 Orientations of graphs

Let $G = (V, E)$ be an undirected graph. Orienting an edge $e = uv$ means the operation that replaces e by one of the two oppositely directed edges (sometimes called arcs) uv or vu. A directed graph arising from G by orienting all of its edges is called an orientation of G. A graph orientation problem consists of finding an orientation of G meeting some specified properties such as in-degree constraints (lower and upper bounds) and/or various connectivity prescriptions. One goal is to characterize undirected graphs for which the requested orientation exists, and a related other one is to design an algorithm for finding the orientation. The first orientation result is due to Robbins [47] who proved that exactly the 2-edge-connected graphs admit strongly connected (strong, for short) orientations. The in-degree constrained orientation problem is equivalent to an in-degree constrained subgraph problem for directed graphs.

An explicit characterization was formulated by Frank and Gyárfás [16], but this may be considered as an appropriate adaptation (or reformulation) of a characterization of degree-constraints for subgraphs of a digraph ([10], Theorem 11.1). The proper novelty of [16] was a solution to an amalgam of the two orientation problems above when one is interested in the existence of a strong orientation which, in addition, complies with upper and lower bounds on the in-degrees of nodes.

The literature is quite rich in orientation results, for a relatively wide overview, see the book [15]. There are, however, other type of requirements for an orientation of G where, rather than having prescribed upper and lower bounds for the in-degrees of nodes (or, sometimes, beside these bounds), one is interested in the global distribution of the in-degrees of nodes. That is, the goal is to find orientations (with possible connectivity expectations) whose in-degree vector (on the node-set) is felt intuitively evenly distributed: 'fair', 'equitable', 'egalitarian', 'uniform'. For example, how can one determine the minimum value β_1 of the largest in-degree of a (k-edge-connected) orientation? Even further, after determining β_1, it might be interesting to minimize the number of nodes with in-degree β_1 among orientations of G with largest in-degree β_1. Or, a more global equitability feeling is captured if we minimize the sum of squares of the in-degrees. For example, the in-degree sequence $(1, 2, 6, 8)$ (with square-sum 105) is felt less 'fair' or 'egalitarian' or 'evenly distributed' than $(4, 4, 4, 5)$ (with square-sum 73).

A formally different definition was recently suggested and investigated by Borradaile et al. [4] who called an orientation of $G = (V, E)$ **egalitarian** if the highest in-degree of the nodes is as small as possible, and within this, the second highest (but not necessarily distinct) in-degree is as small as possible, and within this, the third highest in-degree is as small as possible, and so on. In other words, if we rearrange the in-degrees of nodes in a decreasing order, then the sequence is lexicographically minimal. In order to emphasize that the in-degrees are considered in a decreasing order, we prefer the use of the more expressive term **decreasingly minimal** (**dec-min**, for short) for such an orientation, rather than egalitarian.

This change of terminology is reasonable since one may also consider the mirror problem of finding an **increasingly maximal** (or **inc-max**, for short) orientation that is an orientation of G in which the smallest in-degree is as large as possible, within this, the second smallest in-degree is as large as possible, and so on. Intuitively, such an orientation may equally be felt 'egalitarian' in the informal meaning of the word.

Borradaile et al. [4], however, proved that a graph is decreasingly minimal (egalitarian in their original term) if and only if there is no 'small' improvement, where a small improvement means the reorientation of a dipath from some node s to another node t with in-degrees $\varrho(t) \geq \varrho(s) + 2$. This theorem immediately implies that an orientation is decreasingly minimal if and only if it is increasingly maximal, and therefore we could retain the original terminology 'egalitarian orientation' used in [4].

However, when orientations are considered with specific requirements such as strong (or, more generally, k-edge-) connectivity and/or in-degree bounds on the nodes, the possible equivalence of decreasingly minimal and increasingly maximal orientations had not yet been investigated. Actually, Borradaile et al. conjectured that a strong orientation of a graph is decreasingly minimal (among strong orientations) if and only if there is no small improvement preserving strong connectivity, and this, if true, would imply immediately that decreasing minimality and increasing maximality do coincide for strong orientations, as well.

We shall prove this conjecture in its extended form concerning k-edge-connected and in-degree constrained orientations. This result implies immediately that the notions of decreasing minimality and increasing maximality coincide for k-edge-connected and in-degree constrained orientations, as well. We hasten to emphasize that this coincidence is not at all automatic or inevitable. For example, although Robbins' theorem on strong orientability of undirected graphs nicely extends to mixed graphs, as was pointed out by Boesch and Tindell [2], it is not true anymore that a decreasingly minimal strong orientation of a mixed graph is always increasingly maximal. (For a counterexample, see Section 9.) This discrepancy may be explained by the fact that the in-degree vectors of strong orientations of a graph form an M-convex set while the set of in-degree vectors of strong orientations of a mixed graph is not M-convex anymore: it is the intersection of two M-convex sets. (The definition of an M-convex set was mentioned in the Abstract and will be introduced formally in Section 2.)

Interestingly, it will turn out that for k-edge-connected and in-degree constrained orientations of undirected graphs not only decreasing minimality and increasing maximality coincide but such orientations are exactly those minimizing the square-sum of the in-degrees of the nodes.

1.1.2 A resource allocation problem and network flows

Another source of our investigations is due to Harvey at al. [29] who solved the problem of minimizing $\sum[d_F(s)(d_F(s)+1) : s \in S]$ over the semi-matchings F of a simple bipartite graph $G = (S, T; E)$. Here a semi-matching is a subset F of edges for which $d_F(t) = 1$ holds for every node $t \in T$. This was extended by Bokal et al. [3] to quasi-matchings. It turns out that these problems are strongly related to minimization of a separable convex function over (integral elements of) a base-polyhedron which has been investigated in the literature under the name of 'resource allocation problems under submodular constraints" ([9], [31] [30] [36] [33] [35]). Ghodsi et al. [26] considered the problem of finding a semi-matching F of $G = (S, T; E)$ whose degree-vector restricted to S is increasingly maximal. This problem, under the name 'constrained max-min fairness' originated from a framework to model a fair sharing problem for datacenter jobs. Here T corresponds to the set of available computers while S to the set of users. An edge st belongs to E if user s can run her program on computer t. Ghodsi et al. also consider the fractional version when, instead of finding a semi-matching of G, one is to find a real vector $x : E \to \mathbf{R}_+$ so that $d_x(t) = 1$ for every $t \in T$ and the vector $(d_x(s) : s \in S)$ is increasingly maximal. (Here $d_x(v) := \sum[x(uv) : uv \in E]$). When x is requested to be $(0, 1)$-valued, we are back at the subgraph version.

It should be emphasized that, unlike the well-known situation with ordinary bipartite matchings or b-matchings, in this problem the optima for the subgraph version and for the fractional version may be different. For example, if T consists of a single node t, $S = \{s_1, s_2\}$, and $E = \{s_1t, s_2t\}$, then the original subgraph problem has two inc-max solutions: $F_1 = \{s_1t\}$ and $F_2 = \{s_2t\}$, where the degree-vector in S is $(1, 0)$ in the first case, and $(0, 1)$ in the second case. (Note that both F_1 and F_2 are also dec-min in S.) On the other hand, in the fractional version there is a unique inc-max fractional solution: $x(s_1t) = 1/2$ and $x(s_2t) = 1/2$ (and this happens to be the unique dec-min solution). Here the 'fractional degree-vector' of x in S is $(1/2, 1/2)$. Obviously, the fractional vector $(1/2, 1/2)$ is decreasingly smaller (and increasingly larger) than $(1, 0)$.

We shall solve the following generalization of the subgraph problem. Suppose that we are also given a positive integer γ, a lower bound function $f : V \to \mathbf{Z}_+$ and an upper bound function $g : V \to \mathbf{Z}_+$ with $f \leq g$ where $V := S \cup T$. The problem is to find a subgraph $H = (S, T; F)$ of G with $|F| = \gamma$ for which $f(v) \leq d_F(v) \leq g(v)$ for every node $v \in V$ and the degree-vector of H in S (**!**) is increasingly maximal (or decreasingly minimal). It will turn out that in this case a solution is dec-min if and only if it is inc-max. We emphasize that in this problem the roles of S and T are not symmetric since we require that the restriction on S of the degree-vector of the degree-constrained subgraph H in S should be decreasingly minimal. The symmetric version when the degree-vector requested to be dec-min on the whole node-set $V = S \cup T$ is definitely more difficult, and will be solved in [21]. An explanation for this difference of the two seemingly quite similar problems is that the first one may be viewed as a problem concerning a single base-polyhedron while the second one may be viewed as a problem on the intersection of two base-polyhedra.

There is a much earlier, strongly related problem concerning network flows, due to Megiddo [39], [40]. We are given a digraph $D = (V, A)$ with a source-set $S \subset V$ and a single sink-node $t \in V - S$. (The more general case, when the sink-set $T \subseteq V - S$ may consist of more than one node can easily be reduced to the special case when $T = \{t\}$). Let $g : A \to \mathbf{R}_+$ be a capacity function. By an St-flow, or just a flow, we mean a function $x : A \to \mathbf{R}_+$ for which the net out-flow $\delta_x(v) - \varrho_x(v) = 0$ if $v \in V - (S + t)$ and $\delta_x(v) - \varrho_x(v) \geq 0$ if $v \in S$. (Here $\varrho_x(v) := \sum[x(uv) : uv \in A]$ and $\delta_x(v) := \sum[x(vu) : vu \in A]$.) The flow is feasible if $x \leq g$. The flow **amount** of x is the net inflow $\varrho_x(t) - \delta_x(t)$ of t. Megiddo solved the problem of finding a feasible flow of maximum flow amount which is, with his term, 'source-optimal' at S. Source-optimality is the same as requiring that the net out-flow vector on S is increasingly maximal. Note that there is a formally different but technically equivelent version of Megiddo's problem when $S = \{s\}$, $T = \{t\}$ (with no arcs entering s) and we are interested in finding a feasible flow x of maximum amount for which the restriction of x onto the set of arcs leaving s is increasingly maximal. It must be emphasized that the flow in Megiddo's problem is not requested to be integer-valued.

The integrality property is a fundamental feature of ordinary network flows. It states that in case of an integer-valued capacity function g there always exists a maximum flow which is integer-valued. In this light, it is quite surprising that the integer-valued (or discrete) version of Megiddo's inc-max problem (source-optimal with his term), when the capacity function g is integer-valued and the max flow is required to be integer-valued, has not been investigated in the literature, and we consider the present work as the first such attempt.

We solve this discrete version of Megiddo's problem in a more general form concerning base-polyhedra (or M-convex sets) and this approach gives rise to a strongly polynomial algorithm.

1.1.3 Matroid bases

A third source of discrete decreasing minimality problems is due to Levin and Onn [37] who used the term 'shifted optimization'. They considered the following matroid optimization problem. For a specified integer k, find k bases $Z_1, Z_2, \dots, Z_k$ of a matroid M on S in such a way that the vector $\sum_i \chi_{Z_i}$ be, in our terms, decreasingly minimal, where χ_Z is the incidence (or characteristic) vector of a subset Z. They apply the following natural approach to reduce the problem to classic results of matroid theory. Replace first each element s of S by k copies to be parallel in the resulting matroid M' on the new ground-set $S' = S_1 \cup S_2 \cup \cdots \cup S_k$ where $S_1, \dots, S_k$ are the k copies of S. Assign then a 'rapidly increasing' cost function to the copies. (The paper [37] explicitly describes what rapidly increasing means). Then a minimum cost basis of the matroid M_0 obtained by multiplying M' k-times will be a solution to the problem. (By definition, a basis of M' is the union of k disjoint bases of M').

Our goal is to provide a solution to a natural generalization of this problem when k matroids $M_1, \dots, M_k$ are given on the common ground-set S and we want to select a basis Z_i of each matroid M_i in such a way that $\sum_i \chi_{Z_i}$ should be decreasingly minimal. The approach of Levin and Onn does not seem to work in this more general setting. We can even prescribe upper and lower bounds on the elements s of S to constrain the number of the bases containing s.

1.2 Main goals

Each of the three problems above may be viewed as a special case of a single discrete optimization problem: characterize decreasingly minimal elements of an M-convex set (or, in other words, dec-min integral elements of a base-polyhedron). By one of its equivalent definitions, an M-convex set is nothing but the set of integral elements of an integral base-polyhedron. The notion was introduced and investigated by Murota [42], [43].

We characterize dec-min elements of an M-convex set as those admitting no local improvement, and prove that the set of dec-min elements is itself an M-convex set arising by translating a matroid base-polyhedron with an integral vector. This result implies that decreasing minimality and increasing maximality coincide for M-convex sets. We shall also show that an element of an M-convex set is dec-min precisely if it is a square-sum minimizer. Using the characterization of dec-min elements, we shall derive a novel min-max theorem for the minimum square-sum of elements of an integral member of a base-polyhedron. Furthermore, we describe a strongly polynomial algorithm for finding a dec-min element. The algorithm relies on a subroutine to minimize a submodular function but in the special cases mentioned above this general routine can be replaced by known strongly polynomial network flow and matroid algorithms.

The topic of our investigations may be interpreted as a discrete counter-part of the work by Fujishige [23] from 1980 on the lexicographically optimal base of a base-polyhedron B, where lexicographically optimal is the same as decreasingly minimal. He proved that there is a unique lexicographically optimal member x_0 of B, and x_0 is the unique minimum norm (that is, the minimum square-sum) element of B. This uniqueness result reflects a characteristic difference between the behaviour of the fractional and the discrete versions of decreasing minimization since in the latter case the set of dec-min elements (of an M-convex set) is typically not a singleton, and it actually has, as indicated above, a matroidal structure.

Fujishige also introduced the concept of principal partitions concerning the dual structure of the minimum norm point of a base-polyhedron. Actually, he introduced a special chain of the subsets of ground-set S and his principal partition arises by taking the difference sets of this chain. We will prove that there is an analogous concept in the discrete case, as well. As an extension of the above-mentioned elegant result of Borradaile et al. [5] concerning graphs, we show that there is a canonical chain describing the structure of dec-min elements of an M-convex set. We will point out in Part II [20], that Fujishige's principal partition is a refinement of our canonical partition. Our approach gives rise to a combinatorial algorithm to compute the canonical chain.

1.2.1 Outlook

The present work is the first member of a three-partite series. In Part II [20], we offer a broader structural view from discrete convex analysis (DCA). In particular, min-max formulas will be derived as special cases of general DCA results. Furthermore, the relationship between continuous and discrete problems will also be clarified. Finally, Part III [21] describes a strongly polynomial algorithm for the discrete decreasing minimization problem over the intersection of two M-convex sets, and also over the integral elements of an integral submodular flow. One of the motivations behind these investigation was the observatin mentioned earlier that, for strong orientations of mixed graphs, dec-min orientations and inc-max orientations do not coincide. The reason behind this phenomenon is that the set of in-degree vectors of strong orientations of a mixed graph is not an M-convex set anymore. It is, in fact, the intersection of two M-convex sets, and therefore the results of Part III [21] can be used to solve this special case, ase well.

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continuous availability to answer our questions concerning the paper [4] and the work by Borradaile, Migler, and Wilfong [5], which paper was also a prime driving force in our investigations.

1.3 Notation

Throughout the paper, S denotes a finite non-empty ground-set. For elements $s, t \in S$, we say that $X \subset S$ is an $s\overline{t}$**-set** if $s \in X \subseteq S - t$. For a vector $m \in \mathbf{R}^S$ (or function $m : S \to \mathbf{R}$), the resctriction of m to $X \subseteq S$ is denoted by $m|X$. We also use the notation $\widetilde{m}(X) = \sum[m(v) : v \in X]$. With a small abuse of notation, we do not distinguish between a one-element set $\{s\}$ called a singleton and its only element s. When we work with a chain $\mathcal{C}$ of non-empty sets $C_1 \subset C_2 \subset \cdots \subset C_q$, we sometimes use C_0 to denote the empty set without assuming that C_0 is a member of $\mathcal{C}$.

Two subsets X and Y of S are **intersecting** if $X \cap Y \neq \emptyset$ and **properly intersecting** if none of $X - Y, Y - X$ and $X \cap Y$ is empty. If, in addition, $S - (X \cup Y) \neq \emptyset$, the two sets are **crossing**.

We assume that the occurring graphs or digraphs have no loops but parallel edges are allowed. For a digraph $D = (V, A)$, the **in-degree** of a node v is the number of arcs of D with head v. The in-degree $\varrho_D(Z) = \varrho(Z)$ of a subset $Z \subseteq V$ denotes the number of edges (= arcs) entering Z where an arc uv is said to enter Z if its head v is in Z while its tail u is in $V - Z$. The **out-degree** $\delta_D(Z) = \delta(Z)$ is the number of arcs leaving Z, that is $\delta(Z) = \varrho(V - Z)$. The number of edges of a directed or undirected graph H induced by $Z \subseteq V$ is denoted by $i(Z) = i_H(Z)$. In an undirected graph $G = (V, E)$, the **degree** $d(Z) = d_G(Z)$ of a subset $Z \subseteq V$ denotes the number of edges connecting Z and $V - Z$ while $e(Z) = e_G$ denotes the number of edges with one or two end-nodes in Z. Clearly, $e(Z) = d(Z) + i(Z)$.

The characteristic (or incidence) vector of Z is denoted by χ_Z, that is, $\chi_Z(v) = 1$ if $v \in Z$ and $\chi_Z(v) = 0$ otherwise.

For a polyhedron B, $\ddot{B}$ denotes the set of integral members (elements, vectors, points) of B (pronounce: dotted B).

For a set-function h, we allow to have value $+\infty$ or $-\infty$. Unless otherwise stated, $h(\emptyset) = 0$ is assume throughout. Where $h(S)$ is finite, the **complementary function** $\overline{h}$ is defined by $\overline{h}(X) = h(S) - h(S - X)$. Observe that the complementary function of $\overline{h}$ is h itself.

Let b be a set-function for which $b(X) = +\infty$ is allowed but $b(X) = -\infty$ not. The submodular inequality for subsets $X, Y \subseteq S$ is defined by

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$

We say that b is submodular if the submodular inequality holds for every pair of subsets $X, Y \subseteq S$ with finite b-values. When the submodular inequality is required only for intersecting (crossing) pairs of subsets, we say that b is **intersecting** (**crossing**) **submodular**. When we say that a function b is submodular, this formally always means that b is fully submodular, but in avoid any misunderstanding sometimes we emphasize this by writing (fully) submodular, in particular in an enviroment where intersecting or crossing submodular functions also show up.

A set-function p is (fully, intersecting, crossing) supermodular if $-p$ is (fully, intersecting, crossing) submodular. When a submodular function b and a supermodular function p meet the **cross-inequality**

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X)$$

for every pair $X, Y \subseteq S$, we say that (p, b) is a **paramodular pair** (a **strong pair**, for short.) If b is intersecting submodular, p is intersecting supermodular, and the cross-inequality holds for intersecting pairs of sets, we speak of an **intersecting paramodular pair** (or for short, a **weak pair**.)

For functions $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{+\infty\}$ with $f \leq g$, the polyhedron $T(f, g) = \{x \in \mathbf{R}^S : f \leq x \leq g\}$ is called a **box**. If $g(s) \leq f(s) + 1$ holds for every $s \in S$, we speak of a **small box**. For example, the $(0, 1)$-box is small.

2 Base-polyhedra

2.1 Basic notions, properties, and notation

Let S be a finite non-empty ground-set and let b be a (fully) submodular integer-valued set-function on S for which $b(\emptyset) = 0$ and $b(S)$ is finite. A (possibly unbounded) **base-polyhedron** $B = B(b)$ in $\mathbf{R}^S$ is defined by

$$B = \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S),\ \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subset S\}.$$

A special base-polyhedron is the one of matroids. Given a matroid M, Edmonds proved that the polytope (that is, the convex hull) of the incidence (or characteristic) vectors of the bases of M is the base-polyhedron $B(r)$ defined by the rank function r of M, that is, $B(r) = \{x \in \mathbf{R}^S :\ \widetilde{x}(S) = r(S)$ and $\widetilde{x}(Z) \leq r(Z)$ for every subset $Z \subset S\}$. (Note that there is no need to require explicitly the non-negativity of x, since this follows from the monotonicity of the rank function r: $x(s) = \widetilde{x}(S) - \widetilde{x}(S - s) \geq r(S) - r(S - s) \geq 0$). It can be proved that a kind of converse also holds, namely, every (integral) base-polyhedron in the unit $(0, 1)$-cube is a matroid base-polyhedron.

For a weak pair (p, b), the polyhedron $Q = Q(p, b) := \{x : p(Z) \leq \widetilde{x}(Z) \leq b(Z)$ for every $Z \subseteq S\}$ is called a generalized polymatroid (g-polymatroid, for short). By convention, the empty set is also considered a g-polymatroid. In the special case, when $p \equiv 0$ and b is monotone non-decreasing, we are back at the concept of polymatroids, introduced by Edmonds [6]. G-polymatroids were introduced by Frank [13] who proved that $Q(p, b)$ is a non-empty integral polyhedron, a g-polymatroid uniquely determines its defining strong pair, and the intersection of two integral g-polymatroids $Q(p_1, b_1)$ and $Q(p_2, b_2)$ is an integral polyhedron which is non-empty if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.

The book of Frank [15] includes an overview of basic properties and constructions of base-polyhedra and g-polymatroids. For example, the following operations on a g-polymatroid result in a g-polymatroid: projection along axes, translation (or shifting) by a vector, negation (that is, reflection through the origin), intersection with a box

$T(f,g) := \{x : f \leq x \leq g\}$, intersection with a plank $\{x : \alpha \leq \widetilde{x}(S) \leq \beta\}$, and taking a face.

A base-polyhedron is a special g-polymatroid (where $p(S) = b(S)$) and every g-polymatroid arises from a base-polyhedron by projecting it along a single axis. Each of the operations of translation, intersection with a box, negation, taking a face, when applied to a base-polyhedron, results in a base-polyhedron. The intersection of a g-polymatroid with a hyperplane $\{x : \widetilde{x}(S) = \gamma\}$ is a base-polyhedron. Each of the operations above, when applied to an integral g-polymatroid (base-polyhedron) results in an integral g-polymatroid (base-polyhedron). The (Minkowski) sum of g-polymatroids (base-polyhedra) is a g-polymatroid (base-polyhedron).

We call the translation of a matroid base-polyhedron a **translated matroid base-polyhedron**. It follows that the intersection of a base-polyhedron with a small box is a translated matroid base-polyhedron.

A base-polyhedron $B(b)$ is never empty, and $B(b)$ is known to be an integral polyhedron. (A rational polyhedron is **integral** if each of its faces contains an integral element. In particular, a pointed rational polyhedron is integral if all of its vertices are integral.) By convention, the empty set is also considered a base-polyhedron. Note that a real-valued submodular function b also defines a base-polyhedron $B(b)$ but in the present work we are interested only in integer-valued submodular functions and integral base-polyhedra.

