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Global Rigidity of Unit Ball Graphs

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Abstract

A d-dimensional bar-and-joint framework (G, p), where G is a graph and p maps the vertices of G to points in $\mathbb{R}^d$, is said to be globally rigid if every d-dimensional framework (G, q) with the same graph and same edge lengths is congruent to (G, p). Global rigidity of frameworks and graphs is a well-studied area of rigidity theory with a number of applications, including the localization problem of sensor networks.

Motivated by this application we consider the new notion of unit ball global rigidity, which can be defined similarly, except that (G, p) as well as (G, q) are required to be unit ball frameworks in the above definition. In a unit ball framework two vertices are adjacent if and only if their distance is less than a fixed constant (which corresponds to the sensing radius in a sensor network).

In this paper we initiate a theoretical analysis of this version of global rigidity and prove several structural results. Among others we identify families of frameworks (and corresponding graphs G) in $\mathbb{R}^d$, for all $d \geq 1$, which are unit ball globally rigid without being globally rigid in the usual sense. These families contain minimally rigid graphs, too, which have less edges than any of the globally rigid graphs on the same number of vertices.

1 Introduction

Consider a set of n points in $\mathbb{R}^d$ and fix a subset of the pairwise distances between the pairs of points. It can happen that the fixed subset uniquely determines all pairwise distances, that is, all d-dimensional configurations of the point set with these fixed distances are pairwise congruent. When this happens, we say that the set of points, together with the pairs of points for which the distance is fixed, is globally rigid. It is convenient to model this situation by a so-called framework, which consists of a graph G on n vertices (corresponding to the points) and a map from the vertex set of G to $\mathbb{R}^d$, which defines the configuration. Two vertices are connected by an edge in G if and only if the distance between the corresponding points is fixed.

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The uniqueness problem above emerges naturally e.g. in sensor network localization and in molecular conformation where the goal is to find the locations of a set of n sensors (resp. atoms) in the plane (resp. in three-space) when only a subset of the pairwise distances is available. The global rigidity of frameworks is a central and well-studied notion in rigidity theory. Several results from this field have been used in and motivated by the above applications. Characterizations of global rigidity as well as algorithmic methods for finding the unique configuration are both interesting in this context.

In this paper we shall consider a modified version of global rigidity, motivated by a specific family of sensor networks: the case when a pair of sensors can communicate – and compute their distance by sending radio signals – if and only if they are within a given threshold value r (the sensing radius) from each other. We may assume that $r = 1$. In this case the corresponding framework is a unit ball framework (or unit disk framework, when the network is in $\mathbb{R}^2$).

Although not all sensor networks belong to this category, there is a substantial amount of work dealing with various problems arising in sensor networks where this unit disk model is applicable. Nevertheless, the localization aspects of this model and the corresponding question of "global rigidity within the family of unit ball frameworks" has rarely been studied before.

In this version uniqueness (with respect to a set of fixed pairwise distances) is meant only within the family of unit ball frameworks: point configrations in which there is a pair of points $\{u, v\}$ which are non-adjacent in G but whose distance is less than 1, need not be considered. This key observation was used by Oliva et al. [27] to show that some "implied" edges can be added to the framework without changing the set of unit ball frameworks satisfying the fixed distances. With these new edges a trilateration process may become possible and can be used to localize the network. Kaewprapha et al. [23] used the observation above to show that all configurations consistent with the fixed distances can be found more efficiently by taking into account the unit disk property which is used to substantially reduce the size of the search tree. We are not aware of other related results.

Our goal is to initiate a theoretical analysis of this version of global rigidity and to obtain new structural results which may lead to new concepts and methods in the sensor network localization problem. Among others we identify families of frameworks (and corresponding graphs G) which are globally rigid within the family of unit ball frameworks without G having globally rigid realizations in the usual sense. These families contain minimally rigid graphs, too, which have less edges than any of the globally rigid graphs on the same number of vertices.

The structure of the paper is as follows. In Section 2 we introduce the basic terminology of rigid and globally rigid frameworks and graphs and list some key results. In Section 3 we define unit ball graphs and frameworks and provide some preliminary results, including a complete analyis of our problem in $\mathbb{R}^1$. Section 4 contains our main result: every saturated non-globally rigid unit ball graph has a unit ball globally rigid generic realization. In Section 5 we study special graphs which are minimally rigid and saturated non-globally rigid at the same time. These graphs occur in various other applications, too, and as we shall see, many of them have unit disk realizations.

In Section 6 we describe a different method for identifying unit ball globally rigid frameworks, based on tensegrity frameworks. We provide a number of open questions in the concluding Section 7 and suggest further research problems.

2 Rigid and globally rigid graphs and frameworks

A d-dimensional *bar-and-joint framework* (or simply *framework*) is a pair (G, p), where $G = (V, E)$ is a simple graph and p is a map from V to $\mathbb{R}^d$. We may think of the vertices as universal joints and the edges as rigid (i.e. fixed length) bars connecting certain pairs of joints. A framework (G, p) is said to be a *realization* of G in $\mathbb{R}^d$. We say that two realizations (G, p) and (G, q) of a graph G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, and *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. Here $|| \cdot ||$ denotes the Euclidean norm in $\mathbb{R}^d$.

A d-dimensional framework (G, p) is *globally rigid* in $\mathbb{R}^d$ if every framework in $\mathbb{R}^d$ which is equivalent to (G, p) is congruent to (G, p). In other words, the edge lengths uniquely determine all pairwise distances. We shall also consider the following weaker property of frameworks. We say that (G, p) is *rigid* in $\mathbb{R}^d$ if every continuous motion of its vertices in $\mathbb{R}^d$ which preserves all edge lengths takes the framework to a realization of G which is congruent to (G, p).

It is NP-hard to test whether a given framework is globally rigid (resp. rigid) in $\mathbb{R}^d$ for all $d \geq 1$ (resp. $d \geq 2$), see [30], [1]. However, if we exclude “degenerate” point configurations, these problems may become more tractable. This leads us to the following notion. We say that (G, p) is *generic* if the set of the $d|V|$ coordinates of the vertices is algebraically independent over the rationals.

It is known that both the rigidity and the global rigidity of frameworks in $\mathbb{R}^d$ is a generic property for every fixed dimension $d \geq 1$, that is, the rigidity (resp. global rigidity) of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic, see [3, 12]. Thus we say that the graph G is *generically rigid*, or simply *rigid* (resp. *generically globally rigid*, or simply *globally rigid*), in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid (resp. globally rigid).

The following necessary conditions of global rigidity, due to B. Hendrickson, establish an important link between rigidity and global rigidity. We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all edges e of G. A graph G is called *k-connected* if G has at least $k+1$ vertices and $G - S$ is connected for every $S \subset V(G)$ with $|S| \leq k - 1$.

Theorem 2.1. *[14] Let G be globally rigid in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

It is known that the necessary conditions of Theorem 2.1 together are sufficient to characterize global rigidity in $\mathbb{R}^d$ for $d = 1, 2$. It is a folklore result that G is globally rigid in $\mathbb{R}^1$ if and only if G is isomorphic to a complete graph on at most two vertices, or G is 2-connected. The two-dimensional result is due to B. Jackson and the second author.

