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Discrete Decreasing Minimization, Part III: Network Flows

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Abstract

A strongly polynomial algorithm is developed for finding an integer-valued feasible *st*-flow of given flow-amount which is decreasingly minimal on a specified subset F of edges in the sense that the largest flow-value on F is as small as possible, within this, the second largest flow-value on F is as small as possible, within this, the third largest flow-value on F is as small as possible, and so on. A characterization of the set of these *st*-flows gives rise to an algorithm to compute a cheapest F-decreasingly minimal integer-valued feasible *st*-flow of given flow-amount.

Keywords: discrete convex optimization, lexicographic minimization, network flows, polynomial algorithms.

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Contents

1 Introduction

N. Megiddo [11], [12] introduced and solved the problem of finding a (possibly fractional) maximum flow which is 'lexicographically optimal' on the set of edges leaving the source node. The problem, in equivalent terms, is as follows. Let $D = (V, A)$ be a digraph with a source-node s and a sink-node t, and let S_A denote the set of edges leaving s. We assume that no edge enters s and no edge leaves t. Let $g : A \to \mathbf{R}_+$ be a non-negative capacity function on the edge-set. By the standard definition, an st-flow, or just a flow, is a function $x : A \to \mathbf{R}_+$ for which $\varrho_x(v) = \delta_x(v)$ holds for every node $v \in V - \{s, t\}$. (Here $\varrho_x(v) := \sum[x(uv) : uv \in A]$ and $\delta_x(v) := \sum[x(vu) : vu \in A]$.) The flow is called **feasible** if $x \leq g$. The **flow-amount** of x is $\delta_x(s)$ which is known to be equal to $\varrho_x(t)$. We refer to a feasible flow with maximum flow-amount as a **max-flow**.

Megiddo solved the problem of finding a feasible flow x which is lexicographically optimal on S_A in the sense that the smallest x-value on S_A is as large as possible, within this, the second smallest (though not necessarily distinct) x-value on S_A is as large as possible, and so on. It is a known fact (implied, for example, by the max-flow algorithm of Ford and Fulkerson [3]) that a lexicographically optimal flow is a max-flow. It is a basic property of flows that for an integral capacity function g there always exists a max-flow which is integer-valued. On the other hand, an easy example was shown in [6] in which g is integer-valued (actually identically 1) and the unique max-flow which is lexicographically optimal on S_A is not integer-valued.

In [6] and [7], we called a member x of a set Q of vectors a **decreasingly minimal** (dec-min, for short) element of Q if the largest (but not necessarily distinct) component of x is as small as possible, within this, the next largest component of x is as small as possible, and so on. Analogously, x is an **increasingly maximal** (inc-max) element of Q if its smallest component is as large as possible, within this, the next smallest component of x is as large as possible, and so on. Therefore increasing maximality is the same as Megiddo's lexicographic optimality.

In [6] and [7], we solved the discrete counterpart of Megiddo's problem when the capacity function g is integral and one is interested in finding an integral max-flow whose restriction to the set S_A of edges leaving s is increasingly maximal. This was actually a consequence of the more general result concerning dec-min elements of an M-convex set (where an M-convex set, by definition, is the set of integral elements of an integral base-polyhedron). Among others, we proved that an element z is decreasingly minimal if and only if z is increasingly maximal. We also developed a strongly polynomial algorithm for finding a dec-min element. Since the restrictions of max-flows to S_A form a base-polyhedron, we obtained in this way an algorithm to find an integral max-flow which is decreasingly minimal (and increasingly maximal) when restricted to S_A.

A closely related previous work is due to Kaibel, Onn, and Sarrabezolles [10]. They considered (in an equivalent formulation) the problem of finding an integer-valued uncapacitated st-flow with specified flow-amount K which is decreasingly minimal on the whole edge-set A. They developed an algorithm which is polynomial in the size of D plus the value of K but not polynomial in the size of number K (which is roughly $\lceil \log K \rceil$). This is analogous to the well-known characteristic of the classic Ford–Fulkerson max-flow algorithm [3], where the running time is proportional to the largest value $g_{\max}$ of the capacity function

g, and therefore this algorithm is not polynomial (unless $g_{\max}$ is small in the sense that it is bounded by a polynomial of $|A|$). It should also be mentioned that Kaibel et al. considered only the uncapacitated st-flow problem, where no capacity (upper-bound) restrictions are imposed on the edges. (For example, the flow-value on any edge is allowed to be K.)

In the present work, we consider the more general question when $F \subseteq A$ is an arbitrarily specified subset of edges, and we are interested in finding a feasible integral max-flow whose restriction to F is decreasingly minimal. This problem substantially differs from its special case mentioned above when $F = S_A$ in that the set of restrictions of max-flows to F is not necessarily a base-polyhedron. Therefore, a dec-min max-flow is not necessarily inc-max. Our main goal is to provide a description of the set of integral max-flows which are dec-min on F as well as a strongly polynomial algorithm to find such a max-flow. The description makes it possible to solve algorithmically even the minimum cost dec-min max-flow problem.

Instead of maximum st-flows, we consider the formally more general (though equivalent) setting of modular flows which, however, allows a technically simpler discussion.

2 Decreasingly-minimal integer-valued feasible modular flows

2.1 Modular flows

Let $D = (V, A)$ be a digraph endowed with integer-valued functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ for which $f \leq g$. Here f and g are serving as lower and upper bound functions, respectively. An edge e is called **tight** if $f(e) = g(e)$. The polyhedron $T(f, g) := \{x : f \leq x \leq g\}$ is called a **box**.

We are given a finite integer-valued function m on V for which $\widetilde{m}(V) = 0$. (Here and throughout, $\widetilde{m}(X) := \sum[m(v) : v \in X]$.) A **modular flow** (with respect to m) or, for short, a **mod-flow** x is a finite-valued function on A (or a vector in $\mathbf{R}^A$) for which $\varrho_x(v) - \delta_x(v) = m(v)$ for each node $v \in V$. When we want to emphasize the defining vector m, we speak of an m**-flow.**

A mod-flow x is called (f, g)**-bounded** or **feasible** if $f \leq x \leq g$. A circulation is an m-flow with respect to $m \equiv 0$, and an st-flow of given flow-amount K is also an m-flow with respect to m defined by

$$m(v) := \begin{cases} 0 & \text{if } v \in V - \{s, t\}, \\ K & \text{if } v = t, \\ -K & \text{if } v = s. \end{cases} \tag{2.1}$$

Circulations form a subspace of $\mathbf{R}^A$ while the set of mod-flows is an affine space. The set of feasible mod-flows, which is called a **feasible mod-flow polyhedron,** may be viewed as the intersection of this affine subspace with the box $T(f, g)$. It follows from this definition that the face of such a polyhedron is also a feasible m-flow polyhedron. We note, however, that the projection along axes is not necessarily a feasible mod-flow polyhedron since its description may need an exponential number of inequalities while a feasible mod-flow polyhedron is described by at most $2|A| + |V|$ inequalities.

Suppose that there is an integer-valued (f, g)-bounded m-flow. By Hoffman's theorem [9], this is equivalent to requiring that the Hoffman-condition $\varrho_g - \delta_f \geq \widetilde{m}$ holds, that is,

$$\varrho_g(Z) - \delta_f(Z) \geq \widetilde{m}(Z) \quad \text{whenever} \quad Z \subseteq V. \tag{2.2}$$

It is well-known that there are strongly polynomial algorithms that find a feasible m-flow when it exists or find a subset Z violating (2.2) (see, for example, appropriate variations of the algorithms by Edmonds and Karp [2], Dinits [1], or Goldberg and Tarjan [8]). Actually, when no feasible m-flow exists, not only a violating subset can be computed but the most violating set as well, that is, a set Z^* maximizing $\widetilde{m}(Z) - \varrho_g(Z) + \delta_f(Z)$. Note that this latter function is fully supermodular, and there is a general algorithm to maximize an arbitrary supermodular function. The point here is that for finding Z^* we do not have to rely on this general algorithm since much simpler (and more efficient) flow-techniques do the job.

Let $Q = Q(f, g; m)$ denote the set of (f, g)-bounded m-flows. It is well-known that Q is an integral polyhedron whenever f, g, and m are integral vectors. Let $\ddot{Q} = \ddot{Q}(f, g; m)$ denote the set of integral elements of Q. The notion of decreasing minimality was introduced in Section 1 but we work throughout the paper with the following slightly extended definition. Let F be a specified subset of A. We say that $z \in \ddot{Q}(f, g; m)$ is **decreasingly minimal on** F (or F**-dec-min** for short) if the restriction of z to F is decreasingly minimal. One of our main goals is to prove the following characterization of the subset of elements of $\ddot{Q}$ which are decreasingly minimal on F.

Theorem 2.1. *Let $D = (V, A)$ be a digraph endowed with integer-valued lower and upper bound functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ for which $f \leq g$. Let $m : V \to \mathbf{Z}$ be a function on V with $\widetilde{m}(V) = 0$ such that there exists an (f, g)-bounded m-flow. Let $F \subseteq A$ be a specified subset of edges such that both f and g are finite-valued on F. There exists a pair (f^*, g^*) of integer-valued functions on A with $f \leq f^* \leq g^* \leq g$ (allowing $f^*(e) = -\infty$ and $g^*(e) = +\infty$ for $e \in A - F$) such that an integral (f, g)-bounded m-flow z is decreasingly minimal on F if and only if z is an integral (f^*, g^*)-bounded m-flow. Moreover, the box $T(f^*, g^*)$ is narrow on F in the sense that $0 \leq g^*(e) - f^*(e) \leq 1$ for every $e \in F$.* ■

Our second main goal is to describe a strongly polynomial algorithm to compute f^* and g^*. Once these bounds are available, one is able to compute not only a single (f, g)-bounded integer-valued m-flow which is dec-min on F but a minimum cost dec-min m-flow as well (with the help of a standard min-cost circulation algorithm).

Remark 2.1. In Section 9, we shall consider the general case when f and g are not required to be finite-valued on F. In this case, an F-dec-min (f, g)-feasible m-flow may not exist, and we shall provide a characterization for the existence. In Theorem 9.6, we shall show how Theorem 2.1 can be extended to the case when only the existence of an F-dec-min (f, g)-feasible m-flow is assumed. ■

Remark 2.2. One may also be interested in finding an (integral) (f, g)-bounded m-flow z which is **increasingly maximal** (inc-max) **on** F in the sense that the smallest z-value on F is as large as possible, within this, the second smallest (but not necessarily distinct) z-value on F is as large as possible, and so on. (Megiddo [11], [12], for example, considered the fractional inc-max problem for st-flows when F was the set of edges leaving s.) But an (f, g)-bounded m-flow z is increasingly maximal on F precisely if $-z$ is a $(-g, -f)$-bounded $(-m)$-flow which is dec-min on F, implying that the inc-max and the dec-min problems are equivalent for modular flows. Hence we concentrate throughout only on decreasing minimality. Note that in [6] and [7] we investigated these problems for M-convex sets and proved that the two problems are not only equivalent but they are one and the same in the sense that an element z of an M-convex set is dec-min if and only if z is inc-max. (As mentioned earlier, an M-convex set, by definition, is nothing but the set of integral elements of an integral base-polyhedron). ■

2.2 Approach of the proof of Theorem 2.1

By tightening an edge e we mean the operation that replaces the bounding pair $(f(e), g(e))$ by $(f'(e), g'(e))$ where $f(e) \leq f'(e) \leq g'(e) \leq g(e)$ and $g'(e) - f'(e) < g(e) - f(e)$. The approach of the proof is that we tighten edges as long as possible without loosing any integral m-flow which is dec-min on F, and prove that when no more tightening step is available for the current (f^*, g^*) then every (f^*, g^*)-bounded integral m-flow is dec-min on F.

