Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2019-11. Published by the Egerváry Research Group, Pázmány P. sétány 1/C,
H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Online 2-dimensional rectangular bin packing

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September 2019

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Abstract

The classical one-dimensional bin packing problem has several generalizations. This paper is concerned with a two-dimensional version where the bins are rectangles of fixed size $a \times 1$, and the rectangular items can be rotated by $90°$. We provide an online algorithm for the problem and an upper bound on the asymptotic competitive ratio that goes from ≈ 2.57 at $a = 4/3$ to ≈ 2.46 at $a = 2$, and tends to 1.82 as $a \to \infty$. Additionally, for any constant space bounded online algorithm, a lower bound as a function of a is also given. The lower bound is ≈ 2.15 at $a = 4/3$, remains above 1.8 for $a < 2 + 1/3$, and tends to ≈ 1.69 as $a \to \infty$.

1 Introduction

The simplest version of the bin packing problem is one-dimensional. Several items, each of weight less than 1 unit, need to be grouped in such a way that the sum of the weights is at most 1 in each group, and the aim is to minimize the number of groups. Another way to view the problem is to consider items as line segments shorter than one unit, and the aim is to pack them on a minimal number of unit line segments such that none overlap, and each is placed within a single unit segment.

In the standard two-dimensional generalization of bin packing, several rectangular items need to be placed into unit square shaped bins. Items have sides of length at most 1. They should be placed parallel to the bin's sides without overlap, and they should fit entirely in the bins. The aim is use a minimal number of bins. There are two versions depending on whether $90°$ rotation of the items is allowed.

A natural extension is to consider the case where the bins are not square shaped, but rectangular. If rotation is not allowed, then this can be easily reduced to the square version. However, if rotation is allowed, then there is no simple reduction between the two problems. This article focuses on the latter problem, which was absent from previous literature.

There is a distinction between offline and online bin packing problems. While the item sizes are known in advance in the offline version, the online version has the size of the next item revealed only after the previous item was permanently packed in a

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bin. After an item has been placed, its bin and its position inside the bin cannot be changed. Online algorithms can be further classified based on the number of bins simultaneously open. For each new item, the algorithm has to decide whether to open a new bin for it or pack it into a bin that already has another item. The algorithm can also permanently close bins so that it is not possible to pack more items in them. If there is a constant $k \in \mathbb{N}$ such that at most k bins are simultaneously open at any time, then the algorithm is *k-space-bounded*; if there is no such bound, then the algorithm is *unbounded*.

The performance of an algorithm (online or offline) can be measured based on the ratio of the number of bins they use compared to the number of bins that an optimal offline algorithm would use on the same list of items. For online algorithms, the usual measure is the *asymptotic competitive ratio*, defined below. Let L denote a list of items, and $\mathrm{OPT}(L)$ denote the number of bins that the optimal offline packing would need to pack all items of L. Furthermore, let $A(L)$ denote the number of bins that algorithm A would need to pack the items of L. For $n \in \mathcal{N}$, define

$$R_A^n := \max\left\{\frac{A(L)}{\mathrm{OPT}(L)} \mid \mathrm{OPT}(L) = n\right\}.$$

The *asymptotic competitive ratio* of algorithm A is

$$\mathrm{ACR}(A) := \limsup_{n\to\infty} R_A^n.$$

Usually, the ACR is hard to determine exactly, since computing $\mathrm{OPT}(L)$ for a given L is an NP-hard problem. The quality of a k-bounded online algorithm A can be demonstrated by showing an upper bound on $\mathrm{ACR}(A)$, together with a lower bound on the ACR of any k-bounded online algorithm.

Related work

There is a vast literature on rectangle packing where the bins are unit squares. Here we only focus on the results for the online version of the problem. If rotations are not allowed and there is no bound on the number of open bins, then the best known lower bound on the ACR is 1.91 by Epstein [2], while the best upper bound is 2.55 by Han et al. [5]. There is an optimal k-space-bounded algorithm by Epstein and van Stee [3] (with very large k) that has an ACR of 2.86.

Rotations were first considered in the online setting by Fujita and Hada [4], who gave an algorithm with ACR ≈ 2.63. Later, the gap between the lower and upper bounds for bounded space case was almost closed by Epstein [1], who proved a lower bound of 2.535 and gave an algorithm with ACR ≈ 2.546.

2 Algorithm for rectangular bins

We will provide an online algorithm that works for all bin shapes where the ratio of the two sides is at least 4/3. If the ratio is close to 1, then the bins are closer to being a square and thus any algorithm for square shaped bins can be adapted by leaving the small part of the bin outside of the unit square empty. The ratio 4/3 was chosen for technical reasons: a certain configuration of 4 items that is essential in the square case no longer appears if the ratio is at least 4/3.

The items are packed according to their shape. Typically, each bin will be filled with items of similar sizes. As detailed below, items are classified according to the length of their sides, and these classes also define the types of the active bins. We use the following notation:

- a is the length of the longer side of a bin (the other side is 1). We consider the bins to have height 1 and width a. We assume $a > 4/3$.
- x is the length of the shorter side of the current item,
- y is the length of the longer side of the current item.

In this section we first describe the classification of the items, then the packing for each bin type. A bound on the filling ratio of each bin type is also given. The filling ratio of a given bin type is the minimum possible ratio between the area of all items in a closed bin of that type and the area of the bin.

Our classification and packing method has the property that most bins types have a filling ratio of at least 4/9. This will be important for the upper bound on the ACR of the algorithm. The reciprocal of the worst filling ratio is a (trivial) upper bound, because the offline algorithm can not put more in a bin than its area. To get a better bound, we exploit the fact that some items types are shaped in a way that offline algorithms can not pack them very well either. By giving lower bounds on the number of bins used by any offline algorithm for types with filling ratio less than 4/9, we can get a better upper bound on the ACR than the reciprocal of the worst filling ratio.

2.1 Classification of items

As mentioned above, items are classified in a way that most classes have filling ratios above 4/9. For each class of items, there is a corresponding bin type. The items are assigned to 4 main groups according to the size of their longer side (y):

- H, huge for $2/3 < y$
- B, big for $1/2 < y \leq 2/3$
- S, small for $1/3 < y \leq 1/2$
- T, tiny for $y \leq 1/3$

These main groups are further divided into classes, depending on x, y and a. H is divided into

- $H1$ if $x \leq a/3$

- $H2$ if $a/3 < x \leq a/2$
- $H3$ if $a/2 < x$.

We remark that not all classes exist for all choices of a. For example, $a > 2$ implies that x cannot be larger than $a/2$. Class $H3$ is therefore unnecessary in this case.

B is divided into

- $B1a$ if $x \leq 1/4$ and $1/2 < y \leq 1/\sqrt{3}$
- $B1b$ if $x \leq 1/4$ and $1/\sqrt{3} < y \leq 2/3$
- $B2a$ if $1/4 < x \leq 1/3$ and $1/2 < y \leq 1/\sqrt{3}$
- $B2b$ if $1/4 < x \leq 1/3$ and $1/\sqrt{3} < y \leq 2/3$
- $B3a$ if $1/3 < x \leq 1/2$ and $y \leq a/3$
- $B3b$ if $1/3 < x \leq 1/2$ and $a/3 < y$
- $B4a$ if $1/2 < x \leq 2/3$ and $x \leq a/3$
- $B4b$ if $1/2 < x \leq 2/3$ and $a/3 < x$

S is divided into

- $S1$ if $x \leq a/3$
- $S2$ if $a/3 < x$

T is divided into

- $T3$ if $1/24 < y \leq 1/12$
- $T1$ if not in $T3$ and $\exists k \in \mathbb{N} : y \leq 1/3 \cdot 1/2^k$ and $y > 1/4 \cdot 1/2^k$
- $T2$ otherwise

A classification similar to $T1$ and $T2$ was introduced by Fujita and Hada [4].