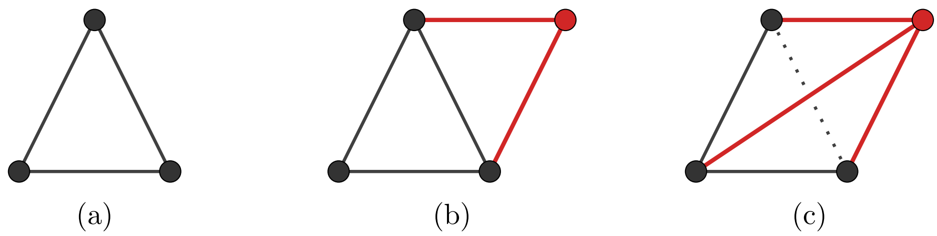


Figure 1: (b) A 0-extension, and (c) a 1-extension of a triangle.

Theorem 2.2. *[15] Let $G = (V, E)$ be a graph on at least four vertices. Then G is globally rigid in $\mathbb{R}^2$ if and only if G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

There are examples showing that these conditions are not sufficient when $d \geq 3$, see [7, 20]. Finding a good characterization for the family of globally rigid graphs in $\mathbb{R}^d$ for $d \geq 3$ is a major open problem.

We recall some key results and definitions concerning rigidity as well. A rigid graph $G = (V, E)$ is called *minimally rigid* (or *Laman*) in $\mathbb{R}^d$ if $G - e$ is not rigid for all $e \in E$. The edge sets of the minimally rigid graphs (in a given dimension) on a vertex set V correspond to the bases of the so-called *rigidity matroid*, defined on the edge set of a complete graph on V, and have the same size. This size is equal to $d|V| - \binom{d+1}{2}$ whenever $|V| \geq d$; in particular, it is $2|V| - 3$ in $\mathbb{R}^2$.

For a subset $X \subseteq V$ we use $i(X)$ to denote the number of edges induced by X. The 0-*extension* operation adds a new vertex v to G and two new edges vx, vy for two distinct vertices $x, y \in V$. The *1-extension* operation on edge uw and vertex z adds a new vertex v, deletes uw, and adds three new edges vu, vw, vz. We say that the graph G is *sparse* if

$$i(X) \leq 2|X| - 3 \text{ for all } X \subseteq V \text{ with } |X| \geq 2. \tag{1}$$

The following characterization of minimally rigid graphs in the plane is due to G. Laman.

Theorem 2.3. *[25] Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$. Then the following are equivalent:*
(i) G is minimally rigid in $\mathbb{R}^2$,
(ii) G is sparse,
(iii) G can be obtained from K_2 by a sequence of 0-extensions and 1-extensions.

3 Unit ball graphs

A d-dimensional framework (G, p) is said to be a *unit ball framework* if for all $u, v \in V$ we have

$$uv \in E \iff \|p(u) - p(v)\| < 1. \tag{2}$$

The graph G is a (d-dimensional) *unit ball graph* if it has a unit ball realization in $\mathbb{R}^d$. In other words, unit ball graphs are the intersection graphs of open unit balls in $\mathbb{R}^d$.

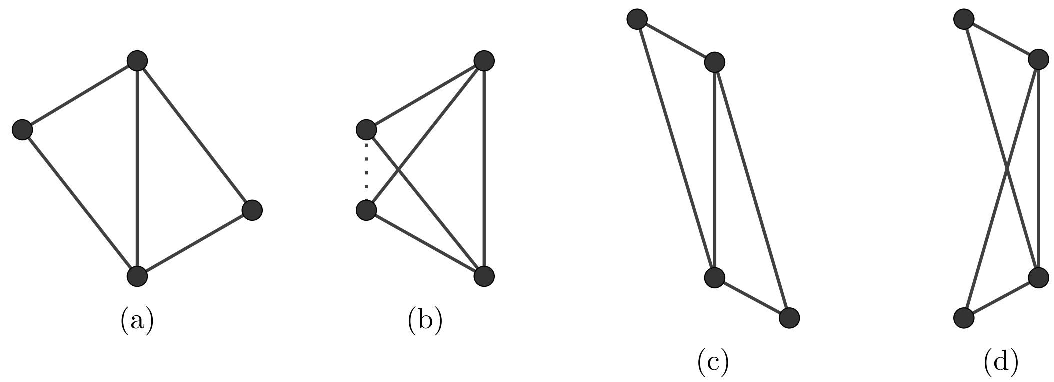


Figure 2: (a) A UBGR and (c) a non-UBGR realization of a rigid but not globally rigid graph, with corresponding equivalent but not congruent realizations.

We follow the standard terminology and in $\mathbb{R}^1$ and $\mathbb{R}^2$ we use the terms *unit interval* and *unit disk*, respectively, in order to emphasize the fixed dimension.

We note that unit ball graphs are often defined as the intersection graphs of *closed* unit balls, as opposed to our definition using open balls. However, as it was shown in [26] for $d = 2$, the two definitions are equivalent. As we shall see, having strict inequality in (2) is more convenient in our context.

Deciding whether G is a unit ball graph in $\mathbb{R}^d$ is NP-hard for all $d \geq 2$ [6, 24]. Surprisingly few structural results are known about unit disk graphs, and unit ball graphs in general, and most of what is known is related to forbidden subgraphs. For example, a unit disk graph cannot contain the complete bipartite graphs $K_{1,6}$ and $K_{2,3}$ as induced subgraphs. Other examples can be found in [5].

The idea of investigating global rigidity within the family of unit ball frameworks leads us to the following definition. We say that a d-dimensional unit ball framework (G, p) is *unit ball globally rigid* in $\mathbb{R}^d$ if whenever (G, q) is an equivalent unit ball realization of G in $\mathbb{R}^d$, it is congruent to (G, p). As in the case of global rigidity, we shall focus on generic unit ball globally rigid realizations in order to capture the typical behaviour of unit ball graphs. The following remark justifies this by showing that the generic unit ball realizations of a graph form a dense subset of its unit ball realizations.

Consider a unit ball realization (G, p) in $\mathbb{R}^d$ of a d-dimensional unit ball graph G. Then there exists an $\epsilon' > 0$ such that for all $0 < \epsilon < \epsilon'$ the scaled realization $(G, (1 + \epsilon)p)$ has an open neighbourhood (in the space $\mathbb{R}^{d|V|}$ of realizations) U for which (G, q) is unit ball for all $q \in U$. Hence G has a generic unit ball realization arbitrarily close to (G, p).

Unlike global rigidity, unit ball global rigidity is not a generic property in $\mathbb{R}^d$ for $d \geq 2$. See Figure 2. Thus we may define two families of unit ball graphs: we say that a unit ball graph G is *unit ball globally rigid* (UBGR) in $\mathbb{R}^d$ if there exists a d-dimensional generic unit ball realization of G which is unit ball globally rigid. We call G *strongly unit ball globally rigid* (SUBGR) in $\mathbb{R}^d$ if every d-dimensional generic unit ball realization of G is unit ball globally rigid.

A necessary condition of unit ball global rigidity is rigidity.

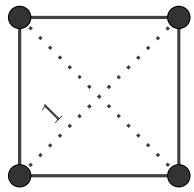


Figure 3: Example of a non-generic realization of C_4 that is UBGR but not rigid. In an equivalent but not congruent realization one of the diagonals must have length less than 1.