A natural reduction step consists of removing a tight edge e from F (where e could be tight originally or may have become tight during a tightening step). This simply means that we replace F by $F' := F - e$ (but keep e in the digraph itself). Obviously, an m-flow z is F-dec-min if and only if z is F'-dec-min. Therefore, we may always assume that F contains no tight edges.

We say that an integral (f, g)-bounded m-flow z is an F-**max minimizer** if the largest component of z in F is as small as possible. Clearly, every F-dec-min m-flow $z \in \ddot{Q}(f, g; m)$ is F-max minimizer. Let β_F denote this smallest maximum value, that is,

$$\beta_F := \min\{\max\{z(a) : a \in F\} : z \in \ddot{Q}(f, g; m)\}. \tag{2.3}$$

Note that β_F may be interpreted as the smallest integer for which there is an integer-valued feasible m-flow after decreasing $g(e)$ to β_F for each $e \in F$ with $g(e) > \beta_F$. In Section 7, we shall describe how β_F can be computed in strongly polynomial time with the help of the Newton–Dinkelbach algorithm and a standard max-flow algorithm, but for the proof of Theorem 2.1 we assume that β_F is available. Therefore, we can assume that $\max\{g(e) : e \in F\} = \beta_F$ which is equivalent to requiring that $Q(f, g; m)$ is non-empty but $Q(f, g^-; m) = \emptyset$ where g^- arises from g by subtracting 1 from $g(e)$ for each $e \in F$ with $g(e) = \beta_F$.

3 Covering a supermodular function by a smallest subgraph

We say that a digraph $D = (V, A)$ (or its edge-set A) **covers** a set-function p if $\varrho_D(Z) \geq p(Z)$ for every subset $Z \subseteq V$, where ϱ_D is the in-degree function of D. Let $p : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be an intersecting supermodular set-function on V and let $D_L = (V, L)$ be a digraph covering

p. We are interested in the minimum cardinality subset of edges of D_L that covers p. Let A_L denote the $(0,1)$-matrix whose rows correspond to subsets X of V for which $p(X) > -\infty$ and the columns correspond to the edges in L. An entry of A_L corresponding to Z and e is 1 if e enters Z and 0 otherwise. The following result was proved in [4] (see, also, Theorem 17.1.1 in the book [5]).

Theorem 3.1. *Let p be an intersecting supermodular set-function on V. The linear inequality system $\{A_L x_L \geq p,\ x_L \leq \underline{1},\ x_L \geq 0\}$ is totally dual integral (TDI). (Hence) the primal linear program*

$$\min\{\underline{1}x_L : \ A_L x_L \geq p,\ x_L \leq \underline{1},\ x_L \geq 0\} \tag{3.1}$$

and the dual linear program

$$\max\{yp - \underline{1}z : yA_L - z \leq \underline{1},\ (y,z) \geq 0\} \tag{3.2}$$

have integer-valued optimal solutions (where $\underline{1}$ denotes the everywhere 1 vector of dimension $|L|$.) Moreover, there is an integer-valued dual optimum (y^*, z^*) for which its support family $\mathcal{L} := \{Z : y^*(Z) > 0\}$ is laminar. ■

For a family $\mathcal{L}$ of subsets, let $\varrho_L(\mathcal{L})$ denote the number of edges entering at least one member of $\mathcal{L}$. The min-max theorem arising from Theorem 3.1 is as follows.

Theorem 3.2. *Given a digraph $D_L = (V, L)$ covering an intersecting supermodular function p, the minimum number of edges of D_L covering p is equal to*

$$\max\{\varrho_L(\mathcal{L}) \ - \sum[\varrho_L(Z) - p(Z) : \ Z \in \mathcal{L}]\} \tag{3.3}$$

where the maximum is taken over all laminar families $\mathcal{L}$ of subsets Z of V with $p(Z) > -\infty$. When p is fully supermodular, the optimal laminar family $\mathcal{L}^*$ may be chosen as a chain of subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ of V.

Proof. Suppose that we remove some edges from L so that the set X of the remaining edges continues to cover p. For each $Z \in \mathcal{L}$, the number of removed edges entering Z is bounded by $\varrho_L(Z) - p(Z)$, and hence the number of removed edges entering at least one member of $\mathcal{L}$ is bounded from above by $\sum[\varrho_L(Z) - p(Z) : Z \in \mathcal{L}]$. On the other hand, the number of removed edges entering at least one member of $\mathcal{L}$ is bounded from below by $\varrho_L(\mathcal{L}) - |X|$. Therefore we have

$$\varrho_L(\mathcal{L}) - |X| \leq \sum[\varrho_L(Z) - p(Z) : Z \in \mathcal{L}],$$

from which the trivial direction max ≤ min follows.

To see the reverse inequality, we have to find a covering $X^* \subseteq L$ of p and a laminar family $\mathcal{L}^*$ for which equality holds. To this end, let x^* be a $(0,1)$-valued optimal solution of the primal problem (3.1) in Theorem 3.1 and let (y^*, z^*) be an integer-valued optimal solution of the dual problem for which its support family $\mathcal{L}^*$ is laminar. Then the subset $X^* := \{e \in L : x^*(e) = 1\}$ is a smallest subset of L covering p.

Observe that y^* uniquely determines z^*, namely, $z^*(e) = 0$ when e enters no member of $\mathcal{L}^*$ and

$$z^*(e) = \sum[y^*(Z) : Z \in \mathcal{L}^*, e \text{ enters } Z] \ - 1 \tag{3.4}$$

when e enters at least one member of $\mathcal{L}^*$.

Claim 3.3. *The optimal y^* may be chosen* $(0, 1)$*-valued.*

Proof. Suppose that (y^*, z^*) is an integer-valued dual optimum in which the sum of y^*-components is as small as possible. We show that y^* is $(0, 1)$-valued. Suppose indirectly that $y^*(Z) \geq 2$ for some set Z. In this case $z^*(e) \geq 1$ for every edge e entering Z. If we decrease $y^*(Z)$ by 1 and decrease $z^*(e)$ by 1 on every edge e entering Z, then the resulting (y', z') is also a dual feasible solution for which

$$y^*p - \underline{1}z^* \;\geq\; y'p - \underline{1}z' \;=\; y^*p - \underline{1}z^* - p(Z) + \varrho_L(Z) \;\geq\; y^*p - \underline{1}z^*,$$

where the last inequality follows from the assumption that D_L covers p and hence $\varrho_L(Z) \geq p(Z)$. Therefore we have equality throughout and hence (y', z') is also an optimal dual solution, contradicting the minimal choice of y^*. ■

By the claim, (3.4) simplifies as follows:

$$z^*(e) = [\text{the number of members of } \mathcal{L} \text{ entered by } e] \;-\; 1. \tag{3.5}$$

Now the dual optimum value is:

$$\begin{aligned}
& y^*p - \underline{1}z^* \\
&= \sum[p(Z) : Z \in \mathcal{L}^*] - \sum[z^*(e) : e \in L \text{ enters a member of } \mathcal{L}^*] \\
&= \sum[p(Z) : Z \in \mathcal{L}^*] \\
&\quad - \sum[(\text{the number of members of } \mathcal{L}^* \text{entered by } e) - 1 :\; e \text{ enters a member of } \mathcal{L}^*] \\
&= \sum[p(Z) : Z \in \mathcal{L}^*] - \sum[\varrho_L(Z) : Z \in \mathcal{L}^*] + \varrho_L(\mathcal{L}^*) \\
&= \varrho_L(\mathcal{L}^*) \;-\; \sum[\varrho_L(Z) - p(Z) :\; Z \in \mathcal{L}^*]\}.
\end{aligned} \tag{3.6}$$

Therefore $|X^*|$ is equal to the value in (3.6), from which the non-trivial direction max $\geq$ min follows, implying the requested min = max.

To see the last statement of the theorem, consider an optimal laminar family $\mathcal{L}$ with a minimum number of members. We claim that $\mathcal{L}$ is a chain of subsets when p is fully supermodular. Suppose, indirectly, that $\mathcal{L}$ has two disjoint members and let X and Y be disjoint members of $\mathcal{L}$ whose union is maximal. Then the family $\mathcal{L}'$ obtained from $\mathcal{L}$ by replacing X and Y with their union $X \cup Y$ is also laminar. By the full supermodularity of p, we have $\sum[p(Z) :\; Z \in \mathcal{L}] \leq \sum[p(Z) :\; Z \in \mathcal{L}']$. Furthermore,

$$\varrho_L(\mathcal{L}) - \sum[\varrho_L(Z) :\; Z \in \mathcal{L}] = \varrho_L(\mathcal{L}') - \sum[\varrho_L(Z) :\; Z \in \mathcal{L}'].$$

Therefore $\mathcal{L}'$ is also a dual optimal laminar family, contradicting the minimal choice of $\mathcal{L}$. ■■

Theorem 3.4. *Let $D_L = (V, L)$ be a digraph covering a fully supermodular function p. There is a chain C^* of subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ of V with $p(V_i) > -\infty$ such that a subset $X \subseteq L$ is a minimum cardinality subset of edges covering p if and only if the following three optimality criteria hold.*

(A) *For every* V_i, $\varrho_X(V_i) = p(V_i)$.
(B) *Every edge in* X *enters at least one* V_i. *(Equivalently, if* $e \in L$ *enters no* V_i*, then* $e \notin X$*.)*
(C) *Every edge in* $L - X$ *enters at most one* V_i. *(Equivalently, if* $e \in L$ *enters at least two* V_i*'s, then* $e \in X$*.)*

Proof. Let C^* denote the optimal chain of subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ given in Theorem 3.2. This corresponded to a special integer-valued solution (y^*, z^*) to the dual linear program (3.2) where y^* was actually $(0, 1)$-valued and y^* (or its support family C^*) determined uniquely z^*. Namely, $z^*(e)$ was 0 when e did not enter any V_i, and $z^*(e)$ was the number of V_i's entered by e minus 1 when e entered at least one V_i.

Since both the primal and the dual variables in the linear programs in Theorem 3.1 are non-negative, the optimality criteria (= complementary slackness conditions) of linear programming require that if a primal variable is positive, then the corresponding dual inequality holds with equality, and symmetrically, if a dual variable is positive, then the corresponding primal inequality holds with equality.

Let x^* be a $(0, 1)$-valued primal solution and let $X^* := \{e \in L : x^*(e) = 1\}$ be the corresponding set of edges that covers p. The optimality criterion concerning the dual variable y^*, requires that if $y^*(Z) = 1$ (that is, if Z is one of the sets V_i), then the corresponding primal inequality holds with equality. That is, $\varrho_{X^*}(V_i) = \varrho_{x^*}(V_i) = p(V_i)$, which is just Criterion (A).

The optimality criterion concerning the primal variable x^* requires that if $x^*(e) = 1$ for an edge e (that is, if $e \in X^*$), then the corresponding dual inequality holds with equality. Hence e must enter at least one V_i (as $z^*(e) \geq 0$), which is just Criterion (B).

