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Rigid realizations of graphs with few locations in the plane

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Abstract

Adiprasito and Nevo (2018) proved that there exists a set of 76 points in $\mathbb{R}^3$ such that every triangulated planar graph has an infinitesimally rigid realization in which each vertex is mapped to a point in this set.

In this paper we show that there exists a set A of 26 points in the plane such that every planar graph which is generically rigid in $\mathbb{R}^2$ has an infinitesimally rigid realization in which each vertex is mapped to a point in this set.

It is known that a similar result, with a set of constant size, does not hold for the family of all generically rigid graphs in R^d, $d \geq 2$. We show that for every positive integer n there is a set of $c(\sqrt{n})$ points in the plane, for some constant c, such that every generically rigid graph in R^2 has an infinitesimally rigid realization on this set. This bound is best possible.

1 Introduction

Adiprasito and Nevo [1] asked the following question: "How generic does the realization of a generically rigid graph need to be to guarantee that it is infinitesimally rigid?" In fact, Adiprasito and Nevo considered a more exact question. Which graph classes have infinitesimally rigid realizations for each of its members on a given subset of $\mathbb{R}^d$ of constant cardinality? They showed that triangulated planar graphs have such realizations on 76 points in $\mathbb{R}^3$. They also gave similar results for 3-dimensional realizations of triangulations of closed surfaces.

The first result of this paper is that every planar graph which is generically rigid in the plane has an infinitesimally rigid realization on a given set of 26 points of the plane. Furthermore, a similar result follows when we change the class of rigid planar graphs to a class of rigid graphs whose members can be embedded in a given closed surface. This implies our second result that states that every graph on n vertices which is rigid in the plane has an infinitesimally rigid realization on a given set of $O(\sqrt{n})$ points of the plane. We note that the above question of Adiprasito and Nevo was also considered before by Fekete and Jordán [2] who proved that instead of using

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generic points one can always find an infinitesimally rigid *injective* realization on a grid of size $(\sqrt{n}+O(1)) \times (\sqrt{n}+O(1))$.

Before introducing the above problems formally, we summarize some basics of rigidity theory. We refer to [3] for more details. A **d-dimensional framework** is a pair (G,p), where $G=(V,E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We will also refer to (G,p) (or less precisely to p) as a **realization** of G and to $p(v)$ as the **location** of v for a vertex $v \in V$.

We assign to (G,p) a matrix, called the **rigidity matrix** $R(G,p) \in \mathbb{R}^{|E|\times d|V|}$ which is defined as follows. We assign a row of $R(G,p)$ to each edge $uv \in E$ and d columns to each $v \in V$. The row of $R(G,p)$ assigned to $uv \in E$ contain the $d+d$ coordinates of $p(u)-p(v)$ and $p(v)-p(u)$ in the d columns assigned to u and in the d columns assigned to v, respectively, while the other entries are zeros.

An **infinitesimal motion** of a bar-joint framework (G,p) is an assignment $m : V \to \mathbb{R}^d$ of infinitesimal velocities to the vertices, such that

$$\langle p(u)-p(v), m(u)-m(v)\rangle = 0 \text{ for all edges } uv \in E, \tag{1}$$

that is, $R(G,p)\,m = 0$. An infinitesimal motion m is **trivial** if $m(v) = S\,p(v)+t$ holds for all $v \in V$, for some $d\times d$ skew-symmetric matrix S and some $t \in \mathbb{R}^d$, that is, if m is in the kernel of $R(K_V,p)$ where K_V is the complete graph on V. (G,p) is **infinitesimally rigid** in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. We also note that the dimension of the vector space of the trivial infinitesimal motions of a d-dimensional framework is $\binom{d+1}{2}$ when the underlying graph has at least d vertices. Thus, assuming that $|V| \geq d$, (G,p) is infinitesimally rigid if and only if $\operatorname{rank}(R(G,p)) = d|V| - \binom{d+1}{2}$.