Lemma 3.1. *Let (G, p) be a generic unit ball globally rigid unit ball realization of G in $\mathbb{R}^d$. Then (G, p) is rigid in $\mathbb{R}^d$ (and hence G is rigid in $\mathbb{R}^d$).*

Proof. Suppose that (G, p) is not rigid and consider a continuous motion which results in an equivalent but non-congruent realization (G, q). Since (G, p) is generic, $||p(u) - p(v)||$ cannot be an integer for any pair $u, v \in V(G)$. Thus $||p(u) - p(v)|| > 1$ for all non-adjacent pairs u, v. This implies that in a small enough range the motion preserves the unit ball property and takes (G, p) to an equivalent but non-congruent unit disk realization (G, q'). This means that (G, p) is not unit ball globally rigid, a contradiction. □

Rigidity is no longer necessary if we drop the genericity assumption. Indeed, a square with unit length diagonals is an example of a non-rigid but unit disk globally rigid (non-generic) unit disk framework in the plane.

Clearly, if a unit ball graph is globally rigid in $\mathbb{R}^d$ (in the usual sense), then it is strongly unit ball globally rigid as well. Thus we have

$$GGR \subseteq SUBGR \subseteq UBGR \subseteq GR \tag{3}$$

within the family of d-dimensional unit ball graphs, for every fixed $d \geq 1$. A natural question is whether the containment relations in (3) are proper - see Section 7 for further comments on this.

3.1 Unit ball global rigidity on the line

To illustrate our notions we start with a simple result which provides a complete characterization of (strong) unit ball global rigidity in $\mathbb{R}^1$. It turns out that unit ball global rigidity is a generic property of unit interval graphs.

Theorem 3.2. *Let G be a unit interval graph. The following properties are equivalent:*
(i) G is connected,
(ii) G is unit ball globally rigid in $\mathbb{R}^1$,
(iii) G is strongly unit ball globally rigid in $\mathbb{R}^1$.

Proof. We show that (i) implies (iii). The implications (iii)→ (ii)→ (i) are straightforward. Let G be a unit interval graph and let (G, p) be a generic unit interval realization of G in $\mathbb{R}^1$. If G is complete or 2-connected then (G, p) is globally rigid in $\mathbb{R}^1$, and thus also strongly unit ball globally rigid in $\mathbb{R}^1$. Otherwise every one-dimensional realization (G, q) of G which is equivalent to (G, p) can be obtained from (G, p) by a sequence of partial reflections about cut-vertices.[1] However, such a partial reflection on the line destroys the unit interval property of the framework. Hence there is no equivalent but non-congruent unit interval realization of (G, p). This implies that G is strongly unit ball globally rigid in $\mathbb{R}^1$. □

A related observation is that if G is a connected unit interval graph then the so-called block-cut-vertex tree of G is a path. We omit the proof.

3.2 Minimally rigid unit ball graphs

In this subsection we collect some preliminary observations concerning minimally rigid unit ball graphs which may be of independent interest, although we shall not use them in the rest of the paper. We consider minimally rigid graphs (or rather a certain subfamily of them) in more depth in Section 5.

We start with a simple structural result in the plane.

Lemma 3.3. *Let $G = (V, E)$ be a unit disk graph with $|V| \geq 3$ and $|E| \geq 2|V| - 3$. Then G contains a triangle.*

Proof. Consider a unit disk realization (G, p) of G in $\mathbb{R}^2$. This realization can be considered as a straight line embedding of G in the plane. If there are no edge crossings and no triangles then it follows from Euler's formula that $|E| \leq 2|V| - 4$, a contradiction.

Thus we may suppose that G has two edges ab, cd, for which the corresponding line segments X, Y cross. By the triangle inequality and the unit disk property at least one of the edges ac, bd (resp. ad, cb) is present in G, showing that G contains a triangle in this case as well. □

As a corollary, we obtain that a minimally rigid unit disk graph in $\mathbb{R}^2$ (on at least three vertices) contains a triangle.

It is easy to see that every rigid subgraph of a minimally rigid graph is minimally rigid. On the other hand it is known (and, again, easy to see) that K_{d+2} is not minimally rigid in d dimensions. These observations together imply that a minimally rigid graph cannot contain K_{d+2} as a subgraph, a fact that we will use in the following proof.

Lemma 3.4. *For every positive integer $d \geq 1$ there exists a constant $f(d)$ such that the maximum degree of a minimally rigid unit ball graph in $\mathbb{R}^d$ is at most $f(d)$. In particular, $f(2) = 11$.*

Proof. First consider the two-dimensional case. Let (G, p) be a unit disk framework and for a contradiction suppose that $d(v) \geq 12$ for some vertex v. Divide the unit disk centered at v into six congruent sectors, overlapping at their boundaries, such that

[1] A fact that can be verified by induction on the number of two-connected blocks.

some neighbour falls onto the boundary of a sector. Then by the pigeonhole principle some sector must contain at least 3 points, and the distance of any two points lying in the same sector (and in the interior of the disk) is less than 1. Thus there is a triangle in G spanned by some neighbours of v. But then G has K_4 as a subgraph, contradicting the fact that G is minimally rigid.

In the general case we can deduce a maximum degree upper bound as follows. Consider a unit ball in $\mathbb{R}^d$ and a maximal collection of pairwise disjoint balls of radius $1/4$ with their centers in the unit ball. Their union is contained in a ball of radius $5/4$, so volume considerations show that we can pack at most 5^d balls. It is easy to see that by doubling the radius of each ball in the collection we obtain a family of balls that cover the unit ball. Hence an argument similar to that of the two-dimensional case shows that the degree of each vertex is at most $d \cdot 5^d$. □

The following lemma can be used to show that every unit ball graph in $\mathbb{R}^d$ can be extended by adding a new vertex of degree d so that the new graph is also unit ball. This will imply that there exist infinitely many minimally rigid unit ball graphs in $\mathbb{R}^d$, for all $d \geq 1$.

Lemma 3.5. *Let $S \subseteq \mathbb{R}^d$ be a finite set of points and for $v \in \mathbb{R}^d$ denote by s_v the number of points in S with distance less than 1 from v (so that $s_v = |\{x \in S : \|x - v\| < 1\}|$). Suppose that there exists some point $v' \in \mathbb{R}^d$ with $s_{v'} \geq d$. Then there exists a point $v \in \mathbb{R}^d - S$ such that $s_v = d$.*

Proof. We first concentrate on the case of $d = 2$. Let I be the union of the pairwise intersections of unit circles centered at the points of S, i.e. $I = \{v \in \mathbb{R}^2 : |\{x \in S : \|x - v\| = 1\}| \geq 2\}$. Choose a point z not contained in $S \cup I$ such that $s_z \geq 2$. Such a point exists, since by the condition on S there is a non-empty open set of points with $s_v \geq 2$, and $S \cup I$ is finite. By the finiteness of the latter set there also exists a line l going through z which avoids the points in $S \cup I$. We claim that l contains a suitable point. Indeed, $s_z \geq 2$ and $s_v = 0$ for all sufficiently distant points of l, so it suffices to show that s_v only changes by increments of 1 as we move the point v along l. To see this, note that s_v (as a function of v) only changes value at points where $\{x \in S : \|x - v\| = 1\}$ is nonempty, and since $l \cap I = \emptyset$, for such v the set must contain exactly one element. So s_v changes by at most 1 at v, as desired.