Finally, the optimality criterion concerning the dual variable $z^*(e)$ requires that if $z^*(e) > 0$ (that is, if e enters at least two V_i's), then the corresponding primal inequality is met by equality, that is, $x^*(e) = 1$ or equivalently $e \in X^*$, which is just Criterion (C). ∎

4 L-upper-minimal m-flows

Let $D = (V, A)$ be a digraph and $m : V \to \mathbf{Z}$ a function with $\widetilde{m}(V) = 0$. Let $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ be bounding functions with $f \leq g$. Let L be a subset of A for which $-\infty < f(e) < g(e) < +\infty$ for every $e \in L$. (That is, $f(e)$ may be $-\infty$ and $g(e)$ may be $+\infty$ only if $e \in A - L$.) We say that an (f, g)-bounded integer-valued m-flow x is L**-upper-minimal** or that x is an L**-upper-minimizer** if the number of g-saturated edges in L is as small as possible, where an edge $e \in L$ is called g**-saturated** if $x(e) = g(e)$. In this section, we are interested in characterizing the L-upper-minimizer integral (f, g)-bounded m-flows. For the proof of Theorem 2.1, however, we will use this characterization only in the special case when $L := \{e : e \in F, g(e) = \beta_F\}$, that is, $g(e)$ is the same value for each element e of L. The only reason for this more general setting is to get a clearer picture of the background.

Theorem 4.1. *The minimum number of* g*-saturated* L*-edges in an* (f, g)*-bounded integer-valued* m*-flow is equal to*

$$\max\{\varrho_L(C) - \sum[\varrho_g(Z) - \delta_f(Z) - \widetilde{m}(Z) : Z \in C]\}, \tag{4.1}$$

where the maximum is taken over all chains C of subsets Z of V with $\varrho_g(Z) - \delta_f(Z) < +\infty$, and $\varrho_L(C)$ denotes the number of L-edges entering at least one member of C. In particular, if the minimum is zero, the maximum is attained at the empty chain.

Proof. Let $g^- := g - \chi_L$, that is,

$$g^-(e) := \begin{cases} g(e) - 1 & \text{if} \quad e \in L \\ g(e) & \text{if} \quad e \in A - L. \end{cases} \tag{4.2}$$

Since $g(e) < +\infty$ for $e \in L$, $g^- \neq g$. By the hypothesis, L contains no tight edges and hence $f \leq g^-$. Define a set-function p as follows:

$$p := \widetilde{m} - \varrho_{g^-} + \delta_f. \tag{4.3}$$

Since $g^- \geq f$, the function $\varrho_{g^-} - \delta_f$ is fully submodular and hence p is fully supermodular. Furthermore, $p(Z) > -\infty$ precisely if $\varrho_g(Z) - \delta_f(Z) < +\infty$.

Lemma 4.2. *An integer-valued (f, g)-bounded m-flow x is an L-upper-minimizer if and only if $X := \{e \in L : x(e) = g(e)\}$ is a smallest subset of L covering p.*

Proof.

Claim 4.3. (A) *If x is an integer-valued (f, g)-bounded m-flow, and $X \subseteq L$ is the set of g-saturated L-edges, (that is, $X := \{e \in L : x(e) = g(e)\}$), then X covers p.* (B) *If a subset $X \subseteq L$ covers p, then there is an integer-valued m-flow which is $(f, g^- + \chi_X)$-bounded.*

Proof. (A) For every subset $Z \subseteq V$, we have

$$\widetilde{m}(Z) = \varrho_x(Z) - \delta_x(Z) \leq [\varrho_{g^-}(Z) + \varrho_X(Z)] - \delta_f(Z),$$

from which

$$\varrho_X(Z) \geq \widetilde{m}(Z) - \varrho_{g^-}(Z) + \delta_f(Z) = p(Z),$$

as required.

(B) It follows form the hypothesis $\varrho_X \geq p = \widetilde{m} - \varrho_{g^-} + \delta_f$ that $\varrho_{g^-} + \varrho_X - \delta_f \geq \widetilde{m}$. Then Hoffman's theorem implies that there is an integer-valued $(f, g^- + \chi_X)$-bounded m-flow. ■

Claim 4.4. *If x is an L-upper-minimizer (f, g)-bounded m-flow, then $X := \{e \in L : x(e) = g(e)\}$ is a smallest subset of L covering p.*

Proof. By Part (A) of Claim 4.3, we know that X covers p. Let $X' \subseteq L$ be an arbitrary cover of p, that is,

$$\varrho_{X'} \geq \widetilde{m} - \varrho_{g^-} + \delta_f,$$

or equivalently,

$$\varrho_{X'} + \varrho_{g^-} - \delta_f \geq \widetilde{m}.$$

By Part (B) of Claim 4.3, there exists an integer-valued m-flow x' which is $(f, g^- + \chi_{X'})$-bounded. Hence every g-saturated L-edge (with respect to x') belongs to X'. Since x is an L-upper-minimizer, it follows that $|X| \leq |X'|$, that is, X is indeed a smallest subset of L covering p. ■

Claim 4.5. *If $X^* \subseteq L$ is a smallest subset of L covering p, then every integer-valued $(f, g^- + \chi_{X^*})$-bounded m-flow x^* is an L-upper-minimizer (f, g)-bounded m-flow.*

Proof. Let $X' := \{e \in L : x^*(e) = g(e)\}$. By Claim 4.3, X' covers p and hence $|X^*| \leq |X'|$. Since x^* is $(f, g^- + \chi_{X^*})$-bounded, it follows that x^* admits at most $|X^*|$ g-saturated L-edges from which $|X^*| \geq |X'|$. Therefore $|X^*| = |X'|$ and thus x^* saturates a minimum number of elements of L, that is, x^* is an L-upper-minimizer. ■

From Claims 4.4 and 4.5, the lemma immediately follows. ■ ■

Let us turn to the proof of Theorem 4.1. Let x be an (f, g)-bounded integer-valued m-flow with a minimum number of g-saturated L-edges. Let $X = \{e \in L : x(e) = g(e)\}$, that is, X is the set of g-saturated L-edges. By Lemma 4.2, X is a smallest subset of L covering p.

Apply Theorem 3.2 to the digraph $D_L = (V, L)$ and to the set-function p defined in (4.3). In this case, p is fully supermodular from which we obtain that

$$\begin{aligned} |X| &= \max\{\varrho_L(C) - \sum[\varrho_L(Z) - p(Z) : Z \in C] : C \text{ a chain of subsets of } V\} \\ &= \max\{\varrho_L(C) - \sum[\varrho_g(Z) - \delta_f(Z) - \widetilde{m}(Z) : Z \in C] : C \text{ a chain of subsets of } V\}, \end{aligned}$$

as required. ■ ■ ■

Our next goal is to obtain optimality criteria for L-upper-minimizer m-flows.

Theorem 4.6. *Let L be a subset of A such that $-\infty < f(e) < g(e) < +\infty$ for every $e \in L$. There is a chain C^* of subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ of V with $\varrho_g(V_i) - \delta_f(V_i) < +\infty$ such that an integer-valued (f, g)-bounded m-flow z is an L-upper-minimizer if and only if the following optimality criteria hold.*
(O1) *$z(e) = f(e)$ for every edge $e \in A$ leaving a set V_i,*
(O2) *$z(e) = g(e)$ for every edge $e \in A - L$ entering a set V_i,*
(O3) *$g(e) - 1 \leq z(e) \leq g(e)$ for every edge $e \in L$ entering exactly one V_i,*
(O4) *$z(e) = g(e)$ for every edge $e \in L$ entering at least two V_i's,*
(O5) *$f(e) \leq z(e) \leq g(e) - 1$ for every edge $e \in L$ neither entering nor leaving any V_i.*

Proof. Apply Theorem 3.4 to the digraph $D_L = (V, L)$ and to the set-function p defined in (4.3), and consider the chain $C^* = \{V_1, \ldots, V_q\}$ ensured by the theorem where $V_1 \supset \cdots \supset V_q$. Since $p(V_i)$ is finite for each $i = 1, \ldots, q$, so is $\varrho_g(V_i) - \delta_f(V_i)$. Note that both $f(e)$ and $g(e)$ are finite for each edge $e \in L$ and for each edge leaving or entering a member of C^*.

To see the necessity of the conditions, suppose that x^* is an integer-valued (f, g)-bounded m-flow which is an L-upper-minimizer. By Lemma 4.2, the set $X^* := \{e \in L : x^*(e) = g(e)\}$ is a smallest subset of L covering p. Hence the optimality criteria (A), (B), and (C) in Theorem 3.4 hold.

By Property (A), $\varrho_{X^*}(V_i) = p(V_i)$ for every V_i, which is equivalent to

$$\varrho_{g^-}(V_i) + \varrho_{X^*}(V_i) - \delta_f(V_i) = \widetilde{m}(V_i), \tag{4.4}$$

from which

$$\widetilde{m}(V_i) = \varrho_{x^*}(V_i) - \delta_{x^*}(V_i) \leq \varrho_{g^-}(V_i) + \varrho_{X^*}(V_i) - \delta_f(V_i) = \widetilde{m}(V_i).$$

Hence we have equality throughout, in particular,

$$\varrho_{x^*}(V_i) = \varrho_{g^-}(V_i) + \varrho_{X^*}(V_i) \quad [= \widetilde{m}(V_i) + \delta_f(V_i)] \tag{4.5}$$

and

$$\delta_{x^*}(V_i) = \delta_f(V_i). \tag{4.6}$$

The equality in (4.6) shows that (O1) holds. Condition (4.5) implies for an edge $e \in A - L$ entering a V_i that $x^*(e) = g^-(e) = g(e)$ and hence (O2) holds. Condition (4.5) implies for an edge $e \in L$ entering a V_i that $g(e) - 1 \leq x^*(e) \leq g(e)$ and hence (O3) holds.

By Property (C), if an edge $e \in L$ enters at least two V_i's, then $e \in X^*$ and hence $x^*(e) = g(e)$, that is, (O4) holds.

To see (O5), let $e \in L$ be an edge neither entering nor leaving any V_i. By Property (B), $e \notin X^*$ and hence $x^*(e) \leq g(e) - 1$, from which (O5) follows.

To see the sufficiency of the conditions, let z be an integer-valued (f, g)-bounded m-flow satisfying the five conditions in the theorem. Let $X := \{e \in L : z(e) = g(e)\}$. By Part (A) of Claim 4.3, X covers p. We claim that X meets the three optimality criteria in Theorem 3.4. Let V_i be a member of chain C^*.

(O2) implies that

$$\sum [z(e) : e \in A - L, e \text{ enters } V_i] = \sum [g(e) : e \in A - L, e \text{ enters } V_i].$$

From the definition of X, we have

$$\sum[z(e) : e \in X, e \text{ enters } V_i] = \sum[g(e) : e \in X, e \text{ enters } V_i].$$

(O3) implies that

$$\sum[z(e) : e \in L - X, e \text{ enters } V_i] = \sum[g(e) - 1 : e \in L - X, e \text{ enters } V_i].$$

By merging these three equalities, we obtain

$$\varrho_z(V_i) = \varrho_{g^-}(V_i) + \varrho_X(V_i).$$

Furthermore, (O1) implies that

$$\delta_z(V_i) = \delta_f(V_i),$$

from which

$$\widetilde{m}(V_i) = \varrho_z(V_i) - \delta_z(V_i) = \varrho_{g^-}(V_i) + \varrho_X(V_i) - \delta_f(V_i),$$

that is,

$$\varrho_X(V_i) = \widetilde{m}(V_i) - \varrho_{g^-}(V_i) + \delta_f(V_i) = p(V_i),$$

showing that Property (A) in Theorem 3.4 holds indeed.