A set of points $A \subseteq \mathbb{R}^d$ is said to be **generic** if the (multi)set of the coordinates of the points in A is algebraically independent over $\mathbb{Q}$. A realization p of G is said to be **generic** if p is injective and its image is a generic set. It follows by the definition of a generic realization that if the determinant of a square submatrix of $R(G,p_0)$ is 0 for a generic realization p_0, then the determinant of the same submatrix of $R(G,p)$ is also 0 for every other realization. Thus $\operatorname{rank}(R(G,p_0)) = \max\{\operatorname{rank}(R(G,p)) : p : V \to \mathbb{R}^d\}$. Therefore, the infinitesimal rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the infinitesimal rigidity of (G,p) depends only on the graph G and not the particular realization p, if (G,p) is generic (see [8]). We say that the graph G is **rigid** in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is infinitesimally rigid. $G=(V,E)$ is said to be **minimally rigid** in $\mathbb{R}^d$ if G is rigid but $G-e$ is not rigid in $\mathbb{R}^d$ for each $e \in E$. It is easy to see that if $G=(V,E)$ is minimally rigid and p is an infinitesimally rigid realization of G, then the rows of $R(G,p)$ are linearly independent. Let $\boldsymbol{E(X)}$ denote the set of edges in a graph $G=(V,E)$ induced by a set $X \subseteq V$, let $\boldsymbol{i_G(X)}{:=}\,|E(X)|$, and let $\boldsymbol{d_G(v)}$ or $\boldsymbol{d_E(v)}$ denote degree of a vertex $v \in V$. We can give the following necessary conditions for minimal rigidity.

Theorem 1.1 ([8]). *Let $G=(V,E)$ be minimally rigid in $\mathbb{R}^d$ with $|V| \geq d$. Then*
(i) $|E| = d|V| - \binom{d+1}{2}$,
(ii) $i_G(X) \leq d|X| - \binom{d+1}{2}$ for every $X \subseteq V$ with $|X| \geq d$. □

Pollaczek-Geiringer [6] (and Laman [4]) showed that these necessary conditions are also sufficient for minimal rigidity when $d=2$.

Theorem 1.2 ([4, 6])**.** *A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if*
(L1) $|E| = 2|V| - 3$,
(L2) $i_G(X) \leq 2|X| - 3$ for every $X \subseteq V$ with $|X| \geq 2$. □

A graph $G = (V, E)$ for which (L2) holds is called **sparse**. A graph for which both (L1) and (L2) hold is called **tight** or **Laman**.

Formally, the problem posed by Adiprasito and Nevo [1] is the following.

Problem 1.3 ([1])**.** Let $\mathcal{G}$ be a graph class and $c \in \mathbb{Z}_+$. $\mathcal{G}$ is called **rigid in $\mathbb{R}^d$ with c locations** if there exists a set $A \subset \mathbb{R}^d$ with $|A| = c$ such that, for each $G = (V, E) \in \mathcal{G}$, there exists an infinitesimally rigid realization $p : V \to A$ of G. Which graph classes are rigid in $\mathbb{R}^d$ with c locations for some constant c?

The main result of Adiprasito and Nevo [1] is the following.

Theorem 1.4 ([1])**.** *Let $A \subseteq \mathbb{R}^3$ be a generic set with $|A| = 76$. Then, for every triangulated planar graph $G = (V, E)$, there exists an infinitesimally rigid realization $p : V \to A$ of G.* □

We note that Fekete and Jordán [2] observed that the class of graphs which are rigid on the line (that is, the class of connected graphs) is rigid on the line with 2 locations. By a simple construction, they also observed that for the 2-dimensional case we cannot give such a constant cardinality set.

Proposition 1.5 ([2])**.** *For every positive integer c there exists a graph G on $c + 1 + \binom{c+1}{2}$ vertices which is rigid in $\mathbb{R}^2$ and has no infinitesimally rigid realization in $\mathbb{R}^2$ with c locations.* □

However, the following result shows that the class of rigid planar graphs is rigid in $\mathbb{R}^2$ with 26 locations.

Theorem 1.6. *Let $A \subseteq \mathbb{R}^2$ be a generic set with $|A| = 26$. Then, for every planar graph $G = (V, E)$ which is rigid in $\mathbb{R}^2$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

We prove Theorem 1.6 in Section 3. In Section 4 we obtain the following result by using some observations on graph embeddings in closed surfaces.

Theorem 1.7. *There exists a constant $c > 0$ such that, for every graph $G = (V, E)$ which is rigid in $\mathbb{R}^2$ and every set A of generic points in $\mathbb{R}^2$ with $|A| = c\sqrt{|V|}$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

Note that Proposition 1.5 shows that the above bound on the cardinality of A is sharp up to a constant factor.