In fact, the same proof works in the d-dimensional case for $d \geq 3$. The only apparent problem is that in this case I may be infinite (it is a union of some $(d-2)$-spheres), so we need to justify the existence of a suitable point z and line l in a different way. The former is evident, since $S \cup I$ has measure zero (under the d-dimensional Lebesgue measure), while the set of points with $s_v \geq d$ is non-empty and open, so it has positive measure. While the existence of a line l going through this point and avoiding $S \cup I$ is intuitively easy to believe, showing it is somewhat cumbersome. The reader can verify that the intersection of a $(d-2)$-sphere and a hyperplane not containing it is either empty, or a $(d-3)$-sphere. This allows us to proceed by induction on d: find a hyperplane through z not containing any of the finitely many $(d-2)$-spheres in I; by the induction hyphotesis, a suitable line can be found within this hyperplane. □

It follows that every unit ball graph can be extended to a larger unit ball graph by adding a new vertex of degree d. This operation (the d-dimensional version of 0-extension) is known to preserve (minimal) rigidity. Hence we can obtain an infinite family by using K_{d+1} as the base graph, which is minimally rigid and unit ball.

4 Saturated non-globally rigid graphs

In this section we introduce a family of non-globally rigid graphs which are, as we shall show, unit ball globally rigid nonetheless. We say that a graph G is *saturated non-globally rigid* (or SNGR, for short) in $\mathbb{R}^d$ if G is not globally rigid in $\mathbb{R}^d$ but for every non-adjacent pair $u, v \in V(G)$ the graph $G+uv$ is globally rigid in $\mathbb{R}^d$. Since all (globally) rigid graphs in $\mathbb{R}^d$ on at most $d+1$ vertices are complete, all SNGR graphs in $\mathbb{R}^d$ on at most $d+1$ vertices are isomorphic to $K_r - e$ for some $2 \leq r \leq d+1$.

For larger graphs Hendrickson's theorem implies the following statement.

Lemma 4.1. *Let G be SNGR in $\mathbb{R}^d$ on at least $d+2$ vertices. Then G is rigid in $\mathbb{R}^d$.*

Lemma 4.2. *Let G be SNGR in $\mathbb{R}^d$ on at least $d+2$ vertices. Then*
(i) if G is not $(d+1)$-connected then G is obtained from two complete graphs of size at least $d+1$ by gluing them along d vertices,
(ii) if G is minimally rigid in $\mathbb{R}^d$ then every rigid proper subgraph of G is isomorphic to a complete graph of size at most $d+1$.

Proof. (i) Let S be a separating vertex set in G with $|S| \leq d$. Theorem 2.1 implies that $G+e$ is $(d+1)$-connected for every edge $e = uv$, where u, v is a non-adjacent pair in G. Hence $|S| = d$ must hold, $G - S$ has exactly two connected components, and for each such component C the subgraph of G on $C \cup S$ is complete. This means that G is obtained from two complete graphs of size at least $d+1$ by gluing along d vertices. Note that if G is the result of such a gluing then it is SNGR: for any new edge e the graph $G+e$ can be obtained from one of the two complete graphs (which is globally rigid) by iteratively adding vertices of degree at least $d+1$ (an operation that is known to preserve global rigidity).

(ii) Let H be a rigid proper subgraph of G. Since G is minimally rigid, H is an induced subgraph. Moreover, there is an edge f in G not induced by H. Suppose that there is a non-adjacent vertex pair u, v in H. Then the edge $e = uv$ is not present in G. Furthermore, since G is minimally rigid and H is rigid, we have that $G + e - f$ is non-rigid. This contradicts the fact that G is SNGR by Theorem 2.1. □

In $\mathbb{R}^2$ Lemma 4.2(ii) gives an exact description of minimally rigid SNGR graphs. This family (the so-called special graphs) will be studied in the next section.

There exist $(d+1)$-connected SNGR graphs which are not minimally rigid in $\mathbb{R}^d$. For example, a copy of K_3 and a copy of K_4 connected by three disjoint edges is SNGR in $\mathbb{R}^2$.

A useful observation is that a graph G is SNGR in $\mathbb{R}^d$ if and only if every non-adjacent vertex pair $\{u, v\}$ in G satisfies that for every generic realization (G, p) and every equivalent non-congruent realization (G, q) we have $||p(u) - p(v)|| \neq ||q(u) - q(v)||$.

We can say more by using the following recent result of S. Gortler, L. Theran, and D. Thurston.

Theorem 4.3. *[13] Let (G, p) be a generic d-dimensional framework for some $d \geq 2$ where G is a globally rigid graph, and let (H, q) be another d-dimensional framework, with H having the same number of vertices and edges as G. If the (unordered) set of edge lengths is the same for the two frameworks, then G and H are isomorphic and p and q are congruent (after relabeling).*

Lemma 4.4. *Let $G = (V, E)$ be SNGR in $\mathbb{R}^d$ and let (G, p) be a generic d-dimensional realization of G. Let (G, q) be another realization of G which is equivalent, but not congruent to (G, p). Then for all pairs of non-adjacent vertex pairs $\{u, v\}$ and $\{u', v'\}$ we have $\|p(u) - p(v)\| \neq \|q(u') - q(v')\|$.*

Proof. Since G is SNGR, it is connected and has at least three vertices. Suppose, for a contradiction, that for some non-adjacent vertex pairs $\{x, y\}$ and $\{x', y'\}$ we have $\|p(x) - p(y)\|$
$= \|q(x') - q(y')\|$. Since G is SNGR, $G + xy$ is globally rigid in $\mathbb{R}^d$. Thus, by applying Theorem 4.3 to $(G + xy, p)$ and $(G + x'y', q)$ we obtain that $G + xy$ and $G + x'y'$ are isomorphic and p and q are congruent after relabeling. In other words, if φ denotes this isomorphism, then we have $\|p(u) - p(v)\| = \|q(\varphi(u)) - q(\varphi(v))\|$ for all $u, v \in V$.

It is sufficient to show that φ is the identity morphism, that is, $\varphi(v) = v$ for every vertex v, as this would mean that (G, p) and (G, q) are congruent frameworks, contradicting the original assumption. Note that, since p is generic and φ is a bijection of V, the distances $\|q(u) - q(v)\| = \|p(\varphi^{-1}(u)) - p(\varphi^{-1}(v))\|$ are different for all pairs $u, v \in V$. Moreover, since (G, p) and (G, q) are equivalent, for any edge $uv \in E$ we have $\|q(u) - q(v)\| = \|p(u) - p(v)\| = \|q(\varphi(u)) - q(\varphi(v))\|$. It follows that $\{u, v\} = \{\varphi(u), \varphi(v)\}$, that is, φ leaves the edges of G in place.