2.2 Packing of the different classes of items

Items from the same class are packed into the same type of bin. Thus, the number of bins simultaneously open during the algorithm is at most the number of classes (it is somewhat less because some classes can share the same type of bin). 14 simultaneously open bins are needed if $a < 2$, 11 if $2 \leq a < 3$ and 10 if $3 \leq a$. This section describes the packing method of each class, as well as their filling ratio. In most cases, the filling ratio depends on a, so it is given as a function of a. The filling ratios are summarized in a table at the end of the section (Table 1). Minimum values are also given for various intervals of a.

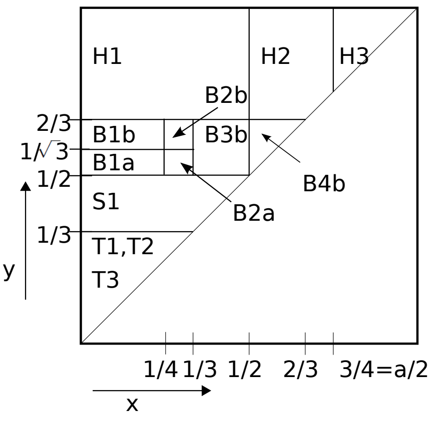

Figure 1: Grouping of items in case of a=3/2 (left side) and a=2 (right side). If an item is put horizontally in the lower left corner of these rectangles, then the upper right corner's position specifies the type of the item.

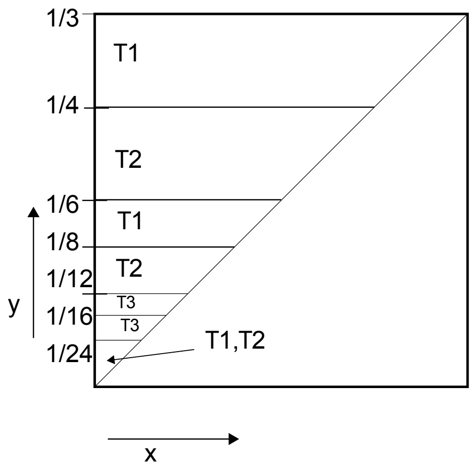

Figure 2: Grouping of the tiny elements

We assume that a is the width of a bin, while its height is 1. We say that an item is placed *vertically* into a bin if it is rotated so that its height is larger than its width; otherwise it is placed *horizontally*. Bins are sometimes divided into containers. Typically, a vertical item touches the bottom of the container (or the bin) it is placed in, and horizontal items are placed on the left side of the container or bin.

2.2.1 $H1$

The first item of type $H1$ is placed vertically in the lower left corner of the active $H1$-bin, and all the following items are placed vertically directly next to it as long as they fit in the active bin. If the current item cannot fit in the active bin, then the active bin is closed, and a new bin is opened, which becomes the active bin of type $H1$.

As $y > 2/3$, the bin is filled vertically at least to the height of $2/3$ where an item is in the bin. As $x < a/3$, the empty space on the right side of the bin cannot be more than $a/3$. This results in a filling ratio of $((2/3) \cdot (a - (1/3)a))/a = \mathbf{4/9}$. If $a > 3$, then $a/3 > 1$, so $x < 1$ is a stricter upper bound, and the filling ratio can be lower bounded by $((2/3) \cdot (a - 1))/a = 2/3 \cdot (1 - (1/a))$.

2.2.2 $H2$

The items in class $H2$ are packed the same way as those in $H1$. Exactly 2 of them fit into one bin. Their size together is $2xy > 2(a/3)(2/3)$, so the filling ratio is at least $4/9$. Class $H2$ does not exist if $a > 3$.

2.2.3 $H3$

Only one item of type $H3$ can be packed into a bin. The filling ratio is $(a/2)(a/2)/a = a/4$. Class $H3$ does not exist if $a > 2$.

2.2.4 $B1a$, $B1b$, $B2a$, $B2b$

These four classes are packed similarly, but into 4 different types of bins. Each type of bin is divided into containers of height 1, but their widths differ: for $B1a$ and $B2a$, their width is $1/\sqrt{3}$, while for $B1b$ and $B2b$ the width is $2/3$. The containers are cut from the left side, and they are right next to each other. In most cases, a small strip remains on the right side. First, the incoming items are placed horizontally in the lowest possible part of any container. If the current item cannot be packed into any of the containers of the active bin, then it is placed vertically into the strip on the right hand side if possible. If neither the containers nor the strip can accommodate the current item, then the active bin is closed, and a new bin of the same type is opened.

Claim 1. *In case of $B1a$, $B1b$, $B2a$, and $B2b$, the filling ratio of the containers is at least $\sqrt{3} \cdot 3/8 \approx 0.649$.*

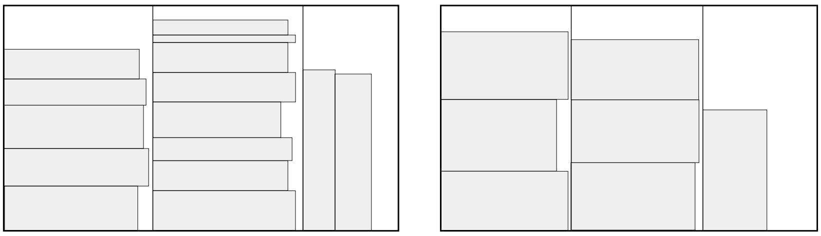

Figure 3: Packing of B1b (left) and B2a (right) items

Proof. In the following it will be shown that the containers are filled to 3/4 of their height, and to $\sqrt{3}/2$ of their width. From these, the claim follows.
Height: For $B1a$ and $B1b$, the height of the rectangles (which is the x side, as they are placed horizontally) is at most 1/4, so if an item cannot be packed into any of the containers, then the remaining area at the top of each container is at most 1/4. Thus, the container is filled to at least 3/4 of its height.

For $B2a$ and $B2b$, the height (x side) of each item is between 1/4 and 1/3, so exactly 3 items fit into one container, resulting in a joint height of 3/4.
Width: For $B1a$ and $B2a$, the width (the y side) of each item is at most 1/2 and the width of the container is $1/\sqrt{3}$, so the $\sqrt{3}/2$ vertical filling is achieved. For $B1b$ and $B2b$, the width (y side) of each item is at most $1/\sqrt{3}$ and the width of the container is 2/3, which again implies $\sqrt{3}/2$ vertical filling. □

A closed bin of type $B1a$, $B1b$, $B2a$ or $B2b$ can be divided into three rectangles of height 1: containers, filled part of the strip, and empty part of the strip. The containers are filled at least to $\sqrt{3} \cdot 3/8$. The filled part of the strip is filled to at least 1/2, the empty part is not filled at all. The containers take up at least $a - 2/3$ area, the filled part takes up at most 1/3 area, and the empty part takes up at most 1/3 area. The filling ratio is lower bounded by $((\sqrt{3} \cdot 3/8)(a-2/3)+(1/3)(1/2)+0)/a$. It is also true that there are at least two containers in each closed bin, so the filling ratio can also be lower bounded by $((\sqrt{3} \cdot 3/8)(2/\sqrt{3})+(1/3)(1/2)+(1/3)0)/(2/\sqrt{3}+2/3) \approx 0.503$. The previous lower bound is monotone increasing, and it is approximately 0.516 for $a = 2$, and $0.560\dot{3}$ for $a = 3$, and it approaches $\sqrt{3} \cdot 3/8$ as $a \to \infty$.