To see Property (B), let $e \in X$ ($\subseteq L$) be an edge. Then $z(e) = g(e)$ and, by (O5), e enters or leaves a V_i. But e cannot leave any V_i since if it did, then (O1) would imply $z(e) = f(e)$ and this would contradict the assumption that L contains no tight edge. Therefore e must enter a V_i, that is, (B) holds indeed.

To see Property (C), let e be an edge in L which enters at least two V_i's. By (O4), $z(e) = g(e)$ and hence $e \in X$, that is, (C) holds.

By Theorem 3.4, X is a smallest subset of L covering p. By Lemma 4.2, x is an L-upper-minimizer (f, g)-bounded m-flow, as stated in the theorem. ■

■

In Section 8, we describe an algorithmic proof of Theorem 4.6. The algorithm will compute in strongly polynomial time an (f, g)-bounded L-upper-minimizer integral m-flow along with the optimal chain described in the theorem.

5 Description of dec-min m-flows: Proof of Theorem 2.1

After preparations in Sections 3 and 4, we turn to our main goal of proving Theorem 2.1. As before, let $D = (V, A)$ be a digraph and $F \subseteq A$ a specified subset of edges. We assume that the underlying undirected graph of D is connected. Let $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ be bounding functions with $f \leq g$. We require $-\infty < f(e) \leq g(e) < +\infty$ for every $e \in F$. Let $m : V \to \mathbf{Z}$ be a function on the node-set for which there is an integer-valued (f, g)-bounded m-flow (that is, $\widetilde{m}(V) = 0$ and Hoffman's condition (2.2) holds). Recall that $\ddot{Q} = \ddot{Q}(f, g; m)$ denoted the set of integer-valued (f, g)-bounded m-flows.

In the proof we shall use induction on $|F|$. Since $f^* := f$ and $g^* = g$ clearly meet the requirements of the theorem when $F = \emptyset$, we can assume that F is non-empty. We observed already in Section 2.2 that it suffices to prove Theorem 2.1 in the special case when F contains no tight edge, therefore we assume throughout that $f(e) < g(e)$ for each edge $e \in F$.

Let $\beta = \beta_F$ denote the smallest integer for which $\ddddot{Q}$ has an element z satisfying $z(e) \leq \beta$ for every edge $e \in F$. In the next section, we shall work out an algorithm to compute β_F in strongly polynomial time. Since we are interested in F-dec-min members of $\ddddot{Q}$, we may assume that the largest g-value of the edges in F is this β. Let $L := \{e \in F : g(e) = \beta\}$. Now Hoffman's condition (2.2) holds but, since F contains no tight edges and since β is minimal, after decreasing the g-value of the elements of L from β to $\beta - 1$, the resulting function $g^- := g - \chi_L$ violates (2.2), that is, $Q(f, g^-; m) = \emptyset$. Summing up, we shall rely on the following notation and assumptions:

$$\begin{cases} F \text{ is non-empty and contains no } (f,g)\text{-tight edges,} \\ \beta := \max\{g(e) : e \in F\}, \\ L := \{e \in F : g(e) = \beta\}, \\ g^- := g - \chi_L, \\ \ddddot{Q} = \ddddot{Q}(f,g;m) \text{ is non-empty,} \\ \ddddot{Q}(f,g^-;m) \text{ is empty.} \end{cases} \tag{5.1}$$

As a preparation for deriving the main result Theorem 2.1, we need the following relaxation of decreasing minimality. We call a member z of $\ddddot{Q}$ **pre-decreasingly minimal** (**pre-dec-min**, for short) **on** F if the number μ of edges e in L with $z(e) = \beta$ is as small as possible. Obviously, if z is F-dec-min, then z is pre-dec-min on F. By applying Theorem 4.6 to the present special case, we obtain the following characterization of pre-dec-min elements.

Theorem 5.1. *Given* (5.1)*, there is a chain $\mathcal{C}'$ of non-empty proper subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ of V with $\varrho_g(V_i) - \delta_f(V_i) < +\infty$ such that a member z of $\ddddot{Q}$ is pre-dec-min on F if and only if the following optimality criteria hold:*

(O1) *$z(e) = f(e)$ for every edge $e \in A$ leaving a member of $\mathcal{C}'$,*

(O2) *$z(e) = g(e)$ for every edge $e \in A - L$ entering a member of $\mathcal{C}'$,*

(O3) *$\beta - 1 \leq z(e) \leq \beta$ for every edge $e \in L$ entering exactly one member of $\mathcal{C}'$,*

(O4) *$z(e) = \beta$ for every edge $e \in L$ entering at least two members of $\mathcal{C}'$,*

(O5) *$f(e) \leq z(e) \leq \beta - 1$ for every edge $e \in L$ neither entering nor leaving any member of $\mathcal{C}'$.* ■

Define the bounding pair $(f'(e), g'(e))$ for each edge e, as follows. For $e \in L$, let

$$(f'(e), g'(e)) := \begin{cases} (\beta, \beta) & \text{if } e \text{ enters at least two members of } \mathcal{C}', \\ (\beta - 1, \beta) & \text{if } e \text{ enters exactly one member of } \mathcal{C}', \\ (f(e), f(e)) & \text{if } e \text{ leaves a member of } \mathcal{C}', \\ (f(e), \beta - 1) & \text{if } e \text{ neither leaves nor enters any member of } \mathcal{C}'. \end{cases} \tag{5.2}$$

For $e \in A - L$, let

$$(f'(e), g'(e)) := \begin{cases} (g(e), g(e)) & \text{if } e \text{ enters a member of } C', \\ (f(e), f(e)) & \text{if } e \text{ leaves a member of } C', \\ (f(e), g(e)) & \text{if } e \text{ neither leaves nor enters any member of } C'. \end{cases} \tag{5.3}$$

It follows from this definition that $f \leq f' \leq g' \leq g$. Let

$$\ddddot{Q}' := \ddddot{Q}(f', g'; m). \tag{5.4}$$

Lemma 5.2. (A) *An m-flow $z \in \ddddot{Q}$ is pre-dec-min on F if and only if $z \in \ddddot{Q}'$.* (B) *An m-flow $z \in \ddddot{Q}$ is F-dec-min if and only if z is an F-dec-min element of $\ddddot{Q}'$.*

Proof. Theorem 5.1 immediately implies the equivalence in Part (A). To see Part (B), suppose first that z is an F-dec-min element of $\ddddot{Q}$. Then z is surely F-pre-dec-min in $\ddddot{Q}$ and hence, by Part (A), z is in $\ddddot{Q}'$. If, indirectly, $\ddddot{Q}'$ had an element z' which is decreasingly smaller on F than z, then z could not have been an F-dec-min element of $\ddddot{Q}$. Conversely, let z' be an F-dec-min element of $\ddddot{Q}'$ and suppose indirectly that z' is not an F-dec-min element of $\ddddot{Q}$. Then any F-dec-min element z of $\ddddot{Q}$ is decreasingly smaller on F than z'. But any F-dec-min element of $\ddddot{Q}$ is pre-dec-min on F and hence, by Part (A), z is in $\ddddot{Q}'$, contradicting the assumption that z' was an F-dec-min element of $\ddddot{Q}'$. ■

Theorem 2.1 will be an immediate consequence of the following result.

Theorem 5.3. *Given* (5.1)*, there is a pair (f', g') of integral functions on A with $f \leq f' \leq g' \leq g$ and there is a set $F' \subset F$ for which an element z of $\ddddot{Q}$ is an F-dec-min member of $\ddddot{Q}$ if and only if z is an F'-dec-min member of $\ddddot{Q}' = \ddddot{Q}(f', g'; m)$. In addition, the box $T(f', g')$ is narrow on $F - F'$ in the sense that* $0 \leq g'(e) - f'(e) \leq 1$ *holds for every $e \in F - F'$.*

Proof. Let C' be the chain ensured by Theorem 5.1, let (f', g') be the pair of bounding functions defined in (5.2) and (5.3), and let $\ddddot{Q}' := \ddddot{Q}(f', g'; m)$.

Claim 5.4. *The subset $L' \subseteq L$ consisting of those elements of L that enter at least one member of C' is non-empty.*

Proof. Let z be an element of $\ddddot{Q}$ which is pre-dec-min on F. By Part (A) of the lemma, $z \in \ddddot{Q}'$. By (5.1), there is an edge e in F, for which $z(e) = \beta = g(e)$, and hence $e \in L$. Since $g(e) = z(e) \leq g'(e) \leq g(e)$ and F contains no (f, g)-tight edges, we have $f(e) < g(e) = g'(e) = \beta$. This and definition (5.2) imply that e enters at least one member of C'. ■

Since $L' \neq \emptyset$ by the claim, we have

$$F' := F - L' \text{ is a proper subset of } F.$$

We are going to show that (f', g') and F' meet the requirements of the theorem. Call two vectors in $\mathbf{Z}^A$ value-equivalent on L' if their restrictions to L' (that is, their projection to $\mathbf{Z}^{L'}$), when both arranged in a decreasing order, are equal.

Lemma 5.5. *The members of $\ddddot{Q}'$ are value-equivalent on L'.*

Proof. By Part (A) of Lemma 5.2, the members of $\ddddot{Q}'$ are exactly those elements of $\ddddot{Q}$ which are pre-dec-min on F. Hence each member z of $\ddddot{Q}'$ has the same number μ of edges in L for which $z(e) = \beta$.

As F contains no (f, g)-tight edges, we have $z(e) \leq g'(e) \leq \beta - 1$ for every edge $e \in L - L'$ and hence each element e of L with $z(e) = \beta$ belongs to L', from which

$$|\{e \in L' : z(e) = \beta\}| = \mu.$$

Furthermore, we have $f'(e) \geq \beta - 1$ for every element e of L' from which L' has exactly $|L'| - \mu$ edges with $z(e) = \beta - 1$, implying that the members of $\ddddot{Q}'$ are indeed value-equivalent on L'. ■

Part (B) of Lemma 5.2 implies that the F-dec-min elements of $\ddddot{Q}$ are exactly the F-dec-min elements of $\ddddot{Q}'$, and hence it suffices to prove that an element z of $\ddddot{Q}'$ is an F-dec-min member of $\ddddot{Q}'$ if and only if z is an F'-dec-min member of $\ddddot{Q}'$. But this latter equivalence is an immediate consequence of Lemma 5.5.

To prove the last part of Theorem 5.3, recall that $F - F' = L'$ and L' consisted of those elements of L that enter at least one member of C'. But the definition of (f', g') in (5.2) implies that $\beta - 1 \leq f'(e) \leq g'(e) = \beta$ for every element e of L', that is, the box $T(f', g')$ is indeed narrow on $F - F'$. ■■

Proof of Theorem 2.1 We use induction on $|F|$. Since $f^* := f$ and $g^* = g$ clearly meet the requirements of the theorem when $F = \emptyset$, we can assume that F is non-empty. As before, we may assume that F contains no (f, g)-tight edges. By Theorem 5.3, it suffices to prove Theorem 2.1 for $\ddddot{Q}(f', g'; m)$ and F'. But this follows by induction since F' is a proper subset of F. ■

Cheapest integral F-dec-min m-flows In Sections 7 and 8, we shall describe an algorithm to compute (f^*, g^*) in Theorem 2.1. Once these bounding functions are available, we can immediately solve the problem of computing a cheapest integral F-dec-min (f, g)-bounded m-flow with respect to a cost-function $c : A \to \mathbf{R}$. By theorem 2.1, this latter problem is nothing but a minimum cost (f^*, g^*)-bounded m-flow problem, which can indeed be solved by a minimum cost feasible circulation algorithm. In the literature there are several strongly polynomial algorithms for the cheapest circulation problem, the first one was due to Tardos [14].