We call the set $\overset{....}{B}$ of integral elements of an integral base-polyhedron B an **M-convex set**. Originally, this basic notion of Discrete Convex Analysis (DCA), introduced by Murota [42] (see, also the book [43]), was defined as a set of integral points in $\mathbf{R}^S$ satisfying certain exchange axioms, and it was proved by Murota that the two properties are equivalent ([43], Theorem 4.15). The set of integral elements of a translated matroid base-polyhedron will be called a **matroidal M-convex set**.

Since in the present work the central notion is that of base-polyhedra, we define M-convex sets via base-polyhedra. The set of integral points of an integral g-polymatroid is called by Murota [43] an M$^\natural$-convex set (pronounce M-natural convex). Since a base-polyhedron is a special g-polymatroid, an M-convex set is a special M$^\natural$-convex set. Note that the original definition of M$^\natural$-convex sets is different (and it is a theorem that the two definitions are equivalent).

A non-empty base-polyhedron B can also be defined by a supermodular function p for which $p(\emptyset) = 0$ and $p(S)$ is finite as follows: $B = B'(p) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = p(S),\ \widetilde{x}(Z) \geq p(Z)$ for every $Z \subset S\}$.

For a set $Z \subset S$, $p|Z\ \ (= p - (S - Z))$ denotes the restriction of p to Z (that is, $p|Z$ is obtained from p by deleting $S - Z$), while $p' = p/Z\ \ (= p \div (S - Z))$ is the set-function on $S - Z$ obtained from p by contracting Z, which is defined for $X \subseteq S - Z$ by $p'(X) = p(X \cup Z) - p(Z)$. Note that p/Z and $\overline{p}|(S - Z)$ are complementary set-functions. It is also known for disjoint subsets Z_1 and Z_2 of S that

$$(p/Z_1)/Z_2 = p/(Z_1 \cup Z_2), \tag{1}$$

that is, contracting first Z_1 and then Z_2 is the same as contracting $Z_1 \cup Z_2$. (Indeed, this follows from $p_1'(X \cup Z_2) - p_1'(Z_2) = p(X \cup Z_1 \cup Z_2) - p(Z_1) - [p(Z_1 \cup Z_2) -$

$p(Z_1)] = p(X \cup Z_1 \cup Z_2) - p(Z_1 \cup Z_2)$.) Furthermore, $(p/Z_1)|Z_2 = (p|(Z_1 \cup Z_2))/Z_1$.

It is known that B uniquely determines both p and b, namely, $b(Z) = \max\{\widetilde{x}(Z) : x \in B\}$ and $p(Z) = \min\{\widetilde{x}(Z) : x \in B\}$. The functions p and b are complementary functions, that is, $b(X) = p(S) - p(S - X)$ or $p(X) = b(S) - b(S - X)$ (where $b(S) = p(S)$).

Let $\{S_1, \dots, S_q\}$ be a partition of S and let p_i be a supermodular function on S_i. Let p denote the supermodular function on S defined by $p(X) := \sum[p_i(S_i \cap X) : i = 1, \dots, q]$ for $X \subseteq S$. The base-polyhedron $B'(p)$ is called the **direct sum** of the q base-polyhedra $B'(p_i)$. Obviously, a vector $x \in \mathbf{R}^S$ is in $B'(p)$ if and only if each x_i is in $B'(p_i)$ $(i = 1, \dots, q)$ where x_i denotes the restriction $x|S_i$ of x to S_i.

Let Z be a subset of S for which $p(Z)$ is finite. The **restriction** of a base-polyhedron $B'(p)$ to Z is the base-polyhedron $B'(p|Z)$.

It is known that a face F of a non-empty base-polyhedron B is also a base-polyhedron. The (special) face of $B'(p)$ defined by the single equality $\widetilde{x}(Z) = p(Z)$ is the direct sum of the base polyhedra $B'(p|Z)$ and $B'(p/Z)$. More generally, any face F of B can be described with the help of a chain $(\emptyset \subset)$ $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ of subsets by $F := \{z : z \in B,\ p(C_i) = \widetilde{z}(C_i) \text{ for } i = 1, \dots, \ell\}$. (In particular, when $\ell = 1$, the face F is B itself.) Let $S_1 := C_1$ and $S_i := C_i - C_{i-1}$ for $i = 2, \dots, \ell$. Then F is the direct sum of the base-polyhedra $B'(p_i)$ where p_i is a supermodular function on S_i defined by $p_i(X) := p(X \cup C_{i-1}) - p(C_{i-1})$ for $X \subseteq S_i$. In other words, p_i is a set-function on S_i obtained from p by deleting C_{i-1} and contracting $S - C_i$. The unique supermodular function p_F defining the face F is given by $\sum[p_i(S_i \cap X) : i = 1, \dots, \ell]$. The polymatroid greedy algorithm of Edmonds [6] along with the proof of its correctness, when adapted to base-polyhedra, shows that F is the set of elements x of B minimizing cx whenever $c : S \to \mathbf{R}$ is a linear cost function such that $c(s) = c(t)$ if $s, t \in S_i$ for some i and $c(s) > c(t)$ if $s \in S_i$ and $t \in S_j$ for some subscripts $i < j$.

The intersection of an integral base-polyhedron $B = B'(p)$ $(= B(\overline{p}))$ and an integral box $T(f, g)$ is an integral base-polyhedron. The intersection is non-empty if and only if

$$p \leq \widetilde{g} \quad \text{and} \quad \widetilde{f} \leq \overline{p} \tag{2}$$

Note that g occurs only in the first inequality and f occurs only in the second inequality, from which it follows that if B has an element $x_1 \geq f$ and B has an element $x_2 \leq g$, then B has an element x with $f \leq x \leq g$. This phenomenon is often called the linking principle or linking property.

For an element m of a base-polyhedron $B = B(b)$ defined by a (fully) submodular function b, we call a subset $X \subseteq S$ m**-tight** (with respect to b) if $\widetilde{m}(X) = b(X)$. Clearly, the empty set and S are m-tight, and m-tight sets are closed under taking union and intersection. Therefore, for each subset $Z \subseteq S$, there is a unique smallest m-tight set $T_m(Z; b)$ including Z. When $Z = \{s\}$ is a singleton, we simply write $T_m(s; b)$ to denote the smallest m-tight set containing s. When the submodular function b in this notation is unambiguous from the context, we abbreviate $T_m(Z; b)$ by $T_m(Z)$.

Analogously, when $B = B'(p)$ is given by a supermodular function p, we call $X \subseteq S$ m**-tight** (with respect to p) if $\widetilde{m}(X) = p(X)$. In this case, we also use the analogous notation $T_m(Z) = T_m(Z;p)$ and $T_m(s) = T_m(s;p)$. Observe that for complementary functions b and p, X is m-tight with respect to b precisely if $S - X$ is m-tight with respect to p.

In applications it is important that weaker set-functions may also define base-polyhedra. For example, if p is an integer-valued crossing supermodular function, then $B'(p)$ is still an integral base-polyhedron, which may, however, be empty. This result was proved independently in [12] and in [24]. To prove theorems on base-polyhedra, it is much easier to work with base-polyhedra defined by fully sub- or supermodular functions. On the other hand, in applications, base-polyhedra are often defined with a crossing sub- or supermodular (or even weaker) function. For example, the in-degree vectors of the k-edge-connected orientations of a $2k$-edge-connected graph are the integral elements of a base-polyhedron defined by a crossing supermodular function, as was pointed out in an even more general setting [11]. It is exactly the combination of these double features that makes it possible to prove a conjecture of Borradaile et al. [4] even in extended form. Details will be discussed in Section 9.

3 Decreasingly minimal elements of base-polyhedra

3.1 Decreasing minimality

For a vector x, let $x{\downarrow}$ or $[x]{\downarrow}$ denote the vector obtained from x by rearranging its components in a decreasing order. For example, $x{\downarrow} = (5,5,4,2,1)$ when $x = (2,5,5,1,4)$. We call two vectors x and y (of same dimension) **value-equivalent** if $x{\downarrow} = y{\downarrow}$. For example, $(2,5,5,1,4)$ and $(1,4,5,2,5)$ are value-equivalent while the vectors $(3,5,5,3,4)$ and $(3,4,5,4,4)$ are not.

A vector x is **decreasingly smaller** than vector y, in notation $x <_{\rm dec} y$ if $x{\downarrow}$ is lexicographically smaller than $y{\downarrow}$ in the sense that they are not value-equivalent and $x{\downarrow}(j) < y{\downarrow}(j)$ for the smallest subscript j for which $x{\downarrow}(j)$ and $y{\downarrow}(j)$ differ. For example, $x = (2,5,5,1,4)$ is decreasingly smaller than $y = (1,5,5,5,1)$ since $x{\downarrow} = (5,5,4,2,1)$ is lexicographically smaller than $y{\downarrow} = (5,5,5,1,1)$.

A vector x is **decreasingly smaller than or equal to** vector y, in notation $x \leq_{\rm dec} y$, if they are either value-equivalent or $x <_{\rm dec} y$. For a set Q of vectors, $x \in Q$ is a **globally decreasingly minimal** or simply **decreasingly minimal** (**dec-min**, for short) if $x \leq_{\rm dec} y$ for every $y \in Q$. Note that the dec-min elements of Q are value-equivalent. Therefore an element m of Q is dec-min if its largest component is as small as possible, within this, its second largest component (with the same or smaller value than the largest one) is as small as possible, and so on. An element x of Q is said to be a **max-minimized** element (a **max-minimizer**, for short) if its largest component is as small as possible. A max-minimizer element x is **pre-decreasingly minimal** (**pre-dec-min**, for short) in Q if the number of its largest components is as small as possible. Obviously, a dec-min element is pre-dec-min, and a pre-dec-min element is max-minimized.

In an analogous way, for a vector x, we let $x{\uparrow}$ or $[x]{\uparrow}$ denote the vector obtained from x by rearranging its components in an increasing order. A vector y is **increasingly**

larger than vector x, in notation $y >_{\rm inc} x$, if they are not value-equivalent and $y{\uparrow}(j) > x{\uparrow}(j)$ holds for the smallest subscript j for which $y{\uparrow}(j)$ and $x{\uparrow}(j)$ differ. We write $y \geq_{\rm inc} x$ if either $y >_{\rm inc} x$ or x and y are value-equivalent. Furthermore, we call an element m of Q **(globally) increasingly maximal** (**inc-max** for short) if its smallest component is as large as possible over the elements of Q, within this its second smallest component is as large as possible, and so on. Similarly, we can use the analogous terms **min-maximized** and **pre-increasingly maximal** (**pre-inc-max**).

It should be emphasized that a dec-min element of a base-polyhedron B is not necessarily integer-valued. For example, if $B = \{(x_1, x_2) : x_1 + x_2 = 1\}$, then $x^* = (1/2, 1/2)$ is a dec-min element of B. In this case, the dec-min members of $\overset{....}{B}$ are $(0,1)$ and $(1,0)$.

Therefore, finding a dec-min element of B and finding a dec-min element of $\overset{....}{B}$ (the set of integral points of B) are two distinct problems, and we shall concentrate only on the second, discrete problem. In what follows, the slightly sloppy term integral dec-min element of B will always mean a dec-min element of $\overset{....}{B}$. (The term is sloppy in the sense that an integral dec-min element of B is not necessarily a dec-min element of B).

We call an integral vector $x \in \mathbf{Z}^S$ **uniform** if all of its components are the same integer ℓ, and **near-uniform** if its largest and smallest components differ by at most 1, that is, if $x(s) \in \{\ell, \ell+1\}$ for some integer ℓ for every $s \in S$. Note that if Q consists of integral vectors and the component-sum is the same for each member of Q, then any near-uniform integral member of Q is obviously both decreasingly minimal and increasingly maximal integral vector.

3.2 Characterizing dec-min elements

Let $B = B(b) = B'(p)$ be a base-polyhedron defined by an integer-valued submodular function b or a supermodular function p (where b and p are complementary set-functions). Let m be an integral member of B, that is, $m \in \overset{....}{B}$. A set $X \subseteq S$ is m-tight with respect to b precisely if its complement $S - X$ is m-tight with respect to p. Recall that $T_m(s; b)$ denoted the unique smallest m-tight set (with respect to b) containing s. In other words, $T_m(s; b)$ is the intersection of all m-tight sets containing s. The easy equivalences in the next claim will be used throughout.

Claim 3.1. *Let s and t be elements of S and let $m' := m + \chi_s - \chi_t$. The following properties are pairwise equivalent.*

(A) $m' \in \overset{....}{B}$.

(P1) *There is no $t\overline{s}$-set which is m-tight with respect to p.*

(P2) $s \in T_m(t; p)$.

(B1) *There is no $s\overline{t}$-set which is m-tight with respect to b.*

(B2) $t \in T_m(s; b)$. •

A **1-tightening step** for $m \in \overset{....}{B}$ is an operation that replaces m by $m' := m + \chi_s - \chi_t$ where s and t are elements of S for which $m(t) \geq m(s) + 2$ and m' belongs to $\overset{....}{B}$. (See Claim 3.1 for properties equivalent to $m' \in \overset{....}{B}$.) Note that m' is both decreasingly smaller and increasingly larger than m.

Since the mean of the components of m does not change at a 1-tightening step while the square-sum of the components of m strictly drops, consecutive 1-tightening steps may occur only a finite number of times (even if B is unbounded).

A member m of $\overset{....}{B}$ is **locally decreasingly minimal** in $\overset{....}{B}$ if there is no 1-tightening step for m, that is, there are no two elements s and t of S with $m(t) \geq m(s)+2$ so that $m' := m + \chi_s - \chi_t \in \overset{....}{B}$. The term refers to the easy observation that if there are such elements s and t, then m' is an element in $\overset{....}{B}$ which is decreasingly smaller than m. Note that in this case m' is also increasingly larger than m. Analogously, m is **locally increasingly maximal** if there are no two elements s and t with $m(t) \geq m(s) + 2$ and $m' := m + \chi_s - \chi_t \in B$.

The equivalence of the properties in the next claim is immediate from the definitions.

Claim 3.2. *For an integral element m of the integral base-polyhedron $B = B(b) = B'(p)$, the following conditions are pairwise equivalent.*

(A1) *There is no 1-tightening step for m.*

(A2) *m is locally decreasingly minimal.*

(A3) *m is locally increasingly maximal.*

(P1) *$m(s) \geq m(t) - 1$ holds whenever $t \in S$ and $s \in T_m(t; p)$.*

(P2) *Whenever $m(t) \geq m(s) + 2$, there is a $t\overline{s}$-set X which is m-tight with respect to p.*

(B1) *$m(s) \geq m(t) - 1$ holds whenever $s \in S$ and $t \in T_m(s; b)$.*

(B2) *Whenever $m(t) \geq m(s) + 2$, there is an $s\overline{t}$-set Y which is m-tight with respect to b.* •

For a given vector m in $\mathbf{R}^S$, we call a non-empty set $X \subseteq S$ an m**-top set** (or a top-set with respect to m) if $m(u) \geq m(v)$ holds whenever $u \in X$ and $v \in S - X$. Both the empty set and the ground-set S are m-top sets, and m-top sets are closed under taking union and intersection. If $m(u) > m(v)$ holds whenever $u \in X$ and $v \in S - X$, we speak of a **strict** m**-top set**. Note that the number of strict non-empty m-top sets is at most n for every $m \in \overset{....}{B}$ while $m \equiv 0$ exemplifies that even all of the non-empty subsets of S can be m-top sets.

THEOREM 3.3. *Let b be an integer-valued submodular function and let $p := \overline{b}$ be its complementary (supermodular) function. For an integral element m of the integral base-polyhedron $B = B(b) = B'(p)$, the following four conditions are pairwise equivalent.*

(A) *There is no 1-tightening step for m (or anyone of the six other equivalent properties holds in Claim 3.2).*

(B) *There is a chain $\mathcal{C}$ of m-top sets ($\emptyset \subset$) $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ which are m-tight with respect to p (or equivalently, whose complements are m-tight with respect to b) such that the restriction $m_i = m|S_i$ of m to S_i is near-uniform for each member S_i of the S-partition $\{S_1, \dots, S_\ell\}$ where $S_1 = C_1$ and $S_i := C_i - C_{i-1}$ $(i = 2, \dots, \ell)$.*

(C1) *m is (globally) decreasingly minimal in $\ddot{\ddot{B}}$.*

(C2) *m is (globally) increasingly maximal in $\ddot{\ddot{B}}$.*

Proof. (B)→(A) If $m(t) \geq m(s) + 2$, then there is an m-tight set C_i containing t and not containing s, from which Property (A) follows from Claim 3.2.

(A)→(B) Let $\mathcal{C}$ be a longest chain consisting of non-empty m-tight and m-top sets $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$. For notational convenience, let $C_0 = \emptyset$ (but C_0 is not a member of of $\mathcal{C}$). We claim that $\mathcal{C}$ meets the requirement of (B). If, indirectly, this is not the case, then there is a subscript $i \in \{1, \dots, \ell\}$ for which m is not near-uniform within $S_i := C_i - C_{i-1}$. This means that the max m-value β_i in S_i is at least 2 larger than the min m-value α_i in S_i, that is, $\beta_i \geq \alpha_i + 2$. Let $Z := \cup[T_m(t;p) : t \in S_i,\ m(t) = \beta_i]$. Then Z is m-tight. Since C_i is m-tight, $T_m(t;p) \subseteq C_i$ holds for $t \in S_i$ and hence $Z \subseteq C_i$. Furthermore, (A) implies that $m(v) \geq \beta_i - 1$ for every $v \in Z \cap S_i$.

Consider the set $C' := C_{i-1} \cup Z$. Then C' is m-tight, and $C_{i-1} \subset C' \subset C_i$. Moreover, we claim that C' is an m-top set. Indeed, if, indirectly, there is an element $u \in C'$ and an element $v \in S - C'$ for which $m(u) < m(v)$, then $u \in Z \cap S_i$ and $v \in C_i - Z$ since both C_{i-1} and C_i are m-top sets. But this is impossible since the m-value of each element of Z is β_i or $\beta_i - 1$ while the m-value of each element of $C_i - Z$ is at most $\beta_i - 1$.

The existence of C' contradicts the assumption that $\mathcal{C}$ was a longest chain of m-tight and m-top sets, and therefore m must be near-uniform within each S_i, that is, $\mathcal{C}$ meets indeed the requirements in (B).

(C1)→(A) and (C2)→(A) Property (A) must indeed hold since a 1-tightening step for m results in an element m' of $\ddot{\ddot{B}}$ which is both decreasingly smaller and increasingly larger than m.

(B)→(C1) We may assume that the elements of S are arranged in an m-decreasing order $s_1, \dots, s_n$ (that is, $m(s_1) \geq m(s_2) \geq \cdots \geq m(s_n)$) in such a way that each C_i in (B) is a starting segment. Let m' be an element of $\ddot{\ddot{B}}$ which is decreasingly smaller than or value-equivalent to m. Recall that $m|X$ denoted the vector m restricted to a subset $X \subseteq S$.

Lemma 3.4. *For each $i = 0, 1, \dots, \ell$, vector $m'|C_i$ is value-equivalent to vector $m|C_i$.*

Proof. Induction on i. For $i = 0$, the statement is void so we assume that $1 \leq i \leq \ell$. By induction, we may assume that the statement holds for $j \leq i - 1$ and we want to prove it for i. Since $m'|C_{i-1}$ is value-equivalent to $m|C_{i-1}$ and C_{i-1} is m-tight, it follows that C_{i-1} is m'-tight, too.

Let β_i denote the max m-value of the elements of $S_i = C_i - C_{i-1}$. By the hypothesis in (B), the maximum and the minimum of the m-values in S_i differ by at most 1. Hence we can assume that there are $r_i > 0$ elements in S_i with m-value β_i and $|S_i| - r_i \geq 0$ elements with m-value $\beta_i - 1$.

As $m|C_{i-1}$ is value-equivalent to $m'|C_{i-1}$ and m' was assumed to be decreasingly smaller than or value-equivalent to m, we can conclude that $m'|(S - C_{i-1})$ is decreasingly smaller than or value-equivalent to $m|(S - C_{i-1})$. Therefore, S_i contains at most r_i elements of m'-value β_i and hence

$$p(C_i) \leq \widetilde{m}'(C_i) = \widetilde{m}'(C_{i-1}) + \widetilde{m}'(S_i) \leq \widetilde{m}'(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1) =$$

$$\widetilde{m}(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1) = p(C_i),$$

from which equality follows everywhere. In particular, S_i contains exactly r_i elements of m'-value β_i and $|S_i| - r_i$ elements of m'-value $\beta_i - 1$, proving the lemma. ■

By the lemma, m' is value-equivalent to m, and hence m is a decreasingly minimal element of $\overset{....}{B}$, that is, (C1) follows.

(B)→(C2) The property in (C1) that m is globally decreasing minimal in $\overset{....}{B}$ is equivalent to the statement that $-m$ is globally increasing maximal in $-\overset{....}{B}$, that is, (C2) holds with respect to $-m$ and $-\overset{....}{B}$. As we have already proved the implications (C2)→(A)→(B)→(C1), it follows that (C1) holds for $-m$ and $-\overset{....}{B}$. But (C1) for $-m$ and $-\overset{....}{B}$ is just the same as (C2) for m and $\overset{....}{B}$. ■ ■

3.2.1 Minimizing the sum of the k largest components

A decreasingly minimal element of $\overset{....}{B}$ has the starting property that its largest component is as small as possible. As a natural extension, one may be interested in finding a member of $\overset{....}{B}$ for which the sum of the k largest components is as small as possible. We refer to this problem as **min-k-largest**.

THEOREM 3.5. *Let B be an integral base-polyhedron and k an integer with $1 \leq k \leq n$. Then any dec-min element m of $\overset{....}{B}$ is a solution to Problem min-k-largest.*

Proof. Observe first that if z_1 and z_2 are dec-min elements of $\overset{....}{B}$, then it follows from the very definition of decreasing minimality that the sum of the first j largest components of z_1 and of z_2 are the same for each $j = 1, \dots, n$. Let K denote the sum of the first k largest components of any dec-min element, and assume indirectly that there is a member $y \in \overset{....}{B}$ for which the sum of its first largest components is smaller than K. Assume that the componentwise square-sum of y is as small as possible. By the previous observation, y is not a dec-min element. Theorem 3.3 implies that there are elements s and t of S for which $y(t) \geq y(s) + 2$ and $y' := y - \chi_t + \chi_s$ is in $\overset{....}{B}$. The sum of the first k largest components of y' is at most the sum of the first k largest components of y, and hence this sum is also smaller than K. But this contradicts the choice of y since the componentwise square-sum of y' is strictly smaller than that of y. ■

Note that the theorem implies that a dec-min element m will be a solution to the min-k-largest problem for each $k = 1, \dots, n$. In [20], we will point out how this result is related to the notion of least majorization, investigated by Tamir [49].