Finally, in Section 5, we show the following theorem by using another idea of Fekete and Jordán [2].

Theorem 1.8. *Let $A \subseteq \mathbb{R}^d$ and let $G = (V, E)$ be a graph. Assume that there exists a map $p : V \to A$ such that (G, p) is infinitesimally rigid. Then there exists a set of integral points $B_G \subseteq \{1, \dots, |V|\}^d$ with $|B_G| \leq |A|$ and a map $p' : V \to B_G$ such that (G, p') is infinitesimally rigid.* □

This result implies that some slightly weaker statements remain true if we change 'generic' in Theorems 1.4, 1.6, or 1.7 to 'integral'. However, note that, in Theorem 1.8, the image set B_G of p' depends on the graph G contrary to the generic case where we have the same image set A for all graphs in the investigated graph classes.

2 Preliminaries

We will use the following property of Laman graphs.

Lemma 2.1. *Let $G = (V, E)$ be a Laman graph on $n \geq 6$ vertices. Then it has at least $n/3 + 2$ vertices of degree at most 4.*

Proof. Let n_i ($n_{\leq i}$, $n_{\geq i}$, respectively) denote the number of vertices in G with degree i (at most i, at least i, respectively). Then

$$2n_{\leq 4} + 5n_{\geq 5} \leq \sum_{i=2}^{n-1} in_i = 2|E| = 4n - 6 = 4n_{\leq 4} + 4n_{\geq 5} - 6.$$

Hence, $n = n_{\leq 4} + n_{\geq 5} \leq 3n_{\leq 4} - 6$. Therefore, $n/3 + 2 \leq n_{\leq 4}$. □

We say that a graph $G' = (V, E \cup F)$ is F**-crossing** if $E \cap F = \emptyset$ and G' can be drawn with continuous curves in the plane such that only edges in F can cross each other and each edge in F can cross at most one other edge in F. It is easy to observe the following property of F-crossing graphs.

Proposition 2.2. *If $G' = (V, E \cup F)$ is F-crossing, then there exists a partition of F into two sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ are planar.* □

We will also need the following property of F-crossing graphs.

Lemma 2.3. *Let $G = (V, E)$ be a Laman graph on n vertices and let $G' = (V, E \cup F)$ be F-crossing and simple. Then there exists at most $n/3 - 1$ vertices $v \in V$ such that $d_F(v) \geq 12$.*

Proof. Since G is Laman, $|E| = 2n - 3$. By Proposition 2.2, there exists a partition of F into two sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ are planar. As G'_i is simple planar, we get $|E \cup F_i| \leq 3n - 6$ for $i = 1, 2$. Hence $|F_1| \leq n - 3$ and $|F_2| \leq n - 3$ and thus $|F| \leq 2n - 6$.

Let $n'_{\geq 12}$ denote the number of vertices $v \in V$ for which $d_F(v) \geq 12$. Now, $12n'_{\geq 12} \leq 2|F| \leq 4n - 12$. Hence $n'_{\geq 12} \leq n/3 - 1$. □

We shall also use the following key observation of Adiprasito and Nevo, implicitly proved in [1] for $\mathbb{R}^3$.

Lemma 2.4. *Assume that (G, p) is an infinitesimally rigid framework in $\mathbb{R}^d$ and v is a vertex of degree c. Let $A \subset \mathbb{R}^d$ be a given set of points with generic coordinates. Assume that $|A| \geq \binom{d+c}{d}$. Then there exists an $a \in A$ such that (G, p') is infinitesimally rigid for the map $p' : V \to \mathbb{R}^d$ defined by $p'(v) := a$ and $p'(u) := p(u)$ for $u \in V - v$.*

Proof. By deleting some edges of G for which the corresponding row of the rigidity matrix $R(G,p)$ is linearly dependent from the other rows of $R(G,p)$, we can assume that (G,p) is minimally infinitesimally rigid. Hence, by Theorem 1.1 $|E| = d|V| - \binom{d+1}{2}$.