6 Characterization by improving di-circuits and by feasible potential-vectors

Let $D = (V, A)$, F, f, g, m be the same as in Theorem 2.1. Let $\ddddot{Q} = \ddddot{Q}(f, g; m)$ denote the set of integral (f, g)-bounded m-flows. We assume that $\ddddot{Q}$ is non-empty but the properties in (5.1) are not a priori expected. For an element $z \in \ddddot{Q}$, let $D_z = (V, A_z)$ denote the standard auxiliary digraph associated with z, that is,

$$A_z := \{uv : uv \in A, z(uv) < g(uv)\} \cup \{vu : uv \in A, z(uv) > f(uv)\}.$$

An edge $uv \in A_z$ is called a forward edge when $z(uv) < g(uv)$ and a backward edge when $z(vu) > f(vu)$.

Theorem 2.1 provided a characterization for the set of F-dec-min elements of $\ddddot{Q}$, namely, an element $z \in \ddddot{Q}$ is F-dec-min precisely if $f^* \leq z \leq g^*$. The goal of this section is to describe a different characterization for $z \in \ddddot{Q}$ to be decreasingly minimal on F, consisting of two equivalent properties. (For a comparison of the previous and this new characterizations, see Remark 6.2.) For the first one, we introduce a simple and natural way to obtain from z a decreasingly smaller feasible m-flow by improving z along an appropriate di-circuit of D_z. For the second property, by extending the standard notion of feasible potentials, we introduce feasible potential-vectors. The main result of the section states (roughly) that the following three properties for z are pairwise equivalent: (A) z is dec-min on F, (B) no di-circuit improving z exists, and (C) there exists a feasible potential-vector.

6.1 Feasible potential-vectors

Let $c : A_0 \to \mathbf{R}$ be a cost-function defined on the edge-set of a digraph $D_0 = (V, A_0)$. A di-circuit C of D_0 is called negative (with respect to c) if the total c-cost $\widetilde{c}(C) = \sum[c(e) : e \in C]$ of C is negative. In the literature, c is called **conservative** if D_0 admits no negative di-circuit. A function $\pi : V \to \mathbf{R}$ is called a **c-feasible potential** if $\pi(v) - \pi(u) \leq c(uv)$ holds for every edge uv of D_0. A classic result of Gallai is as follows.

Theorem 6.1 (Gallai)**.** *Given a digraph $D_0 = (V, A_0)$ and a cost-function $c : A_0 \to \mathbf{R}$, there exists a c-feasible potential $\pi : V \to \mathbf{R}$ if and only if c is conservative. If c is conservative and integer-valued, then π can be chosen integer-valued, as well.* ■

Given two k-dimensional vectors $\underline{x} = (x_1, x_2, \dots, x_k)$ and $\underline{y} = (y_1, y_2, \dots, y_k)$, we say that $\underline{x}$ is **lexicographically smaller** than $\underline{y}$, in notation $\underline{x} \prec \underline{y}$, if $\underline{x} \neq \underline{y}$ and $x_i < y_i$ where i denotes the first component in which they differ. We write $\underline{x} \preceq \underline{y}$ if $\underline{x} = \underline{y}$ or $\underline{x} \prec \underline{y}$. Note that the relation $\preceq$ is a total ordering of the elements of $\mathbf{R}^k$.

Let $\underline{c} : A_0 \to \mathbf{R}^k$ be a vector-valued function on the edge-set of $D_0 = (V, A_0)$ that assigns a vector $\underline{c}(e) = (c_1(e), c_2(e), \dots, c_k(e))$ to each edge e of D_0. We call a vector-valued function $\underline{\pi} : V \to \mathbf{R}^k$ on the node-set V **$\underline{c}$-feasible** or just feasible if

$$\underline{\pi}(v) - \underline{\pi}(u) \preceq \underline{c}(uv) \tag{6.1}$$

holds for every edge uv of D_0.

A di-circuit C is said to be **$\underline{c}$-negative** if the sum $\widetilde{\underline{c}}(C) = (\widetilde{c}_1(C), \widetilde{c}_2(C), \dots, \widetilde{c}_k(C))$ of the $\underline{c}$-vectors assigned to its edges is lexicographically smaller than the k-dimensional zero vector $\underline{0}_k$. The vector-valued function $\underline{c}$ is **conservative** if D_0 has no $\underline{c}$-negative di-circuit.

The following Gallai-type theorem specializes to Theorem 6.1 in case $k = 1$, but in its proof we rely on Theorem 6.1.

Theorem 6.2. *Given a digraph $D_0 = (V, A_0)$ and a vector-valued function $\underline{c} : A_0 \to \mathbf{R}^k$ on its edge-set, there exists a $\underline{c}$-feasible potential-vector $\underline{\pi} : V \to \mathbf{R}^k$ if and only if $\underline{c}$ is conservative, that is, D_0 admits no $\underline{c}$-negative di-circuit. If $\underline{c}$ is integer vector-valued and conservative, then a $\underline{c}$-feasible $\underline{\pi}$ can be chosen to be integer vector-valued.*

Proof. Let C be a di-circuit of D_0 whose nodes, in cyclic order, are $v_1, v_2, \dots, v_q$. Accordingly the edges of C are $e_1 = v_1v_2, e_2 = v_2v_3, \dots, e_q = v_qv_1$. Let $\underline{\pi}$ be a $\underline{c}$-feasible potential-vector. Then

$$\begin{aligned}\underline{0}_k &= [\underline{\pi}(v_2) - \underline{\pi}(v_1)] + [\underline{\pi}(v_3) - \underline{\pi}(v_2)] + \cdots + [\underline{\pi}(v_1) - \underline{\pi}(v_q)]\\ &\leq \sum[\underline{c}(e_i) : i = 1, \dots, q] = \widetilde{\underline{c}}(C).\end{aligned}$$

To see the reverse direction, we apply induction on k. When $k = 1$, we are back at Theorem 6.1. Suppose now that $k \geq 2$, and assume that D_0 admits no $\underline{c}$-negative di-circuit.

Consider the functions $c_i : A_0 \to \mathbf{R}$ formed by the i-th components of $\underline{c}$ $(i = 1, \dots, k)$. As $\underline{c}$ is conservative, so is c_1, that is $\widetilde{c_1}(C) \geq 0$ for every di-circuit C. By Theorem 6.1, there exists a c_1-feasible potential $\pi_1 : V \to \mathbf{R}$ (which is integer-valued when c_1 is integer-valued). Let A_1 denote the set of tight edges, that is

$$A_1 = \{uv \in A_0 : \pi_1(v) - \pi_1(u) = c_1(uv)\}.$$

Let $k' := k - 1$ and $\underline{c}' := (c_2, c_3, \dots, c_k)$. Then $\underline{c}'$ is conservative in $D_1 = (V, A_1)$ since $\underline{c}$ is conservative and $\pi_1(v) - \pi_1(u) = c_1(uv)$ holds for every edge uv in A_1. By induction, there is a $(k-1)$-dimensional potential-vector, $\underline{\pi}' = (\pi_2, \dots, \pi_k)$ which is $\underline{c}'$-feasible on the edges in A_1. Let $\underline{\pi} := (\pi_1, \pi_2, \dots, \pi_k)$. Then $\underline{\pi}$ is $\underline{c}$-feasible on the edges in A_1. Moreover, $\pi_1(v) - \pi_1(u) < c_1(uv)$ for every edge $uv \in A_0 - A_1$, and hence $\underline{\pi}$ is $\underline{c}$-feasible on these edges, as well. ∎

6.2 Improving di-circuits

Let A_+ and A_- be two disjoint sets and let $A_* := A_+ \cup A_-$. Let x be an integer-valued function on A_*. As a preparatory lemma, we develop an equivalent condition for the function

$$x' := x + \chi_{A_+} - \chi_{A_-} \tag{6.2}$$

to be decreasingly smaller than x. To this end, define $x^* : A_* \to \mathbf{Z}$, as follows:

$$x^* := x - \chi_{A_-}. \tag{6.3}$$

Let $\lambda_1 > \lambda_2 > \cdots > \lambda_h$ denote the distinct values of the components of x^*. We assign a h-dimensional vector $\underline{c}'(e)$ to every element $e \in A_*$, as follows:

$$\underline{c}'(e) := \begin{cases} \underline{\varepsilon}'_i & \text{if } e \in A_+ \text{ and } x^*(e) = \lambda_i, \\ -\underline{\varepsilon}'_i & \text{if } e \in A_- \text{ and } x^*(e) = \lambda_i, \end{cases} \tag{6.4}$$

where $\underline{\varepsilon}'_i$ is the h-dimensional unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ whose i-th component is 1.

Lemma 6.3. *$x' <_{\mathrm{dec}} x$ if and only if $\underline{\widetilde{c}}'(A_*) \prec \underline{0}_h$.*

Proof. Induction on $|A_*|$. If $|A_*| = 0$, then the statement of the lemma is void, so suppose that $A_* \neq \emptyset$. If $A_- = \emptyset$, then $x' \geq_{\mathrm{dec}} x$ and $\underline{\widetilde{c}}'(A_*) \succeq \underline{0}_h$, and hence we can suppose that $A_- \neq \emptyset$. If $A_+ = \emptyset$, then $x' <_{\mathrm{dec}} x$ and $\underline{\widetilde{c}}'(A_*) \prec \underline{0}_h$, and we can suppose that $A_+ \neq \emptyset$.

Let e_+ be an element of A_+ for which $\lambda_i = x^*(e_+)$ is maximum, and let e_- be an element of A_- for which $\gamma_j = x^*(e_-)$ is maximum. If $\lambda_i > \lambda_j$, then $x' >_{\mathrm{dec}} x$ and $\underline{\widetilde{c}}'(A_*) \succ 0_h$. If $\lambda_i < \lambda_j$, then $x' <_{\mathrm{dec}} x$ and $\underline{\widetilde{c}}'(A_*) \prec \underline{0}_h$.

In the remaining case, when $\lambda_i = \lambda_j$, we have $x(e_+) + 1 = x(e_-)$. Define $A'_+ := A_+ - e_+$, $A'_- := A_- - e_-$, and let $A'_* := A_* - \{e_-, e_+\}$. Observe that the restriction of x' to A'_* is decreasingly smaller than the restriction of x to A'_* precisely if $x' <_{\mathrm{dec}} x$. On the other hand, $\underline{\widetilde{c}}'(A'_*) = \underline{\widetilde{c}}'(A_*)$ and hence $\underline{\widetilde{c}}'(A'_*) \prec \underline{0}_h$ precisely if $\underline{\widetilde{c}}'(A_*) \prec \underline{0}_h$. Since $|A'_*| < |A_*|$, we are done by induction. ∎

After this preparation, we return to $D = (V, A)$, $F \subseteq A$, and $z \in \ddot{\ddot{Q}} = \ddot{\ddot{Q}}(f, g; m)$. Let $D_z = (V, A_z)$ be the auxiliary digraph associated with z. We call a di-circuit C of D_z ***z*-improving on** F (or just z-improving) if $z' \in \ddot{\ddot{Q}}$ is decreasingly smaller than z on F, where $z'(uv)$ is defined for $uv \in A$, as follows:

$$z'(uv) := \begin{cases} z(uv) + 1 & \text{if } uv \text{ is a forward edge of } C, \\ z(uv) - 1 & \text{if } vu \text{ is a backward edge of } C, \\ z(uv) & \text{otherwise.} \end{cases} \tag{6.5}$$

Note that the definition of D_z implies that z' is indeed in $\ddot{\ddot{Q}}$.