3.3 An example for the intersection of two base-polyhedra

We proved for a single base-polyhedron that an integral element is decreasingly minimal if and only if it is increasingly maximal (and therefore the two properties could jointly be called egalitarian). The following example shows that the two properties may differ if the polyhedron is the intersection of two (integral) base-polyhedra.

Let $S = \{s_1, s_2, s_3, s_4\}$ be the common ground-set of two rank-2 matroids M_1 and M_2 which are described by their circuits. Both matroids have two 2-element circuits. Namely, the circuits of M_1 are $\{s_1, s_4\}$ and $\{s_2, s_3\}$ while the circuits of M_2 are $\{s_1, s_3\}$ and $\{s_2, s_4\}$. Both matroids have four bases:

$$\begin{aligned} \mathcal{B}_1 &:= \{\{s_1, s_2\},\ \{s_3, s_4\},\ \{s_1, s_3\}, \{s_2, s_4\}\}, \\ \mathcal{B}_2 &:= \{\{s_1, s_2\},\ \{s_3, s_4\},\ \{s_1, s_4\}, \{s_2, s_3\}\}. \end{aligned}$$

Let B_1 and B_2, respectively, denote the base-polyhedra of M_1 and M_2. (That is, B_i is the convex hull of the incidence vectors of the four members of $\mathcal{B}_i$.) The common bases are as follows.

$$\mathcal{B} := \mathcal{B}_1 \cap \mathcal{B}_2 = \{\{s_1, s_2\},\ \{s_3, s_4\}\}.$$

Let B denote the convex hull of the incidence vectors of the two members of $\mathcal{B}$. That is, B is the line segment connecting $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. Note that $B = B_1 \cap B_2$ since the intersection of two matroid base-polyhedra is an integral polyhedron, by a theorem of Edmonds [6].

Let B'_i denote the base-polyhedron obtained from B_i by adding the vector $(1, -1, 0, 0)$ to B_i. Let B' be obtained from B in the same way. Then we have:

$$\begin{aligned} \overset{....}{B'_1} &= \{(2, 0, 0, 0),\ (1, -1, 1, 1),\ (2, -1, 1, 0),\ (1, 0, 0, 1)\}, \\ \overset{....}{B'_2} &= \{(2, 0, 0, 0), (1, -1, 1, 1), (2, 1, 0, 1), (1, 0, 1, 0)\}, \\ \overset{....}{B} &= \{(2, 0, 0, 0),\ (1, -1, 1, 1)\}. \end{aligned}$$

Now $x = (2, 0, 0, 0)$ is an increasingly maximal element of $\overset{....}{B}$ while $y = (1, -1, 1, 1)$ is decreasingly minimal.

Therefore finding a decreasingly minimal integral element and an increasingly maximal integral element of the intersection B of two base-polyhedra are two different problems (unlike the analogous problems for a single base-polyhedron). The two problems, however, are equivalent in the sense that an element x of $\overset{....}{B}$ is decreasingly minimal if and only if $-x$ is an increasingly maximal element of $-\overset{....}{B}$, and if B is the intersection of two base-polyhedra, then so is $-B$.

Furthermore, we claim that the intersection B has no integral least majorized element. Indeed, we have only the two possible choices $x = (2, 0, 0, 0)$ and $y = (1, -1, 1, 1)$, but x is not least majorized since the largest component of y is smaller than the largest component of x, and y is neither a least majorized element of $\overset{....}{B}$ since the sum of the 3 largest components of x is smaller than the sum of the 3 largest components of y.

In a forthcoming paper [21], we shall consider the even more general problem of finding an increasingly minimal (integral) submodular flow. Note that that submodular flows not only generalize ordinary flows and circulations but the intersection of two g-polymatroids is also a special submodular flow polyhedron.

4 Characterizing the set of pre-decreasingly minimal elements

We continue to assume that p is an integer-valued (with possible $-\infty$ values but with finite $p(S)$) supermodular function which implies that $B = B'(p)$ is a non-empty integral base-polyhedron. Previously we proved that an integral element m of B (that is, an element of $\overset{....}{B}$) is decreasingly-minimal (= dec-min) precisely if m is increasingly-maximal (= inc-max).

One of our main goals is to prove that the set $\mathrm{dm}(\overset{....}{B})$ of all dec-min elements of $\overset{....}{B}$ is an M-convex set, meaning that there exists an integral base-polyhedron $B^\bullet \subseteq B$ such that $\mathrm{dm}(\overset{....}{B})$ is the set of integral elements of $B^\bullet$. In addition, we shall show that $\mathrm{dm}(\overset{....}{B})$ is actually a matroidal M-convex set, that is, $B^\bullet$ is a special base-polyhedron which is obtained from a matroid base-polyhedron by translating it with an integral vector.

The base-polyhedron $B^\bullet$ will be obtained with the help of a decomposition of B along a certain 'canonical' partition $\{S_1, S_2, \dots, S_q\}$ of S into non-empty sets. To this end, we start by introducing the first member S_1 of this partition along with a matroid on S_1. The set S_1, depending only on B, will be called the peak-set of S.

4.1 Max-minimizers and pre-dec-min elements

Recall that an element of $\overset{....}{B}$ was called a max-minimizer if its largest component was as small as possible, while a max-minimizer was called a pre-dec-min element of $\overset{....}{B}$ if the number of its maximum components was as small as possible. As a dec-min element of $\overset{....}{B}$ is automatically pre-dec-min (in particular, a max-minimizer), we start our investigations by studying max-minimizers and pre-dec-min elements of $\overset{....}{B}$. For a number β, we say that a vector is β-**covered** if each of its components is at most β. Throughout our discussions,

$$\beta_1 := \beta(B) \tag{3}$$

denotes the smallest integer for which $\overset{....}{B}$ has a β_1-covered element. In other words, β_1 is the largest component of a max-minimizer of $\overset{....}{B}$. Therefore β_1 is the largest component of any pre-dec-min (and hence any dec-min) element of $\overset{....}{B}$. Note that an element m of $\overset{....}{B}$ is β_1-covered precisely if m is a max-minimizer.

THEOREM 4.1. *For the largest component β_1 of a max-minimizer of $\overset{....}{B}$, one has*

$$\beta_1 = \max\{\lceil \frac{p(X)}{|X|} \rceil : \emptyset \neq X \subseteq S\}. \tag{4}$$

Proof. It follows from formula (2) that B has a β-covered element if and only if

$$\beta|X| \geq p(X) \text{ whenever } \ X \subseteq S. \tag{5}$$

Moreover, if β is an integer and (5) holds, then B has an integral β-covered element. As $\beta|X| \geq p(X)$ holds for an arbitrary β when $X = \emptyset$, it follows that the smallest integer β meeting this (5) is indeed $\max\{\lceil \frac{p(X)}{|X|} \rceil : \emptyset \neq X \subseteq S\}$. ∎

For a β_1-covered element m of $\overset{....}{B}$, let $r_1(m)$ denote the number of β_1-valued components of m. Recall that for an element $s \in S$ we denoted the unique smallest m-tight set containing s by $T_m(s) = T_m(s;p)$ (that is, $T_m(s)$ is the intersection of all m-tight sets containing s). Furthermore, let

$$S_1(m) := \cup\{T_m(t) : m(t) = \beta_1\}. \tag{6}$$

Then $S_1(m)$ is m-tight and $S_1(m)$ is actually the unique smallest m-tight set containing all the β_1-valued elements of m.

THEOREM 4.2. *A β_1-covered element m of $\overset{....}{B}$ is pre-dec-min if and only if $m(s) \geq \beta_1 - 1$ for each $s \in S_1(m)$.*

Proof. Necessity. Let m be a pre-dec-min element of $\overset{....}{B}$. For any β_1-valued element $t \in S$ and any element $s \in T_m(t)$, we claim that $m(s) \geq \beta_1 - 1$. Indeed, if we had $m(s) \leq \beta_1 - 2$, then the vector m' arising from m by decreasing $m(t)$ by 1 and increasing $m(s)$ by 1 belongs to B (since $T_m(t)$ is the smallest m-tight set containing t) and has one less β_1-valued components than m has, contradicting the assumption that m is pre-dec-min.

Sufficiency. Let m' be an arbitrary β_1-covered integral element of B. Abbreviate $S_1(m)$ by Z and let h' denote the number of elements $z \in Z$ for which $m'(z) = \beta_1$. Then

$$\begin{aligned} &|Z|(\beta_1 - 1) + r_1(m) = \widetilde{m}(Z) = p(Z) \leq \widetilde{m}'(Z) \\ &\leq h'\beta_1 + (|Z| - h')(\beta_1 - 1) = |Z|(\beta_1 - 1) + h' \\ &\leq |Z|(\beta_1 - 1) + r_1(m') \end{aligned}$$

from which $r_1(m) \leq r_1(m')$, as required. ∎

Define the set-function h_1 on S as follows.

$$h_1(X) := p(X) - (\beta_1 - 1)|X| \quad \text{for} \quad X \subseteq S. \tag{7}$$

THEOREM 4.3. *For the minimum number r_1 of β_1-valued components of a β_1-covered member of $\ddot{B}$, one has*

$$r_1 = \max\{h_1(X) : X \subseteq S\}. \tag{8}$$

Proof. Let m be an element of $\ddot{B}$ for which the maximum of its components is β_1, and let X be an arbitrary subset of S. Suppose that X has ℓ β_1-valued components. Then

$$p(X) \leq \widetilde{m}(X) \leq \ell\beta_1 + (|X| - \ell)(\beta_1 - 1) = |X|(\beta_1 - 1) + \ell \leq |X|(\beta_1 - 1) + r_1(m) \tag{9}$$

from which $r_1(m) \geq p(X) - (\beta_1 - 1)|X| = h_1(X)$, implying that

$$r_1 = \min\{r_1(m) : m \in \ddot{B},\ m \text{ is } \beta_1\text{-covered}\} \geq \max\{h_1(X) : X \subseteq S\}.$$

In order to prove the reverse inequality, we have to find a β_1-covered integral element m of B and a subset X of S for which $r_1(m) = h_1(X)$, which is equivalent to requiring that each of the three inequalities in (9) holds with equality. That is, the following three **optimality criteria** hold: (a) X is m-tight, (b) X contains all β_1-valued components of m, and (c) $m(s) \geq \beta_1 - 1$ for each $s \in X$.

Let m be a pre-dec-min element of B. Then $S_1(m)$ is m-tight, $S_1(m)$ contains all β_1-valued elements and, by Theorem 4.2, $m(s) \geq \beta_1 - 1$ for all $s \in S_1(m)$, therefore m and $S_1(m)$ satisfy the three optimality criteria. ∎

Note that r_1 is the number of β_1-valued components of any pre-dec-min element (and in particular, any dec-min element) of $\ddot{B}$.

4.2 The peak-set S_1

Since the set-function h_1 introduced in (7) is supermodular, the maximizers of h_1 are closed under taking intersection and union. Let S_1 denote the unique smallest subset of S maximizing h_1. In other words, S_1 is the intersection of all sets maximizing h_1. We call this set S_1 the **peak-set** of B (and of $\ddot{B}$),

THEOREM 4.4. *For every pre-dec-min (and in particular, for every dec-min) element m of $\ddot{B}$, the set $S_1(m)$ introduced in* (6) *is independent of the choice of m and $S_1(m) = S_1$, where S_1 is the peak-set of B.*

Proof. It follows from Theorem 4.3 that, given a pre-dec-min element m of B, a subset X is maximizing h_1 precisely if the three optimality criteria mentioned in the proof hold. Since $S_1(m)$ meets the optimality criteria, it follows that $S_1 \subseteq S_1(m)$. If, indirectly, there is an element $s \in S_1(m) - S_1$, then $m(s) = \beta_1 - 1$ since S_1 contains all the β_1-valued elements. By the definition of $S_1(m)$, there is a β_1-valued element $t \in S_1(m)$ for which the smallest m-tight set $T_m(t)$ contains s, but this is impossible since S_1 is an m-tight set containing t but not s. ∎

Since $S_1 = S_1(m)$ is m-tight and near-uniform, we obtain that

$$\beta_1 = \lceil \frac{\widetilde{m}_1(S_1)}{|S_1|} \rceil = \lceil \frac{p(S_1)}{|S_1|} \rceil,$$

and the definitions of S_1 and r_1 imply that

$$r_1 = p(S_1) - (\beta_1 - 1)|S_1|. \tag{10}$$

Proposition 4.5. $S_1 = \{s \in S :$ *there is a pre-dec-min element* $m \in \overset{....}{B}$ *with* $m(s) = \beta_1\}$. *In particular,* $m(s) \leq \beta_1 - 1$ *for every* $s \in S - S_1$ *and for every pre-dec-min element* m *of* $\overset{....}{B}$.

Proof. If $m(s) = \beta_1$ for some pre-dec-min m, then $s \in S_1(m) = S_1$. Conversely, let $s \in S_1$ and let m be a pre-dec-min element. We are done if $m(s) = \beta_1$. If this is not the case, then $m(s) = \beta_1 - 1$ by Theorem 4.2. By the definition of $S_1(m)$, there is an element $t \in S_1(m)$ for which $m(t) = \beta_1$ and $s \in T_m(t)$. But then $m' := m + \chi_{\mathbf{s}} - \chi_{\mathbf{t}}$ is in $\overset{....}{B}$, $m'(s) = \beta_1$ and m' is also pre-dec-min as it is value-equivalent to m. ■

4.2.1 Separating along S_1

Let S_1 be the peak-set occurring in Theorem 4.4 and let $S_1' := S - S_1$. Let $p_1 = p|S_1$ denote the restriction of p to S_1, and let $B_1 \subseteq \mathbf{R}^{S_1}$ denote the base-polyhedron defined by p_1, that is, $B_1 := B'(p_1)$. Suppose that $S_1' \neq \emptyset$ and let $p_1' := p/S_1$, that is, p_1' is the set-function on S_1' obtained from p by contracting S_1 $(p_1'(X) = p(S_1 \cup X) - p(S_1)$ for $X \subseteq S_1')$.

Consider the face F of B determined by S_1, that is, F is the direct sum of the base-polyhedra $B_1 = B'(p_1)$ and $B_1' = B'(p_1')$. Then the dec-min elements of $\overset{....}{B}_1$ are exactly the integral elements of the intersection of B_1 and the box given by $\{x : \beta_1 - 1 \leq x(s) \leq \beta_1$ for every $s\}$. Hence the dec-min elements of $\overset{....}{B}_1$ are near-uniform.

THEOREM 4.6. *An integral vector* $m = (m_1, m_1')$ *is a dec-min element of* $\overset{....}{B}$ *if and only if* m_1 *is a dec-min element of* $\overset{....}{B}_1$ *and* m_1' *is a dec-min element of* $\overset{....}{B}_1'$.

Proof. Suppose first that m is a dec-min element of $\overset{....}{B}$. Then $S_1 = S_1(m)$ by Theorem 4.4 and m is a max-minimizer implying that every component of m in S_1 is of value $\beta_1 - 1$ or value β_1, and m has exactly r_1 components of value β_1. Therefore m_1 is in $\overset{....}{B}_1$. Moreover, each of the components of m_1 is $\beta_1 - 1$ or β_1, that is, m_1 is near-uniform implying that m_1 is actually dec-min in $\overset{....}{B}_1$.

Since $\widetilde{m}(S_1) = p(S_1)$, for a set $X \subseteq S_1'$, we have

$$\widetilde{m}_1'(X) = \widetilde{m}(X) = \widetilde{m}(S_1 \cup X) - \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup X) - p(S_1) \geq p(S_1 \cup X) - p(S_1) = p_1'(X).$$

Furthermore

$$\widetilde{m}_1'(S_1') = \widetilde{m}(S_1') = \widetilde{m}(S_1 \cup S_1') - \widetilde{m}(S_1) = p(S_1 \cup S_1') - p(S_1) = p_1'(S_1'),$$

that is m_1' is in $\overset{....}{B}{}_1'$. If, indirectly, m_1' is not dec-min, then, by applying Theorem 3.3 to S_1', m_1', and p_1', we obtain that there are elements t and s of S_1' for which $m_1'(t) \geq m_1'(s) + 2$ and $(*)$ no $t\overline{s}$-set exists which is m_1'-tight with respect to p_1'. On the other hand, m is a dec-min element of $\overset{....}{B}$ for which

$$m(t) = m_1'(t) \geq m_1'(s) + 2 = m(s) + 2,$$

and hence there must be a $t\overline{s}$-set Y which is m-tight with respect to p.

Since S_1 is m-tight with respect to p, the set $S_1 \cup Y$ is also m-tight with respect to p. Let $X := S_1' \cap Y$. Then

$$\widetilde{m}(X) + \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y) = p(S_1 \cup X),$$

and hence

$$\widetilde{m}_1'(X) = \widetilde{m}(X) = p(S_1 \cup X) - \widetilde{m}(S_1) = p(S_1 \cup X) - p(S_1) = p_1'(X),$$

that is, X is $t\overline{s}$-set which is m_1'-tight with respect to p_1', in a contradiction with statement $(*)$ above that no such set exists.

To see the converse, assume that m_1 is a dec-min element of $\overset{....}{B}_1$ and m_1' is a dec-min element of $\overset{....}{B}{}_1'$. This immediately implies that m is in the face F of B determined by S_1. Suppose, indirectly, that m is not a dec-min element of $\overset{....}{B}$. By Theorem 3.3, there are elements t and s of S for which $m(t) \geq m(s) + 2$ and $(**)$ no $t\overline{s}$-set exists which is m-tight with respect to p. If $t \in S_1$, then s cannot be in S_1 since the m-value of each element of S_1 is β_1 or $\beta_1 - 1$. But S_1 is m_1-tight with respect to p and hence it is m-tight with respect to p, contradicting property $(**)$. Therefore t must be in S_1', implying that s is also in S_1'.

Since m_1' is a dec-min element of $\overset{....}{B}{}_1'$, there must be a $t\overline{s}$-set $Y \subset S_1'$ which is m_1'-tight with respect to p_1'. It follows that

$$\widetilde{m}(Y) = \widetilde{m}_1'(Y) = p_1'(Y) = p(S_1 \cup Y) - p(S_1) \leq \widetilde{m}(S_1 \cup Y) - \widetilde{m}(S_1) = \widetilde{m}(Y),$$

from which $\widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y)$, contradicting property $(**)$ that no $t\overline{s}$-set exists which is m-tight with respect to p. ■

An important consequence of Theorem 4.6 is that, in order to find a dec-min element of $\overset{....}{B}$, it will suffice to find separately a dec-min element of $\overset{....}{B}_1$ (which was shown above to be a near-uniform vector) and a dec-min element of $\overset{....}{B}{}_1'$. The algorithmic details will be discussed in Section 7.

THEOREM 4.7. *Let S_1 be the peak-set of $\overset{....}{B}$. For an element m_1 of $\overset{....}{B}_1$, the following properties are pairwise equivalent.*

(A1) *m_1 has r_1 $(= p(S_1) - (\beta_1 - 1)|S_1| \ > 0)$ components of value β_1 and $|S_1| - r_1$ (≥ 0) components of value $\beta_1 - 1$.*

(A2) *m_1 is near-uniform.*

(A3) *m_1 is dec-min in $\ddot{\ddot{B}}_1$.*

(B1) *m_1 is the restriction of a dec-min element m of $\ddot{\ddot{B}}$ to S_1.*

(B2) *m_1 is the restriction of a pre-dec-min element m of $\ddot{\ddot{B}}$ to S_1.*

Proof. The implications (A1)→(A2)→(A3) and (B1)→(B2) are immediate from the definitions.

(A3)→(B1) Let m_1' be an arbitrary dec-min element of $\ddot{\ddot{B}}_1'$. By Theorem 4.6, $m := (m_1, m_1')$ is a dec-min element of $\ddot{\ddot{B}}$ and hence m_1 is indeed the restriction of a dec-min element of $\ddot{\ddot{B}}$ to S_1.

(B2)→(A1) By Theorems 4.2 and 4.4, we have $m_1(s) \geq \beta_1 - 1$ for each $s \in S_1(m) = S_1$, that is, $\beta_1 - 1 \leq m_1(s) \leq \beta_1$. By letting r' denote the number of β_1-valued components of m_1, we obtain by (10) that

$$r_1 + (\beta_1 - 1)|S_1| = p_1(S_1) = \widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + r'$$

and hence $r' = r_1$. ■

Theorem 4.6 implies that, in order to characterize the set of dec-min elements of $\ddot{\ddot{B}}$, it suffices to characterize the set of dec-min elements of $\ddot{\ddot{B}}_1'$.

THEOREM 4.8. *Let β_2 denote the smallest integer for which $\ddot{\ddot{B}}_1'$ has a β_2-covered element, that is, $\beta_2 = \beta(B_1')$. Then*

$$\beta_2 = \max\{\lceil \frac{p_1'(X)}{|X|} \rceil : \emptyset \neq X \subseteq S - S_1\}, \tag{11}$$

where $p_1'(X) = p(X \cup S_1) - p(S_1)$. Furthermore, β_2 is the largest component in $S - S_1$ of every dec-min element of $\ddot{\ddot{B}}$, and $\beta_2 < \beta_1$.

Proof. Formula (11) follows by applying Theorem 4.1 to base-polyhedron B_1' ($= B'(p_1')$) in place of B. By Theorem 4.6, the largest component in $S - S_1$ of any dec-min element m of $\ddot{\ddot{B}}$ is β_2. By Theorem 4.4, $S_1(m) = S_1$, and the definition of $S_1(m)$ shows that $m(s) \leq \beta_1 - 1$ holds for every $s \in S - S_1$, from which $\beta_2 < \beta_1$ follows. ■

4.3 The matroid M_1 on S_1

It is known from the theory of base-polyhedra that the intersection of an integral base-polyhedron with an integral box is an integral base-polyhedron. Morover, if the box in question is small, then the intersection is actually a translated matroid base-polyhedron (meaning that the intersection arises from a matroid base-polyhedron by translating it with an integral vector). This result is a consequence of the theorem that (∗) any integral base-polyhedron in the unit $(0, 1)$-cube is the convex hull of (incidence vectors of) the bases of a matroid.

Consider the special small integral box $T_1 \subseteq \mathbf{Z}^{S_1}$ defined by

$$T_1 := \{x : \beta_1 - 1 \leq x(s) \leq \beta_1\}$$

and its intersection $B_1^{\bullet} := B_1 \cap T_1$ with the base-polyhedron B_1 investigated above. Therefore $B_1^{\bullet}$ is a translated matroid base-polyhedron and Theorem 4.7 implies the following.