Let us consider the rigidity matrix $R(G,p_v)$ of another realization p_v of G which arises by taking $p_v(u) := p(u)$ for each $u \in V - v$ and considering $p_v(v)$ as a vector with d variable entries $(x_1,\dots,x_d)$. (G,p_v) is not infinitesimally rigid if and only if $\text{rank}(R(G,p_v)) < d|V| - \binom{d+1}{2} = |E|$, that is, the determinant of every $|E| \times |E|$ submatrix of $R(G,p_v)$ is 0. Each such determinant is a polynomial with variables $x_1,\dots,x_d$ of degree at most c (as $d_G(v) = c$). One can look at the polynomials over $\mathbb{R}$ with d variables and maximum degree at most c as a $\binom{d+c}{d}$-dimensional vector space over $\mathbb{R}$ whose bases are the monomials with d variables and maximum degree at most c. As (G,p) is infinitesimally rigid, at least one of the polynomials corresponding to the submatrices of $R(G,p_v)$, say P, must be not identically zero.

We claim that no choice of $\binom{d+c}{d}$ points from A makes P vanish on each of these points. To see this, put the coefficients of P into a vector $u \in \mathbb{R}^{\binom{d+c}{d}}$ where the jth coordinate corresponds to the coefficient of the jth monomial with d variables and maximum degree at most c in the lexicographical order of these monomials. Next consider the $\binom{d+c}{d} \times \binom{d+c}{d}$ matrix M where the jth entry in the ith row is the value of the jth monomial in the lexicographical order (which has coefficient u_j in P) computed on the coordinates of the ith point $(a_1^i,\dots,a_d^i)$ in A. Since A is generic and the determinant of M is a not identically zero polynomial on the coordinates of the points in A with integer coefficients, $\det(M) \neq 0$. If P vanishes on each of our $\binom{d+c}{d}$ points, then it means that $Mu = 0$ and hence, as $\det(M) \neq 0$, $u = 0$ contradicting our assumption that P is not identically 0. Therefore, we can extend $p'|_{V-v} \equiv p|_{V-v}$ with $p'(v) \in A$ such that (G,p') is infinitesimally rigid. □

We note that Lemma 2.4 immediately implies the following.

Theorem 2.5. *Let $G = (V,E)$ be a generically rigid graph in $\mathbb{R}^d$ with maximum degree Δ and let $A \subset \mathbb{R}^d$ be a given set of points with generic coordinates. Assume that $|A| \geq \binom{d+\Delta}{d}$. Then there exists a realization $p : V \to A$ such that (G,p) is infinitesimally rigid.* □

In what follows, we will say that a **set $X \subseteq V$ is tight in a sparse graph** $G = (V,E)$ if the subgraph $G[X]$ induced by X is tight. The following two lemmas simply follow from the supermodularity of the function i_G see [3].

Lemma 2.6. *Let $G = (V,E)$ be a sparse graph and let $X,Y \subset V$ be two tight sets in G with $|X \cap Y| \geq 1$. Then $i_G(X \cup Y) \geq 2|X \cup Y| - 4$.* □

Lemma 2.7. *Let $G = (V,E)$ be a sparse graph and let $X,Y,Z \subset V$ be three tight sets in G such that $X \cap Y - Z \neq \emptyset$, $X \cap Z - Y \neq \emptyset$, and $Y \cap Z - X \neq \emptyset$. Then $X \cup Y \cup Z$ is also tight in G.* □

2.1 Operations preserving rigidity

For our proof, we shall use some operations that preserve the rigidity of frameworks. The **Henneberg-0 extension**, or simply **0-extension**, on G adds a new vertex and connects it to 2 distinct vertices of G. The **1-extension**, deletes an edge $uw \in E$, adds a new vertex v and connects it to u, w and 1 other vertex of G. The following two lemmas show that 0- and 1-extensions preserve rigidity.

Lemma 2.8 ([8]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ with $p(v_1) \neq p(v_2)$. Let G^+ be a 0-extension of G that arises by adding a new vertex v with two incident edges vv_1 and vv_2 and let us take $p(v) \in \mathbb{R}^2$ such that it is not on the line through $p(v_1)$ and $p(v_2)$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* □

Lemma 2.9 ([8]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ where the set $\{p(v_1), p(v_2), p(v_3)\}$ affinely spans the plane and v_1v_2 forms an edge. Let G^+ be a 1-extension of G that arises by deleting v_1v_2 and adding a new vertex v with three incident edges vv_1, vv_2 and vv_3 and let us take $p(v) \in \mathbb{R}^2 - \{p(v_1), p(v_2)\}$ such that it is on the line through $p(v_1)$ and $p(v_2)$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* □

The following well-known result was a key in the proof of Theorem 1.2 in [4, 6].