Let F_z denote the subset of A_z corresponding to F (that is, for $uv \in F$, if $z(uv) < g(uv)$, then the forward edge uv belongs to F_z, while if $z(uv) > f(uv)$, then the backward edge vu belongs to F_z). The sets of forward and backward edges in F_z are denoted by $F_{\mathbf{f}}$ and $F_{\mathbf{b}}$, respectively. (The subscripts **f** and **b** refer to **f**orward and **b**ackward.)

Define the function z^* on F_z, as follows:

$$z^*(uv) := \begin{cases} z(uv) & \text{if } uv \in F_{\mathbf{f}}, \\ z(vu) - 1 & \text{if } uv \in F_{\mathbf{b}}. \end{cases} \tag{6.6}$$

Let $\gamma_1 > \gamma_2 > \cdots > \gamma_k$ denote the distinct values of the components of z^*. Let $\underline{\varepsilon}_i$ denote the k-dimensional unit-vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ whose i-th component is 1. We assign a k-dimensional vector $\underline{c}(e)$ to every edge e of D_z, as follows:

$$\underline{c}(e) := \begin{cases} \underline{0}_k & \text{if } e \in A_z - F_z, \\ \underline{\varepsilon}_i & \text{if } e \in F_{\mathbf{f}} \text{ and } z^*(e) = \gamma_i, \\ -\underline{\varepsilon}_i & \text{if } e \in F_{\mathbf{b}} \text{ and } z^*(e) = \gamma_i. \end{cases} \tag{6.7}$$

Lemma 6.4. *A di-circuit C of D_z is z-improving on F if and only if $\underline{\widetilde{c}}(C) \prec \underline{0}_k$.*

Proof. Let $A_+ := \{uv : uv \in F_{\mathbf{f}} \cap C\}$, $A_- := \{uv : vu \in F_{\mathbf{b}} \cap C\}$, and $A_* := A_+ \cup A_-$. Note that $A_* \subseteq A$. Let x denote the restriction of z to A_*. Then x' defined in (6.2) is the restriction of z' to A_*, and x^* defined in (6.3) is the restriction of z^* to A_*. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_h$ denote the distinct values of x^*, and consider the vector $\underline{c}'$ defined in (6.4). Note that $\{\lambda_1, \lambda_2, \dots, \lambda_h\}$ is a subsequence of $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$, in particular, $h \leq k$. Observe that C is z-improving if and only if x' is decreasingly smaller than x. Also observe that $\widetilde{\underline{c}}(C) \prec \underline{0}_k$ if and only if $\widetilde{\underline{c}}'(A_*) \prec \underline{0}_h$. Then we are done by Lemma 6.3. ∎

The main result of this section is as follows.

Theorem 6.5. *For an element $z \in \ddot{Q} = \ddot{Q}(f, g; m)$, the following properties are equivalent.*
(A) *z is decreasingly minimal on F.*
(B) *There is no z-improving di-circuit in the auxiliary digraph D_z.*
(C) *There is an integer-valued potential-vector function $\underline{\pi}$ on V which is $\underline{c}$-feasible, that is, $\underline{\pi}(v) - \underline{\pi}(u) \leq \underline{c}(uv)$ for every edge $uv \in A_z$.*

Proof. The equivalence of (A) and (B) is exactly Lemma 6.4. The equivalence of (B) and (C) is a consequence of Theorem 6.2. ∎

Remark 6.1. As indicated in Part II [7] of this series of papers, the decreasing minimization on a discrete set can be formulated as a separable convex function minimization on that set. Accordingly, discrete convex analysis often offers effective tools for investigating discrete decreasing minimization. Indeed, alternative proofs of Theorems 2.1 and 6.5 can be constructed on the basis of the DCA results summarized in Section 7.3 of Part II [7]. ∎

Remark 6.2. From a theoretical computer science point of view, a slight drawback of the characterization in Theorem 2.1 is that, in order to be convinced that z is indeed F-dec-min, one must believe the correctness of (f^*, g^*). In this respect, Property (C) in Theorem 6.5 is more convincing since it provides a certificate for z to be F-dec-min whose validity can be checked immediately.

Just for an analogy to understand better this aspect of certificates, consider the well-known maximum weight perfect matching problem in a bipartite graph $G = (S, T; E)$ endowed with a weight-function w on E. On one hand, one can prove the characterization that there is a subgraph $G' = (S, T; E')$ of G such that a perfect matching M of G is of maximum w-weight if and only if $M \subseteq E'$. (This result intuitively corresponds to Theorem 2.1). This certificate E', however, is convincing (for the optimality of M) only if we can check that it has been correctly computed. On the other hand, Egerváry's classic theorem provides an immediately checkable certificate for M to be of maximum w-weight: a function $\pi : S \cup T \to \mathbf{R}$ for which $\pi(s) + \pi(t) \geq w(st)$ for every edge $st \in E$ and $\pi(s) + \pi(t) = w(st)$ for every edge $st \in M$. (This result intuitively corresponds to the equivalence of (A) and (C) in Theorem 6.5). ∎

7 Algorithm for minimizing the largest m-flow value on F

Our remaining task is to describe a strongly polynomial algorithm to compute the bounding pair (f^*, g^*) described in Theorem 2.1. To this end, it suffices to compute the bounding pair (f', g') and the proper subset F' of F satisfying the requirements in Theorem 5.3 since after repeating this reduction at most $|F|$ times we arrive at the trivial case $F = \emptyset$.

Since the pair (f', g') is defined with the help of β and the chain C' in Lemma 5.1, the computation of (f', g') and F' consists of two parts. The present section describes an algorithm to compute β, the smallest integer for which $\ddddot{Q}$ has an element z satisfying $z(e) \leq \beta$ for every edge $e \in F$. The next section shall include an algorithm for computing the chain C' in Lemma 5.1.

As before, we suppose that there is an (f, g)-bounded m-flow, and also that F contains no (f, g)-tight edges. Our first goal is to find the smallest integer β such that by decreasing $g(e)$ to β for each edge $e \in F$ for which $g(e) > \beta$, the resulting g' and the unchanged f continue to meet the inequality $f \leq g'$ and the Hoffman-condition. The first requirement implies that β is at least the largest f-value on the edges in F, which is denoted by f_1.

Let $g_1 > g_2 > \cdots > g_q$ denote the distinct g-values of the edges in F, and let $L := \{e \in F : g(e) = g_1\}$. Let $\beta_1 := \max\{f_1, g_2\}$.

By an m-flow feasibility computation, we can check whether the g-value g_1 on the elements of L can be uniformly decreased to β_1 without destroying (2.2). If this is the case, then either $\beta_1 = f_1$ in which case a tight edge arises in F and we can remove this tight edge from F, or $\beta_1 = g_2$ in which case the number of distinct g_i-values becomes one smaller. Clearly, as the total number of distinct g_i-values in F is at most $|F|$, this kind of reduction may occur at most $|F|$ times.

Therefore, we are at a case when g_1 cannot be decreased to β_1 without violating (2.2). Let us try to figure out the lowest integer value β to which g_1 can be decreased without violating (2.2).

Recall that $L = \{e \in F : g(e) = g_1\}$ and let $A_0 := A - L$ (that is, A_0 is the complement of L with respect to the whole edge-set A). Let g' denote the function arising from g by reducing $g(e)$ on the elements of L (where $g(e) = g_1$) to β_1. Since $g' \geq f$ holds and $\varrho_{g'} - \delta_f$ is submodular, the set-function p' defined by

$$p'(Z) := \widetilde{m}(Z) - \varrho_{g'}(Z) + \delta_f(Z)$$

is supermodular.

Since g_1 in the present case cannot be decreased to β_1 without violating (2.2), there is a subset Z^* violating $\varrho_{g'}(Z) - \delta_f(Z) \geq \widetilde{m}(Z)$, or for short, $p'(Z^*) > 0$. The original g meets (2.2), meaning that $\varrho_g - \delta_f \geq \widetilde{m}$, which is equivalent to

$$(g_1 - \beta_1)\varrho_L(Z) + \varrho_{g'}(Z) - \delta_f(Z) = \varrho_g(Z) - \delta_f(Z) \geq \widetilde{m}(Z)$$

holds for every $Z \subseteq V$. Therefore our goal is to find the smallest integer μ for which

$$\mu\varrho_L(Z) + \varrho_{g'}(Z) - \delta_f(Z) \geq \widetilde{m}(Z)$$

for every $Z \subseteq V$, that is,

$$\mu b(Z) \geq p'(Z),$$

where $b(Z) := \varrho_L(Z)$. Clearly

$$\mu = \max\left\{\left\lceil \frac{p'(Z)}{b(Z)} \right\rceil : b(Z) > 0\right\}. \tag{7.1}$$

Since b is submodular, p' is supermodular, and we have $\max\{b(Z) : Z \subseteq V\} \leq |L| \leq |A|$, we can apply the Newton–Dinkelbach algorithm for this case, as described in [6]. That algorithm needs a subroutine to compute a subset of V maximizing $p'(Z) - \mu b(Z)$ $(Z \subseteq V)$ for any fixed integer $\mu \geq 0$. This subroutine is applied at most M times where M denotes the largest value of b. Since in the present case of flows, the largest value of b is at most $|L| \leq |A|$, the subroutine is applied at most $|A|$ times. Furthermore, by the definition of p' and b, the equivalent subroutine to minimize $\mu b(Z) - p'(Z) = \mu\varrho_L(Z) + \varrho_{g'}(Z) - \delta_f(Z) - \widetilde{m}(Z)$ can be realized with the help of a straightforward reduction to a max-flow min-cut computation in a related edge-capacitated digraph on node-set $V \cup \{s, t\}$ with extra source-node s and sink-node t.

Therefore, by relying on an efficient max-flow computation, the smallest μ can be computed in strongly polynomial time, and hence the smallest β $(= \beta_1 + \mu)$ is available for which $\beta > \beta_1 = \max\{f_1, g_2\}$ and the value g_1 can be reduced to β on the edges in L without violating (2.2).

8 Computing an L-upper-minimizer m-flow and the dual optimum chain

In this section, we describe an alternative, algorithmic proof of Theorems 4.1 and 4.6. In this light, their original proof in Section 4 may seem superfluous but we keep both proofs because the first one is more transparent and technically simpler than the algorithmic approach to be presented here.

The algorithm computes an integer-valued L-upper-minimizer (f, g)-bounded m-flow as well as a maximizer chain C in (4.1) meeting the optimality criteria in Theorem 4.6. As before, $D = (V, A)$ is a digraph and we assume that L is a subset of A for which $-\infty < f(e) < g(e) < \infty$ for each edge $e \in L$. (For edges in $A - L$, $f(e) = -\infty$ and $g(e) = +\infty$ are allowed.) Our primal goal is to find an integral (f, g)-bounded m-flow g-saturating a minimum number of elements of L. To this end, we introduce a parallel copy e' of each $e \in L$. Let L' denote the set of new edges. We shall refer to the edges in A as old or original edges. Let $A_1 := A \cup L'$, $D' = (V, L')$, and $D_1 = (V, A \cup L')$. Define g^- on A by $g^- := g - \chi_L$, that is, we reduce $g(e)$ by 1 for each $e \in L$.