Corollary 4.9. *The dec-min elements of $\ddot{B}_1$ are exactly the integral elements of the translated matroid base-polyhedron $B_1^{\bullet}$.* ■

Our next goal is to reprove Corollary 4.9 by concretely describing the matroid in question and not relying on the background theorem (∗) mentioned above. For a dec-min element m_1 of $\ddot{B}_1$, let

$$L_1(m_1) := \{s \in S_1 : m_1(s) = \beta_1\}.$$

We know from Theorem 4.7 that $|L_1(m_1)| = r_1$. Define a set-system $\mathcal{B}_1$ as follows.

$$\mathcal{B}_1 := \{L \subseteq S_1 : L = L_1(m_1) \text{ for some dec-min element } m_1 \text{ of } \ddot{B}_1\}. \tag{12}$$

We need the following charecterization of $\mathcal{B}_1$.

Proposition 4.10. *An r_1-element subset L of S_1 is in $\mathcal{B}_1$ if and only if*

$$|L \cap X| \geq p_1'(X) := p_1(X) - (\beta_1 - 1)|X| \quad \text{whenever } X \subseteq S_1. \tag{13}$$

Proof. Suppose first that $L \in \mathcal{B}_1$, that is, there is a dec-min element m_1 of $\ddot{B}_1$ for which $L = L_1(m_1)$. Then

$$(\beta_1 - 1)|X| + |X \cap L| = \widetilde{m}_1(X) \geq p_1(X),$$

for every subset $X \subseteq S_1$ from which (13) follows.

To see the converse, let $L \subseteq S_1$ be an r_1-element set meeting (13). Let

$$m_1(s) := \begin{cases} \beta_1 & \text{if } \ s \in L \\ \beta_1 - 1 & \text{if } \ s \in S - L. \end{cases} \tag{14}$$

Then obviously $L = L_1(m_1)$. Furthermore,

$$\widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + |L| = (\beta_1 - 1)|S_1| + r_1 = p(S_1)$$

and

$$\widetilde{m}_1(X) = (\beta_1 - 1)|X| + |L \cap X| \geq p_1(X) \text{ whenever } \ X \subset S_1,$$

showing that $m_1 \in B_1$. Since $m_1 \in T_1$, we conclude that m_1 is a dec-min element of $\ddot{B}_1$. ■

THEOREM 4.11. *The set-system $\mathcal{B}_1$ defined in (12) forms the set of bases of a matroid M_1 on ground-set S_1.*

Proof. The set-system $\mathcal{B}_1$ is clearly non-empty and all of its members are of cardinality r_1.

It is widely known [6] that for an integral submodular function b on a ground-set S_1 the set-system

$$\{L \subseteq S_1 : \ |L \cap X| \leq b(X) \text{ whenever } X \subset S_1, \ |L| = b(S_1)\},$$

if non-empty, satisfies the matroid basis axioms. This implies for the supermodular function p_1' that the set-system $\{L : |L \cap X| \geq p_1'(X) \text{ whenever } X \subset S_1, \ |L| = p_1'(S_1)\}$, if non-empty, forms the set of bases of a matroid. By applying this fact to the supermodular function p_1' defined by $p_1'(X) := p_1(X) - (\beta_1 - 1)|X|$, one obtains that $\mathcal{B}_1$ is non-empty and forms the set of bases of a matroid. ■ ■

In this way, we proved the following more explicit form of Corollary 4.9.

Corollary 4.12. *Let $\Delta_1 : S_1 \to \mathbf{Z}$ denote the integral vector defined by $\Delta_1(s) := \beta_1 - 1$ for $s \in S_1$. A member m_1 of $\overset{....}{B_1}$ is decreasingly minimal if and only if there is a basis B_1 of M_1 so that $m_1 = \chi_{B_1} + \Delta_1$.* ■

4.3.1 Value-fixed elements of S_1 versus co-loops of matroid M_1

We say that an element $s \in S$ is **value-fixed** with respect to $\overset{....}{B}$ if $m(s)$ is the same for every dec-min element m of $\overset{....}{B}$. In Section 6.2, we will need a description of value-fixed elements of $\overset{....}{B}$. In the present section, we consider the value-fixed elements with respect to B_1, that is, $s \in S_1$ is value-fixed if $m_1(s)$ is the same for every dec-min element $m_1 \in \overset{....}{B_1}$. Recall that $m_1 \in \overset{....}{B_1}$ was shown to be dec-min precisely if $\beta_1 - 1 \leq m_1(s) \leq \beta_1$ for each $s \in S_1$.

A **loop** of a matroid is an element $s \in S_1$ not belonging to any basis. (Often the singleton $\{s\}$ is called a loop, that is, $\{s\}$ is a one-element circuit). A **co-loop** (or cut-element or isthmus) of a matroid is an element s belonging to all bases.

Proposition 4.13. *M_1 has no loops.*

Proof. By Proposition 4.5, for every $s \in S_1$ there is a pre-dec-min element m of $\overset{....}{B}$ for which $m(s) = \beta_1$. Then $m_1 := m|S_1$ is a pre-dec-min element of $\overset{....}{B_1}$ by Theorem 4.7 from which s_1 belongs to a basis of M_1 by Corollary 4.12. ■

The proposition implies that:

Proposition 4.14. *If $s \in S_1$ is value-fixed (with respect to B_1), then $m_1(s) = \beta_1$ for every dec-min element m_1 of $\overset{....}{B_1}$.* ■

By Corollary 4.12, an element $s \in S_1$ is a co-loop of M_1 if and only if $m_1(s) = \beta_1$ holds for every dec-min element m_1 of $\overset{....}{B_1}$. This and Theorem 4.6 imply the following.

THEOREM 4.15. *For an element $s \in S_1$, the following properties are pairwise equivalent.*

(A) *s is a co-loop of M_1.*

(B) *s is value-fixed.*

(C) *$m(s) = \beta_1$ holds for every dec-min element m of $\overset{....}{B}$.* ∎

Our next goal is to characterize the set of value-fixed elements of S_1. Consider the family of subsets S_1 defined by

$$\mathcal{F}_1 := \{X \subseteq S_1 : \ \beta_1|X| = p_1(X)\}. \tag{15}$$

The empty set belongs to $\mathcal{F}_1$ and it is possible that $\mathcal{F}_1$ has no other members. By standard submodularity arguments, $\mathcal{F}_1$ is closed under taking union and intersection. Let F_1 denote the unique largest member of $\mathcal{F}_1$. It is possible that $F_1 = S_1$ in which case we call S_1 **degenerate**.

THEOREM 4.16. *An element $s \in S_1$ is value-fixed if and only if $s \in F_1$.*

Proof. Let m_1 be a dec-min member of $\overset{....}{B}_1$. Then

$$\beta_1|F_1| \geq \widetilde{m}_1(F_1) \geq p_1(F_1) = \beta_1|F_1|$$

and hence we must have $\beta_1 = m_1(s)$ for every $s \in F_1$, that is, the elements of F_1 are indeed value-fixed.

Conversely, let s be value-fixed, that is, $m_1(s) = \beta_1$ for each dec-min element m_1 of $\overset{....}{B}_1$. Let m_1 be a dec-min member of $\overset{....}{B}_1$. Let Z denote the unique smallest set containing s for which $\widetilde{m}_1(Z) = p_1(Z)$. (That is, $Z = T_{m_1}(s; p_1)$.) We claim that $m_1(t) = \beta_1$ for every element $t \in Z$. For if $m_1(t) = \beta_1 - 1$ for some t, then $m_1' := m_1 - \chi_{\mathbf{s}} + \chi_{\mathbf{t}}$ would also be a dec-min member of $\overset{....}{B}_1$, contradicting the assumption that s is value-fixed. Therefore $p_1(Z) = \widetilde{m}_1(Z) = \beta_1|Z|$ from which the definition of F_1 implies that $Z \subseteq F_1$ and hence $s \in F_1$. ∎

5 The set of dec-min elements of an M-convex set

Let $B = B'(p)$ denote again an integral base-polyhedron defined by the (integer-valued) supermodular function p. As in the previous section, $\overset{....}{B}$ continues to denote the M-convex set consisting of the integral vectors (points, elements) of B. Our present goal is to provide a complete description of the set of decreasingly-minimal (= egalitarian) elements of $\overset{....}{B}$.

5.1 Canonical partition and canonical chain

In Section 4 we introduced the integer β_1 as the minimum of the largest component of the elements of $\overset{....}{B}$ as well as the notion of peak-set S_1 of S. We considered the face of B defined by S_1 that was the direct sum of base-polyhedra $B_1 = B'(p_1)$ and $B_1' = B'(p_1')$, where p_1 denoted the restriction of p to S_1 while p_1' arose from p by contracting S_1 (that is, $p_1'(X) = p(S_1 \cup X) - p(S_1)$).

A consequence of Theorem 4.6 is that, in order to characterize the set of dec-min elements of $\ddot{B}$, it suffices to characterize separately the dec-min elements of $\ddot{B}_1$ and the dec-min elements of $\ddot{B}'_1$. In Theorem 4.7, we characterized the dec-min elements of $\ddot{B}_1$ as those belonging to the small box $T_1 := \{x \in \mathbf{R}^{S_1} : \beta_1 - 1 \leq x(s) \leq \beta_1$ for $s \in S_1\}$. We also proved that the set $\ddot{B}_1^\bullet$ of dec-min elements of $\ddot{B}_1$ can be described with the help of matroid M_1. If the peak-set S_1 happens to be the whole ground-set S, then the characterization of the set of dec-min elements of $\ddot{B}$ is complete. If $S_1 \subset S$, then our remaining task is to characterize the set of dec-min elements of $\ddot{B}'_1$. This can be done by repeating iteratively the separation procedure to the base-polyhedron $B'_1 = B'(p'_1) \subseteq \mathbf{R}^{S-S_1}$ described in Section 4 for B.

In this iterative way, we are going to define a partition $\mathcal{P}^* = \{S_1, S_2, \dots, S_q\}$ of S which determines a chain $\mathcal{C}^* = \{C_1, C_2, \dots, C_q\}$ where $C_i := S_1 \cup S_2 \cup \dots \cup S_i$ (in particular $C_q = S$), and the supermodular function

$$p'_i := p/C_i \text{ on set } \overline{C_i} := S - C_i$$

which defines the base-polyhedron $B'_i = B'(p'_i)$ in $\mathbf{R}^{\overline{C}_i}$. Moreover, we define iteratively a decreasing sequence $\beta_1 > \beta_2 > \dots > \beta_q$ of integers, a small box

$$T_i := \{x \in \mathbf{R}^{S_i} :\ \beta_i - 1 \leq x(s) \leq \beta_i \text{ for } s \in S_i\}, \tag{16}$$

and the supermodular function p_i on S_i where

$$p_i := p'_{i-1}|S_i \ \ (= (p/C_{i-1})|S_i) \tag{17}$$

which defines the base-polyhedron $B_i = B'(p_i) \subseteq \mathbf{R}^{S_i}$.

In the general step, suppose that the pairwise disjoint non-empty sets $S_1, S_2, \dots, S_{j-1}$ have already been defined, along with the decreasing sequence $\beta_1 > \beta_2 > \dots > \beta_{j-1}$ of integers. If $S = S_1 \cup \dots \cup S_{j-1}$, then by taking $q := j-1$, the iterative procedure terminates. So suppose that this is not the case, that is, $C_{j-1} \subset S$. We assume that p_{j-1} on S_{j-1} has been defined as well as p'_{j-1} on $\overline{C_{j-1}}$.

Let

$$\beta_j = \max\{\lceil \frac{p'_{j-1}(X)}{|X|} \rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\}, \tag{18}$$

that is,

$$\beta_j = \max\{\lceil \frac{p(X \cup C_{j-1}) - p(C_{j-1})}{|X|} \rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\}. \tag{19}$$

Note that, by the iterative feature of these definitions, Theorem 4.8 implies that

$$\beta_j < \beta_{j-1}.$$

Furthermore, let h_j be a set-function on $\overline{C_{j-1}}$ defined as follows.

$$h_j(X) := p'_{j-1}(X) - (\beta_j - 1)|X| \ \text{ for } \ \ X \subseteq \overline{C_{j-1}}, \tag{20}$$

and let $S_j \subseteq \overline{C_{j-1}}$ be the peak-set of $\overline{C_{j-1}}$ assigned to $B'_{j-1} := B'(p'_{j-1})$, that is, S_j is the smallest subset of $\overline{C}_{j-1}$ maximizing h_j. Finally, let $p_j := p'_{j-1}|S_j$ and let $p'_j := p'_{j-1}/S_j$. Observe by (1) that $p'_j = p/C_j$. Therefore p_j is a set-function on S_j while p'_j is defined on $\overline{C_j}$.

We shall refer to the partition $\mathcal{P}^*$ and the chain $\mathcal{C}^*$ defined above as the **canonical partition** and **canonical chain** of S, respectively, assigned to B, while the sequence $\beta_1 > \cdots > \beta_q$ will be called the **essential value-sequence** of $\overset{....}{B}$. Let $B^\oplus$ denote the face of B defined by the canonical chain $\mathcal{C}^*$, that is, $B^\oplus$ is the direct sum of the q base-polyhedra $B'(p_i)$ $(i = 1, \dots, q)$. Finally, let T^* be the direct sum of the small boxes T_i $(i = 1, \dots, q)$, that is, T^* is the integral box defined by the essential value-sequence as follows.

$$T^* := \{x \in \mathbf{R}^S : \ \beta_i - 1 \leq x(s) \leq \beta_i \ \text{ whenever } s \in S_i \ (i = 1, \dots, q)\}, \tag{21}$$

and let

$$B^\bullet := B^\oplus \cap T^*.$$

Note that the intersection of an integral base-polyhedron with an integral box is always an integral base-polyhedron and hence $B^\bullet$ is an integral base-polyhedron.

THEOREM 5.1. *Let $B = B'(p)$ be an integral base-polyhedron on ground-set S. The set of decreasingly-minimal (= egalitarian) elements of $\overset{....}{B}$ is (the* M*-convex set) $\overset{....}{B^\bullet}$. Equivalently, an element $m \in \overset{....}{B}$ is decreasingly-minimal if and only if its restriction $m_i := m|S_i$ to S_i belongs to $B_i \cap T_i$ for each $i = 1, \dots, q$, where $\{S_1, \dots, S_q\}$ is the canonical partition of S belonging to B, T_i is the small box defined in* (16)*, and B_i is the base-polyhedron $B'(p_i)$ belonging to the supermodular set-function p_i defined in* (17)*.*

Proof. We use induction on q. Suppose first that $q = 1$, that is, $S_1 = S$ and $B_1 = B$. If m is a dec-min element of B, then the equivalence of Properties (A1) and (A3) in Theorem 4.7 implies that m is in $\overset{....}{B^\bullet}$. If, conversely, $m \in \overset{....}{B^\bullet}$, then m is near-uniform and, by the equivalence of Properties (A1) and (A3) in Theorem 4.7 again, m is dec-min.

Suppose now that $q \geq 2$ and consider the base-polyhedron $B'_1 = B'(p'_1)$ appearing in Theorem 4.6. The iterative definition of the canonical partition $\mathcal{P}^*$ implies that the canonical partition of $S - S_1$ assigned to B'_1 is $\{S_2, \dots, S_q\}$ and the essential value-sequence belonging to B'_1 is $\beta_2 > \beta_3 > \cdots > \beta_q$. Also, the canonical chain $\mathcal{C}' := \{C'_2, \dots, C'_q\}$ of B'_1 consists of the sets $C'_i = S_2 \cup \cdots \cup S_i = C_i - S_1$ $(i = 2, \dots, q)$.

By applying the inductive hypothesis to B'_1, we obtain that an integral element m'_1 of B'_1 is dec-min if and only if m'_1 is in the face of B'_1 defined by chain $\mathcal{C}'$ and m'_1 belongs to the box $T' := \{x \in \mathbf{R}^{S-S_1} : \ \beta_i - 1 \leq x(s) \leq \beta_i \ \text{ whenever } s \in S_i \ (i = 2, \dots, q)\}$. By applying Theorem 4.6, we are done in this case as well. ■

5.2 Matroidal description of the set of dec-min elements

Let m be an element of $\dddot{B}$. We called a set $X \subseteq S$ m-tight if $\widetilde{m}(X) = p(X)$. Recall from Section 2.1 that, for a subset $Z \subseteq S$, $T_m(Z) = T_m(Z;p)$ denoted the unique smallest m-tight set including Z, that is, $T_m(Z)$ is the intersection of all the m-tight sets including Z. Obviously,

$$T_m(Z) = \cup(T_m(z) : z \in Z). \tag{22}$$

Let m be an arbitrary dec-min element of $\dddot{B}$. We proved that m is in the face $B^{\oplus}$ of B defined by the canonical chain $\mathcal{C}^* = \{C_1, \dots, C_q\}$ belonging to B. Therefore each C_i is m-tight with respect to p. Furthermore $m_i := m|S_i$ belongs to the box T_i defined in (16). This implies that $m(s) \geq \beta_i - 1$ for every $s \in C_i$ and $m(s') \leq \beta_{i+1}$ for every $s' \in \overline{C}_i$. (The last inequality holds indeed since $s' \in \overline{C}_i$ implies that $s' \in S_j$ for some $j \geq i+1$ from which $m(s') \leq \beta_j \leq \beta_{i+1}$.) Since $\beta_{i+1} \leq \beta_i - 1$, we obtain that each C_i is an m-top set.

Since m_i is near-uniform on S_i with values β_i and possibly $\beta_i - 1$, we obtain

$$\beta_i = \lceil \frac{\widetilde{m}_i(S_i)}{|S_i|} \rceil = \lceil \frac{p_i(S_i)}{|S_i|} \rceil = \lceil \frac{p(C_i) - p(C_{i-1})}{|S_i|} \rceil.$$

Let $L_i := \{s \in S - C_{i-1} : m(s) = \beta_i\}$ and let $r_i := |L_i|$. Then $p_i(S_i) = \widetilde{m}_i(S_i) = (\beta_i - 1)|S_i| + r_i$ and hence

$$r_i = p(C_i) - p(C_{i-1}) - (\beta_i - 1)|S_i|. \tag{23}$$

The content of the next lemma is that, once C_{i-1} is given, the next member C_i of the canonical chain (and hence S_i, as well) can be expressed with the help of m. Recall that $T_m(L_i) = T_m(L_i;p)$ denoted the smallest m-tight set including L_i.

Lemma 5.2. $C_i = C_{i-1} \cup T_m(L_i;p)$.

Proof. Recall the definition of function h_i given in (20). We have

$$h_i(S_i) = r_i \tag{24}$$

since $h_i(S_i) = p'_{i-1}(S_i) - (\beta_i - 1)|S_i| = p(S_i \cup C_{i-1}) - p(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(C_i) - \widetilde{m}(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(S_i) - (\beta_i - 1)|S_i| = r_i$.

Since $L_i \subseteq C_i$ and each of C_{i-1}, C_i, and $T_m(L_i)$ are m-tight, we have $C_{i-1} \cup T_m(L_i;p) \subseteq C_i$. For $X' := T_m(L_i) \cap \overline{C_{i-1}}$ we have

$$\begin{aligned} h_i(X') &= p(C_{i-1} \cup T_m(L_i)) - p(C_{i-1}) - (\beta_i - 1)|X'_i| \\ &= \widetilde{m}(C_{i-1} \cup T_m(L_i)) - \widetilde{m}(C_{i-1}) - (\beta_i - 1)|X'_i| \\ &= \widetilde{m}(X') - (\beta_i - 1)|X'_i| = |L_i| = r_i = h_i(S_i), \end{aligned}$$

that is, X' is also a maximizer of $h_i(X)$. Since S_i was the smallest maximizer of h_i, we conclude that $C_{i-1} \cup T_m(L_i;p) \supseteq C_i$. ∎

The lemma implies that both the essential value-sequence $\beta_1 > \cdots > \beta_q$ and the canonical chain $\mathcal{C}^*$ belonging to $\dddot{B}$ may be directly obtained from m.

Corollary 5.3. *Let m be an arbitrary dec-min element of $\overset{....}{B}$. Let $\beta_1 > \cdots > \beta_q$ be the essential value-sequence and $\mathcal{C}^* = \{C_1, \dots, C_q\}$ the canonical chain belonging to $\overset{....}{B}$. Then β_1 is the largest m-value and C_1 is the smallest m-tight set containing all β_1-valued elements. Moreover, for $i = 2, \dots, q$, β_i is the largest value of $m|\overline{C_{i-1}}$ and C_i is the smallest m-tight set containing each element of m-value at least β_i.* ■

(Note that a dec-min element m of $\overset{....}{B}$ may have more than q distinct values. For example, if $q = 1$ and $L_1 \subset C_1 = S$, then m has two distinct values, namely β_1 on the elements of L_1 and $\beta_1 - 1$ on the elements of $S - L_1$, while its essential value sequence consists of the single member β_1.)

In Section 4.3, we introduced a matroid M_1 on S_1 and proved in Corollary 4.9 that the dec-min elements of $\overset{....}{B_1}$ are exactly the integral elements of the translated base-polyhedron of M_1, where the translation means the addition of the constant vector $(\beta_1 - 1, \dots, \beta_1 - 1)$ of dimension $|S_1|$. The same notions and results can be applied to each subscript $i = 2, \dots, q$.

THEOREM 5.4. *The set-system $\mathcal{B}_i := \{L \subseteq S_i : L = L_i(m_i)$ for some dec-min element m_i of $\overset{....}{B_i}\}$ forms the set of bases of a matroid M_i on ground-set S_i.*

Note that m_i is in $\overset{....}{B_i}$ precisely if it is the restriction of a dec-min element m of $\overset{....}{B}$. By formulating Lemma 4.11 for subscript i in place of 1, we obtain that an r_i-element subset L of S_i is a basis of M_i if and only if

$$|L \cap X| \geq p_i'(X) := p_i(X) - (\beta_i - 1)|X| \tag{25}$$

holds for every $X \subseteq S_i$.

Let M^* denote the direct sum of matroids $M_1, \dots, M_q$ and let $\Delta^* \in \mathbf{Z}^S$ denote the translation vector defined by

$$\Delta^*(s) := \beta_i - 1 \quad \text{whenever} \quad s \in S_i,\ i = 1, \dots, q.$$

By integrating these results, we obtain the following characterization.

THEOREM 5.5. *Let $B = B'(p)$ be an integral base-polyhedron. An element m of (the M-convex set) $\overset{....}{B}$ is decreasingly minimal (= egalitarian) if and only if m can be obtained in the form $m = \chi_L + \Delta^*$ where L is a basis of the matroid M^*. The base-polyhedron $B^\bullet$ arises from the base-polyhedron of M^* by adding the translation vector Δ^*. Concisely, the set of dec-min elements of $\overset{....}{B}$ is a matroidal M-convex set.* ■

6 Integral square-sum minimization

6.1 Separable convex functions

Let S be a non-empty ground-set of n elements. Let $B = B'(p)$ be a base-polyhedron defined by an integer-valued supermodular function p on S, and let $\overset{....}{B}$ denote the set of integral elements of B.