Lemma 2.10 ([4, 6]). *A graph is Laman if and only if it arises from K_2 by using 0- and 1-extensions.*

In this paper we will also need to use the following operation which is called an **X-replacement**. Let $G = (V, E)$ and $v_1v_2, v_3v_4 \in E$ be two independent edges. The X-replacement deletes v_1v_2, v_3v_4, adds a new vertex v and connects it to v_1, v_2, v_3 and v_4. The following lemma shows that X-replacement preserves rigidity.

Lemma 2.11 ([7]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ where any three element of the set $\{p(v_1), p(v_2), p(v_3), p(v_4)\}$ affinely span the plane, v_1v_2 and v_3v_4 both form edges, and the lines through $p(v_1)$ and $p(v_2)$, and $p(v_3)$ and $p(v_4)$ are intersecting. Let G^+ be an X-replacement of G that arises by deleting v_1v_2 and v_3v_4 and adding a new vertex v with four incident edges vv_1, vv_2, vv_3 and vv_4 and let us define $p(v)$ as the intersection point of the line through $p(v_1)$ and $p(v_2)$ and the line through $p(v_3)$ and $p(v_4)$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* □

The inverse operation of 1-extension is called a **1-reduction**. The following lemma is also well-known.

Lemma 2.12 ([7]). *Let G be a Laman graph and v be a vertex of G with exactly 3 neighbors v_1, v_2 and v_3. Then there exists some $1 \leq i < j \leq 3$ such that the 1-reduction of G, $G - v + v_iv_j$ is Laman.* □

Tay and Whiteley [7] showed that a degree 4 vertex can always be removed from a Laman graph along with adding two, possibly not independent, new edges between its neighbors such that the resulting graph is Laman. The following lemma shows when we can get a Laman graph after an inverse X-replacement.

Lemma 2.13. *Let $G = (V, E)$ be a Laman graph and v be a vertex in G with exactly 4 neighbors v_1, v_2, v_3 and v_4. Then $G' = G - v + v_1v_2 + v_3v_4$ is Laman if and only if there is no tight set $X \subseteq V - v$ in G with $v_1, v_2 \in X$ or $v_3, v_4 \in X$.*

Proof. Since the necessity of the condition is obvious, we only prove its sufficiency. Observe that G' has $2|V| - 3 - 4 + 2 = 2|V - v| - 3$ edges, hence we only need to prove its sparsity. Assume for a contradiction that there is a set $X \subseteq V - v$ such that $i_{G'}(X) > 2|X| - 3$. If $\{v_1, v_2, v_3, v_4\} \subseteq X$, then $i_G(X \cup \{v\}) > 2|X \cup \{v\}| - 3$, a contradiction. If $\{v_1, v_2\} \nsubseteq X$ and $\{v_3, v_4\} \nsubseteq X$ both hold, then $i_G(X) = i_{G'}(X) > 2|X| - 3$, a contradiction. Hence, by relabeling the neighbors of v, we can assume that $\{v_1, v_2\} \subseteq X$ and $v_4 \notin X$. Thus $2|X| - 2 \leq i_{G'}(X) = i_G(X) + 1 \leq 2|X| - 2$. Therefore, equality holds in the last inequality implying that X is tight in G. □

3 Rigid planar graphs with few locations

In this section we prove Theorem 1.6. Observe that it is enough to prove Theorem 1.6 for planar Laman graphs. In fact, we will prove a slightly stronger result, as follows.

Theorem 3.1. *Let $G = (V, E)$ be a Laman graph and let us assume that $G' = (V, E \cup F)$ is an F-crossing graph. Let A be a set of generic points in the plane with $|A| = 26$. Then there exists a map $p : V \to A$ such that the framework (G, p) is infinitesimally rigid in the plane and $p(u) \neq p(v)$ holds for every edge $uv \in E \cup F$.*

Proof. We prove by induction on $|V|$. By Theorem 1.2 the statement is true when $|V| \leq 26$.

By Lemmas 2.1 and 2.3, there exists a vertex $v \in V$ with $d_G(v) \leq 4$ and $d_F(v) \leq 11$.