2.2.5 $B3a$, $B3b$

Items of type $B3a$ and $B3b$ are packed similarly, but into two different types of bins. Both bins are horizontally divided into two containers of height 1/2. The items are placed horizontally into the containers starting from the left side. If neither of the containers can accommodate the next item, then the bin is closed and a new bin is opened for the current item. All containers are filled to 1/3 height, and an area of at most $a/3$ is left empty at the right end of any container. Indeed, $y \leq a/3$ for $B3a$, the empty area at the end of a strip is most $a/3$; while for $B3b$, $y > a/3$ and hence two items can fit in one container, making them both at least $2a/3$ long. This results in a filling ratio of 4/9 for both types. Class $B3b$ does not exist if $a > 2$. Another lower bound for the filling ratio is $(2/3)(a - 2/3)/a$; this is larger than 4/9 if $a > 2$.

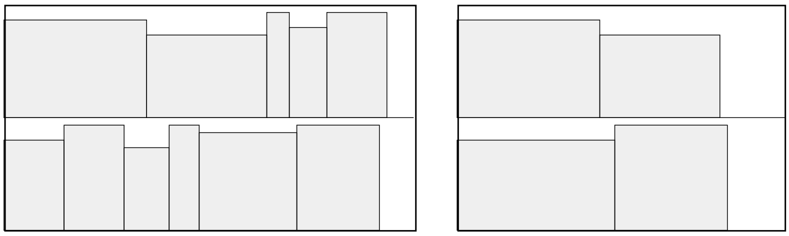

Figure 4: Packing of B3a and S1 items (left side), and B3b and S2 items (right side)

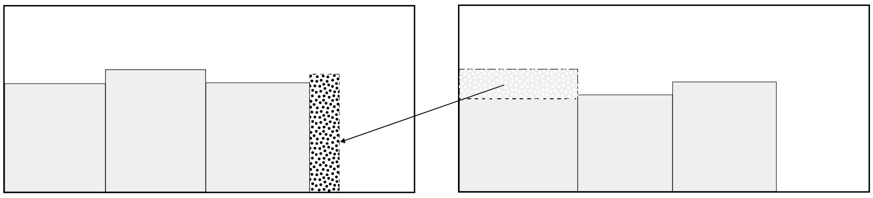

Figure 5: The packing of two consecutive B4a bins, showing the extra part of the first item of the latter bin, which is counted as part of the previous bin

2.2.6 *B4a, B4b*

The packing method of classes $B4a$ and $B4b$ are the same, but they are placed into two different types of bins. Each item is placed vertically in the corresponding bin, starting from the left side. If the current item would not fit in the active bin, then the bin is closed, and a new one is opened. Class $B4a$ only exists if $a > 3/2$, while class $B4b$ exists if $a < 2$.

For $B4b$, two lower bounds are given. On one hand, exactly two items fit in a bin, and their size together is at least $2(a/3)^2$ as $y \geq x > a/3$, resulting in a filling ratio of $2a/9$. Another lower bound for the filling ratio is $1/(2a)$, as the area of the two items is at least $2 \cdot (1/2)^2 = 1/2$.

For $B4a$, also two lower bounds are given. As $x \leq a/3$, at least three items fit in a bin, so the filling ratio is at least $3(1/2)^2/a = 3/(4a)$. The filling ratio is also lower bounded by $(1/2)(a - 2/3)/a$, as there is an empty space at most $2/3$ wide at the right end of a closed bin. This filling ratio can be improved to $(1/2)(a - 1/2)/a$ if we take an asymptotic approach, also illustrated by Figure 5. If the empty space on the right side of the closed bin is wider than $1/2$, then the item which could not be packed there is wider than $1/2$, and also higher than $1/2$ – it is at least as high as the width of the empty space, since these items are placed vertically. When bounding the filling ratio, we only care that the bin is filled to at least $1/2$ of its height, so we can take the upper part (over the $1/2$ height) of the new item, and virtually place it vertically into the empty space in the right side of the previous bin. If we cut this part from the top of the item, it still stays $1/2$ high, and the empty space at the right side of the previous bin becomes at most $1/2$ wide. The same can be done to the next bin, and so on. Thus, an asymptotic filling ratio of $(1/2)(a - 1/2)/a$ can be achieved.

2.2.7 $S1$, $S2$

Items of type $S1$ can be packed similarly to those of type $B3a$, but they have to be placed vertically in the containers. This results in the same filling ratio, and also the $S1$ and $B3a$ items can be mixed and placed in the same bins, still having the same filling ratio of $4/9$ and $((2/3)(a-2/3))/a$.

The situation is similar for class $S2$: these items are packed similarly to those of type $B3b$, but they are placed vertically. Items of type $S1$ and $B3a$ can also be mixed, still having the same filling ratio of $4/9$. Class $S2$ exists if $a < 3/2$.

2.2.8 $T1$

The tiny elements need to be further categorized before packing. This idea comes from [4] as mentioned in the Classification section.

For an item of type $T1$, define $g(y) := \min\{1/(3 \cdot 2^k) | 1/(3 \cdot 2^k) \geq y, k \in \mathbb{N}\}$. If an item's height is y, it will be put into a $g(y)$-container. Each bin will contain several differently shaped containers. The space allocated for the largest $T1$ items ($1/4 < y \leq 1/3$) will be referred to as containers; the smaller items ($1/8 < y \leq 1/6$) will be placed into subcontainers, and there will be several mini-containers for the smallest items ($y \leq 1/24$).

The bin is divided into three 1/3-containers upon each other. The lowest container is provisionally divided into two 1/6-subcontainers from the right side. This means that as long as we pack this container from the right, it behaves as if it was divided, but if we want to pack into the container from the left, it behaves as if it was not divided. We can pack such a mixed container from both sides as long as the two sides do not interfere with each other. From the left end of the upper container, a 1/12 wide part is reserved for the items for which $y \leq 1/24$. We do not need a strip for the 1/12 items as they are in class $T3$.

The items for the 1/3-containers are placed vertically from the left directly next to each other in the upper container. If the next item cannot fit, and there is enough space in the middle container, then the items are placed there, starting from the left. If the next item does not fit the middle container, and there is enough space in the lowest container, then the items are packed into the lowest container. If there is not enough place in the lower container either, then a new bin is opened. The 1/6-items are vertically placed into the 1/6 subcontainers, starting from the right side. The next item is always placed in the subcontainer that is filled less. If the next item cannot be placed into any of these, and there is still space in the container above, then that container is divided into subcontainers, and the items are placed there. If the 1/3- and 1/6-items would collide, or there is no place in the next container to fit the current item, then the bin is closed. The 1/12-wide upper left reserved part is divided into $1/(12 \cdot 2^k)$-wide vertical mini-containers. The very small items are put into their corresponding mini-container, in a way that their longer side is parallel to the lower and upper side of the bin. If a mini-container cannot accommodate the next item, then a new mini-container is reserved as if it was an 1/3-item.