A function $\varphi : \mathbf{Z} \to \mathbf{R}$ (in one variable) is said to be **discrete convex** if

$$2\varphi(z) \leq \varphi(z-1) + \varphi(z+1)$$

holds for every integer z. If $2\varphi(z) < \varphi(z-1) + \varphi(z+1)$ holds for every integer z, we speak of a strictly discrete convex function. For example, $\varphi(z) = z^2$ is strictly discrete convex while $\varphi(z) = |z|$ is discrete convex but not strictly. Given a function φ in one variable, define Φ by

$$\Phi(m) := \sum[\varphi(m(s)) : s \in S]$$

where $m \in \mathbf{Z}^S$. Such a function Φ is called a symmetric separable convex function. When φ is strictly convex, Φ is also called strictly convex. In the special case when $\varphi(z) = z^2$, the resulting Φ will be called the square-sum function, and will be denoted by $w(m)$.

We emphasize that there is a fundamental difference between the problems of finding a minimum square-sum element over a base-polyhedron B and over the M-convex set $\overset{....}{B}$ (the set of integral elements of B). In the first case (investigated by Fujishige [23]), there alway exists a single, unique solution, while in the second case, the square-sum minimizer elements of $\overset{....}{B}$ have an elegant matroidal structure. Namely, we prove that they are exactly the dec-min elements of $\overset{....}{B}$ and hence, by Theorem 5.5, the set of square-sum minimizers of an M-convex set arises from the bases of a matroid by translating their incidence vectors with a vector.

Proposition 6.1. *Let B be an integral base-polyhedron and Φ a symmetric separable discrete convex function. Then each dec-min element of $\overset{....}{B}$ is a minimizer of Φ over $\overset{....}{B}$.*

Proof. First we show that there is a dec-min element of $\overset{....}{B}$ which minimizes Φ. Let m be a Φ-minimizer element of $\overset{....}{B}$. Property (A) in Theorem 3.3 implies that if m is not a dec-min element of $\overset{....}{B}$, then there is a 1-tightening step for m resulting in a decreasingly smaller member m' of $\overset{....}{B}$. The discrete convexity of φ implies that $\Phi(m') \leq \Phi(m)$, from which $\Phi(m') = \Phi(m)$ follows by the assumption that m is a Φ-minimizer. After a finite number of 1-tightening steps, we arrive at a Φ-minimizer of $\overset{....}{B}$ which is a dec-min element.

Since the dec-min elements of $\overset{....}{B}$ are value-equivalent, the symmetry of Φ implies that each dec-min element of $\overset{....}{B}$ is a Φ-minimizer. ■

Note that if φ is convex but not strictly convex, then Φ may have minimizers which are not dec-min elements. This is exemplified by the identically zero function φ for which every member of $\overset{....}{B}$ is a minimizer. However, for strictly convex functions we have the following characterization.

THEOREM 6.2. *Given an integral base-polyhedron B and a symmetric separable strictly discrete convex function Φ, an element m of $\overset{....}{B}$ is a minimizer of Φ if and only if m is a dec-min element of $\overset{....}{B}$.*

Proof. If m is a dec-min element, then m is a Φ-minimizer by Proposition 6.1. To see the converse, let m be a Φ-minimizer of $\overset{....}{B}$. If, indirectly, m is not a dec-min element, then Property (A) in Theorem 3.3 implies that there is a 1-tightening step for m resulting in decreasingly smaller member m' of $\overset{....}{B}$. The strict discrete convexity of φ implies that $\Phi(m') < \Phi(m)$ contradicting the assumption that m is a Φ-minimizer. ∎

An immediate consequence of these results is that a square-sum minimizer of $\overset{....}{B}$ minimizes an arbitrary symmetric separable discrete convex function. Note, however, that this consequence immediately follows from a much earlier result by Groenevelt [27] who characterized the elements of $\overset{....}{B}$ minimizing a (not-necessarily symmetric) discrete convex function Φ defined for $m \in \mathbf{Z}^S$ by $\Phi(m) := \sum[\varphi_s(m(s)) : s \in S]$ (where φ_s is a discrete convex function for each $s \in S$).

6.1.1 Dec-min elements versus integral square-sum minimizers

It follows from Theorem 6.2 that an element m of $\overset{....}{B}$ is dec-min precisely if m is a square-sum minimizer and the proof relied on Theorem 3.3. One may feel that it would have been a more natural approach to derive this equivalence by showing that $x \leq_{\rm dec} y$ holds precisely if $w(x) \leq w(y)$. Perhaps surprisingly, however, this equivalence fails to hold, that is, the square-sum is not order-preserving with respect to the quasi-order $\leq_{\rm dec}$. To see this, consider the following four vectors in increasing order.

$$m_1 = (2,3,3,1) <_{\rm dec} m_2 = (3,3,3,0) <_{\rm dec} m_3 = (2,2,4,1) <_{\rm dec} m_4 = (3,2,4,0).$$

Their square-sums admit a different order.

$$w(m_1) = 23,\ w(m_2) = 27,\ w(m_3) = 25,\ w(m_4) = 29.$$

It is also worth mentioning that the four vectors m_i $(i = 1,2,3,4)$ arise from the matroid M_1 given in Section 3.3 by adding the vector $(2,2,3,0)$ to the incidence vectors of the four bases of M_1. In other words, if B_1' denotes the base-polyhedron obtained from the base-polyhedron of M_1 by adding $(2,2,3,0)$, then $\overset{....}{B}{}_1' = \{m_1, m_2, m_3, m_4\}$. Among these four elements, m_1 is the unique dec-min element and the unique square-sum minimizer but the decreasing-order and the square-sum order of the other three elements are different.

By adding the same vector $(2,2,3,0)$ to the base-polyhedron of matroid M_2 in Section 3.3, we obtain a base-polyhedron B_2' whose integral elements are $\{3,3,3,0\}$, $\{2,2,4,1\}$, $\{3,2,3,1\}$, $\{2,3,4,0\}$. Let B denote the intersection of base-polyhedra B_1 and B_2. Then $\overset{....}{B} = \{\{3,3,3,0\},\{2,2,4,1\}\}$. Here $\{3,3,3,0\}$ is the unique dec-min element while $\{2,2,4,1\}$ is the unique square-sum minimizer, demonstrating that the two notions of optima may differ for the intersection of two base-polyhedra.

Finally, we remark that if φ is not only strictly discrete convex but 'rapidly' increasing as well, then $x <_{\rm dec} y$ can be proved to be equivalent to $\Phi(x) < \Phi(y)$. Here the intuitive notion of rapid increase can be formalized (for example) by

$$\varphi(k+1) \geq (n+1)\varphi(k) \quad (k \in \mathbf{Z})$$

where $n = |S|$. For example, $\varphi(k) = n^k$ is rapidly increasing in this sense, but $\varphi(k) = k^2$ is (of course) not.

6.2 Min-max theorem for integral square-sum

For a vector $z : S \to \mathbf{Z}$, we introduced the notation

$$w(z) := \sum [z(u)^2 : u \in S]$$

and called the value $w(z)$ the **square-sum** of z. We say that an element $m \in \overset{....}{B}$ is a **square-sum minimizer** (over $\overset{....}{B}$) or that m is an **integral square-sum minimizer** of B if $w(m) \leq w(z)$ holds for each $z \in \overset{....}{B}$. The main goal of this section is to derive a min-max formula for the minimum integral square-sum of an element of $\overset{....}{B}$, along with a characterization of (integral) square-sum minimizers.

A set-function h can be considered as a function defined on $(0,1)$-vectors. It is known that h can be extended in a natural way to every vector π in $\mathbf{R}^S$, as follows. For the sake of this definition, we may assume that the elements of S are indexed in a decreasing order of the components of π, that is, $\pi(s_1) \geq \cdots \geq \pi(s_n)$ (where the order of the subsequent components of π with the same value is arbitrary). For $j = 1, \dots, n$, let $I_j := \{s_1, \dots, s_j\}$ and let

$$\hat{h}(\pi) := h(I_n)\pi(s_n) + \sum_{j=1}^{n-1} h(I_j)[\pi(s_j) - \pi(s_{j+1})].$$

Obviously, $h(Z) = \hat{h}(\chi_Z)$. The function $\hat{h}$ is called the **linear extension** of h. (The linear extension was first considered by Edmonds [6] who proved for a polymatroid $P = P(b)$ defined by a monotone, non-decreasing submodular function b that $\max\{\pi x : x \in \overset{....}{P}\} = \hat{b}(\pi)$ when π is non-negative. The same approach shows for a base-polyhedron $B = B'(p)$ defined by a supermodular function p that $\min\{\pi x : x \in \overset{....}{B}\} = \hat{p}(\pi)$. Another basic result is due to Lovász [38] who proved that h is submodular if and only if $\hat{h}$ is convex. We do not, however, explicitly need these results, only remark that in the literature the linear extension is often called Lovász extension.)

The min-max formula in the next theorem concerning min square-sum over the integral elements of an integral base-polyhedron can be derived from the much more general Fenchel-type duality theorem in DCA (Discrete Convex Analysis), due to Murota (see [42] and also Theorem 8.21, page 222, in the book [43]). The advantage of the present min-max formula is that it does not need the notion of conjugate functions which is an essential part of the general result in [42]. In addition, our proof relies on the characterization of dec-min elements described in Theorem 3.3 and does not need the general tools of DCA.

THEOREM 6.3. *Let $B = B'(p)$ be a base-polyhedron defined by an integer-valued supermodular function p. Then*

$$\min\{\sum_{s\in S} m(s)^2 : m \in \overset{....}{B}\} = \max\{\hat{p}(\pi) - \sum_{s\in S} \lfloor \frac{\pi(s)}{2} \rfloor \lceil \frac{\pi(s)}{2} \rceil : \pi \in \mathbf{Z}^S\}. \qquad (26)$$

Proof. We start with two easy estimations.

Claim 6.4. *For $m, \pi \in \mathbf{Z}^S$, one has*

$$\sum_{s\in S} \lfloor \frac{\pi(s)}{2} \rfloor \lceil \frac{\pi(s)}{2} \rceil \geq \sum_{s\in S} m(s)[\pi(s) - m(s)]. \qquad (27)$$

Moreover, equality holds if and only if

$$m(s) \in \{\lfloor \frac{\pi(s)}{2} \rfloor, \lceil \frac{\pi(s)}{2} \rceil\} \text{ for every } s \in S. \qquad (28)$$

Proof. The claim follows by observing that $\lfloor \frac{a}{2} \rfloor \lceil \frac{a}{2} \rceil \geq b(a-b)$ holds for any pair of integers a and b, where equality holds precisely if $b \in \{\lfloor \frac{a}{2} \rfloor, \lceil \frac{a}{2} \rceil\}$. ■

Recall that a non-empty subset $X \subseteq S$ was called a strict π-top set if $\pi(u) > \pi(v)$ held whenever $u \in X$ and $v \in S - X$. In what follows, for an $m \in \overset{....}{B}$, m-tightness is always meant with respect to p.

Claim 6.5. *For $m \in \overset{....}{B}$ and $\pi \in \mathbf{Z}^S$, one has*

$$\hat{p}(\pi) \leq \sum_{s\in S} m(s)\pi(s). \qquad (29)$$

Moreover, equality holds if and only if each (of the at most n) strict π-top set is m-tight.

Proof. Suppose that the elements of S are indexed in such a way that $\pi(s_1) \geq \pi(s_2) \geq \cdots \geq \pi(s_n)$. For $j = 1, \dots, n$, let $I_j := \{s_1, \dots, s_j\}$. Then

$$\begin{aligned}
\hat{p}(\pi) &= p(I_n)\pi(s_n) + \sum_{j=1}^{n-1} p(I_j)[\pi(s_j) - \pi(s_{j+1})] \\
&\leq \widetilde{m}(I_n)\pi(s_n) + \sum_{j=1}^{n-1} \widetilde{m}(I_j)[\pi(s_j) - \pi(s_{j+1})] \\
&= \sum_{1\leq i\leq j\leq n} m(s_i)\pi(s_j) - \sum_{1\leq i\leq j\leq n-1} m(s_i)\pi(s_{j+1}) \\
&= \sum_{1\leq i\leq j\leq n} m(s_i)\pi(s_j) - \sum_{1\leq i< j'\leq n} m(s_i)\pi(s_{j'}) \\
&= \sum_{j=1}^{n} m(s_j)\pi(s_j),
\end{aligned}$$

from which (29) follows. Furthermore, we have equality in (29) precisely if $\widetilde{m}(I_j) = p(I_j)$ holds whenever $\pi(s_j) - \pi(s_{j+1}) > 0$. But this latter condition is equivalent to requiring that each strict π-top set is m-tight. ■

To prove max ≤ min, let m be an element of $\ddot{\ddot{B}}$ and let $\pi \in \mathbf{Z}^S$. By the two preceding claims,

$$\sum_{s\in S} m(s)^2 = \sum_{s\in S} m(s)\pi(s) - \sum_{s\in S} m(s)[\pi(s) - m(s)] \geq \hat{p}(\pi) - \sum_{s\in S} \lfloor \frac{\pi(s)}{2} \rfloor \lceil \frac{\pi(s)}{2} \rceil \quad (30)$$

from which max ≤ min indeed follows.

In order to prove max = min, we show that there are an $m \in \ddot{\ddot{B}}$ and an integral vector π for which (30) holds with equality. By the claims again, (30) holds with equality if

(O1) (28) holds and **(O2)** each strict π-top set is m-tight with respect to p.

We refer to these requirements as **optimality criteria**.

Let m be an arbitrary dec-min element of $\ddot{\ddot{B}}$. By Property (B) of Theorem 3.3, there is a chain $(\emptyset \subset)\ C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ of m-tight and m-top sets for which the restrictions of m onto the difference sets $S_i := C_i - C_{i-1}$ $(i = 1, \dots, \ell)$ are near-uniform in S_i (where $C_0 := \emptyset$). Note that $\{S_1, \dots, S_\ell\}$ is a partition of S.

Define $\pi_m : S \to \mathbf{Z}$ by

$$\pi_m(s) := 2\beta_i(m) - 1 \ \text{ if } \ s \in S_i \quad (i = 1, \dots, \ell).$$

We have

$$\lfloor \pi_m(s)/2 \rfloor = \beta_i(m) - 1 \leq m(s) \leq \beta_i(m) = \lceil \pi_m(s)/2 \rceil$$

for every $s \in S_i$, and hence the first optimality criterion (28) holds for m and π_m.

We claim that each strict π_m-top set Z is a member of chain $\mathcal{C}$. Indeed, as π_m is uniform in each S_j, if Z contains an element of S_j, then Z includes the whole S_j. Furthermore, since each C_i is an m-top set, we have $\beta_1(m) \geq \beta_2(m) \geq \cdots \geq \beta_\ell(m)$, and hence if Z includes S_j, then it includes each S_i with $i < j$. Therefore every strict π_m-top set is indeed a member of the chain, implying Optimality criterion (O2). ■ ■

It should be noted that the optimal dual solution π_m obtained in the proof of the theorem is actually an **odd** vector in the sense that each of its component is an odd integer.

Corollary 6.6. *There is an odd dual optimizer π in the min-max formula* (26) *for which the min-max formula in Theorem* 6.3 *is as follows.*

$$\min\{\sum_{s\in S} m(s)^2 : m \in \ddot{\ddot{B}}\} = \max\{\hat{p}(\pi) - \sum_{s\in S} \frac{\pi(s)^2 - 1}{4} :\ \pi \in \mathbf{Z}^S,\ \pi \textit{ is odd}\}. \quad (31)$$

We emphasize that for the proof of Theorem 6.3 and Corollary 6.6 we relied only on Theorem 3.3 and did not need the charcterization of the set of dec-min elements of $\overset{....}{B}$ given in Section 5.

In the proof of Theorem 6.3, we chose an arbitrary dec-min element m of $\overset{....}{B}$ and an arbitrary chain of m-tight and m-top sets for which the difference sets are near-uniform. In Section 5, we proved that there is a single canonical chain $\mathcal{C}^*$ which meets these properties for every dec-min element of $\overset{....}{B}$. Therefore the dual optimal π^* assigned to $\mathcal{C}^*$ is also independent of m. Namely, consider the canonical S-partition $\{S_1, \dots, S_q\}$ and the essential value-sequence $\beta_1 > \cdots > \beta_q$. Define π^* by

$$\pi^*(s) := 2\beta_i - 1 \text{ if } s \in S_i \quad (i = 1, \dots, q). \tag{32}$$

As we pointed out in the proof of Theorem 6.3, this π^* is also a dual optimum in (26). We shall prove in the next section that π^* is actually the unique smallest dual optimum in Theorem 6.3.

6.3 The set of optimal duals to integral square-sum minimization

We proved earlier that an element $m \in \overset{....}{B}$ is a square-sum minimizer precisely if it is a dec-min element. This and Theorem 5.1 imply that the square-sum minimizers of $\overset{....}{B}$ are the integral members of a base-polyhedron $B^\bullet$ obtained by intersecting a particular face of B with a special small box. This means that the integral square-sum minimizers form an M-convex set.

One of the equivalent definitions of an $\mathrm{L}^\natural$-convex set L (pronounce L-natural convex) in Discrete Convex Analysis is that L is the set of integer-valued feasible potentials. Formally, $L = \{\pi \in \mathbf{Z}^S : \pi(v) - \pi(u) \leq g(uv)\}$ where g is an integer-valued function on the ordered pairs of elements of S. (By a theorem of Gallai, L is non-empty if and only if there is no dicircuit of negative total g-weight.)

Our next goal is to show that the dual optima in Theorem 6.3 form an $\mathrm{L}^\natural$-convex set Π, and we provide a description of Π as the integral solution set of feasible potentials in a box.

Recall that the optimality criteria for a dec-min element m of $\overset{....}{B}$ and for an integral vector π were as follows:

$$\text{(O1)} \qquad m(s) \in \{\lfloor \frac{\pi(s)}{2} \rfloor, \lceil \frac{\pi(s)}{2} \rceil\} \ \text{ whenever } s \in S, \tag{33}$$

$$\text{(O2)} \qquad \text{each strict } \pi\text{-top-set is } m\text{-tight with respect to } p. \tag{34}$$

These immediately imply the following.

Proposition 6.7. *For an integral vector π, the following are equivalent.*

(A) *π is a dual optimum (that is, π belongs to* Π.

(B) *There is a dec-min element m of $\overset{....}{B}$ such that m and π meet the optimality criteria.*

(C) *For every dec-min m of $\overset{....}{B}$, m and π meet the optimality criteria.* ■

Consider the canonical S-partition $\{S_1, \dots, S_q\}$, the essential value sequence $\{\beta_1 > \beta_2 > \cdots > \beta_q\}$ and the matroids M_i on S_i $(i = 1, \dots, q)$. We can use the notions and apply the results of Section 4.3.1 formulated for M_1 to each M_i $(i = 1, \dots, q)$. To follow the pattern of $\mathcal{F}_1$ introduced in (15), let

$$\mathcal{F}_i := \{X \subseteq S_i : \ \beta_i|X| = p_i(X)\} \tag{35}$$

where p_i was defined by $p_i(X) = p(C_{i-1} \cup X) - p(C_{i-1})$ for $X \subseteq S_i$. Since $\beta_i|X| \geq p_i(X)$ for every $X \subseteq S_i$ and p_i is supermodular, $\mathcal{F}_i$ is closed under taking intersection and union. Let F_i denote the unique largest member of $\mathcal{F}_i$, that is, F_i is the union of the members of $\mathcal{F}_i$. Both $F_i = \emptyset$ and $F_i = S_i$ are possible.

THEOREM 6.8. *For an element $s \in S_i$ $(i = 1, \dots, q)$, the following properties are pairwise equivalent.*

(A) *s is value-fixed.*

(B) *$m(s) = \beta_i$ holds for every dec-min element m of $\overset{....}{B}$.*

(C) *$s \in F_i$.*

(D) *s is a co-loop of M_i.* ■

Define a digraph $D_i = (F_i, A_i)$ on node-set F_i in which st is an arc if $s, t \in F_i$ and there is no $t\overline{s}$-set in $\mathcal{F}_i$. This implies that no arc of D_i enters any member of $\mathcal{F}_i$.

THEOREM 6.9. *An integral vector $\pi \in \mathbf{Z}^S$ is an optimal dual solution to the integral minimum square-sum problem (that is, $\pi \in \Pi$ if and only if the following three conditions hold for each $i = 1, \dots, q$.*

$$\pi(s) = 2\beta_i - 1 \ \ \text{for every } s \in S_i - F_i, \tag{36}$$

$$2\beta_i - 1 \leq \pi(s) \leq 2\beta_i + 1 \ \ \text{for every } s \in F_i, \tag{37}$$

$$\pi(s) - \pi(t) \geq 0 \ \ \text{whenever } s, t \in F_i \text{ and } st \in A_i. \tag{38}$$

Proof.

Claim 6.10. *Optimality criterion (O1) is equivalent to*

$$(O1') \qquad 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1 \ \ \text{for} \ \ s \in S \tag{39}$$

Proof. When $\pi(s)$ is even, we have the following equivalences.

$$m(s) \in \{\lfloor \frac{\pi(s)}{2} \rfloor, \lceil \frac{\pi(s)}{2} \rceil \ \Leftrightarrow \ \pi(s) = 2m(s)$$
$$\Leftrightarrow \ 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1.$$

When $\pi(s)$ is odd, we have the following equivalences:

$$m(s) \in \{\lfloor \frac{\pi(s)}{2} \rfloor, \lceil \frac{\pi(s)}{2} \rceil\} \Leftrightarrow \pi(s) - 1 \leq 2m(s) \leq \pi(s) + 1$$
$$\Leftrightarrow 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1. \bullet$$

Suppose first that $\pi \in \mathbf{Z}^S$ is an optimal dual solution. Then the Optimality criteria (O1$'$) and (O2) formulated in (39) and (34) hold for every dec-min element m of $\ddot{\ddot{B}}$.

Let s be an element of $S_i - F_i$. Since s is not value-fixed, there are dec-min elements m and m' of $\ddot{\ddot{B}}$ for which $m(s) = \beta_i - 1$ and $m'(s) = \beta_i$. By applying (39) to m and to m', we obtain that

$$2\beta_i - 1 = 2m'(s) - 1 \leq \pi(s) \leq 2m(s) + 1 = 2(\beta_i - 1) + 1 = 2\beta_i - 1$$

from which $\pi(s) = 2\beta_i - 1$ follows, and hence (36) holds indeed.

Let s be an element of F_i. As s is value-fixed, $m(s) = \beta_i$ holds for any dec-min element m of $\ddot{\ddot{B}}$. We obtain from (39) that

$$2\beta_i - 1 = 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1 = 2\beta_i + 1$$

and hence (37) holds.