Case 1: $d_G(v) = 2$. Let us denote the neighbors of v in G by v_1 and v_2. $G - v$ is a planar Laman graph by Lemma 2.10. Furthermore, $G'' = (V - v, E(V - v) \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_2\} - E(V - v)$ since G' is F-crossing and G'' arises from G' by deleting v and adding the F'-edge v_1v_2 which can be drawn by joining the curves corresponding to the edges v_1v and vv_2. By induction, there exists an infinitesimally rigid realization p of $G - v$ in A such that the two endvertices of each edge in $E(V - v) \cup F(V - v)$ have different locations and $p(v_1) \neq p(v_2)$. By Lemma 2.8, any position of v outside the line through $p(v_1)$ and $p(v_2)$ results in an infinitesimally rigid realization of G. Since A is generic, $A - \{p(v_1), p(v_2)\}$ is outside this line. Hence we can find an infinitesimally rigid realization of G on A such that the two endvertices of each edge in $E \cup F$ have different location by choosing $p(v)$ outside the locations of the at most 13 neighbors of v in G' by $|A| = 26 \geq 14$.

Case 2: $d_G(v) = 3$. Let us denote the neighbors of v in G by v_1, v_2 and v_3. By Lemma 2.12, we can perform a 1-reduction on v resulting in a Laman graph. By relabeling the neighbors of v, we can assume that $G - v + v_1v_2$ is Laman. It is easy to observe that $G - v + v_1v_2$ is also planar and $G'' = (V - v, E(V - v) \cup \{v_1v_2\} \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_3, v_2v_3\} - E(V - v)$ (see Figure 1). By induction, there exists an infinitesimally rigid realization p of $G - v + v_1v_2$ on A such that the two endvertices of each edge in $E(V - v) \cup F(V - v)$ have different locations

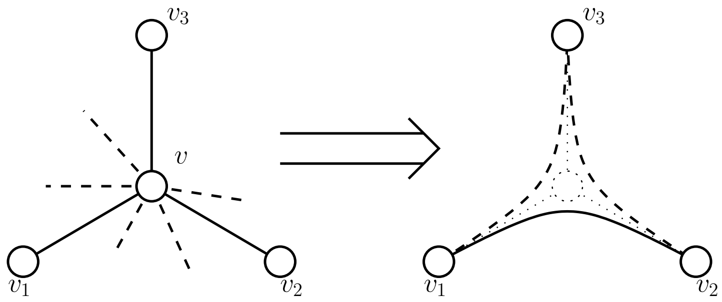

Figure 1: A planar 1-reduction at v used in Case 2. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges.

and $|\{p(v_1), p(v_2), p(v_3)\}| = 3$. Since A is generic, the latter statement implies that $p(v_1), p(v_2), p(v_3)$ affinely span the plane. Lemma 2.9 implies that we can define $p(v)$ in such a way that (G, p) is infinitesimally rigid, however, at this point we cannot guarantee that $p(v) \in A$ but we have $p(u) \in A$ for every $u \in V - v$. Now, by Lemma 2.4, we can define a map $p' : V \to A$ such that $p'(u) = p(u)$ for $u \in V - v$, $p'(v) \in A$, (G, p') is infinitesimally rigid, and $p'(v)$ is not equal to the location of any of its F-neighbors since $|A| = 26 \geq \binom{5}{2} + 11$. Note that $p'(v)$ is not equal to the location of any of the neighbors of v in G' since otherwise one of the $2|V| - 6$ rows of the rigidity will be 0, contradicting the infinitesimal rigidity of (G, p').

Case 3: $d_G(v) = 4$. Let us denote the neighbors of v in G by v_1, v_2, v_3 and v_4, such that this is the order of the outgoing edges in E from v in some fixed F-crossing drawing of G'. We have the following two subcases:

Subcase 3.1: $G - v + v_1v_2 + v_3v_4$ or $G - v + v_1v_4 + v_2v_3$ is Laman. By relabeling the neighbors of v we can assume that $G - v + v_1v_2 + v_3v_4$ is Laman. It is easy to see that $G - v + v_1v_2 + v_3v_4$ is planar and $G'' = (V - v, E(V - v) \cup \{v_1v_2, v_3v_4\} \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_3, v_1v_4, v_2v_3, v_2v_4\} - E(V - v)$ (see Figure 2). By