For $T1$, all containers, subcontainers and mini-containers are filled to at least 3/4

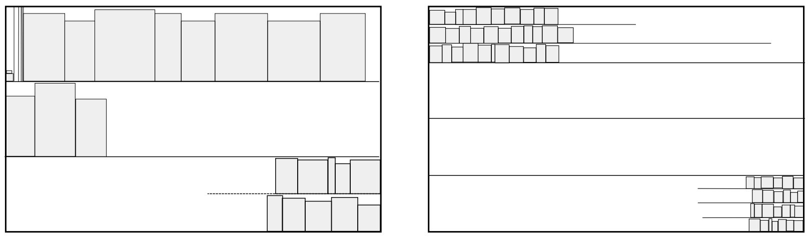

Figure 6: Packing of T1 items (left side) and T3 items (right side)

of their shorter side, as the items that would be smaller than this are in classes $T2$ or $T3$. At the end of a 1/3-container only a 1/3-long part can be empty, and at the end of a 1/6-subcontainer a 1/6-long part can be empty. At the meeting point of the 1/3-containers and 1/6-subcontainers, at most $(1/6)^2 + (1/3)^2 = 5/36$ space can be left empty. The empty space described up to now is largest if there is one container with a collision, and two containers with only 1/3-items. This results in $2(1/3)^2 + 5/36 = 13/36$ of empty space. The opened but not filled mini-containers take up at most $(1/3)(1/12) = 1/36$ space, which could be empty. The only remaining empty space is that of the filled mini-containers; this is largest for the 1/24-mini-container, where it can take $(1/24)/(1/3) = 1/8$ of the mini-container, but this can be reduced to $(3/4)(1/8) = 3/32$ by the same asymptotic approach as explained for the $B4a$ items. Even if the whole bin is filled with 1/24-mini-containers, it will only give an $1 - 3/32 = 29/32$ factor to the whole bin's filling ratio. So the filling ratio is at least $(29/32)(3/4)(a - 13/36 - 1/36)/a$. This is an increasing function, being > 0.48 at $a = 4/3$, > 0.54. at $a = 2$, and > 0.59 at $a = 3$.

2.2.9 $T2$

The $T2$ items are packed similarly to the $T1$ items, only the constants change. The sorting function is $g(y) := \min\{1/(4 \cdot 2^k) | 1/(4 \cdot 2^k) \geq y, k \in \mathbb{N}\}$, and the bin is divided into 4 containers. The mini-containers take up a 1/16 wide part of the upper container, and the lower container is divided into two 1/8-subcontainers from the right.

For $T2$ all containers, subcontainers and mini-containers are filled to 2/3 of their shorter side. At the end of a 1/4-container only a 1/4 long part can be empty, at the end of a 1/8-subcontainer a 1/8 part can be empty. At the collision of the 1/4-containers and 1/8-subcontainers, at most $(1/8)^2 + (1/4)^2 = 5/64$ space can be left empty. The largest empty space of these kinds occur if there is one container which is mixed, and three containers of only 1/4-items. This results in $3*(1/4)^2+5/64 = 17/64$ of empty space. There can be at most $(1/4)(1/16) = 1/64$ empty area of not-closed mini-containers. The empty spaces at the end of the mini-containers are at most 1/32, which is $(1/32)(1/4) = 1/8$ of the mini-container's length. This can be reduced to $(2/3)(1/8) = 1/12$ by the asymptotic approach. Even if the whole bin is packed with 1/24-mini-containers, it will only give an 11/12 factor to the whole bin's filling ratio. So the filling ratio is at least $(11/12)(2/3)(a - (17/64) - (1/64))/a$. This is an increasing function, and it is > 0.48 at $a = 4/3$, > 0.52. at $a = 2$, and > 0.55 at $a = 3$.

Ratio:	$4/3 \leq a < 2$		$2 \leq a < 3$		$3 \leq a$		$a \to \infty$
	function	min	function	min	function	min	
H1	4/9	4/9	4/9	4/9	(2/3)(1-1/a)	4/9	2/3
H2	4/9	4/9	4/9	4/9	-	-	-
H3	$a/4$	1/3	-	-	-	-	-
B1a	$\frac{\frac{\sqrt{3}\cdot 3}{8}(a-\frac{2}{3})+\frac{1}{6}}{a}$	>0.5	$\frac{\frac{\sqrt{3}\cdot 3}{8}(a-\frac{2}{3})+\frac{1}{6}}{a}$	0.516	same	0.56	$\sqrt{3}\cdot 3/8$
B1b		>0.5		0.516		0.56	$\sqrt{3}\cdot 3/8$
B2a		>0.5		0.516		0.56	$\sqrt{3}\cdot 3/8$
B2b		>0.5		0.516		0.56	$\sqrt{3}\cdot 3/8$
B3a & S1	4/9	4/9	$((2/3)(a-2/3))/a$	4/9	same	>0.51	2/3
B3b & S2	4/9	4/9	-	-	-	-	-
B4a	$3/4a$	3/8	$(1/2)(a-1/2)/a$	3/8	same	5/12	1/2
B4b	$\max\{1/(2a), 2a/9\}$	1/3	-	-	-	-	-
T1	$\frac{29}{32}\frac{3}{4}\frac{a-14/36}{a}$	>0.48	same	>0.54	same	>0.59	$\frac{29}{32}\frac{3}{4} > 0.67$
T2	$\frac{11}{12}\frac{2}{3}\frac{a-18/64}{a}$	>0.48	same	>0.52	same	>0.55	$\frac{11}{12}\frac{2}{3} > 0.61$
T3	$\frac{2}{3}\frac{a-19/192}{a}$	>0.61	same	>0.63	same	>0.64	2/3>0.66

Table 1: Overview of the filling ratios for different type of items, and different side-ratios of the bin. For each interval of the ratio there is a function, and the minimum value of that function in the interval is indicated. The text “same” means that the function is the same as for the previous interval (two cells to the left).

2.2.10 *T*3

For the $T3$ items, the bin is divided into four 1/4-containers, which are further divided. The upper container is provisionally divided into three 1/12-subcontainers from the left, and the lower container is provisionally divided into four 1/16-subcontainers from the right. Items for which $1/16 < y \leq 1/12$ are vertically put into the prior ones (always into the emptiest ones), the other ones ($1/24 < y \leq 1/16$) are put into the 1/16-subcontainers. Again, if one container is filled, then the container above or below is provisionally divided into the missing subcontainers if possible. If the next item can not be packed, then a new bin gets opened.

In a $T3$ bin, the subcontainers are filled to at least 2/3 or 3/4 height of the subcontainer – we use 2/3 as a lower bound. At the end of a container with only 1/12-subcontainers, there is at most (1/4)(1/12) of empty space. At the end of a container with only 1/16-subcontainers, the empty space is at most (1/4)(1/16), while in a mixed container it is at most $(1/4)(1/12) + (1/4)(1/16)$. In total, there is at most $3 \cdot (1/4)(1/12) + (1/4)(1/12) + (1/4)(1/16) = 19/192$ of empty space at the end of the containers. Altogether the filling ratio is at least $(2/3)(a - 19/192)/a$, which is an increasing function, and it is > 0.61 at $a = 4/3$, > 0.63 at $a = 2$, and > 0.64 at $a = 3$.

3 Upper bound for the algorithm

In this section we compute upper bounds for the asymptotic competitive ratio (ACR) of the algorithm. As the algorithm works for differently shaped bins, the ACR is a function which depends on the ratio of the bins' sides, so the upper bound is also given as a function. This function does not have the same formula for all sides-ratios, but there are intervals where the formula remains the same. The formula is given for these intervals in the following subsections, and also a rounded minimal and maximal value is given for each interval, for better comprehension. The upper bounds are illustrated on 7 along with the lower bounds from the next section for different values of a.