Take now a vertex u with degree at least two and let $uv, uw \in E$ be two edges incident to u. Then

$$\{u\} = \{u, v\} \cap \{u, w\} = \{\varphi(u), \varphi(v)\} \cap \{\varphi(u), \varphi(w)\} = \{\varphi(u)\},$$

so $u = \varphi(u)$. But G is a connected graph on at least three vertices, so every edge has an endpoint with degree at least two. This means that φ leaves the edges of G in place, and leaves at least one endpoint of each edge in place, so it must be the identity. $\square$

4.1 Unit ball global rigidity of SNGR graphs

In order to deduce our main result, in this subsection we also need the notions of infinitesimal motion and infinitesimal rigidity of frameworks. Infinitesimal rigidity, which is stronger than rigidity, is a well-known concept in rigidity theory. We refer the reader to e.g. [3, 8, 19] for the definition.

The proof of the next theorem is based on the so-called Averaging Lemma[2] and is inspired by the proof of [9, Theorem 13].

Theorem 4.5. *Let (G, p) be a unit ball realization of G in $\mathbb{R}^d$ that is both infinitesimally rigid and UBGR. Then p has an open neighbourhood $U_p \subseteq \mathbb{R}^{d|V|}$ such that for every unit ball realization $p' \in U_p$, the framework (G, p') is UBGR.*

Proof. For a contradiction suppose that there exists a sequence $(p^i)_{i\in\mathbb{N}}$ of configurations converging to p, such that for all $i \in \mathbb{N}$, (G, p^i) is a unit ball framework which is not UBGR. Then there exists a sequence $(q^i)_{i\in\mathbb{N}}$ of configurations such that (G, q^i) is unit ball, (G, p^i) is equivalent to (G, q^i), but p^i and q^i are not congruent. By applying a translation to each q^i we can assume that for some fixed vertex $v_0 \in V$, $q^i(v_0) = p(v_0)$ for all $i \in \mathbb{N}$. Then the sequence $(q^i)_{i\in\mathbb{N}}$ is bounded, so compactness implies that it has a convergent subsequence $(q^{n_i})_{i\in\mathbb{N}}$.

Note that $(G, \lim q^{n_i})$ is a unit ball realization of G that is equivalent to $(G, \lim p^{n_i}) = (G, p)$. Since (G, p) is UBGR, this means that the limit is congruent to p. Hence, by applying a congruence to each q^{n_i}, we obtain a sequence $(r^{n_i})_{i\in\mathbb{N}}$ that converges to p. Since p^{n_i} is equivalent, but not congruent to r^{n_i}, the Averaging Lemma implies that the framework $(G, \frac{p^{n_i}+r^{n_i}}{2})$ is not infinitesimally rigid. But this, combined with the fact that $\frac{p^{n_i}+r^{n_i}}{2}$ converges to p, is a contradiction, since (G, p) is infinitesimally rigid and the set of infinitesimally rigid realizations of a graph is open. $\square$

We are now ready to verify our main result about unit ball SNGR graphs.

Theorem 4.6. *Let G be a unit ball SNGR graph in $\mathbb{R}^d$. Then G is UBGR in $\mathbb{R}^d$, that is, G has a generic unit ball globally rigid realization in $\mathbb{R}^d$.*

Proof. Let (G, p) be a generic UB realization of G. It is well-known that, since G is rigid, it has only finitely many congruence classes of equivalent realizations. Furthermore, each congruence class contains a generic realization, see [17, Corollary 3.7]. Let $p_1, \ldots, p_k$ be a maximal set of pairwise non-congruent generic UB realizations of G. Intuitively, we would like to shrink each realization by the same amount until only one of them remains a unit ball realization; then it will be UBGR. This can be made precise as follows. For each $1 \leq i \leq k$, let α_i denote the smallest distance between non-adjacent vertices in (G, p_i). Note that, by Lemma 4.4, $\alpha_i \neq \alpha_j$ for $i \neq j$. Take i such that α_i is maximal. We claim that $p = \frac{1}{\alpha_i} p_i$ (that is, p_i scaled by $\frac{1}{\alpha_i}$) is a unit ball globally rigid realization of G. Clearly it is a unit ball realization, since the minimal distance of non-adjacent vertices is exactly one, and the distance of adjacent vertices remains smaller than one after scaling. Moreover, equivalent realizations of p correspond to scaled equivalent realizations of p_i. It is easy to see that scaling by a factor of at most one does not create new unit ball realizations, and any of the original UB realizations (G, p_j) with $j \neq i$ will have a pair u, v of non-adjacent vertices with $||p_j(u) - p_j(v)|| = \frac{\alpha_j}{\alpha_i} < 1$ after scaling, hence the scaled version will not be a unit ball

[2] The lemma says that if (G, p) and (G, q) are equivalent realizations of G then $p - q$ is an infinitesimal motion of the average framework $(G, \frac{p+q}{2})$. Furthermore, if the two frameworks are not congruent then $p - q$ is a non-trivial infinitesimal motion.

realization. This shows that (G, p) is UBGR. Since G is rigid and (G, p_i) is a generic framework, it is infinitesimally rigid. Scaling does not affect infinitesimal rigidity, so (G, p) is infinitesimally rigid as well. Thus we can apply Theorem 4.5 to obtain a generic UBGR realization of G. $\square$

5 Families of SNGR graphs in the plane

Following [17] we say that a minimally rigid graph G in $\mathbb{R}^d$ is *special* if every rigid proper subgraph of G is complete. The only special graphs in $\mathbb{R}^1$ are K_2 and $K_{1,2}$. In $\mathbb{R}^2$ we have infinitely many special graphs, as it was observed e.g. in [17]. The next lemma, which is implicit in the aforementioned paper, shows that special graphs are relevant in our context, too. In the rest of this section we assume that $d = 2$.

Lemma 5.1. *Let G be a minimally rigid graph in $\mathbb{R}^2$ on at least four vertices. Then G is SNGR if and only if G is special.*

Proof. Lemma 4.2(ii) implies necessity. To verify sufficiency first consider the case when G has exactly four vertices. Then G can be obtained from K_4 by deleting an edge, hence G is SNGR. So we may assume that G has at least five vertices. If $G-S$ is disconnected for some vertex set S with $|S| \leq 2$ then either G is non-rigid or there is a component C of $G-S$ for which the subgraph of G on $S \cup C$ is rigid and non-complete, a contradiction. Thus G is 3-connected.

Furthermore, it follows from Theorem 2.3 and the fact that G is special that for every edge f of G and every new edge e the graph $G+e-f$ is (minimally) rigid. Thus $G + e$ is 3-connected and redundantly rigid for every new edge e. Now Theorem 2.2 implies that G is SNGR. $\square$

(a) Prism (b) $K_{3,3}$

Figure 4: Examples of special graphs on 6 vertices.

Our main goal is to construct families of unit disk special graphs. By Lemma 5.1 and Theorem 4.6 these graphs are UBGR, even though they are not globally rigid. We remark that no inductive construction is known for special graphs, although they occur in a number of other rigidity related problems, too, see e.g. [17, 18, 28]. Thus our construction methods may be of interest in other applications, too.

In order to build special graphs the extension operations seem useful, due to Theorem 2.3(iii). Note, however, that an extension of a special graph G may not be special. In fact, if G is not an edge or a triangle, then a 0-extension of G is never special.

Let us say that a pair $\{u, v\}$ of vertices in a graph G is *non-triangular* if $\{u, v\}$ is not part of any triangle in G (this is trivially true if uv is not an edge of G). The following lemma characterizes 1-extensions that preserve the property of being special.