To derive (38), suppose indirectly that st is an arc in A_i for which $\pi(t) > \pi(s) \geq 2\beta_i - 1$. Let $Z := \{v \in S : \pi(v) \geq \pi(t)\}$. Then Z is a strict π-top set and hence $C_{i-1} \subseteq Z \subseteq C_{i-1} \cup F_i - s$. By Optimality criterion (O2), Z is m-tight with respect to p. Let $X := Z \cap S_i$. Then $X \subseteq F_i$ and hence

$$p(Z) = \widetilde{m}(Z) = \widetilde{m}(C_{i-1}) + \widetilde{m}(X) = p(C_{i-1}) + \beta_i|X|$$

from which

$$\beta_i|X| = p(Z) - p(C_{i-1}) = p_i(X),$$

that is, X is in $\mathcal{F}_i$, in a contradiction with the definition of A_i which requires that st enters no member of $\mathcal{F}_i$.

Suppose now that π meets the three properties formulated in Theorem 6.9. Let $m \in \ddot{\ddot{B}}$ be an arbitrary dec-min element. Consider an element s of S_i. If $s \in F_i$, that is, if s is value-fixed, then $m(s) = \beta_i$. By (37), we have $2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1$, that is, Optimality criterion (O1$'$) holds. If $s \in S_i - F_i$, then $\pi(s) = 2\beta_i - 1$ by (36), from which

$$\lfloor \frac{\pi(s)}{2} \rfloor = \frac{\pi(s) - 1}{2} = \beta_i - 1 \leq m(s) \leq \beta_i = \frac{\pi(s) + 1}{2} = \lceil \frac{\pi(s)}{2} \rceil,$$

showing that Optimality criterion (O1) holds.

To prove optimality criterion (O2), let Z be a strict π-top set and let $\mu := \min\{\pi(v) : v \in Z\}$. Let i denote the largest subscript for which $X := Z \cap S_i \neq \emptyset$. Then $\mu \leq 2\beta_i + 1 \leq 2\beta_{i-1} - 1 \leq \pi(u)$ holds for every $u \in C_{i-1}$ from which $C_{i-1} \subseteq Z$ as Z is a π-top set.

If $\mu = 2\beta_i - 1$, then $S_i \subseteq Z$ as Z is a strict π-top set from which $Z = S_i$ implying that Z is an m-tight set in this case. Therefore $\mu \geq 2\beta_i$ from which $X \subseteq F_i$ follows. Now $X \in \mathcal{F}_i$ for otherwise there is an arc $st \in A_i$ $(s, t \in F_i)$ entering X. But $\pi(t) \leq \pi(s)$ holds by Property (38) and this contradicts the assumption that Z is a strict π-top set. By $X \in \mathcal{F}_i$ we have $\beta_i|X| = p_i(X)$ and hence

$$\begin{aligned} \widetilde{m}(Z) &= \widetilde{m}(X) + \widetilde{m}(C_{i-1}) = \beta_i|X| + p(C_{i-1}) \\ &= p_i(X) + p(C_{i-1}) = p(X \cup C_{i-1}) - p(C_{i-1}) + p(C_{i-1}) = p(Z), \end{aligned}$$

that is, Z is indeed m-tight. ■ ■

In (32), we defined a special dual optimal solution π^* by $\pi^*(s) = 2\beta_i - 1$ whenever $s \in S_i$ $(i = 1, \dots, q)$. Theorem 6.9 and the definition we use for $\mathrm{L}^\natural$-convex sets immediately implies the following.

Corollary 6.11. *The set* Π *of optimal dual integral vectors* π *in the min-max formula* (26) *of Theorem* 6.3 *is an* $\mathrm{L}^\natural$*-convex set. The unique smallest element of* Π *(that is, the unique smallest dual optimum) is* π^*. ■

7 Algorithms

In this section, we consider algorithmic aspects of the problems investigated so far, and show how to compute efficiently a decreasingly minimal element of an M-convex set along with its canonical chain (or S-partition).

Let B be a non-empty integral base-polyhedron. As mentioned earlier, B can be given in the form $B(b)$ where b is a (fully) submodular function or in the form $B'(p)$ where p is a (fully) supermodular function. Here b and p are complementary functions (that is, $p(X) = b(S) - b(S - X)$) and hence an algorithm described for one of them can easily be transformed to work on the other. In the present description, we use supermodular functions with the remark that in applications base-polyhedra are often given with b.

There is a one-to-one correspondence between B and p but, as mentioned earlier, for an intersecting or crossing supermodular function p, $B'(p)$ is also a (possibly empty) base-polyhedron which is integral if p is integer-valued. As already indicated, for obtaining and proving results for B (or for $\ddot{B}$), it is much easier to work with a fully supermodular p while in applications base-polyhedra are often arise from—intersecting or crossing (or even weaker) —supermodular functions. Therefore in describing and analysing algorithms, we must consider these weaker functions as well.

One of the most fundamental algorithms of discrete optimization is for minimizing a submodular function, that is, for finding a subset Z of S for which $b(Z) = \min\{b(X) : X \subseteq S\}$. There are strongly polynomial algorithms for this problem (for example, Schrijver [48] and Iwata et al. [34] are the first, while Orlin [45] is one of the fastest), and we shall refer to such an algorithm as a **submod-minimizer** subroutine. The complexity of Orlin's algorithm [45], for example, is $O(n^6)$ (where $n = |S|$) and the algorithm calls $O(n^5)$ times a routine which evaluates the submodular function in question. (An evaluation routine outputs the value $b(X)$ for any input subset $X \subseteq S$). This complexity bound is definitely attractive from a theoretical point of view but in concrete applications it is always a challenge to develop faster algorithms for the special case. Naturally, submodular function minimization and supermodular function maximization are equivalent.

7.1 The basic algorithm for computing a dec-min element

Our first goal is to describe a natural approach —the **basic algorithm**— for finding a decreasingly minimal element of an M-convex set $\overset{....}{B}$. The basic algorithm is polynomial in $n + |p(S)|$, and hence it is polynomial in n when $|p(S)|$ is small in the sense that it can be bounded by a polynomial of n. This is the case, for example, in an application when we are interested in strongly connected decreasingly minimal (=egalitarian) orientations. In the general case, where typical applications arise by defining p with a 'large' capacity function, a (more complex) strongly polynomial-time algorithm will be described in the next section.

In order to find a dec-min element of an M-convex set $\overset{....}{B}$, we assume that a subroutine is available to

$$\text{compute an integral element of } B. \tag{40}$$

When $B = B'(p)$ and p is (fully) submodular, a variation of Edmonds' polymatroid greedy algorithm finds an integral member of B. (Namely, take any ordering $s_1, \dots, s_n$ of S, and define $m(s_1) := p(s_1)$ and, for $i = 2, \dots, n$, $m(s_i) = p(Z_i) - p(Z_{i-1})$ where $Z_i = \{s_1, s_2, \dots, s_i\}$. Edmonds [6] proved that vector m is indeed in B). This algorithm needs only a subroutine to evaluate $p(Z_i)$ for $i = 1, \dots, n$. If p is intersecting supermodular, then Frank and Tardos [22] described an algorithm which needs n applications of a submod-minimizer routine. For crossing supermodular p, a more complex algorithm is given in [22] which terminates after at most n^2 applications of a submod-minimizer. Note that the latter problem of finding an integral element of a base-polyhedron $B'(p)$ defined by a crossing supermodular function p covers such non-trivial problems as the one of finding a degree-constrained k-edge-connected orientation of an undirected graph, a problem solved first in [11].

Suppose now that an initial integral member m of B is available. The algorithm needs a subroutine to

$$\text{decide for } m \in \overset{....}{B} \text{ and for } s, t \in S \text{ if } m' := m + \chi_{\mathbf{s}} - \chi_{\mathbf{t}} \text{ belongs to } B. \tag{41}$$

Observe that Subroutine (41) is certainly available if we can

$$\text{decide for any } m' \in \mathbf{Z}^S \text{ whether or not } m' \text{ belongs to } B, \tag{42}$$

though applying this more general subroutine is clearly slower than a direct algorithm to realize (41).

Note that $m' = m + \chi_s - \chi_t$ is in B precisely if there is no m-tight $t\overline{s}$-set (with respect to p), and this is true even if B is defined by a crossing supermodular function p. Subroutine (41) can be carried out by a single application of a submod-minimizer.

As long as possible, apply the 1-tightening step (as described in Section 3.2). Recall that a 1-tightening step replaces m by $m' := m + \chi_s - \chi_t$ where s and t are elements of S for which $m(t) \geq m(s) + 2$ and m' belongs to $\ddddot{B}$.

By Theorem 3.3, when no more 1-tigthening step is available, the current m is a decreasingly minimal member of $\ddddot{B}$ and the algorithm terminates. In order to estimate the number of 1-tightening steps, observe that a single 1-tightening step decreases the square-sum of the components. Since the largest square-sum of an arbitrary integral vector z with $\widetilde{z}(S) = p(S)$ is $p(S)^2$ and $\widetilde{z}(S) = p(S)$ holds for all members z of $\ddddot{B}$, we conclude that the number of 1-tightening steps is at most $p(S)^2$. Therefore if $|p(S)|$ is bounded by a polynomial of n, then the basic algorithm to compute a dec-min element of $\ddddot{B}$ is strongly polynomial.

At this point, we postpone the description of the algorithm for arbitrary p, and show how the canonical chain can be computed.

7.2 Computing the essential value-sequence and the canonical chain

Once a dec-min element m of $\ddddot{B}$ is available, the essential value-sequence $\beta_1 > \beta_2 > \cdots > \beta_q$ and the canonical chain $\mathcal{C}^* = \{C_1, \dots, C_q\}$ associated with $\ddddot{B}$ can be computed iteratively by relying on Corollary 5.3, as follows. The first member β_1 is the largest value of m while $C_1 (= S_1)$ is the (unique) smallest m-tight set containing every element with m-value β_1, that is,

$$C_1 = \cup(T_m(u) : m(u) = \beta_1),$$

where $T_m(u)$ is the unique smallest m-tight set containg u with respect to the unique (fully) supermodular function p defining $B = B'(p)$.

It is fundamental to emphasize that $T_m(u)$ can be computed even in the case when the unique fully supermodular function defining B is not explicitly given and B is defined by a weaker function, for example, by a crossing supermodular function. Namely, recall from Claim 3.1 that $T_m(u) = \{s : m - \chi_s + \chi_u \in B\}$ and hence $T_m(u)$ is indeed computable by at most n applications of routine (41).

In the general case, suppose that $\beta_1 > \cdots > \beta_{i-1}$ as well as $C_1 \subset \cdots \subset C_{i-1}$ have already been computed. Then, by Corollary 5.3, β_i is the largest value of m over the elements of $S - C_{i-1}$. By Corollary 5.3 again, C_i is the (unique) smallest m-tight set containing all the elements with m-value at least β_i, that is,

$$C_i = \cup(T_m(u) : m(u) \geq \beta_i),$$

and hence C_i is also computable with n applications of routine (41). With this observation, the algorithm to compute the essential value-sequence and the canonical chain is complete.

We emphasize that the basic algorithm above for computing a dec-min element m of $\ddot{\ddot{B}}$ is polynomial only in $|p(S)|$, meaning that it is polynomial in n only if $|p(S)|$ is small (that is, $|p(S)|$ is bounded by a power of n). On the other hand, the previous algorithm to compute the essential value-sequence and the canonical chain is strongly polynomial for arbitrary p (independently of the magnitude of $|p(S)|$), provided that a dec-min element m of $\ddot{\ddot{B}}$ is already available.

Our next goal is to describe a strongly polynomial algorithm to compute a dec-min element of $\ddot{\ddot{B}}$ in the general case when no restriction is imposed on the magnitude of $|p(S)|$. To this end, we need an algorithm to maximize $\lceil \frac{p(X)}{|X|} \rceil$. We describe an algorithm for a more general case but this generality will be needed only in a forthcoming paper [21] dealing with dec-min elements of the intersection of two M-convex sets.

7.3 Maximizing $\lceil \frac{p(X)}{b(X)} \rceil$ with the Newton-Dinkelbach (ND) algorithm

On a ground-set S with $n \geq 1$ elements, we are given an integer-valued set-function p with $p(\emptyset) = 0$. (Here $p(S)$ is finite but $p(X)$ may otherwise be $-\infty$. However, $p(X)$ is never $+\infty$.) Moreover, we are also given a non-negative, finite integer-valued set-function b. Both p and b are integer-valued. Our present goal is to describe a variation of the Newton-Dinkelbach algorithm to compute the maximum $\lceil \frac{p(X)}{b(X)} \rceil$ over the subsets X of S. An excellent overview by Radzik [46] analyses this method concerning (among others) the special problem of maximizing $\frac{p(X)}{|X|}$ and describes a strongly polynomial algorithm. We present a variation of the ND-algorithm whose specific feature is that it works throughout with integers $\lceil \frac{p(X)}{b(X)} \rceil$. This has the advantage that the proof is simpler than the original one working with the fractions $\frac{p(X)}{b(X)}$.

Let M denote the largest value of b. The algorithm works if a subroutine is available to

$$\text{find a subset of } S \text{ maximizing } p(X) - \mu b(X) \quad (X \subseteq S) \quad \text{for any fixed integer } \mu \geq 0. \tag{43}$$

This routine will actually be needed only for special values of μ when $\mu = \lceil p(X)/\ell \rceil$ (where $X \subseteq S$ and $1 \leq \ell \leq M$). Note that we do not have to assume that p is supermodular and b is submodular, the only requirement for the ND-algorithm is that Subroutine (43) be available. Via a submod-minimizer, this is certainly the case when p happens to be supermodular and b submodular, since then $\mu b - p$ is submodular when $\mu \geq 0$. But (43) is also available in the more general case when the function p' defined by $p'(X) := p(X) - \mu b(X)$ is only crossing supermodular. Indeed, for a given ordered pair of elements $s, t \in S$, the restriction of p' on the family of $s\bar{t}$-sets is fully supermodular, and therefore we can apply a submod-minimizer to each of the $n(n-1)$ ordered pairs (s, t) to get the requested maximum of p'.

We call a value μ **good** if $\mu b(X) \geq p(X)$ [i.e., $p(X) - \mu b(X) \leq 0$] for every $X \subseteq S$. A value that is not good is called **bad**. We assume that there is a good μ, which is equivalent to requiring that $p(X) \leq 0$ whenever $b(X) = 0$. We also assume that $\mu = 0$ is bad.

Our goal is to compute the minimum $\mu_{\min}$ of the good integers. In other words, we want to maximize $\lceil\frac{p(X)}{b(X)}\rceil$ over the subsets of S with $b(X) > 0$.

The algorithm starts with the bad $\mu_0 := 0$. Let

$$X_0 \in \arg\max\{p(X) - \mu_0 b(X) : \; X \subseteq S\},$$

that is, X_0 is a set maximizing the function $p(X) - \mu_0 b(X) = p(X)$. Note that the badness of μ_0 implies that $p(X_0) > 0$. Since, by the assumption, there is a good μ, it follows that $\mu b(X_0) \geq p(X_0)$, and hence $b(X_0) > 0$.

The procedure determines one by one a series of pairs (μ_j, X_j) for subscripts $j = 1, 2, \ldots$ where each integer μ_j is an (intermediate) tentative candidate for μ while X_j is a non-empty subset of S. Suppose that the pair (μ_{j-1}, X_{j-1}) has already been determined for a subscript $j \geq 1$. Let μ_j be the smallest integer for which $\mu_j b(X_{j-1}) \geq p(X_{j-1})$, that is,

$$\mu_j := \lceil\frac{p(X_{j-1})}{b(X_{j-1})}\rceil.$$

If μ_j is bad, that is, if there is a set $X \subseteq S$ with $p(X) - \mu_j b(X) > 0$, then let

$$X_j \in \arg\max\{p(X) - \mu_j b(X) : \; X \subseteq S\},$$

that is, X_j is a set maximizing the function $p(X) - \mu_j b(X)$. (If there are more than one maximizing sets, we can take any). Since μ_j is bad, $X_j \neq \emptyset$ and $p(X_j) - \mu_j b(X_j) > 0$.

Claim 7.1. *If μ_j is bad for some subscript $j \geq 0$, then $\mu_j < \mu_{j+1}$.*

Proof. The badness of μ_j means that $p(X_j) - \mu_j b(X_j) > 0$ from which

$$\mu_{j+1} = \lceil\frac{p(X_j)}{b(X_j)}\rceil = \lceil\frac{p(X_j) - \mu_j b(X_j)}{b(X_j)}\rceil + \mu_j > \mu_j. \; \blacksquare$$

Since there is a good μ and the sequence μ_j is strictly monotone increasing by Claim 7.1, there will be a first subscript $h \geq 1$ for which μ_h is good. The algorithm terminates by outputting this μ_h (and in this case X_h is not computed or needed anymore).

THEOREM 7.2. *If h is the first subscript during the run of the algorithm for which μ_h is good, then $\mu_{\min} = \mu_h$ (that is, μ_h is the requested smallest good μ-value) and $h \leq M$, where M denotes the largest value of b.*

Proof. Since μ_h is good and μ_h is the smallest integer for which $\mu_h b(X_{h-1}) \geq p(X_{h-1})$, the set X_{h-1} certifies that no good integer μ can exist which is smaller than μ_h, that is, $\mu_{\min} = \mu_h$.

Proposition 7.3. *If μ_j is bad for some subscript $j \geq 1$, then $b(X_j) < b(X_{j-1})$.*

Proof. As $\mu_j \ \ (= \lceil \frac{p(X_{j-1})}{b(X_{j-1})} \rceil)$ is bad, we obtain that

$$p(X_j) - \mu_j b(X_j) > 0 = p(X_{j-1}) - \frac{p(X_{j-1})}{b(X_{j-1})} b(X_{j-1}) \geq$$

$$p(X_{j-1}) - \lceil \frac{p(X_{j-1})}{b(X_{j-1})} \rceil b(X_{j-1}) = p(X_{j-1}) - \mu_j b(X_{j-1})$$

from which we get

$$\text{(A)} \qquad p(X_j) - \mu_j b(X_j) > p(X_{j-1}) - \mu_j b(X_{j-1}).$$

Since X_{j-1} maximizes $p(X) - \mu_{j-1} b(X)$, it follows that

$$\text{(B)} \qquad p(X_{j-1}) - \mu_{j-1} b(X_{j-1}) \geq p(X_j) - \mu_{j-1} b(X_j).$$

By adding up (A) and (B), we obtain

$$(\mu_j - \mu_{j-1}) b(X_{j-1}) > (\mu_j - \mu_{j-1}) b(X_j).$$

As μ_j is bad, so is μ_{j-1}, and hence, by applying Claim 7.1 to $j-1$ in place of j, we obtain that $\mu_j > \mu_{j-1}$, from which we arrive at $b(X_j) < b(X_{j-1})$, as required. ■

Proposition 7.3 implies that $M \geq b(X_0) > b(X_1) > \cdots > b(X_{h-1})$ from which $1 \leq b(X_{h-1}) \leq M - (h-1)$, and hence $h \leq M$ follows. ■ ■

Note that the ND-algorithm is definitely polynomial in the special case when M is bounded by a a power of n, since the number of phases is bounded by M. In the special case when $b(X) = |X|$, $M = n$.

In several applications, the requested general purpose submod-minimizer can be superceded by a direct and more efficient algorithm such as the ones for network flows or for matroid partition.

7.4 Computing a dec-min element in strongly polynomial time

In the present context, we need the ND-algorithm above only in the special case when b is the cardinality function, that is, $b(X) = |X|$ for each $X \subseteq S$. Note that in this special case $M = |S|$, and hence the sequence of bad μ_i values has at most $|S|$ members by Theorem 7.2.

After at most $|S|$ applications of Subroutine (43), the ND-algorithm terminates with the smallest integer β_1 for which $\ddddot{B}$ has a β_1-covered member m. It is well-known that such an m can easily be computed with a greedy-type algorithm, as follows. Since there is a β_1-covered member of B, the identical vector $(\beta_1, \beta_1, \dots, \beta_1)$ vector belongs to the so-called supermodular polyhedron $S'(p) := \{x : \widetilde{x}(X) \geq p(X)$ for every $X \subseteq S\}$. Consider the elements of S in an arbitrary order $\{s_1, \dots, s_n\}$. Let

$m(s_1) := \min\{z : (z, \beta_1, \beta_1, \dots, \beta_1) \in S'(p)\}$. In the general step, if the components $m(s_1), \dots, m(s_{i-1})$ have already been determined let

$$m(s_i) := \min\{z : (m(s_1), m(s_2), \dots, m(s_{i-1}), z, \beta_1, \beta_1, \dots, \beta_1) \in S'(p)\}. \quad (44)$$

This computing can be carried out by n applications of a subroutine for a submodular function minimization. (Note that the previous algorithm to compute a β_1-covered integral member of B is nothing but a special case of the algorithm that finds an integral member of a base-polyhedron given by an intersecting submodular function.

Given a β_1-covered integral element of B, our next goal is to obtain a pre-dec-min element of $\overset{....}{B}$. To this end, we apply 1-tightening steps. That is, as long as possible, we pick two elements s and t of S for which $m(t) = \beta_1$ and $m(s) \leq \beta_1 - 2$ so that there is no m-tight $t\overline{s}$-set, reduce $m(t)$ by 1 and increase $m(s)$ by 1. In this way, we obtain another integral element of B for which the largest component continues to be β_1 (as β_1 was chosen to be the smallest upper bound) but the number of β_1-valued components is strictly smaller. Therefore, after at most $|S|-1$ such 1-tightening steps, we arrive at a vector for which no 1-tightening step (with $m(t) = \beta_1$ and $m(s) \leq \beta_1 - 2$) is possible anymore, and hence this final vector is a pre-decreasingly minimal element of $\overset{....}{B}$ by Theorem 4.2. We use the same letter m to denote this pre-dec-min element.

Recall that $T_m(t)$ denoted the unique smallest tight set containing t when p is fully supermodular. But $T_m(t)$ can be described without explicitly referring to p since an element $s \in S$ belongs to $T_m(t)$ precisely if $m' := m - \chi_t + \chi_s$ is in B, and this is computable by subroutine (41). Therefore we can compute $S_1(m)$ (defined in (6)). It was proved in Theorem 4.4 that $S_1(m)$ is the first member S_1 of the canonical S-partition associated with $\overset{....}{B}$.

The restriction $m_1 := m|S_1$ is a near-uniform member of the restriction of $\overset{....}{B}$ to S_1, and by Theorem 4.6, if m_1' is a dec-min element of $\overset{....}{B}_1'$, then (m_1, m_1') is a dec-min element of $\overset{....}{B}$, where B_1' is the base-polyhedron obtained from B by contracting S_1. Such a dec-min element m_1' can be computed by applying iteratively the computation described above for computing m_1.