There are two reasons why the function differs among the intervals: there are item types which do not exist for all side ratios; besides this, the filling ratios for the same type of items also change as the side ratios change. The core idea of the way to get the upper bound comes form Fujita and Hada, [4] who gave an upper bound for the square shaped bin version of the online algorithm.

In each subsection, the main idea behind the proof is the same: an upper bound is given for the number of bins the algorithm uses on any sequence of n items. There are two kinds of items: items for which the filling ratio is higher than a given value (for example 4/9): for these items the algorithm does not need more space than 9/4 times their actual area. So these items need THE SUM OF THEIR AREA$\cdot(9/4)/a$ bins asymptotically. Later on we will regard the area of the items measured in bin sizes, e.a. divided by a. The other kind of item types have filling ratio lower than the given value; here an upper bound is given by explicitly considering the number of bins the algorithm needs. The sum of these values gives an upper bound for the number of bins the online algorithm uses.

The lower bound on the number of bins the optimal offline algorithm uses is given by a maximum of several possible lower bounds. The algorithm needs at least as much area of bins as the sum of the areas of all items. It also needs as many bins as the number of $H3$ items in the list. These constraints both give a lower bound on the number of bins needed, and as both of the constraints have to be fulfilled, at least the maximum of the calculated bin numbers is needed. There are more complicated constraints on the number of bins needed for items with low filling ratios; these will be detailed in the following subsections.

The ratio of the upper bound and the lower bound gives an upper bound on the ACR.

3.1 Case $4/3 \leq a < 3/2$

If $4/3 \leq a < 3/2$, then there are only two types of items for which the filling ratio is less than 4/9: $H3$ and $B4b$. ($B4a$ items do not exist, because $a < 3/2$ and so $a/3 < 1/2 < x$.) Let L be a sequence of n items. Let n_{H3} denote the number of bins that the algorithm would use for the $H3$ items, and n_{B4b} that it would use for $B4b$ items. Let S denote the sum of the area of all items in the sequence L (using the bin size as unit), and let $S_{-H3-B4b}$ denote the area of the items without classes $H3$ and $B4b$. As the filling ratio of these items is at least 4/9, these items are packed into at

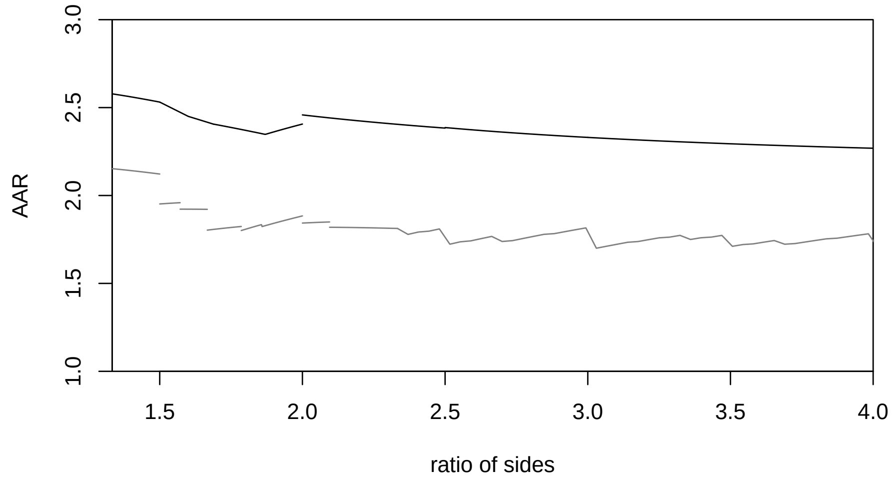

Figure 7: Upper bounds for the ACR of the algorithm, and lower bounds for the ACR of any online k-space bounded ($k > 1$) algorithm solving the 2-dimensional online bin packing problem, using rectangle shaped bins, where rotation is allowed. The function depends on the ratio a of the two sides of the bin.

most $\frac{9}{4}S_{-H3-B4b} + O(1)$ bins by the algorithm. So altogether the algorithm uses at most $n_{H3} + n_{B4b} + \frac{9}{4}S_{-H3-B4b} + O(1)$ bins.

The optimal offline packing utilizes at least

- as many bins as the size of all items(S)
- n_{H3} bins, as only one $H3$ item fits in a bin, and the online algorithm puts also exactly one $H3$ item in a bin
- $n_{H3} + (n_{B4b} - (1/2)n_{H3}) = (1/2)n_{H3} + n_{B4b}$ bins, as one bin for each $H3$ item is needed, but not all $B4b$ items need separate bins: some can be put in a bin with an $H3$ item, so less than n_{B4b} bins are needed for them. At most one $B4b$ fits next to an $H3$ item, which means 1/2 of the items of a $B4b$ bin (both in the online and in the offline case). So if there are n_{H3} $H3$ bins, and each can fit one $B4b$ item as well, then $(1/2)n_{H3}$ bins of $B4b$ items can be packed next to the $H3$s, and the remaining $B4b$ items, which fill $(n_{B4b} - (1/2)n_{H3})$ online bins, must to be put in other bins by the offline packing.

So at least $\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}$ bins are used by the optimal offline packing.

The ACR therefore satisfies

$$\mathrm{ACR}_{4/3<a<3/2} \leq \frac{n_{H3} + n_{B4b} + (9/4)S_{-H3-B4b}}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}}.$$

Let $F(a, X)$ denote the minimal filling ratio of type X bins, if a is the ratio of the sides (these values were calculated in the previous section). Then $S \geq S_{-H3-B4b} +$

$F(a, H3) \cdot n_{H3} + F(a, B4b) \cdot n_{B4b}$. Reformulating this equation gives $S_{-H3-B4b} \le S - F(a, H3) \cdot n_{H3} - F(a, B4b) \cdot n_{B4b}$. For $F(a, B4b)$, there are two functions, but on the interval $[4/3, 3/2)$ the function $1/(2a)$ gives the stricter lower bound. So

$$
\begin{aligned}
&\mathrm{ACR}_{4/3<a<3/2} \\
&\le \frac{(1-(9/4)F(a,H3)) \cdot n_{H3} + (1-(9/4)F(a,B4b)) \cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\le \frac{(1-(9/4)(a/4)) \cdot n_{H3} + (1-(9/4)(1/(2a))) \cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&= \frac{(1-(9/4)(1/(2a))) \cdot ((1/2)n_{H3} + n_{B4b})}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\quad + \frac{(1-(9/4)(a/4) - (1/2)(1-(9/4)(1/(2a)))) \cdot n_{H3} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\le 1 - (9/(8a)) + 1 - (9a/16) - 1/2 + (9/16a) + (9/4) = 3.75 - 9/(16a) - (9a/16)
\end{aligned}
$$

This function monotonically decreases on the interval, $\mathrm{ACR}_{4/3} \le 2.5781$ and $\mathrm{ACR}_{a \nearrow 3/2} \le 2.531$.