Lemma 5.2. *Let G be a special graph on at least four vertices, and let u, v, w be three different vertices of G with $uv \in E$. Let G' be the 1-extension of G on uv and w. Then G' is special if and only if $\{u, w\}$ and $\{v, w\}$ are both non-triangular in G.*

Proof. Let us denote the new vertex added by the 1-extension by x. To see that the condition is necessary, note that if $\{u, w, z\}$ is a triangle in G for some vertex z, then the graph induced by u, w, z and x in G' is a proper rigid subgraph. Indeed, it is either an 1-extension or a 0-extension of the triangle, depending on whether z is equal to v. The case when $\{v, w, z\}$ is a triangle in G is analogous.

Next, we show sufficiency. Take a proper subset $X \subset V(G')$ of vertices with $|X| > 3$. We need to show that $i_{G'}(X) \leq 2|X| - 4$. If $x \notin X$ then we are done, since G is special and $i_{G'}(X) \leq i_G(X)$, with strict inequality when $X = V$. Suppose now that $x \in X$. Notice that $i_{G'}(X) \leq i_G(X - x) + 2 \leq 2(|X| - 1) - 3 + 2 = 2|X| - 3$, and the second inequality is strict unless $X - x$ is a triangle in G. But in that case the first inequality is strict, since equality could only occur if the triangle spanned uw or vw, which would contradict the assumption that the vertex pairs $\{u, w\}$ and $\{v, w\}$ are non-triangular. □

Corollary 5.3. *Every special graph $G = (V, E)$ on at least five vertices has a 1-extension such that the resulting graph is special.*

Proof. It is easy to check that there are no special graphs on five vertices. One can also check that the only special graphs on six vertices are those that appear in Figure Figure 4. Both of these graphs have suitable 1-extensions. Thus we may assume that $|V| \geq 7$.

By Lemma 5.2 it is enough to show that there exists an edge uv and a vertex w such that w is not a neighbour of either u or v. Let u be a vertex of degree three. Such a vertex exists since the average degree in a minimally rigid graph is less than four and in a special graph on five or more vertices every vertex has degree at least three. We claim that there is a neighbour v of u and a non-neighbour w which are suitable. To see this suppose, for a contradiction, that every neighbour of u is a neighbour of all the $n - 4$ non-neighbours of u. Using this assumption and the fact that the minimum degree is equal to three, we obtain the following inequality for the sum of the vertex degrees in G:

$$4|V| - 6 \geq 3 \cdot (|V| - 3) + (|V| - 3) \cdot 3 = 6|V| - 18.$$

Thus $|V| \leq 6$, a contradiction. This proves the claim and completes the proof of the lemma. □

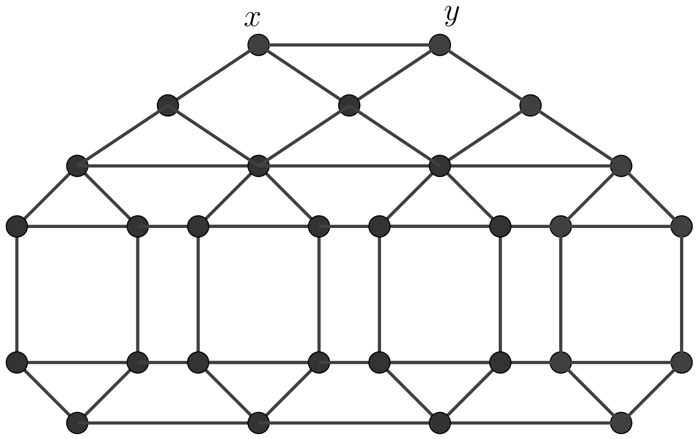


Figure 5: G_4

Next, we show that there exists an infinite family of unit disk special (and hence SNGR) graphs in $\mathbb{R}^2$. We describe the graphs only informally and refer the reader to Figures 5, 9 and 10 for examples. Let us call the graph on 6 vertices obtained by connecting two disjoint triangles by two disjoint edges a *block*. See Figure 6. For any $n \geq 3$ we define the graph G_n as follows. First, connect n blocks in a row, as in the aforementioned figure. Then add a grid of $n-1$ rows, with the first row formed by the topmost vertices of the blocks, and with each vertex in a row connected to two vertices in the preceeding row. Finally, connect the two vertices of the topmost row. Figure 10 shows that we can also construct G_{n+1} from G_n using the extension operations: three 0-extensions and 1-extensions each to add a new block, then one 0-extension for each row besides the first, and finally a 1-extension on the topmost edge of G_n and the vertex that was added last. A proof of the next theorem, stating that these graphs form an infinite family of unit disk special graphs, can be found in the appendix.

Theorem 5.4. *G_n is a unit disk special graph for all $n \geq 4$.*

We remark that other constructions are known as well; see e.g. Figure 7.

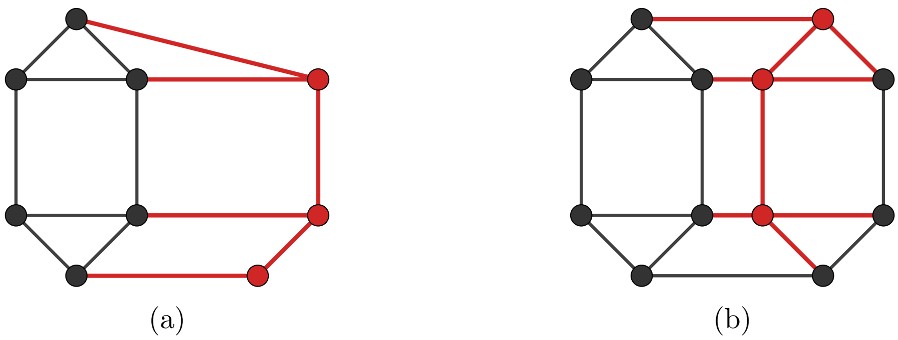


Figure 6: Adding a new block by (a) first making three 0-extensions (top to bottom), and then (b) three 1-extensions (bottom to top).

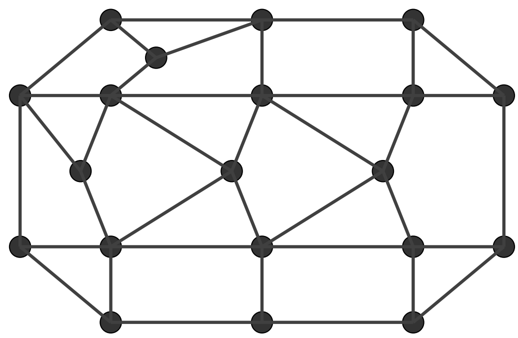


Figure 7: A different example of a unit disk special graph. In fact, this example gives rise to an infinite family of such graphs as well.

6 Tensegrities

In this section we describe another method of finding and constructing UBGR graphs, based on tensegrity frameworks.

A *tensegrity graph* $T = (V; B, C, S)$ is a graph on vertex set V whose edge set is partitioned into three sets B, C, and S, called *bars*, *cables*, and *struts*, respectively. While bars represent fixed distances, cables and struts impose upper (resp. lower) bounds on the distances between their end vertices. A d-dimensional *tensegrity framework* is a pair (T, p), where T is a tensegrity graph and p is a map from V to $\mathbb{R}^d$.