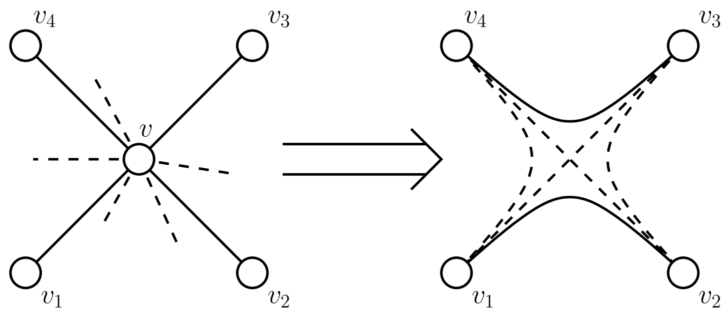

Figure 2: A planar inverse X-replacement at v used in Subcase 3.1. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges except v_1v_3 and v_2v_4 which cross each other.

induction, there exists an infinitesimally rigid realization p of $G - v + v_1v_2 + v_3v_4$ on A such that the two endvertices of each edge in $E(V - v) \cup F(V - v)$ have different locations and $|\{p(v_1), p(v_2), p(v_3), p(v_4)\}| = 4$. Since A is generic, we can use Lemma

2.11 to prove that there exists some placement of $p(v)$ in $\mathbb{R}^2$ such that (G,p) is infinitesimally rigid, however, we need to get it from the set A. By Lemma 2.4, we can define a map $p' : V \to A$ such that $p'(u) = p(u)$ for $u \in V - v$, $p'(v) \in A$, (G,p') is infinitesimally rigid, and $p'(v)$ is not equal to the location of any of its F-neighbors since $|A| = 26 \geq \binom{6}{2} + 11$. Note that, similarly to Case 2, $p'(v)$ is not equal to the location of any of its neighbors in G'.

Subcase 3.2: If neither $G - v + v_1v_2 + v_3v_4$ nor $G - v + v_1v_4 + v_2v_3$ is Laman, then, by Lemma 2.13, there exists some $i \in \{1,2,3,4\}$ such that there are tight sets $X, Y \subset V - v$ in G with $v_i, v_{i+1} \in X$ and $v_i, v_{i-1} \in Y$ where $v_0 := v_4$ and $v_5 := v_1$. By relabeling the vertices cyclically we can assume that $i = 1$. Note that $v_3, v_4 \notin X$ and $v_2, v_3 \notin Y$ since otherwise $X \cup \{v\}$ (or $Y \cup \{v\}$, respectively) induces at least $2|X|-3+3 > 2|X \cup \{v\}|-3$ (or $2|Y|-3+3 > 2|Y \cup \{v\}|-3$, respectively) edges in G, contradicting the sparsity condition (L2). We will use the following two observations.

Claim 3.2. *There exists no tight set $Z \subset V - v$ in G with $v_2, v_4 \in Z$.*

Proof. For the sake of contradiction, suppose that $Z \subset V - v$ is a tight set in G with $v_2, v_4 \in Z$. Note that $v_1, v_3 \notin Z$ since otherwise $Z \cup \{v\}$ induces at least $2|Z| - 3 + 3 > 2|Z \cup \{v\}| - 3$ edges in G, contradicting the sparsity condition (L2). Hence $v_1 \in X \cap Y - Z$, $v_2 \in X \cap Z - Y$, and $v_4 \in Y \cap Z - Y$. Thus $X \cup Y \cup Z$ is tight in $G - v$ by Lemma 2.7. Since the 4 neighbors of v are in $X \cup Y \cup Z$, the tightness of $X \cup Y \cup Z$ implies $i_G(X \cup Y \cup Z \cup \{v\}) > 2|X \cup Y \cup Z \cup \{v\}| - 3$, contradicting the sparsity condition. □

Claim 3.3. *There exists no set $Z' \subset V - v$ with $v_2, v_3, v_4 \in Z'$ and $i_G(Z') \geq 2|Z'| - 4$.*