3.2 Case $3/2 \le a < 27/16$

On the interval $(3/2, 27/16]$, only $H3$ and $B4b$ have worse filling ratio than $4/9$. (As $B4a$'s filling function $F(B4a, a) = 3/(4a)$ is larger than $4/9$ on this interval.) So we can use the same method as above, with a slight modification: For $F(B4b, a)$, the function $2a/9$ gives a stricter bound than $1/(2a)$.

$$
\begin{aligned}
&\mathrm{ACR}_{3/2<a<27/16} \\
&\le \frac{(1-(9/4)F(a,H3)) \cdot n_{H3} + (1-(9/4)F(a,B4b)) \cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\le \frac{(1-(9/4)(a/4)) \cdot n_{H3} + (1-(9/4)(2a/9)) \cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&= \frac{(1-(9/4)(2a/9)) \cdot (n_{B4b} + (1/2)n_{H3})}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\quad + \frac{(1-(9/4)(a/4) - (1/2)(1-(9/4)(2a/9))) \cdot n_{H3} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3} + n_{B4b}\}} \\
&\le 1 - (a/2) + \max(0, 1 - (9a/16) - 1/2 + (a/4)) + (9/4) = 3.25 - a/2 + \max(0, 1/2 - (5a/16))
\end{aligned}
$$

This function monotonically decreases on the interval, $\mathrm{ACR}_{3/2} \le 2.531$ and $\mathrm{ACR}_{a \nearrow 27/16} \le 2.40625$.

3.3 Case $27/16 \le a < 2$

On the interval $(27/16, 16/9]$, $H3$, $B4b$ and $B4a$ have worse filling ratios than $4/9$. We will use a similar method as previously, but an additional constraint on the offline packing is needed, which makes the formula more complicated. The optimal offline packing utilizes at least

- as many bins as the size of all items(S)
- n_{H3} bins
- $(1/2)n_{H3} + n_{B4b}$
- EITHER $n_{H3} + \frac{3n_{B4a}-(n_{H3}-2n_{B4b})}{3} = \frac{2}{3}n_{H3} + \frac{2}{3}n_{B4b} + n_{B4a}$ (if $n_{H3} > 2n_{B4b}$)
  OR $n_{H3} + \frac{2n_{B4b}-n_{H3}}{2} + \frac{3n_{B4a}-(2n_{B4b}-n_{H3})/2}{3} = \frac{2}{3}n_{H3} + \frac{2}{3}n_{B4b} + n_{B4a}$ (if $n_{H3} < 2n_{B4b}$)

The first three constraints are straightforward. For the fourth one we consider two cases.

Let us see the case $n_{H3} < 2n_{B4b}$. In any offline packing, all the $H3$ items get their own bins, and in the best case each of these items may get an additional $B4b$ item in their bin. No further $H3$, $B4b$ or $B4a$ items can be placed in these bins. Thus, we have n_{H3} bins packed with all the $H3$ items and n_{H3} items of class $B4b$, and number of remaining items are $2n_{B4b} - n_{H3}$ for $B4b$ and $3n_{B4a}$ for $B4a$. In the best case scenario two $B4b$ and one $B4a$ items can fit into one bin. So the remaining $B4b$ items can fit into $n_{B4b} - n_{H3}/2$ bins, along with $n_{B4b} - n_{H3}/2$ items of class $B4a$. The remaining $3n_{B4a} - n_{B4b} + n_{H3}/2$ items from class $B4a$ fit into $(3n_{B4a} - n_{B4b} + n_{H3}/2)3 = n_{B4a} - n_{B4b}/3 + n_{H3}/6$ bins. This is altogether $n_{H3} + n_{B4b} - n_{H3}/2 + n_{B4a} - n_{B4b}/3 + n_{H3}/6 = \frac{2}{3}n_{H3} + \frac{2}{3}n_{B4b} + n_{B4a}$ bins.

The other case is $n_{H3} > 2n_{B4b}$. In any offline packing, all the $H3$ items get their own bins, which is n_{H3} bins in total. All the $B4b$ items can be placed next to a $H3$ item, so there will be $2n_{B4b}$ bins packed with a $H3$ and a $B4b$ items, and $n_{H3} - 2n_{B4b}$ bins with only one $H3$ item in them; the latter ones may accommodate an additional $B4a$ item. Thus, from the original $3n_{B4a}$ items of class $B4a$, there will be only $3n_{B4a} - n_{H3} + 2n_{B4b}$ left to pack. These can be placed into $(3n_{B4a} - n_{H3} + 2n_{B4b})/3 = n_{B4a} - n_{H3}/3 + (2/3)n_{B4b}$ bins. This altogether means $n_{H3} + n_{B4a} - n_{H3}/3 + (2/3)n_{B4b} = \frac{2}{3}n_{H3} + \frac{2}{3}n_{B4b} + n_{B4a}$ bins.

For the upper bound on the ACR we get a more complicated formula, but its derivation is similar to the previous cases:

$$\begin{aligned}
&\mathrm{ACR}_{27/16<a<16/9} \\
&\le \frac{(1-(9/4)F(a,H3))\cdot n_{H3} + (1-(9/4)F(a,B4a))\cdot n_{B4a} + (1-(9/4)F(a,B4b))\cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3}+n_{B4b}, \frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a}\}} \\
&\le \frac{(1-(9/4)(a/4))\cdot n_{H3} + (1-(9/4)(3/(4a)))\cdot n_{B4a} + (1-(9/4)(2a/9))\cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3}+n_{B4b}, \frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a}\}} \\
&= \frac{(1-(9a/16))\cdot n_{H3} + (1-(27/(16a)))\cdot n_{B4a} + (1-(a/2))\cdot n_{B4b} + (9/4)S}{\max\{S, n_{H3}, (1/2)n_{H3}+n_{B4b}, \frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a}\}} \\
&= \frac{(\frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a})\cdot(1-(27/(16a))) + ((1/2)n_{H3}+n_{B4b})\cdot((1-(a/2))-\frac{2}{3}(1-27/(16a)))}{\max\{S, n_{H3}, (1/2)n_{H3}+n_{B4b}, \frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a}\}} \\
&+ \frac{n_{H3}\cdot((1-(9a/16)) - \frac{1}{2}(1-(a/2)) - \frac{1}{3}(1-27/(16a))) + 9/4S}{\max\{S, n_{H3}, (1/2)n_{H3}+n_{B4b}, \frac{2}{3}(n_{H3}+n_{B4b})+n_{B4a}\}} \\
&\le 1-(27/(16a)) + \max\{0, 1-(a/2)-2/3(1-(27/(16a)))\} \\
&+ \max\{1-9a/16-(1/2)(1-a/2)-(1/3)\cdot(1-27/(16a))\} + (9/4) \\
&= 1-(27/(16a)) + \max\{0, 1-(a/2)-2/3+(9/(8a))\} \\
&+ \max\{0, 1-9a/16-1/2+a/4-(1/3)+9/(16a)\} + (9/4) \\
&= 1-(27/(16a)) + \max\{0, 1/3-(a/2)+(9/(8a))\} + \max\{0, 1/6-5a/16+9/(16a)\} + (9/4)
\end{aligned}$$

$\mathrm{ACR}_{27/16} = 2.40625$, and also $\mathrm{ACR}_{a\nearrow 2} = 2.40625$, all other values are smaller than this, and the function has a minimum around 1.8699: $\mathrm{ACR}_{1/3+\sqrt{85}/6} \le 7/2-\sqrt{85}/8 \approx 2.347557$.