Let p and q be two realizations of the tensegrity graph T. We say that (T, p) *dominates* (T, q) (denoted by $(T, q) \prec (T, p)$) if the corresponding bar lengths are the same and the struts (cables, resp.) in (T, q) are not shorter (resp. longer) than in (T, p). A d-dimensional tensegrity framework (T, p) is called *globally rigid* in $\mathbb{R}^d$ if for any other d-dimensional realization (T, q) of T, $(T, q) \prec (T, p)$ implies that (T, p) is congruent to (T, q). See [8, 29] for more on globally rigid tensegrities.

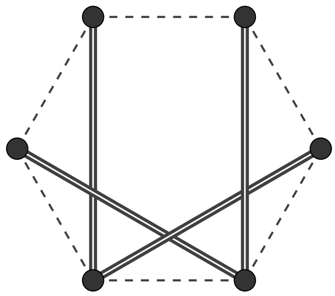
Our method is based on the following simple observation.

Theorem 6.1. *Let (G, p) be a unit ball framework in $\mathbb{R}^d$ with $||p(u) - p(v)|| = 1$ for some pair $u, v \in V(G)$ and let (T, p) be the tensegrity framework obtained from the all-bar tensegrity (G, p) by adding the strut uv. Then (G, p) is unit ball globally rigid in $\mathbb{R}^d$ if (T, p) is globally rigid in $\mathbb{R}^d$.*

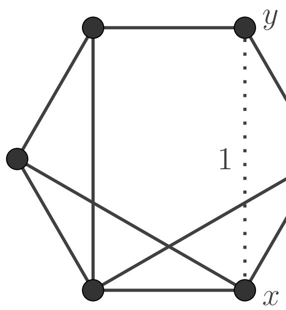
Proof. Suppose that (G, p) is not unit ball globally rigid in $\mathbb{R}^d$. Then there exists a unit ball framework (G, q), which is equivalent but not congruent with (G, p). If $||q(u) - q(v)|| \geq ||p(u) - p(v)||$ then (T, p) dominates (T, q). This is impossible since (T, p) is globally rigid. Thus we must have $||q(u) - q(v)|| < ||p(u) - p(v)|| = 1$. But this implies that (G, q) is not unit ball, a contradiction. □

In fact the theorem remains valid even if we have several pairs of vertices at distance one in (G, p), and hence more struts in (T, p) (just like in the case of the four-cycle with unit diagonals). However, this cannot happen in a (scaled) generic framework.

With the help of Theorem 6.1 (and Theorem 4.5) we can construct (generic) minimally rigid and unit disk globally rigid frameworks from certain globally rigid tensegrities with a unit disk underlying framework. We illustrate this by the so-called *Cauchy*



(a) A Cauchy polygon on 6 vertices.



(b) Theorem 6.1 implies that the above graph, when realized in the plane as a unit disk framework such that the distance of x and y is 1, is UBGR.

Figure 8

polygons, see Figure 8. These are two-dimensional tensegrity frameworks in which the vertices and the cables form a convex polygon and the struts are all diagonals connecting second neighbours. On n vertices we have $n-2$ struts in total. Cauchy polygons are globally and infinitesimally rigid. Let xy be a fixed strut. It is easy to construct a Cauchy polygon (T, p) for which the underlying graph G of $T - xy$ is unit disk and xy has length one. It follows that G is UBGR in $\mathbb{R}^2$. Note that G is far from being special.

More examples may be constructible by using the results of [10].

7 Concluding remarks

7.1 Related notions

By definition, a unit ball framework (G, p) is UBGR if every equivalent unit ball realization of G is congruent to (G, p). In other words, in every equivalent but non-congruent realization (G, q) there exists a non-adjacent pair u, v with $||q(u) - q(v)|| < 1$. This property makes sense for all frameworks, not just for unit ball frameworks, and may lead to extensions and further questions. We may, for example, study frameworks (G, p) with the property that in every equivalent but non-congruent realization (G, q) there exists a non-adjacent pair u, v with $||q(u) - q(v)|| < m$, where m is equal to $\min\{||p(u) - p(v)|| : u, v \in V, uv \notin E\}$.

A different question in the same vein is the following. Suppose that (G, p), viewed as a straight line embedding of G in the plane, has no edge crossings. Can we decide whether every equivalent realization (G, q) satisfies this property?

7.2 The chain of containment relations

By Theorem 3.2 and the well-known characterizations of rigid and globally rigid graphs in $\mathbb{R}^1$ we can deduce that for $d = 1$ we have $GGR \subset SUBGR = UBGR = GR$ within the family of unit interval graphs, where the first relation is strict. In higher dimensions it remains open whether the first and last relations are strict, even for minimally rigid unit ball graphs. Figure 2 (and its generalizations) show that not every UBGR graph is SUBGR in $\mathbb{R}^d$, for $d \geq 2$.

7.3 SNGR graphs in $\mathbb{R}^3$

Maximal planar graphs (or triangulations, for short) are minimally rigid in $\mathbb{R}^3$ by a result of H. Gluck [11]. W. Whiteley showed in [31] that if G is a four-connected triangulation and u, v is a non-adjacent vertex pair then $G + uv$ is redundantly rigid. This implies that four-connected triangulations are special in $\mathbb{R}^3$. A recent result of S. Tanigawa and the second author [21] implies that they are also SNGR in three-space. Hence every unit ball four-connected triangulation is UBGR in $\mathbb{R}^3$. It would be interesting to find families of unit ball four-connected triangulations in $\mathbb{R}^3$.

8 Acknowledgements

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Appendix A An infinite family of special graphs

In what follows we prove that the graphs G_n, presented in Section 5, are special in $\mathbb{R}^2$ for $n \geq 4$. In order to do this we state two lemmas regarding the exact effect of the extension operations on rigid subgraphs. Lemma A.2 can be seen as a generalization of Lemma 5.2, and the proof uses essentially the same ideas. Lemma A.1 is the analogous statement for 0-extensions. We omit the proofs.

Lemma A.1. *Let G be a minimally rigid graph and let $G' = G + x$ be the 0-extension of G on vertices u and v. Then $X \subseteq G'$ induces a rigid subgraph of G' if and only if one of the following cases holds:*

- *$x \notin X$ and X induces a rigid subgraph of G.*
- *$x \in X$ and $X - x$ induces a rigid subgraph of G containing u and v.*

- *X is an edge incident to x or $X = \{x\}$.* □

Lemma A.2. *Let G be a minimally rigid graph and let $G' = G + x$ be the 1-extension of G on an edge uv and vertex w. Then $X \subseteq G'$ induces a rigid subgraph of G' if and only if one of the following cases holds:*

- *$x \notin X$ and X induces a rigid subgraph of G that does not span uv.*
- *$x \in X$ and $X - x$ induces a rigid subgraph of G that contains u, w or v, w.*
- *X is an edge incident to x or $X = \{x\}$.* □

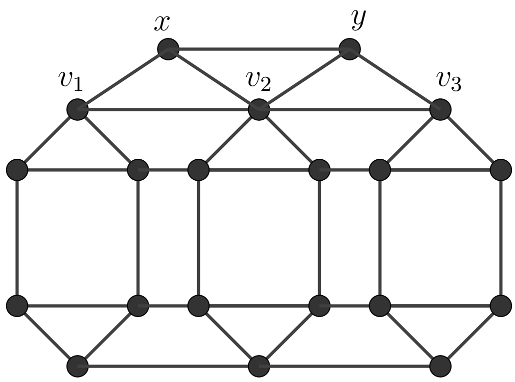

Figure 9: G_3

Now we have the tools to sketch a proof of Theorem 5.4. The details we leave out can be easily (and mostly mechanically) checked.