Let f_1 and g_1 be bounding functions on A_1 defined by

$$g_1(e) := \begin{cases} g^-(e) & \text{if } e \in A, \\ 1 & \text{if } e \in L', \end{cases}$$

$$f_1(e) := \begin{cases} f(e) & \text{if } e \in A, \\ 0 & \text{if } e \in L'. \end{cases}$$

Let c_1 be a $(0, 1)$-valued cost-function on A_1 defined by

$$c_1(e) := \begin{cases} 0 & \text{if } e \in A, \\ 1 & \text{if } e \in L'. \end{cases}$$

Our goal is to find an (f, g)-bounded integer-valued m-flow in D admitting a minimum number of g-saturated L-edges. We claim that this problem is equivalent to finding a minimum c_1-cost (f_1, g_1)-bounded integer-valued m-flow in D_1. Indeed, let x be an (f, g)-bounded m-flow in D and let $X := \{e \in L : x(e) = g(e)\}$ be the set of g-saturated members of L. Let X' denote the subset of L' corresponding to X. Define an m-flow x_1 in D_1 as follows:

$$x_1(e) := \begin{cases} x(e) & \text{if } e \in A - X, \\ g(e) - 1 & \text{if } e \in X, \\ 1 & \text{if } e \in X', \\ 0 & \text{if } e \in L' - X'. \end{cases}$$

Then x_1 is an (f_1, g_1)-bounded m-flow in D_1 whose c_1-cost is $|X|$. Conversely, let x_1 be a minimum cost integer-valued (f_1, g_1)-bounded m-flow in D_1. Observe that if $x_1(e') = 1$ for some $e' \in L'$, then $x_1(e) = g_1(e) = g(e) - 1$ where e is the edge in L corresponding to e'. Indeed, if we had $x_1(e) \leq g(e) - 2$, then the m-flow obtained from x_1 by adding 1 to $x_1(e)$ and subtracting 1 from $x_1(e')$ would be of smaller cost. It follows that the m-flow x in D defined by

$$x(e) := \begin{cases} x_1(e) + x_1(e') & \text{if } e \in L, \\ x_1(e) & \text{if } e \in A - L \end{cases} \tag{8.1}$$

is an (f, g)-bounded m-flow in D, for which the number of g-saturated L-edges is exactly the c_1-cost of x_1.

Therefore, we concentrate on finding an integer-valued min-cost (f_1, g_1)-bounded m-flow in D_1. In order to describe the dual optimization problem, let N denote the node-edge signed incidence matrix of D, that is, the entry of N corresponding to a node v and to an edge $e \in A$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise. Let N' denote the analogous signed incidence matrix of D', and let $N_1 = [N, N']$. Note that N_1 is the signed incidence matrix of D_1 and hence it is totally unimodular. The primal linear program is as follows:

$$\min\{c_1 x_1 : \ N_1 x_1 = m, \ x_1 \geq f_1, \ -x_1 \geq -g_1\}. \tag{8.2}$$

The dual linear program is as follows:

$$\max\{ym + z_1 f_1 - w_1 g_1 : \ yN_1 + z_1 - w_1 = c_1, \ z_1 \geq 0, \ w_1 \geq 0\}. \tag{8.3}$$

Note that the components of $z_1 = (z, z')$ correspond to the edges in A and in L', respectively, and the analogous statement holds for $w_1 = (w, w')$. Since N_1 is totally unimodular, both the primal and the dual optimal solution can be chosen integer-valued.

If (y, z_1, w_1) is a dual solution and both $z_1(e)$ and $w_1(e)$ are positive on an edge $e \in A_1$, then reducing both $z_1(e)$ and $w_1(e)$ by $\min\{z_1(e), w_1(e)\}$ we obtain another dual solution whose dual cost is larger by $g_1(e) - f_1(e) \geq 0$ than the dual cost $ym + z_1 f_1 - w_1 g_1$ of (y, z_1, w_1). Therefore it suffices to consider only those optimal dual solutions (y, z_1, w_1) for which $\min\{z_1(e), w_1(e)\} = 0$ for every edge $e \in A_1$. Observe that for such an optimal dual solution (y, z_1, w_1), since z_1 and w_1 are non-negative, y uniquely determines z_1 and w_1.

Namely, for an edge $e = uv \in A$, we have $c_1(e) = 0$ and hence

$$z_1(e) := \begin{cases} 0 & \text{if } y(v) - y(u) \geq 0, \\ y(u) - y(v) & \text{if } y(v) - y(u) < 0, \end{cases} \tag{8.4}$$

$$w_1(e) := \begin{cases} 0 & \text{if } y(v) - y(u) \leq 0, \\ y(v) - y(u) & \text{if } y(v) - y(u) > 0. \end{cases} \tag{8.5}$$

For an edge $e' = uv \in L'$, we have $c_1(e') = 1$ and hence

$$z_1(e') := \begin{cases} 0 & \text{if } y(v) - y(u) \geq 1, \\ y(u) - y(v) + 1 & \text{if } y(v) - y(u) < 1, \end{cases} \tag{8.6}$$

$$w_1(e') := \begin{cases} 0 & \text{if } y(v) - y(u) \leq 1, \\ y(v) - y(u) - 1 & \text{if } y(v) - y(u) > 1. \end{cases} \tag{8.7}$$

Let x_1 be an integer-valued primal optimum, that is, x_1 is a minimum c_1-cost (f_1, g_1)-bounded m-flow in D_1. Let x be the (f, g)-bounded m-flow in D defined in (8.1). As noted above, x is L-upper-minimizer. Let (y, z_1, w_1) be an integer-valued dual optimum.

Note that the minimum cost flow algorithm of Ford and Fulkerson [3] computes a minimum-cost feasible flow of given amount along with the optimal dual solution. This algorithm relies on a max-flow algorithm as a subroutine. If one uses the strongly polynomial max-flow algorithm of Edmonds and Karp [2], that is, if the augmentation is made always along a shortest path in the corresponding auxiliary digraph, and, furthermore, if the cost-function is $(0, 1)$-valued, then the min-cost flow algorithm of Ford and Fulkerson is strongly polynomial. (In other words, we do not need to use a more sophisticated strongly polynomial algorithm—the first one found by Tardos [14]—for the general min-cost flow problem when the cost-function is arbitrary.) With a standard reduction technique, the min-cost flow algorithm of Ford and Fulkerson can easily be transformed to one for computing a feasible min-cost m-flow. Therefore, we conclude that the integer-valued optimal solutions to the primal and dual linear programs above can be computed in strongly polynomial time via the Ford-Fulkerson min-cost flow algorithm.

Since $\widetilde{m}(V) = 0$, by adding a constant to the components of y, we obtain another optimal dual solution. Therefore we may assume that the smallest component of y is 0. Let $0 = y_0 < y_1 < y_2 < \cdots < y_q$ be the distinct values of the components of y, and consider the chain of subsets $V_1 \supset V_2 \supset \cdots \supset V_q$ of V where $V_i := \{v \in V : y(v) \geq y_i\}$. (In the special case when $y \equiv 0$, the chain in question is empty, that is, $q = 0$).

Note that

$$ym = \sum_{i=1}^{q} (y_i - y_{i-1})\widetilde{m}(V_i). \tag{8.8}$$

We may assume that the difference of subsequent y_i values is 1. Indeed, if $y_{i+1} - y_i \geq 2$ for some i, then by subtracting 1 from $y(v)$ for each $v \in V_{i+1}$, by subtracting 1 from $z_1(e)$ for each $e \in A_1$ leaving V_{i+1}, and by subtracting 1 from $w_1(e)$ for each $e \in A_1$ entering V_{i+1}, we obtain another dual feasible solution (y', z_1', w_1'). By (8.8), $y'm = ym - \widetilde{m}(V_{i+1})$. For the revised z_1' and w_1', we have

$$z_1' f_1 = z_1 f_1 - \delta_{f_1}(V_{i+1}) = z_1 f_1 - \delta_f(V_{i+1}),$$
$$w_1' g_1 = w_1 g_1 - \varrho_{g_1}(V_{i+1}) = w_1 g_1 - \varrho_g(V_{i+1}).$$

Therefore

$$y'm + z_1' f_1 - w_1' g_1 = ym + z_1 f_1 - w_1 g_1 - [\widetilde{m}(V_{i+1}) + \delta_f(V_{i+1}) - \varrho_g(V_{i+1})].$$

Since $\varrho_g(V_{i+1}) - \delta_f(V_{i+1}) \geq \widetilde{m}(V_{i+1})$ by (2.2) and since (y, z_1, w_1) is an optimal dual solution, we obtain

$$\begin{aligned} &ym + z_1 f_1 - w_1 g_1 \geq y'm + z_1' f_1 - w_1' g_1 \\ &= ym + z_1 f_1 - w_1 g_1 - [\widetilde{m}(V_{i+1}) + \delta_f(V_{i+1}) - \varrho_g(V_{i+1})] \geq ym + z_1 f_1 - w_1 g_1. \end{aligned}$$

Therefore, equality must hold everywhere and hence (y', z_1', w_1') is another optimal dual solution. This reduction technique shows that we can assume that

$$y_i = i \ \text{ for } \ i = 1, \ldots, q. \tag{8.9}$$

Note that from an algorithmic point of view, we get immediately the optimal dual y given in (8.9) once the chain $V_1 \supset V_2 \supset \cdots \supset V_q$ belonging to an arbitrary optimal dual solution is available.

By (8.9), (8.4), and (8.5), we have for an edge $e = uv \in A$,

$$z_1(e) = \text{ the number of } V_i\text{'s left by } e, \tag{8.10}$$
$$w_1(e) = \text{ the number of } V_i\text{'s entered by } e. \tag{8.11}$$

For an edge $e' = uv \in L'$, by (8.6) and (8.7), we have

$$z_1(e') = \begin{cases} 0 & \text{if } e' \text{ enters a } V_i, \\ [\text{the number of } V_i\text{'s left by } e'] + 1 & \text{if } e' \text{ enters no } V_i, \end{cases} \tag{8.12}$$

$$w_1(e') = \begin{cases} 0 & \text{if } e' \text{ enters no } V_i, \\ [\text{the number of } V_i\text{'s entered by } e'] - 1 & \text{if } e' \text{ enters a } V_i. \end{cases} \tag{8.13}$$

The optimality criteria (complementary slackness conditions) for the primal and dual linear programs (8.2) and (8.3) are as follows:

$$\text{if } z_1(e) > 0 \text{ for some } e \in A_1, \text{ then } x_1(e) = f_1(e), \tag{8.14}$$
$$\text{if } w_1(e) > 0 \text{ for some } e \in A_1, \text{ then } x_1(e) = g_1(e). \tag{8.15}$$

Lemma 8.1. *The chain $V_1 \supset V_2 \supset \cdots \supset V_q$ and the m-flow x defined in* (8.1) *meet the five optimality criteria in Theorem* 4.6. *Furthermore, $\varrho_g(V_i) - \delta_f(V_i) < +\infty$ holds for each $i = 1, \ldots, q$.*

Proof. (O1) Let $e \in A$ be an edge leaving a V_i. Then $z_1(e) > 0$ by (8.10). By (8.14), $x_1(e) = f_1(e) = f(e)$, from which $x(e) = x_1(e) = f(e)$ follows whenever $e \in A - L$. If $e \in L$, then (8.12) implies $z_1(e') > 0$ for the corresponding parallel edge e' in L'. By (8.14), $x_1(e') = f_1(e') = 0$, and hence $x(e) = x_1(e) + x_1(e') = f(e)$, as required for Criterion (O1).