3.4 Case $2 \le a < 5/2$

In this case only the $B4a$ items have a smaller filling ratio than 4/9. Let S_{-B4a} denote the total area of all items except those of class $B4a$. The online algorithm uses at most $n_{B4a} + S_{-B4a}$ bins. The offline packing uses at least S bins, and also at least $(3/4)n_{B4a}$ bins. The latter is because at least 3 and at most 4 $B4a$ items fit in one bin, so the online algorithm places at least 3 items in each bin of type $B4a$, while an offline packing cannot place more than 4. Thus

$$\begin{aligned}
&\mathrm{ACR}_{2<a<5/2} \le \frac{n_{B4a}+(9/4)S_{-B4a}}{\max\{S,(3/4)n_{B4a}\}} \\
&\le \frac{(1-(9/4)F(a,B4a))\cdot n_{B4a}+(9/4)S}{\max\{S,(3/4)n_{B4a}\}} \le \frac{(1-(9/4)(1/2-1/(4a)))\cdot n_{B4a}+(9/4)S}{\max\{S,(3/4)n_{B4a}\}} \\
&\le (4/3)(1-(9/4)(1/2-1/(4a))) + (9/4) = 25/12 + 3/(4a)
\end{aligned}$$

This function monotonically decreases, $\mathrm{ACR}_2 \le 2.458\dot{3}$ and $\mathrm{ACR}_{a\nearrow 5/2} \le 2.38\dot{3}$.

3.5 Case $5/2 \leq a$

The reasoning is similar as in the previous case: only $B4a$ has smaller filling ratio than 4/9. The difference is in the estimation of the number of bins that the offline packing needs for $B4a$ items compared to the online algorithm. In the online packing, the width of the empty space left on the right side of a bin of type $B4a$ is 2/3. In the best case, the offline packing can place items vertically in a way that no space is left at the left or right end of any bin. Thus, the offline packing needs at least $(a-2/3)/a \cdot n_{B4a}$ bins.

$$\mathrm{ACR}_{2<a} \leq \frac{n_{B4a} + (9/4)S_{-B4a}}{\max\{S, (a-2/3)/a \cdot n_{B4a}\}}$$
$$\leq \frac{(1-(9/4)F(a,B4a)) \cdot n_{B4a} + (9/4)S}{\max\{S, (a-2/3)/a \cdot n_{B4a}\}} \leq \frac{(1-(9/4)(1/2-1/(4a))) \cdot n_{B4a} + (9/4)S}{\max\{S, (a-2/3)/a \cdot n_{B4a}\}}$$
$$\leq (a-2/3)/a \cdot (1-(9/4)(1/2-1/(4a))) + (9/4) = 9/4 - (a-2/3)/(8a) + 9/(16(a-2/3))$$

This function monotonically decreases. $\mathrm{ACR}_{5/2} \leq 2.3863$, $\mathrm{ACR}_3 \leq 2.33$ and $\mathrm{ACR}_4 \leq 2.26$.

3.6 Case $a \to \infty$

As a grows to ∞, the smallest filling ratio is 1/2, so a trivial asymptotic upper bound for the ACR is 2. If we take into account the second smallest filling ratio ≈ 0.61 of $T2$ items, we can use these to lower bound all filling ratios expect for $B4a$, and thus we can achieve a better bound. The offline packing needs at least S bins, and also as many bins as the online algorithm needs for $B4a$ items. Using $1/0.61 \approx 1.64$, we get

$$\mathrm{ACR}_{a\to\infty} \leq \frac{n_{B4a} + 1.64S_{-B4a}}{\max\{S, n_{B4a}\}}$$
$$\leq \frac{(1-1.64F(a,B4a)) \cdot n_{B4a} + 1.64S}{\max\{S, n_{B4a}\}} \leq \frac{(1-1.64(1/2)) \cdot n_{B4a} + 1.64S}{\max\{S, n_{B4a}\}}$$
$$\leq (1-0.82) + (1.64) = 1.82$$

4 Lower bounds for the problem

This section provides a lower bound on the ACR of any online algorithm solving the same problem. Moreover, the lower bound is valid for k-bounded algorithms for any constant k. Recall that the bins are rectangle-shaped with sides of length 1 and $a > 4/3$, and the items can be rotated, but need to be placed parallel to the bin's sides. Similarly to the previous section, the lower bound is given by a function which depends on the ratio a of the sides.

This section is further divided into three parts, as the different ratios of the sides of the bins require different approaches for the lower bounds. The first part ($4/3 < a < 3/2$) is easy, and somewhat different from the others. The second part ($3/2 \leq a <$

$2+1/3$) is much longer and has several subcases, but they are all modifications of a simple packing. Considering so many subcases may seem unnecessarily complicated, but the aim was to show a common lower bound of 1.8. The last part ($a \leq 2+1/3$) is much shorter and also gives a lower bound if $a \to \infty$.

In all subsections, $\epsilon > 0$ is chosen to be small enough so that the described items fit in a bin.

4.1 Case $4/3 \leq a < 3/2$

If $4/3 < a < 3/2$, then the following items can fit into one bin, as illustrated on Figure 8: one square of side length $a/2+\epsilon$, one square of $1/2+\epsilon$, one square of $a/3+\epsilon$, two squares of $a/7+\epsilon$, and one rectangle of $(1/4+\epsilon)\cdot(3/4+\epsilon)$. In addition, the remaining area can be filled with small rectangles. Let the objects of n such bins (where n is much larger than k) be ordered in the following way: the list starts with all the $(a/2+\epsilon)$-squares grouped together, then all the $(1/2+\epsilon)$-squares come, and so on; so all the same shaped rectangles are grouped together.

Let us see how an online algorithm works for such a list. For the squares of $a/2+\epsilon$, any online algorithm needs to open n bins. If the algorithm is k-bounded, then k bins, (each with a single $(a/2+\epsilon)$-square) remain open. Let us consider these k bins empty, and so $n-k$ bins were used for these kind of items. Similarly, the $(1/2+\epsilon)$-squares need $n/2-k$ bins, as at most two can fit into one bin, and we can disregard the last k bins, which remain opened. The squares of $a/3+\epsilon$ need $n/4-k$ bins, the $2n$ squares of $a/7+\epsilon$ need $2n\cdot 1/30-k$ or $2n\cdot 1/24-k$ depending on a, but this can be lower bounded by $n/15-k$. The rectangles of $(1/4+\epsilon)\cdot(3/4+\epsilon)$ need $n/5-k$ bins, while the remaining small rectangles need $n\cdot(a-(a/2)^2-(a/3)^2-(1/2)^2-((1/4)\cdot(3/4))-2\cdot(a/7)^2)/a$ bins (their total area divided by one bin's size).

This means altogether $n\cdot(2,01\dot{6})-5k+n\cdot(a-(a/2)^2-(a/3)^2-(1/2)^2-((1/4)\cdot(3/4))-2\cdot(a/7)^2)/a$ bins. To get the ACR, we need to divide this by n (the number of offline bins), and if n goes to infinity then k can be disregarded as it is a constant.

$$\text{ACR}_{4/3\leq a<3/2} \leq \lim_{n\to\infty} \frac{n\cdot 2,01\dot{6}-5k+n(a-(a/2)^2-(a/3)^2-(1/2)^2-(1/4\cdot 3/4)-2\cdot(a/7)^2)/a}{n}$$

which decreases from $\approx$ **2.15** at 4/3 to $\approx$ **2.12** at 3/2.