8 Applications

8.1 Background

There are two major sources of applicability of the results in the preceding sections. One of them relies on the fact that the class of integral base-polyhedra is closed under several operations. For example, a face of a base-polyhedron is also a base-polyhedron, and so is the intersection of an integral box with a base-polyhedron B. Also, the sum of integral base-polyhedra $B_1, \dots, B_k$ is a base-polyhedron B which has, in addition, the integer decomposition property meaning that any integral element of B can be obtained as the sum of k integral elements by taking one from each B_i. This latter property implies that the sum of M-convex sets is M-convex. We also mention the important operation of taking an aggregate of a base-polyhedron, to be introduced below in Section 8.2.

The other source of applicability is based on the fact that not only fully super- or submodular functions can define base-polyhedra but some weaker functions as well. For example, if p is an integer-valued crossing (in particular, intersecting) supermodular function with finite $p(S)$, then $B = B'(p)$ is a (possibly empty) integral base-polyhedron (and $\overset{....}{B}$ is an M-convex set). This fact will be exploited in solving dec-min orientation problems when both degree-constraints and edge-connectivity requirements must be fulfilled. In some cases even weaker set-functions can define base-polyhedra. This is why we can solve dec-min problems concerning edge- and node-connectivity augmentations of digraphs.

8.2 Matroids

Levin and Onn [37] solved algorithmically the following problem. Find k bases of a matroid M on a ground-set S such that the sum of their characteristic vectors be decreasingly minimal. Their approach, however, does not seem to work in the following natural extension. Suppose we are given k matroids $M_1, \dots, M_k$ on a common ground-set S, and our goal is to find a basis B_i of each matroid M_i in such a way that the vector $\sum[\chi_{B_i} : i = 1, \dots, k]$ is decreasingly minimal. Let $B_{\sum}$ denote the sum of the base-polyhedra of the k matroids. By a theorem of Edmonds, the integral elemenst of $B_{\sum}$ are exactly the vectors of form $\sum[\chi_{B_i} : i = 1, \dots, k]$ where B_i is a basis of M_i. Therefore the problem is to find a dec-min element of $\overset{....}{B}_{\sum}$. This can be found by the basic algorithm described in Section 7. Let us see how the requested subroutines are available in this special case. The algorithm starts with an arbitrary member m of $\overset{....}{B}_{\sum}$ which is obtained by taking a basis B_i from each matroid M_i, and these bases define $m := \sum_i \chi_{B_i}$.

To realize Subroutine (41), we mentioned that it suffices to realize Subroutine (42), which requires for a given integral vector m' with $\widetilde{m}'(S) = \sum_i r_i(S)$ to decide whether m' is in $\overset{....}{B}_{\sum}$ or not. But this can simply be done by Edmonds' matroid intersection algorithm. Namely, let $S_1, \dots, S_k$ be disjoint copies of S and M'_i an isomorphic copy of M_i on S_i. Let N_1 be the direct sum of matroids M'_i on ground-set $S' := S_1 \cup \dots \cup S_k$. Let N_2 be a partition matroid on S' in which a subset Z is a basis if it contains exactly $m'(s)$ members of the k copies of s for each $s \in S$. Then m' is in $\overset{....}{B}_{\sum}$ precisely if N_1 and N_2 have a common basis.

In conclusion, with the help of Edmonds' matroid intersection algorithm, Subroutine (41) is available, and hence the basic algorithm can be applied. (Actually, the algorithm can be speeded up by looking into the details of the matroid intersection algorithm for N_1 and N_2.)

Another natural problem concerns a single matroid M on a ground-set T. Suppose we are given a partition $\mathcal{P} = \{T_1, \dots, T_n\}$ of T and we consider the intersection vector $(|Z \cap T_1|, \dots, |Z \cap T_n|)$ assigned to a basis Z of M. The problem is to find a basis for which the intersection vector is decreasingly minimal.

To solve this problem, we recall an important construction of base-polyhedra, called the aggregate. Let T be a ground-set and B_T an integral base-polyhedron in $\mathbf{R}^T$. Let $\mathcal{P} = \{T_1, \dots, T_n\}$ be a partition of T into non-empty subsets and let $S = \{s_1, \dots, s_n\}$ be a set whose elements correspond to the members of $\mathcal{P}$. The aggregate B_S of B_T is defined as follows.

$$B_S := \{(y_1, \dots, y_n) : \text{there is an } x \in B_T \text{ with } y_i = \widetilde{x}(T_i)\ (i = 1, \dots, n)\}. \tag{45}$$

A basic theorem concerning base-polyhedra states that B_S is a base-polyhedron, moreover, for each integral member $(y_1, \dots, y_n)$ of B_S, the vector x in (45) can be chosen integer-valued. In other words,

$$\overset{....}{B}_S := \{(y_1, \dots, y_n) : \text{there is an } x \in \overset{....}{B}_T \text{ with } y_i = \widetilde{x}(T_i)\ (i = 1, \dots, n)\}. \tag{46}$$

We call $\overset{....}{B}_S$ the **aggregate** of $\overset{....}{B}_T$.

Returning to our matroid problem, let B_T denote the base-polyhedron of matroid M. Then the problem is nothing but finding a dec-min element of $\overset{....}{B}_S$.

We can apply the basic algorithm (concerning M-convex sets) for this special case since the requested subroutines are available through standard matroid algorithms. Namely, Subroutine (40) is available since for any basis Z of M, the intersection vector assigned to Z is nothing but an element of $\overset{....}{B}_S$.

To realize Subroutine (41), we mentioned that it suffices to realize Subroutine (42). Suppose we are given a vector $y \in \mathbf{Z}_+^S$ (Here y stands for m' in (42)). Suppose that $\widetilde{y}(S) = r(T)$ (where r is the rank-function of matroid M) and that $y(s_i) \leq |T_i|$ for $i = 1, \dots, n$

Let $G = (S, T; E)$ denote a bipartite graph where $E = \{ts_i : t \in T_i, i = 1, \dots, n\}$. By this definiton, the degree of every node in T is 1 and hence the elements of E correspond to the elements of M. Let M_1 be the matroid on E corresponding to M (on T). Let M_2 be a partition matroid on E in which a set $F \subseteq E$ is a basis if $d_F(s_i) = y(s_i)$. By this construction, the vector y is in $\overset{....}{B}_S$ precisely if the two matroids M_1 and M_2 have a common basis. This problem is again tractable by Edmonds' matroid intgersection algorithm.

As a special case, we can find a spanning tree of a (connected) directed graph for which its in-degree-vector is decreasingly minimal. Since the family of unions of k disjoint bases of a matroid forms also a matroid, we can also compute k edge-disjoint spanning trees in a digraph whose union has a decreasingly minimal in-degree vector.

Another special case is when we want to find a spanning tree of a connected bipartite graph $G = (S, T; E)$ whose in-degree vector restricted to S is decreasingly minimal.

8.3 Flows

Let $D = (V, A)$ be a digraph endowed with integer-valued bounding functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ for which $f \leq g$. We call a vector (or function) z on A **feasible** if $f \leq z \leq g$. The **net-in-flow** Ψ_z of z is a vector on V and defined by $\Psi_z(v) = \varrho_z(v) - \delta_z(v)$, where $\varrho_z(v) := \sum[z(uv) : uv \in A]$ and $\delta_z(v) := \sum[z(vu) : uv \in A]$. If m is the net-in-flow of a vector z, then we also say that z is an m**-flow**.

A variation of Hoffman's classic theorem on feasible circulations [32] is as follows.

Lemma 8.1. *An integral vector $m : V \to \mathbf{Z}$ is the net-in-flow of an integral feasible vector (or in other words, there is an integer-valued feasible m-flow) if and only $\widetilde{m}(V) = 0$ and*

$$\varrho_f(Z) - \delta_g(Z) \leq \widetilde{m}(Z) \quad \textit{holds whenever} \quad Z \subseteq V, \tag{47}$$

where $\varrho_f(Z) := \sum[f(a) : a \in A$ and a enters $Z]$ and $\delta_g(Z) := \sum[g(a) : a \in A$ and a leaves $Z]$. ■

Define a set-function p_{fg} by

$$p_{fg}(Z) := \varrho_f(Z) - \delta_g(Z).$$

Then p_{fg} is (fully) supermodular (see, e.g. Proposition 1.2.3 in [15]). Consider the base-polyhedron $B_{fg} := B'(p_{fg})$ and the M-convex set $\overset{....}{B}_{fg}$. By the algorithm described in Section 7, we can compute a decreasingly minimal element of $\overset{....}{B}_{fg}$ in strongly polynomial time. By relying on a strongly polynomial push-relabel algorithm, we can check whether or not (47) holds. If it does not, then the push-relabel algorithm can compute a set most violating (47) (that is, a maximizer of $\varrho_f(Z) - \delta_g(Z) - \widetilde{m}(Z)$) while if (47) does hold, then the push-relable algorithm computes an integral valued feasible m-flow. Therefore the requested oracles in the general algorithm for computing a dec-min element are available through a network flow algorithm, and we do not have to rely on a general-purpose submodular function minimizing oracle.

For the sake of an application of this algorithm to capacitated dec-min orientations in Section 9.3, we remark that the algorithm can also be used to compute a dec-min element of the M-convex set obtained from $\overset{....}{B}_{fg}$ by translating it with a given integral vector.

8.3.1 Discrete version of Megiddo's flow problem

Megiddo [39], [40] considered the following problem. Let $D = (V, A)$ be a digraph endowed with a non-negative capacity function $g : A \to \mathbf{R}+$. Let S and T be two disjoint non-empty subsets of V. Megiddo described an algorithm to compute a feasible flow from S to T with maximum flow amount M for which the net-in-flow vector restricted on S is (in our terms) increasingly maximal. Here a feasible flow is a vector x on A for which $\Psi_x(v) \leq 0$ for $v \in S$, $\Psi_x(v) \geq 0$ for $v \in T$, and $\Psi_x(v) = 0$ for $v \in V - (S \cup T)$. The flow amount x is $\sum[\Psi_x(t) : t \in T]$.

We emphasize that Megiddo solved the continuous (fractional) case and did not consider the corresponding discrete (or integer-valued) flow problem. To our knowledge, this natural optimization problem has not been investigated so far.

To provide a solution, suppose that g is integer-valued. Let $f \equiv 0$ and consider the net-in-flow vectors belonging to feasible vectors. These form a base-polyhedron B_1 in $\mathbf{R}^V$. Let B_2 denote the base polyhedron obtained from B_1 by intersecting it with the box defined by $z(v) \leq 0$ for $v \in S$, $z(v) \geq 0$ for $v \in T$ and $z(v) = 0$ for $v \in V - (S \cup T)$. The restriction of B_2 to S is a g-polymatroid Q in $\mathbf{R}^S$. And finally, we can consider the face of Q defined by $\widetilde{z}(S) = -M$. This is a base-polyhedron B_3 in $\mathbf{R}^S$, and the discrete verson of Megiddo's flow problem is equivalent to finding an inc-max element of $\ddot{B}_3$. (Recall that an element of an M-convex set is dec-min precisely if it is inc-max.)

It can be shown that in this case again the general submodular function minimizing subroutine used in the algorithm to find a dec-min element of an M-convex set can be replaced by a max-flow min-cut algorithm.

8.4 Further applications

8.4.1 Root-vectors of arborescences

Another graph-example comes from packing arborescences. Let $D = (V, A)$ be a digraph and $k > 0$ an integer. We say that a non-negative integral vector $m : V \to \mathbf{Z}_+$ is a **root-vector** if there are k edge-disjoint spanning arborescences such that each node $v \in V$ is the root of $m(v)$ arborescences. Edmonds [7] classic result on disjoint arborescences implies that m is a root-vector if and only if $\widetilde{m}(V) = k$ and $\widetilde{m}(X) \geq k - \varrho(X)$ holds for every subset X with $\emptyset \subset X \subset V$. Define set-function p by $p(X) := k - \varrho(X)$ if $\emptyset \subset X \subseteq V$ and $p(\emptyset) := 0$. Then p is intersecting supermodular, so $B'(p)$ is an integral base-polyhedron. The intersection B of $B'(p)$ with the non-negative orthant is also a base-polyhedron, and the theorem of Edmonds is equivalent to stating that a vector m is a root-vector if and only if m is in $\ddot{B}$.

Therefore the general results on base-polyhedra can be specialized to obtain k disjoint spanning arborescences whose root-vector is decreasingly minimal.

8.4.2 Connectivity augmentations

Let $D = (V, A)$ be a directed graph and $k > 0$ an integer. We are interested in finding a so-called augmenting digraph $H = (V, F)$ of γ arcs for which $D + H$ is k-edge-connected or k-node-connected. In both cases, the in-degree vectors of the augmenting digraphs are the integral elements of an integral base-polyhedron [14], [17]. Obviously, the in-degree vectors of the augmented digraphs are the integral elements of an integral base-polyhedron.

Again, our results on general base-polyhedra can be specialized to find an augmenting digraph whose in-degree vector is decreasingly minimal.

9 Orientations of graphs, I

Let $G = (V, E)$ be an undirected graph. For $X \subseteq V$, let $i_G(X) = i(X)$ denote the number of edges induced by X while $e_G(X) = e(X)$ is the number of edges with at least one end-node in X. Then i_G is supermodular, e_G is submodular, and they are complementary functions, that is, $i(X) = e(V) - e(V - X)$. Let $B_G := B(e_G) = B'(i_G)$ denote the base-polyhedron defined by e_G or i_G.

We say that a function $m : V \to \mathbf{Z}$ is the **in-degree vector** of an orientation D of G if $\varrho_D(v) = m(v)$ for each node $v \in V$. An in-degree vector m obviously meets the equality $\widetilde{m}(V) = |E|$. The following basic result, sometimes called the Orientation lemma, is due to Hakimi [28].

Lemma 9.1 (Orientation lemma). *Let $G = (V, E)$ be an undireted graph and $m : V \to \mathbf{Z}$ an integral vector for which $\widetilde{m}(V) = |E|$. Then G has an orientation with in-degree vector m if and only*

$$\widetilde{m}(X) \leq e_G(X) \text{ for every subset } X \subseteq V, \tag{48}$$

which is equivalent to

$$\widetilde{m}(X) \geq i_G(X) \text{ for every subset } X \subseteq V. \blacksquare \tag{49}$$

This immediately implies the following claim.

Claim 9.2. *The in-degree vectors of orientations of G are precisely the integral elements of base-polyhedron B_G $(= B(e_G) = B'(i_G))$, that is, the set of in-degree vectors of orientations of G is the M-convex set $\overset{....}{B}_G$.* $\blacksquare$

9.1 Decreasingly minimal orientations

Due to Claim 9.2, we can apply the earlier results on dec-min elements to the special base-polyhedron B_G. Recall that Borradaile et al. [4] called an orientation of G egalitarian if its in-degree vector is decreasingly minimal but we prefer the term **dec-min orientation** since an orientation with an increasingly maximal in-degree vector also has an intuitive egalitarian feeling. Such an orientation is called **inc-max**. For example, Theorem 3.3 immediately implies the following.

Corollary 9.3. *An orientation of G is dec-min if and only if it is inc-max.* $\blacksquare$

Note that the term dec-min orientation is asymmetric in the sense that it refers to in-degree vectors. One could also aspire for finding an orientation whose out-degree vector is decreasingly minimal. But this problem is clearly equivalent to the in-degree version and hence in the present work we do not consider out-degree vectors, with a single exception in Section 9.1.2.

By Theorem 3.3, an element m of $\overset{....}{B}_G$ is decreasingly minimal if and only if there is no 1-tightening step for m. What is the meaning of a 1-tightening step in terms of orientations?

Claim 9.4. *Let D be an orientation of G with in-degree vector m. Let t and s be nodes of G. The vector $m' := m + \chi_s - \chi_t$ is in B_G if and only if D admits a dipath from s to t.*

Proof. $m' \in B_G$ holds precisely if there is no $t\overline{s}$-set X which is tight with respect to i_G, that is, $\widetilde{m}(X) = i_G(X)$. Since $\varrho(Y) + i_G(Y) = \sum[\varrho(v) : v \in Y] = \widetilde{m}(Y)$ holds for any set $Y \subseteq V$, the tightness of X is equivalent to requiring that $\varrho(X) = 0$. Therefore $m' \in B_G$ if and only if $\varrho(Y) > 0$ holds for every $t\overline{s}$-set Y, which is equivalent to the existence of a dipath of D from s to t. ∎

Recall that a 1-tightening step at a member m of B_G consists of replacing m by m' provided that $m(s) \geq m(t) + 2$ and $m' \in B_G$. By Claim 9.4, a 1-tigthening step at a given orientation of G corresponds to reorienting an arbitrary dipath from a node s to node t for which $\varrho(s) \geq \varrho(t) + 2$. Therefore Theorem 3.3 immediately implies the following basic theorem of Borradaile et al. [4].

THEOREM 9.5 (Borradaile et al. [4])**.** *An orientation D of a graph $G = (V, E)$ is decreasingly minimal if and only if no dipath exists from a node s to a node t for which $\varrho(t) \geq \varrho(s) + 2$.* ∎

Note that this theorem also implies Corollary 9.3. It immediately gives rise to an algorithm for finding a dec-min orientation. Namely, we start with an arbitrary orientation of G. We call a dipath **feasible** if $\varrho(t) \geq \varrho(s) + 2$ holds for its starting node s and end-node t. The algorithm consists of reversing feasible dipaths as long as possible. Since the sum of the squares of in-degrees always drops when a feasible dipath is reversed, and originally this sum is at most $|E|^2$, the dipath-reversing procedure terminates after at most $|E|^2$ reversals. By Theorem 9.5, when no more feasible paths exists, the current orientation is dec-min. The general basic algorithm concerning general base-polyheda in Section 7 is nothing but extension of the algorithm of Borradaile et al.

It should be noted that they suggested to choose at every step the current feasible dipath in such a way that the in-degree of its end-node t is as high as possible, and they proved that the algorithm in this case terminates after at most $|E||V|$ dipath reversals.

Note that we obtained Corollary 9.3 as a special case of a result on M-convex sets but it is also a direct consequence of Theorem 9.5.

9.1.1 Canonical chain

We briefly remark, that the results on the canonical chain associated with a general base-polyhedron can also be applied to the special base-polyhedron B_G. In this way, we arrive at the decomposition theorem of Borradaile et al. [5] concerning graphs.

9.1.2 Orientation with dec-min in-degree vector and dec-min out-degree vector

We mentioned that dec-min and inc-max orientations always concern in-degree vectors. As an example to demonstrate the advantage of the general base-polyhedral view, we outline here one exception when in-degree vectors and out-degree vectors play a symmetric role. The problem is to characterize undirected graphs admitting an orientation which is both dec-min with respect to its in-degree vector and dec-min with respect to its out-degree vector.

For the present purposes, we let d_G denote the degree vector of G, that is, $d_G(v)$ is the number of edges incident to $v \in V$. (This notation differs from the standard set-function meaning of d_G.)

Let $B_{\rm in}$ denote the convex hull of the in-degree vectors of orientations of G, and $B_{\rm out}$ the convex hull of out-degree vectors of orientations of G. (Earlier $B_{\rm in}$ was denoted by B_G but now we have to deal with both out-degrees and in-degrees.) As before, $\ddddot{B}_{\rm in}$ is the set of in-degree vectors of orientations of G, and $\ddddot{B}_{\rm out}$ is the set of out-degree vectors of orientations of G. Let $\ddddot{B}^{\bullet}_{\rm in}$ denote the set of dec-min in-degree vectors of orientations of G, and $\ddddot{B}^{\bullet}_{\rm out}$ the set of dec-min out-degree vectors of orientations of G. By Theorem 5.5 both $\ddddot{B}^{\bullet}_{\rm in}$ and $\ddddot{B}^{\bullet}_{\rm out}$ are matroidal M-convex sets.

Note that the negative of a (matroidal) M-convex set is also a (matroidal) M-convex set, and the translation of a (matroidal) M-convex set by an integral vector is also a (matroidal) M-convex set. Therefore $d_G - \ddddot{B}^{\bullet}_{\rm out}$ is a matroidal M-convex set. Clearly, a vector $m_{\rm in}$ is the in-degree vector of an orientation D of G precisely if $d_G - m_{\rm in}$ is the out-degree vector of D.

We are interested in finding an orientation whose in-degree vector is dec-min and whose out-degree vector is dec-min. This is equivalent to finding a member $m_{\rm in}$ of $\ddddot{B}^{\bullet}_{\rm in}$ for which the vector $m_{\rm out} := d_G - m_{\rm in}$ is in the matroidal M-convex set $\ddddot{B}^{\bullet}_{\rm out}$. But this latter is equivalent to requiring that $m_{\rm in}$ is in the M-convex set $d_G - \ddddot{B}^{\bullet}_{\rm out}$. That is, the problem is equivalent to finding an element of the intersection of the matroidal M-convex sets $\ddddot{B}^{\bullet}_{\rm in}$ and $d_G - \ddddot{B}^{\bullet}_{\rm out}$. This latter problem can be solved by Edmonds matroid intersection algorithm [8].

9.2 In-degree constrained dec-min orientation

Let $f : V \to \mathbf{Z} \cup \{-\infty\}$ be a lower bound function and $g : V \to \mathbf{Z} \cup \{+\infty\}$ an upper bound function for which $f \leq g$. We are interested in in-degree constrained orientations of G by which we mean that $f(v) \leq \varrho(v) \leq g(v)$ for every $v \in V$. Since the in-degree vectors are the integral elements of a base-polyhedron B_G by Claim 9.2, the in-degree vectors of in-degree constrained orientations are the integral elements of the integral base-polyhedron B_{fg} obtained by intersecting B_G with the box $T(f, g)$. By applying Theorem 3.3 to B_{fg}, we obtain the following extension of Theorem 9.5

THEOREM 9.6. *An in-degree-constrained orientation D of G is decreasingly minimal if and only if there are no two nodes s and t for which $\varrho(t) \geq \varrho(s)+2$, $\varrho(t) > f(t)$, $\varrho(s) < g(s)$, and there is a dipath in D from s to t.* ■

9.3 Capacitated orientation

Consider the following capacitated version of the basic dec-min orientation problem of Borradaile et al. Suppose that a positive integer $\ell(e)$ is assigned to each edge e of G. Denote by G^+ the graph arising from G by replacing each edge e of G with

$\ell(e)$ parallel edges. Our goal is to find a dec-min orientation of G^+. In this case, an orientation of G^+ is described by telling that, among the $\ell(e)$ parallel edges connecting the end-nodes u and v of e how many are oriented toward v (implying that the rest of the $\ell(e)$ edges are oriented toward u). In principle, this problem can be solved by applying the algorithm described above to G^+, and this algorithm is satisfactory when ℓ is small in the sense that its largest value can be bounded by a power of $|E|$. The difficulty in the general case is that the algorithm will be polynomial only in the number of edges of G^+, that is, in $\widetilde{\ell}(E)$, and hence this algorithm is not polynomial in $|E|$.