(O2) Let $e = A - L$ be an edge entering a V_i. Then $w_1(e) > 0$ by (8.11). By (8.15), we have $x(e) = x_1(e) = g_1(e) = g(e)$, as required for Criterion (O2).

(O3) Let $e \in L$ be an edge entering V_i and let e' be the corresponding parallel edge in L'. Then $w_1(e) > 0$ by (8.11). By (8.15), we have $x_1(e) = g_1(e) = g(e) - 1$. Since $0 = f_1(e') \leq x_1(e') \leq g_1(e') = 1$ and $x(e) = x_1(e) + x_1(e')$, we obtain that $g(e) - 1 \leq x(e) \leq g(e)$, as required for Criterion (O3).

(O4) Let $e \in L$ be an edge entering at least two V_i's, and let e' be the corresponding parallel edge in L'. By (8.11), we have $w_1(e) > 0$, from which (8.15) implies that $x_1(e) = g_1(e) = g(e) - 1$. By (8.13), we have $w_1(e') > 0$, from which (8.15) implies $x_1(e') = g_1(e') = 1$. Therefore $x(e) = x_1(e) + x_1(e') = g(e)$, as required for Criterion (O4).

(O5) Let $e \in L$ be an edge neither entering nor leaving any V_i, and let e' be the corresponding parallel edge in L'. Since x is (f, g)-bounded, we have $f(e) \leq x(e)$. By (8.12), $z_1(e') = 1$, from which (8.14) implies that $x_1(e') = f_1(e') = 0$. Hence $x(e) = x_1(e) + x_1(e') \leq g_1(e) = g(e) - 1$, as required for Criterion (O5).

To see the second part of the lemma, observe that Criterion (O1) implies that $\delta_f(V_i) = \delta_z(V_i) > -\infty$. As $g(e) < +\infty$ for every edge $e \in L$, and, by Criterion (O2) $g(e) = z(e) < +\infty$ for every edge $e \in A - L$ entering V_i, we conclude that $\varrho_g(V_i) < +\infty$, from which $\varrho_g(V_i) - \delta_f(V_i) < +\infty$, as required. ■

9 Existence of an F-dec-min m-flow

In the previous sections, we assumed that the bounding functions f and g were finite-valued on F. In the more general case, where we allow edges in F as well to have $f(e) = -\infty$ or $g(e) = +\infty$, it may occur that no dec-min feasible m-flow exists at all. For example, if D is a di-circuit, $F = A$, $m \equiv 0$, $f \equiv -\infty$, and $g \equiv 0$, then $z \equiv k$ is a feasible m-flow for each integer $k \leq 0$, implying that in this case there is no F-dec-min feasible m-flow. The main goal of this section is to describe a characterization for the existence of an F-dec-min feasible m-flow. As a consequence of this characterization, we show how Theorem 2.1 and its algorithmic approach can be extended to this more general case.

As before, let $D = (V, A)$ be a digraph and $F \subseteq A$ a non-empty subset of edges. Let $m : V \to \mathbf{Z}$ be a function on V and let $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ be bounding functions on A such that there is a feasible (that is, (f, g)-bounded) m-flow in D. Recall that $\ddot{Q}(f, g; m)$ denoted the set of integral (f, g)-bounded m-flows. In what follows, all the occurring functions (bounds, flows) are assumed to be integer-valued even if this is not mentioned explicitly.

We start by exhibiting an easy reduction by which we can assume that g is finite-valued on F.

Lemma 9.1. *There is a function g' on A which is finite-valued on F such that the (possibly empty) set of F-dec-min elements of $\ddot{Q} := \ddot{Q}(f, g; m)$ is equal to the set of F-dec-min* *elements of* $\ddddot{Q}' := \ddddot{Q}(f, g'; m)$.

Proof. Let z_1 be an element of $\ddddot{Q}$ and let β denote the maximum value of its components. Define g' as follows:

$$g'(e) := \begin{cases} \min\{g(e), \beta\} & \text{if } \ e \in F, \\ g(e) & \text{if } \ e \in A - F. \end{cases} \tag{9.1}$$

As $g' \leq g$, we have $\ddddot{Q}' \subseteq \ddddot{Q}$. In particular, an F-dec-min element z' of $\ddddot{Q}'$ is in $\ddddot{Q}$, and we claim that z' is actually F-dec-min in $\ddddot{Q}$. Indeed, if we had an element $z'' \in \ddddot{Q}$ which is decreasingly smaller on F than z', then z'' is not in $\ddddot{Q}'$, that is, z'' is not (f, g')-bounded. Therefore there is an edge $a \in F$ for which $z''(a) > \beta$, implying that $\max\{z''(e) : e \in F\} > \beta \geq \max\{z'(e) : e \in F\}$. But this contradicts the assumption that z'' is decreasingly smaller on F than z'.

Conversely, suppose that z is an F-dec-min element of $\ddddot{Q}$. Since the largest component of z_1 is β, the largest component of z is at most β, and hence $z \in \ddddot{Q}'$. This and $\ddddot{Q}' \subseteq \ddddot{Q}$ imply that z is an F-dec-min element of $\ddddot{Q}'$. ■

Theorem 9.2. *Let $D = (V, A)$ be a digraph and $F \subseteq A$ a non-empty subset of edges. Let $m : V \to \mathbf{Z}$ be a function on V and let $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ be bounding functions on A such that there is a feasible (that is, (f, g)-bounded) m-flow in D. There exists an F-dec-min (f, g)-bounded integral m-flow if and only if there is no di-circuit C with $C \cap F \neq \emptyset$ in the digraph $D^\infty = (V, A^\infty)$ defined by*

$$A^\infty := \{e : e \in A, f(e) = -\infty\} \cup \{vu : uv \in A - F, g(uv) = +\infty\}. \tag{9.2}$$

Proof. Suppose first that D^∞ includes a di-circuit C intersecting F, and assume, indirectly, that there exists an F-dec-min feasible m-flow z. For $uv \in A$, define $z'(uv)$ as follows:

$$z'(uv) := \begin{cases} z(uv) - 1 & \text{if } \ uv \in C, \ uv \in A, \\ z(uv) + 1 & \text{if } \ vu \in C, \ vu \in A - F, \\ z(uv) & \text{otherwise.} \end{cases} \tag{9.3}$$

Then z' is also a feasible m-flow in D, which is decreasingly smaller on F than z, a contradiction.

To see the converse, suppose that there is no di-circuit of D^∞ intersecting F. We want to prove that there is an F-dec-min feasible m-flow.

Claim 9.3. *The theorem follows from its special case when $g(e)$ is finite for each $e \in F$.*

Proof. Consider the function g' introduced in (9.1). As $g' \leq g$, there is no di-circuit described in the theorem with respect to (f, g'). By assuming the truth of the theorem in this case, we have an F-dec-min (f, g')-bounded m-flow z. By Lemma 9.1, z is an F-dec-min (f, g)-bounded m-flow. ■

By Claim 9.3, henceforth we can assume that g is finite-valued on F. Note that in this case

$$A^\infty = \{e : e \in A, f(e) = -\infty\} \cup \{vu : uv \in A, g(uv) = +\infty\}. \tag{9.4}$$

Claim 9.4. *Let $S \subset V$ be a set for which $\delta_{A^\infty}(S) = 0$, and let $e_0 \in F$ entering S. Then, for any (f, g)-feasible m-flow z,*

$$z(e_0) \geq \widetilde{m}(S) - [\varrho_g(S) - g(e_0)] + \delta_f(S), \tag{9.5}$$

and the right-hand side is finite.

Proof. Since $z \leq g$ and e_0 enters S, we have

$$\varrho_z(S) - z(e_0) \leq \varrho_g(S) - g(e_0),$$

from which

$$\widetilde{m}(S) = \varrho_z(S) - \delta_z(S) = z(e_0) + [\varrho_z(S) - z(e_0)] - \delta_z(S) \leq z(e_0) + [\varrho_g(S) - g(e_0)] - \delta_f(S),$$

implying (9.5).

Furthermore, $\delta_{A^\infty}(S) = 0$ implies that $f(e) > -\infty$ for every edge e of D leaving S and that $g(e) < +\infty$ for every edge e of D entering S, from which the finiteness of the right-hand side of (9.5) follows. ■

Assume indirectly that no F-dec-min (f, g)-bounded m-flow exists, that is, for every (f, g)-bounded m-flow, there exists another one which is decreasingly smaller on F. This implies that there is an edge $e_0 = ts$ in F for which there is an (f, g)-bounded m-flow with $z(e_0) \leq K$ for an arbitrarily small integer K.

Claim 9.5. *There exists an st-dipath P in D^∞.*

Proof. Suppose, indirectly, that the set S of nodes reachable from s in D^∞ does not contain t. Since no edge of D^∞ leaves S and e_0 enters S, it follows from Claim 9.4 that there is a finite lower bound for $z(e_0)$, a contradiction. ■

The di-circuit formed by $e_0 = ts$ and the st-dipath P ensured by Claim 9.5 meets the requirement of the theorem. ■■

Extension of Theorem 2.1 With the help of Theorem 9.2 and Lemma 9.1, Theorem 2.1 can be extended to the case when (f, g) is not assumed to be finite-valued on F, only the existence of a di-circuit in D^∞ intersecting F is excluded (which is equivalent, by Theorem 9.2, to the existence of an F-dec-min (f, g)-bounded m-flow).

Theorem 9.6. *Let $D = (V, A)$ be a digraph endowed with integer-valued lower and upper bound functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ for which $f \leq g$. Let $m : V \to \mathbf{Z}$ be a function on V with $\widetilde{m}(V) = 0$ such that there exists an (f, g)-bounded m-flow. Let $F \subseteq A$ be a specified subset of edges. Assume that there exists an F-dec-min (f, g)-bounded integral m-flow. There exists a pair (f^*, g^*) of integer-valued functions on A with $f \leq f^* \leq g^* \leq g$ (allowing $f^*(e) = -\infty$ and $g^*(e) = +\infty$ for $e \in A - F$, but $f^*(e)$ and $g^*(e)$ are finite for $e \in F$) such that an integral (f, g)-bounded m-flow z is decreasingly minimal on F if and only if z is an integral (f^*, g^*)-bounded m-flow. Moreover, the box $T(f^*, g^*)$ is narrow on F in the sense that $0 \leq g^*(e) - f^*(e) \leq 1$ for every $e \in F$.*

Proof. By Lemma 9.1, we can assume that g is finite-valued on F. Furthermore, the non-existence of a di-circuit C in D^∞ with $C \cap F \neq \emptyset$ implies that, for every edge $e = ts \in F$, the set S_e reachable in D^∞ from s meets the inequality (9.5) for any (f, g)-bounded m-flow z. As the right-hand side of (9.5) is finite by Claim 9.4, there is a finite lower bound

$$f'(e) := \widetilde{m}(S_e) - [\varrho_g(S_e) - g(e)] + \delta_f(S_e) \tag{9.6}$$

for $z(e)$. In this way, each $-\infty$-valued lower bound on the edges in F can be made finite, and the original Theorem 2.1 applies. ■

We emphasize that for each $e \in F$ the set S_e occurring in the proof is easily computable and hence so is the finite lower bound $f'(e)$ given in (9.6). Therefore this reduction to the case when (f, g) is finite-valued on F is algorithmic.

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