Proposition A.3. *a) G_n is a unit disk graph for all $n \geq 3$,*

b) G_3 is minimally rigid and its non-complete, minimally rigid proper subgraphs are spanned by some of $\{x, y, v_1, v_2, v_3\}$,

c) G_n is special for all $n \geq 4$.

Proof. a, It is not difficult to see that the frameworks shown in the figures (e.g. Figure 10) are unit disk frameworks with the appropriate choice of vertex positions (and edge lengths).

b, Figure 11 shows that G_3 can be constructed from a minimally rigid graph using 0- and 1-extensions, thus it is minimally rigid. The rest of the claim can be verified by following the changes in the set of minimally rigid subgraphs throughout this construction (using Lemma A.1 and Lemma A.2).

c, It is clear by the inductive construction and the fact that G_3 is minimally rigid that G_n is minimally rigid as well for all $n \geq 4$. We prove that G_n is special by induction on n. See Figure 10 for the case $n = 5$. By using the previous lemmas and the proposition concerning G_3, we may observe that before the 1-extension by x', every non-complete, minimally rigid proper subgraph spans the edge xy, and no such subgraph contains the vertex y'. It follows from Lemma A.2 that the resulting graph is special. The induction step is analogous to the case $n = 4$. □

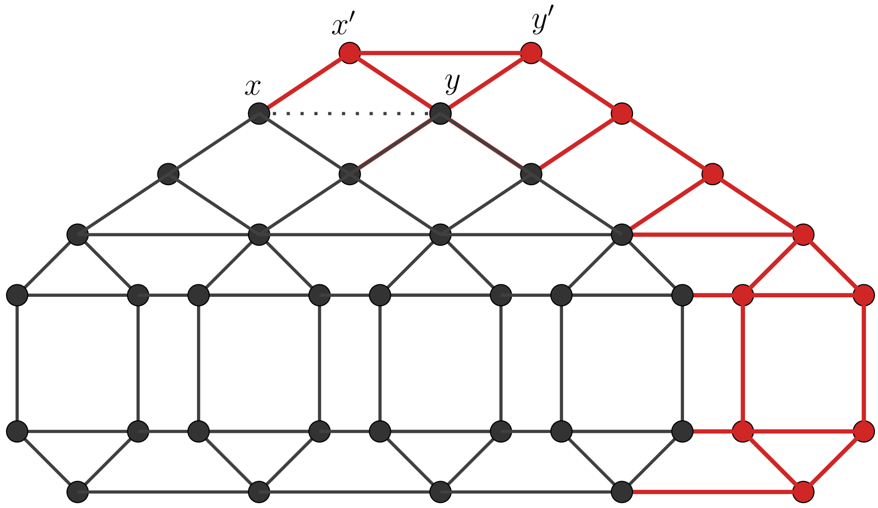

Figure 10: Construction of G_5 from G_4

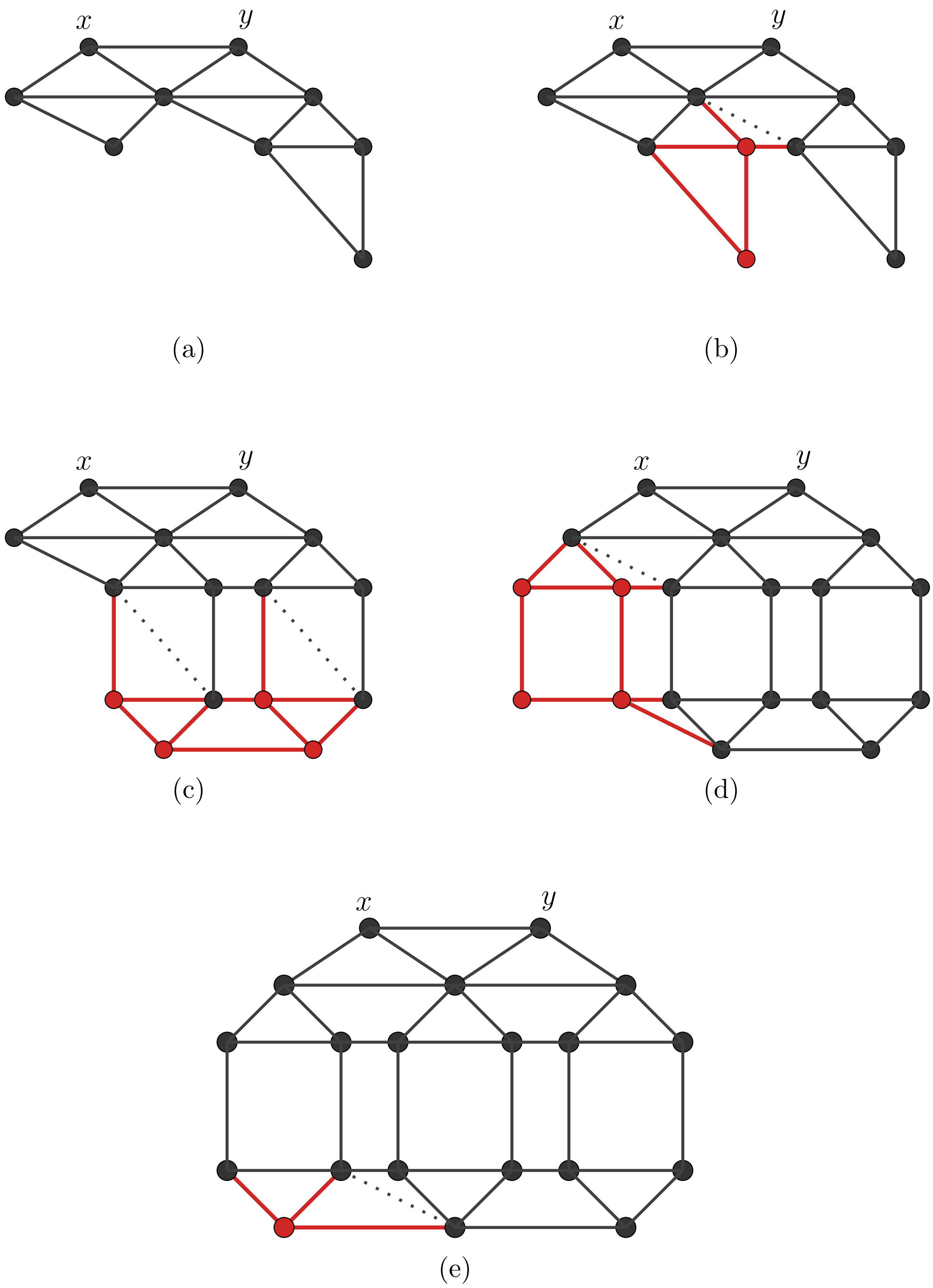

Figure 11: The construction of G_3 by extensions.

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