We show how the algorithm in Section 8.3 can be used to solve the decreasingly minimal orientation problem in the capacitated case in strongly polynomial time. To this end, let $D = (V, A)$ be an arbitrary orientation of G serving as a reference orientation. Define a capacity function g on A by $g(\vec{e}) := \ell(e)$ where $\vec{e}$ denotes the arc of D obtained by orienting e.

We associate an orientation of G^+ with an integral vector $z : A \to \mathbf{Z}_+$ with $z \leq g$ as follows. For an arc uv of D, orient $z(uv)$ parallel copies of $e = uv \in E$ toward v and $g(uv) - z(uv)$ parallel copies toward u. Then the in-degree of a node v is $m_z(v) := \varrho_z(v) + \delta_{g-z}(v) = \varrho_z(v) - \delta_z(v) + \delta_g(v)$. Therefore our goal is to find an integral vector z on A for which $0 \leq z \leq g$ and the vector m_z on V is dec-min. Consider the set of net-in-flow vectors $\{(\Psi_z(v) : v \in V) : 0 \leq z \leq g\}$. In Section 8.3, we proved that this is a base-polyhedron B_1. Therefore the set of vectors $(m_z(v) : v \in V)$ is also a base-polyhedron B arising from B_1 by translating B_1 with the vector $(\delta_D(v) : v \in V)$.

As remarked at the end of Section 8.3, a dec-min element of $\ddot{B}$ can be computed in strongly polynomial time by relying on a push-relabel subroutine for network flows (and not using a general-purpose submodular function minimizer).

Note that the capacitated version of the dec-min in-degree constrained orientation problem can be managed in an analogous way.

9.4 Application to resource allocation

We proved for general M-convex sets (Theorem 6.2) that the dec-min elements are exactly those minimizing the square-sum of components. This can be applied to the special cases described above, that is, our approach provides a solution to finding an in-degree constrained and k-edge-connected orientation for which the total sum of in-degree squares is minimum.

As an application, we show first how a result of Harvey et al. [29] follows immediately. They introduced the notion of a semi-matching of a simple bipartite graph $G = (S, T; E)$ as a subset F of edges for which $d_F(t) = 1$ holds for every node $t \in T$.

Harvey et al. solved the problem of finding a semi-matching F for which $\sum[d_F(s) \cdot (d_F(s) + 1) : s \in S]$ is minimum. Bokal at al. [3] extended the results to quasi-matchings, subgraphs of G meeting a general degree-spcification on T. These problems were motivated by practical applications in the area of resource allocation in computer science. Note that

$$\sum[d_F(s)(d_F(s)+1) : s \in S] = \sum[d_F(s)^2 : s \in S] + \sum[d_F(s) : s \in S] =$$

$$\sum[d_F(s)^2 : s \in S] + |F| = \sum[d_F(s)^2 : s \in S] + |T|,$$

and therefore the problem of Harvey et al. is equivalent to finding a semi-matching F of G that minimizes the square-sum of degrees in S.

By orienting each edge in F toward S and each edge in $E - F$ toward T, a semi-matching can be interpreted as an orientation of $G = (S, T; E)$ in which the out-degree of every node $t \in T$ is 1 (that is, $\varrho(t) = d_G(t) - 1$). In this case, the set of arcs oriented toward S corresponds to a semi-matching and $d_F(s) = \varrho(s)$ for each $s \in S$. Since $\varrho(t)$ for $t \in T$ is the same in these orientations, it follows the total sum of $\varrho(v)^2$ over $S \cup T$ is minimized precisely if $\sum[\varrho(s)^2 : s \in S] = \sum[d_F(s)^2 : s \in S]$ is minimized.

But this is a special degree-constrained orientation problem (where the degree-constraint is actually a degree-specification on the elements of T), and we mentioned above that exactly the dec-min orientations minimize the sum of the in-degree squares. The same approach applies in the more general setting, when we have upper and lower bounds $g_S(s)$ and $f_S(s)$, respectively, for $d_F(s)$ on the elements s of S. Rather than the identically 1 function, one may impose an arbitrary degree-specification m_T on T satisfying $0 \leq m_T(t) \leq d_G(t)$ $(t \in T)$, and consider subsets F of E for which $d_F(t) = m_T(t)$ for each $t \in T$ and $f_S(s) \leq d_F(s) \leq g_S(s)$ for each $s \in S$. Such an F may be called a degree-constrained m_T-semi-matching (degree-constrained in S and degree-specified in T). This corresponds to an orientation of G in which $\varrho(t) = d_G(t) - m_T(t)$ for $t \in T$ and $f(s) \leq \varrho(s) \leq g(s)$ for $s \in S$. Again, an egalitarian degree-constrained orientation is a solution to the problem of minimizing $\sum[\varrho(v)^2 : v \in S \cup T]$ which is equivalent to minimizing $\sum[\varrho(s)^2 : s \in S] = \sum[d_F(s)^2 : s \in S]$.

The degree-specification on T can be relaxed even further by requiring only degree-constraints on T but imposing a specified cardinality γ of F. That is, we are given a positive integer γ, and integral-valued lower and upper bounds f and g on $V = S \cup T$. We consider degree-constrained subgraphs $(S, T; F)$ of G for which $|F| = \gamma$, and want to find such a subgraph for which $\sum[d_F(s)^2 : s \in S]$ is minimum (Notice the asymmetric role of S and T.) This is equivalent to finding an in-degree constrained orientation of G for which $\varrho(S) = \gamma$ and $\sum[\varrho(s)^2 : s \in S]$ is minimum. Here the corresponding in-degree constraints on S are $f_S := f|S$ and $g_S := g|S$ while f_T and g_T on T are defined for $t \in T$ by

$$f_T(t) := d_G(t) - g(t) \quad \text{and} \quad g_T(t) := d_G(t) - f(t).$$

Let B denote the base-polyhedron spanned by the in-degree vectors of the degree-constrained orientations of G. Then the restriction of B to S is a g-polymatroid Q. By intersecting Q with the hyperplane $\{x : \widetilde{x}(S) = \gamma\}$ we obtain an integral base-polyhedron B_S in $\mathbf{R}^S$, and then the elements of $\dddot{B}_S$ are exactly the in-degree vectors of the requested orientations restricted to S. That is, the elements of $\dddot{B}_S$ are the restriction of the degree-vectors of the requested subgraphs of G to S. Since B_S

is a base-polyhedron, a dec-min element of $\ddddot{B}_S$ will be a solution to our minimum degree-square sum problem.

Algorithmic aspects of minimum degree square-sum problems for general graphs were discussed by Apollonio and Sebő [1].

Finally, we briefly indicate that a capacitated version of the problem above can also be solved in polynomial time. Let $G = (S, T; E)$ be again a bipartite graph, γ a positive integer, and f_V and g_V integer-valued bounding functions on $V := S \cup T$ for which $f_V \leq g_V$. In addition, an integer-valued capacity function g_E is also given on the edge-set E, and we are interested in finding an integral vector $z : E \rightarrow \mathbf{Z}_+$ for which $\widetilde{z}(E) = \gamma$, $z \leq g_E$ and $f_V(v) \leq d_z(v) \leq g_V(v)$ for every $v \in V$. We call such a vector **feasible**. The problem is to find a feasible vector z whose degree vector restricted to S (that is, $(d_z(s_1), \dots, d_z(s_n))$ is decreasingly minimal.

By replacing each edge e with $g_E(e)$ parallel edges, it follows from the uncapacitated case above that the vectors $\{(d_z(s_1), \dots, d_z(s_n)) : z$ is a feasible vector$\}$ form an M-convex set. In this case, however, the basic algorithm is not necessarily polynomial since the values of g_E may be large. Therefore we need the general strongly polynomial algorithm described in Section 7.4. In this case the general-purpose Subroutine (43) can be realized by max-flow min-cut computations.

We close this section with some historical remarks. The problem of Harvey et al. is closely related to earlier investigations in the context of minimizing a separable convex function over (integral elements of) a base-polyhedron. For example, Federgruen and Groenevelt [9] provided a polynomial time algorithm in 1988. Hochbaum and Hong [31] in 1995 developed a strongly polynomial algorithm. Their proof, however, included a technical gap which was fixed by Moriguchi, Shioura, and Tsuchimura [41] in 2011. For an early book on resource allocation, see the one by Ibaraki and Katoh [33] while three more recent surveys are due to Katoh and Ibaraki [35] from 1998, to Hochbaum [30] from 2007, and to Katoh, Shioura, and Ibaraki [36] from 2013.

10 Orientations of graphs, II

In this section, we investigate various edge-connectivity requirements for the orientations of G. The main motivation behind these investigations is a conjecture of Borradaile et al. [4] on decreasingly minimal strongly connected orientations. Our goal is to prove their conjecture in a more general form.

10.1 Strongly connected orientations

Suppose that G is 2-edge-connected implying that it has a strong orientation by a theorem of Robbins [47]. We are interested in dec-min strong orientations meaning that the in-degree vector is decreasingly minimal over the strong orientations of G. This problem of Borradaile et al. was motivated by a practical application concerning optimal interval routing schemes.

Analogously to Theorem 9.5, they described a natural way to improve a strong orientation D to get another one whose in-degree vector is decreasingly smaller. Suppose that there are two nodes s and t for which $\varrho(t) \geq \varrho(s) + 2$ and there are two edge-disjoint dipaths from s to t in D. Then reorienting an arbitrary st-dipath of D results in another strongly connected orientation of D which is clearly decreasingly smaller than D. Borradaile et al. conjectured the truth of the converse (and this conjecture was the starting point of our investigations). The next theorem states that the conjecture is true.

THEOREM 10.1. *A strongly connected orientation D of $G = (V, E)$ is decreasingly minimal if and only if there are no two arc-disjoint st-dipaths in D for nodes s and t with $\varrho(t) \geq \varrho(s) + 2$.*

Proof. Suppose first that there are nodes s and t with $\varrho(t) \geq \varrho(s) + 2$ such that there are two arc-dijoint st-dipaths of D. Let P be any st-dipath in D and let D' denote the digraph arising from D by reorienting P. Then D' is strongly connected, since if it had a node-set Z $(\emptyset \subset Z \subset V)$ with no entering arcs, then Z must be a $t\overline{s}$-set and P enters Z exactly once. But then $0 = \varrho_{D'}(Z) = \varrho_D(Z) - 1 \geq 2 - 1 = 1$, a contradiction. Therefore D' is indeed strong and its in-degree vector is decreasingly smaller then that of D.

To see the non-trivial part, define a set-function p_1 as follows.

$$p_1(X) := \begin{cases} 0 & \text{if} \quad X = \emptyset \\ |E| & \text{if} \quad X = V \\ i_G(X) + 1 & \text{if} \quad \emptyset \subset X \subset V. \end{cases} \tag{50}$$

Then p_1 is crossing supermodular and hence $B_1 := B'(p_1)$ is a base-polyhedron.

Claim 10.2. *An integral vector m is the in-degree vector of a strong orientation of G if and only if m is in $\overset{....}{B}_1$.*

Proof. If m is the in-degree vector of a strong orientation of G, then $\widetilde{m}(V) = |E| = p_1(V)$, $\widetilde{m}(\emptyset) = 0 = p_1(\emptyset)$, and

$$\widetilde{m}(Z) = \sum[\varrho(v) : v \in Z] = \varrho(Z) + i_G(Z) \geq 1 + i_G(Z) = p_1(Z)$$

for $\emptyset \subset Z \subset V$, that is, $m \in \overset{....}{B}_1$.

Conversely, let $m \in \overset{....}{B}_1$. Then $m \in B_G$ and hence by Claim 9.2, G has an orientation D with in-degree vector m. We claim that D is strongly connected. Indeed,

$$\varrho(Z) = \sum[\varrho(v) : v \in Z] - i_G(Z) = \widetilde{m}(Z) - i_G(Z) \geq p_1(Z) - i_G(Z) = 1$$

whenever $\emptyset \subset Z \subset V$. ■

Claim 10.3. *Let D be a strong orientation of G with in-degree vector m. Let t and s be nodes of G. The vector $m' := m + \chi_s - \chi_t$ is in B_1 if and only if D admits 2 arc-disjoint dipaths from s to t.*

Proof. $m' \in B_1$ holds precisely if there is no $t\overline{s}$-set X which is tight with respect to p_1, that is, $\widetilde{m}(X) = i_G(X) + 1$. Since $\varrho(Y) + i_G(Y) = \sum[\varrho(v) : v \in Y] = \widetilde{m}(Y)$ holds for any set $Y \subset V$, the tightness of X (that is, $\widetilde{m}(X) = i_G(X) + 1$) is equivalent to requiring that $\varrho(X) = 1$. Therefore $m' \in B_1$ if and only if $\varrho(Y) > 0$ holds for every $t\overline{s}$-set Y, which is, by Menger's theorem, equivalent to the existence of two arc-disjoint st-dipaths of D. ■

By Theorem 3.3, m is a dec-min element of $\ddot{B}_1$ if and only if there is no 1-tighening step for m. By Claim 10.3 this is just equivalent to the condition in the theorem that there are no two arc-disjoint st-dipaths in D for nodes s and t for which $\varrho(t) \geq \varrho(s)+2$. • ■

An immediate consequence of Claim 10.2 and Theorem 3.3 is the following.

Corollary 10.4. *A strong orientation of G is dec-min if and only if it is inc-max.* ■

We indicated above how in-degree constrained dec-min orientations can be managed due to the fact that the intersection of an integral base-polyhedron B with an integral box T is an integral base-polyhedron. The same approach works for degree-constrained strong orientations. For example, in this case dec-min and inc-max again coincide and one can formulate the in-degree constrained versions of Theorems 9.5 and 10.1. In the next section, we overview more general cases.

10.1.1 A counter-example for mixed graphs

Although Robbins' theorem on strong orientability of undirected graphs easily extends to mixed graphs, as was pointed out by Boesch and Tindell [2], it is not true anymore that a decreasingly minimal strong orientation of a mixed graph is always increasingly maximal. Actually, one may consider two natural variants.

In the first one, decreasing minimality and increasing maximality concern the total in-degree of the directed graph obtained from the initial mixed graph after orienting its undirected edges. Let $V = \{a, b, c, d\}$. Let $E = \{ab, cd\}$ denote the set of undirected edges and let $A = \{ad, ad, ad, da, da, bc, bc, cb\}$ denote the set of directed edges of a mixed graph $M = (V, A + E)$. There are two strong orientations of M. In the first one, the orientations of the elements of E are ba and dc, in which case the total in-degree vector is $(3, 1, 3, 3)$. In the second one, the orientations of the elements of E are ab and cd, in which case the total in-degree vector is $(2, 2, 2, 4)$. Now $(3, 1, 3, 3)$ is dec-min while $(2, 2, 2, 4)$ is inc-max.

In the second variant, we are interested in the in-degree vector of the digraph obtained by orienting the originally undirected part E. For this version the counterexample is as follows. Let $V = \{a, b, c, d, x, y, u, v\}$. Let $E = \{ab, cd, au, au, av, av, dy, dy, bx, bx\}$ denote the set of undirected edges and let $A = \{ad, da, bc, cb\}$ denote the set of directed edges of a mixed graph $M = (V, A + E)$. The undirected part of M is denoted by $G = (V, E)$.

In any strong orientation of $M = (V, A+E)$, the orientations of the undirected parallel edge-pairs $\{au, au\}$, $\{av, av\}$, $\{dy, dy\}$, $\{bx, bx\}$ are oriented oppositely, and hence their contribution to the in-degrees (in the order of a, b, c, d, u, v, x, y) is $(2, 1, 0, 1, 1, 1, 1, 1)$.

Therefore there are essentially two distinct strong orientations of M. In the first one, the undirected edges ab, cd are oriented as ba, dc, while in the second one the undirected edges ab, cd are oriented as ab, cd. Hence the in-degree vector of the first strong orientation corresponding to the orientation of G (in the order of a, b, c, d, u, v, x, y) is $(3, 1, 1, 1, 1, 1, 1, 1)$. The in-degree vector of second strong orientation corresponding to the orientation of G is $(2, 2, 0, 2, 1, 1, 1, 1)$. The first vector is inc-max while the second vector is dec-min.

These examples give rise to the question: what is behind the phenomenon that while dec-min and inc-max coincide for strong orientations of undirected graphs, they differ for strong orientations of mixed graph? The explanation is, as we pointed out earlier, that for an M-convex set the two notions coincide and the set of in-degree vectors of strong orientations of an indirected graph is an M-convex set, while the set of in-degree vectors of a mixed graph is, in general, not an M-convex set. It is actually the intersection of two M-convex sets. In [21], we shall describe an algorithm for computing a dec-min element of the intersection of two M-convex sets.

10.2 Higher edge-connectivity

An analogous approach works in a much more general setting. We say that a digraph covers a set-function h if $\varrho(X) \geq h(X)$ holds for every set $X \subseteq V$. The following result was proved in [11].

THEOREM 10.5. *Let h be a finite-valued, non-negative crossing supermodular function with $h(\emptyset) = h(V) = 0$. A graph $G = (V, E)$ has an orientation covering h if and only of*

$$e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V_i) \quad \textit{and} \quad e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V - V_i)$$

hold for every partition $\mathcal{P} = \{V_1, \dots, V_q\}$ of V, where $e_{\mathcal{P}}$ denotes the number of edges connecting distinct parts of $\mathcal{P}$. ■

This theorem easily implies the classic orientation result of Nash-Williams [44] stating that a graph G has a k-edge-connected orientation precisely if G is $2k$-edge-connected. Even more, call a digraph (k, ℓ)-edge-connected ($\ell \leq k$) (with respect to a root-node r_0) if $\varrho(X) \geq k$ whenever $\emptyset \subset X \subseteq V - r_0$ and $\varrho(X) \geq \ell$ whenever $r_0 \in X \subset V$. Then Theorem 10.5 implies:

THEOREM 10.6. *A graph $G = (V, E)$ has a (k, ℓ)-edge-connected orientation if and only if*

$$e_{\mathcal{P}} \geq k(q - 1) + \ell$$

holds for every q-partite partition $\mathcal{P}$ of V. ■

A more general special case of Theorem 10.5 characterizes graphs admitting in-degree-constrained and (k,ℓ)-edge-connected orientations ($\ell \leq k$), where a digraph is called (k,ℓ)-edge-connected with respect to a root-node r_0 if there are k arc-disjoint dipaths from r_0 to every node and there are ℓ arc-disjoint dipaths from every node to r_0.

It is important to emphasize that however general is Theorem 10.5, it does not say anything about strong orientations of mixed graphs. In particular, it does not imply the pretty but easily provable theorem of Boesch and Tindell [2]. The problem of finding decreasingly minimal in-degree constrained k-edge-connected orientation of mixed graphs will be solved in [21].

The next lemma shows why the set of in-degree vectors of orientations of G covering the set-function h appearing in Theorem 10.5 is an M-convex set, ensuring in this way the possibility for applying the results on decreasing minimization over M-convex sets to general graph orientation problems.

Lemma 10.7. *An orientation D of G covers h if and only if its in-degree vector m is in the base-polyhedron $B = B'(p)$ where $p := h + i_G$ is a crossing supermodular function.*

Proof. Suppose first that m is the in-degree vector of a digraph covering h. Then $h(X) \leq \varrho(X) = \widetilde{m}(X) - i_G(X)$ for $X \subset V$ and $h(V) = 0 = \varrho(V) = \widetilde{m}(V) - i_G(V)$, that is, m is indeed in B.

Conversely, suppose that $m \in B$. Since h is finite-valued and non-negative, we have $\widetilde{m}(X) \geq p(X) \geq i_G(X)$ for $X \subset V$ and $\widetilde{m}(V) = i_G(V)$ and hence, by the Orientation lemma, there is an orientation D of G with in-degree vector m. Moreover, this digraph D covers h since $\varrho_D(X) = \widetilde{m}(X) - i_G(X) \geq p(X) - i_G(X) = h(X)$ holds for $X \subset V$. ■

By Lemma 10.7, Theorem 3.3 can be applied again to the general orientation problem covering a non-negative and crossing supermodular set-function h in the same way as it was applied in the special case of strong orientation above, but we formulate the result only for the special case of in-degree constrained and k-edge-connected orientations.

THEOREM 10.8. *Let $G = (V,E)$ be an undirected graph endowed with a lower bound function $f : V \to \mathbf{Z}$ and an upper bound function $g : V \to \mathbf{Z}$ with $f \leq g$. A k-edge-connected and in-degree-constrained orientation D of G is decreasingly minimal if and only if there are no two nodes s and t for which $\varrho(t) \geq \varrho(s) + 2$, $\varrho(t) > f(t)$, $\varrho(s) < g(s)$, and there are $k+1$ arc-disjoint st-dipaths.* ■

The theorem can be extended even further to in-degree-constrained and (k,ℓ)-edge-connected orientations ($\ell \leq k$), where a digraph is called (k,ℓ)-edge-connected with respect to a root-node r_0 if there are k arc-disjoint dipaths from r_0 to every node and there are ℓ arc-disjoint dipaths from every node to r_0.

10.2.1 An extension

We say that a digraph $D = (V,A)$ is k-edge-connected in a specified subset S of nodes if there are k-arc-disjoint dipaths in D from any node of S to any other node of S.

By relying on Lemma 10.7, one can derive the following.

THEOREM 10.9. *Let $G = (V, E)$ be an undirected graph with a specified subset S of V. Let m_0 be an in-degree specification on $V - S$. The set of in-degree vectors of those orientations of G which are k-edge-connected in S and in-degree specified in $V - S$ is an M-convex set.* ■

By this theorem, we can determine a decreasingly minimal orientation among those which are k-edge-connected in S and in-degree specified in $V - S$. Even additional in-degree constraints can be imposed on the elements of S.

10.2.2 Hypergraph orientation

Let $H = (V, \mathcal{E})$ be a hypergraph for which we assume that each hyperedge has at least 2 nodes. Orienting a hyperedge Z means that we designate an element z of Z as its head-node. A hyperedge Z with a designated head-node $z \in Z$ is a directed hyperedge denoted by (Z, z). Orienting a hypergraph means the operation of orienting each of its hyperedges. We say that a directed hyperedge (Z, z) enters a subset X of nodes if $z \in X$ and $Z - X \neq \emptyset$. A directed hypergraph is called k-edge-connected if the in-degree of every non-empty proper subset of nodes is at least k.

The following result was proved in [19] (see, also Theorem 2.22 in the survey paper [18]).

THEOREM 10.10. *The set of in-degree vectors of k-edge-connected and in-degree constrained orientations of a hypergraph forms an M-convex set.* ■

Therefore we can apply the general results obtained for decreasing minimization over base-polyhedra.

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