4.2 Case $3/2 \leq a < 2+1/3$

The basic idea remains the same: we construct a list from the items that an offline algorithm could put into n identically packed bins. In this list the same items are grouped together. We check how many bins are needed for an online algorithm to pack all the items in the list. From now on we will not mention the k bins remaining opened, as they can be disregarded at the end when calculating the ACR as $n\to\infty$. The descriptions will be less precise compared to the above discussed case. For sake

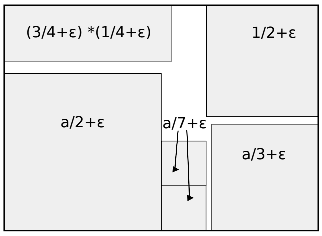

Figure 8: Example packing of on offline bin where $4/3 \leq a < 3/2$. The white area is also filled, but the size of these rectangles do not matter

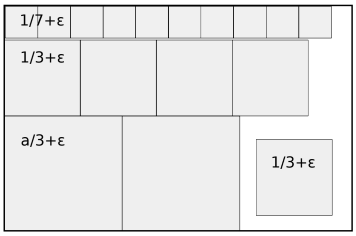

Figure 9: Example packing of on offline bin where $3/2 \leq a < 3/2 + 1/14$. The white area is also filled, but the size of these rectangles do not matter

of simplicity, we say that a group of items fills a fraction r of an online bin, if a group of n such items would fill $n \cdot r$ online bins.

This interval is split to 6 subintervals, but they are all modified cases of the same packing. An offline bin is split into three homogenous rows: a row of squares of $1/2+\epsilon$, another row of squares of $1/3+\epsilon$, and a third row containing squares of $1/7+\epsilon$. This would result in a ratio of $1 + 1/2 + 1/6 = 1.\dot{6}$ without any other rectangles filling the empty places. In the 6 subcases, sometimes the squares are blown up a little bit, so that the last square of the row can not fit. In one case even the row of $a/7+\epsilon$ needs to be omitted, because of the enlargement of another row's items. Also some additional squares are added in to the empty space at the end of the rows. Due to the slightly different filling structures, an overall lower bound of **1.8** can be achieved on the whole interval $(3/2, 2 + 1/3)$. Figure 9 shows the offline bin's packing of the first case; the other cases are similar.

4.2.1 $3/2 \leq a < 3/2 + 1/14$

The filling of one offline bin consists of

- row of $a/3 + \epsilon$ squares which fills 1 online bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 online bin,
- row of $1/7 + \epsilon$ squares which fills 1/6 online bin,
- one extra $1/3 + \epsilon$ square which fills 1/8 online bin,

and the remaining area is filled with small rectangles, which fill $(a-2(a/3)^2-4(1/3)^2-10(1/7)^2-(1/3)^2)/a$ of a bin's area. This is illustrated on Figure 9. This altogether results in an ACR of $1.791\dot{6} + (a - 2(a/3)^2 - 4(1/3)^2 - 10(a/7)^2 - (1/3)^2)/a$, which is $\approx$ **1.952** at 3/2, and increases to $\approx$ **1.959**.

4.2.2 $3/2 + 1/14 \leq a < 5/3$

The filling of one offline bin consists of

- row of $a/3 + \epsilon$ squares which fills 1 online bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 online bin,
- one extra $1/3 + \epsilon$ square which fills 1/8 online bin,

and the remaining area is filled with small rectangles, which fill $(a-2(a/3)^2-4(1/3)^2-(1/3)^2)/a$ of a bin's area. The ACR is at least $1.625+(a-2(a/3)^2-4(1/3)^2-(1/3)^2)/a$ which stays between $\approx$ **1.922** ($a = 3/2 + 1/14$) and $\approx$ **1.921** ($a = 5/3$)

4.2.3 $5/3 \leq a < 5/3 + 5/42$

The filling of one offline bin consists of

- row of $1/2 + \epsilon$ squares which fills 1 online bin,
- row of $a/5 + \epsilon$ squares which fills 1/2 online bin,
- row of $1/7 + \epsilon$ square which fills 1/6 online bin,

and the remaining area is filled with small rectangles, which fill at least $(a-3(1/2)^2-4(a/5)^2-12(1/7)^2)/a$ of a bin's area (as the row of $a/7+\epsilon$ squares consists of 11 or 12 squares depending on a). The ACR in this case is at least $1.\dot{6}+(a-3(1/2)^2-4(a/5)^2-12(1/7)^2)/a$ which stays between $\approx$ **1.803** ($a = 5/3$) and $\approx$ **1.824** ($a = 5/3 + 5/42$).

4.2.4 $5/3 + 5/42 \leq a < 2$

The filling of one offline bin consists of

- row of $1/2 + \epsilon$ squares which fills 1 online bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 online bin,
- row of $1/7 + \epsilon$ square which fills 1/6 online bin,
- one extra $1/7 + \epsilon$ square which fills $1/(6 \cdot 12)$ or $1/(6 \cdot 13)$ online bin

and the remaining area is filled with small rectangles, which fill $(a - 3(1/2)^2 - 5(1/3)^2 - \lfloor a/(1/7+\epsilon) \rfloor (1/7)^2 - (1/7)^2)/a$ of a bin's area. The ACR is at least $1.\dot{6} + \lfloor a/(1/7+\epsilon) \rfloor + (a - 3(1/2)^2 - 5(1/3)^2 - \lfloor a/(1/7+\epsilon) \rfloor (1/7)^2 - (1/7))^2/a$ which stays between $\approx$ **1.801** ($a = 5/3 + 5/42$) and $\approx$ **1.884** ($a = 2$).

4.2.5 $2 < a < 2 + 2/21$

The filling of one offline bin consists of

- row of $a/4 + \epsilon$ squares which fills 1 online bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 online bin,
- row of $1/7 + \epsilon$ square which fills 1/6 online bin,
- one extra $1/3 + \epsilon$ square which fills 1/12 online bin

and the remaining area is filled with small rectangles, which fill $(a-3(a/4)^2-6(1/3)^2-14(1/7)^2-(1/3)^2)/a$ of a bin's area. The ACR is at least $1.75+(a-3(a/4)^2-6(1/3)^2-14(1/7)^2-(1/3))^2/a$ which stays between $\approx$ **1.843** ($a = 2$) and $\approx$ **1.850** ($a = 2+2/21$).

4.2.6 $2 + 2/21 \leq a < 2 + 1/3$

The filling of one offline bin consists of

- row of $a/4 + \epsilon$ squares which fills 1 online bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 online bin,
- one extra $1/3 + \epsilon$ square which fills 1/12 online bin

and the remaining area is filled with small rectangles, which fills $(a-3(a/4)^2-6(1/3)^2-(1/3)^2)/a$ of a bin's area. So the ACR is at least $1.58\dot{3} + (a - 3(a/4)^2 - 6(1/3)^2 - (1/3))^2/a$ which stays between $\approx$ **1.819** ($a = 2 + 2/21$) and $\approx$ **1.813** ($a = 2 + 1/3$).

4.3 Case $2 + 1/3 \leq a$

For all $a \geq 2 + 1/3$ the filling of one offline bin is similar:

- row of $1/2 + \epsilon$ squares which fills 1 bin,
- row of $1/3 + \epsilon$ squares which fills 1/2 bin,
- row of $1/7 + \epsilon$ square which fills 1/6 bin

and the remaining area is filled with small rectangles. There is at least $(1/42 - 3\epsilon)a$ area left for them, so the ACR is at least $1 + 1/2 + 1/6 + 1/42 \approx$ **1.690**. This ratio holds as $n \to \infty$. On Figure 7, the whole remaining area is taken into account, not just the $(1/42 - 3\epsilon)a$. The space at the end of each row grows with a, but drops to zero when a new item can be added to the row; this results in the serrated look of the lower bound's function.

Acknowledgement

The research was supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002), and by Thematic Excellence Programme, Industry and Digitization Subprogramme, NRDI Office, 2019.

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