Proof. For the sake of contradiction, suppose that $Z' \subset V - v$ is a set with $v_2, v_3, v_4 \in Z'$ and $i_G(Z') \geq 2|Z'| - 4$. Then, in $G - vv_1$, v has exactly three neighbors in Z' and hence $Z' \cup \{v\}$ is tight in $G - vv_1$. Note that $v_1 \notin Z$ since otherwise $Z' \cup \{v\}$ induces at least $2|Z| - 4 + 4 > 2|Z \cup \{v\}| - 3$ edges in G, contradicting the sparsity condition (L2). Hence $v_1 \in X \cap Y - Z$, $v_2 \in X \cap Z - Y$, and $v_4 \in Y \cap Z - Y$. Thus $X \cup Y \cup (Z' \cup \{v\})$ is tight in $G - vv_1$ by Lemma 2.7. Since vv_1 is induced by $X \cup Y \cup Z' \cup \{v\}$ in G, this implies $i_G(X \cup Y \cup Z' \cup \{v\}) > 2|X \cup Y \cup Z' \cup \{v\}| - 3$, contradicting the sparsity condition. □

Now, it is impossible to have two tight sets $Z_1, Z_2 \subset V - v$ with $v_2, v_3 \in Z_1$ and $v_3, v_4 \in Z_2$, since otherwise $i_G(Z_1 \cup Z_2) \geq 2|Z_1 \cup Z_2| - 4$ and $v_2, v_3, v_4 \in Z_1 \cup Z_2$, contradicting Claim 3.3. By swapping v_2 and v_4, we can assume that there is no tight set $Z_2 \subset V - v$ with $v_3, v_4 \in Z_2$. This along with Claims 3.2 and 3.3 implies that $G - v \cup \{v_2v_4, v_3v_4\}$ is Laman. Furthermore, $G - v \cup \{v_2v_4, v_3v_4\}$ is planar and $G'' = (V - v, E(V - v) \cup \{v_2v_4, v_3v_4\} \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_2, v_1v_4, v_2v_3\} - E(V - v)$ (see Figure 3). By induction, there exists an infinitesimally rigid realization p of $G - v + v_2v_4 + v_3v_4$ on A such that the two endvertices of each edge in $E(V - v) \cup F(V - v)$ have different locations and either $|\{p(v_1), p(v_2), p(v_3), p(v_4)\}| = 4$, or $= 3$ and $p(v_1) = p(v_3)$.

Next we add v to $G - v + v_2v_4 + v_3v_4$ by a 1-extension on v_2v_4 along with the edges vv_1, vv_2 and vv_4. By using the proof of Case 2, we can see that from any 10 points in

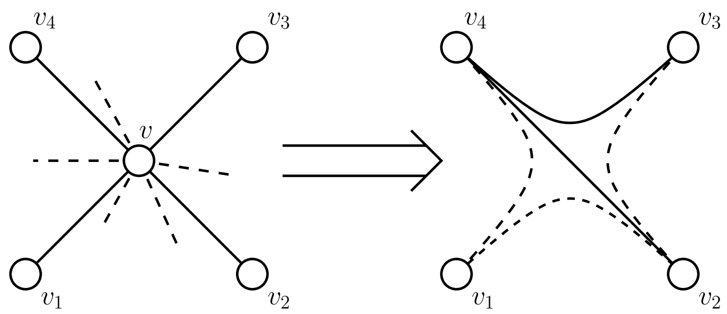

Figure 3: The reduction at v used in Subcase 3.2. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges.

A we can find at least one, say a, for which the extension p^a of p with $p^a(v) := a$ is an infinitesimally rigid realization of $G - vv_3 + v_3v_4$. Furthermore, this also implies that from any 11 points in A we can find at least two, say a and b, for which the extensions p^a and p^b of p with $p^a(v) := a$ and $p^b(v) := b$ are both infinitesimally rigid realizations of $G - vv_3 + v_3v_4$. As $|A| = 26 \geq 11 + 11 + 4$, we can choose such a and b in such a way that $p(u) \neq a$ and $p(u) \neq b$ both hold for every $u \in V$ for which $uv \in E \cup F$.

Note that, in an infinitesimally rigid realization of a Laman graph G^* on vertex set V, any tight set in G^* induces an infinitesimally rigid subframework since otherwise the corresponding rows of the rigidity matrix are not linearly independent and hence the rigidity matrix has at most $2|V| - 4$ linearly independent rows contradicting to the infinitesimal rigidity of G^*. Thus both of $((G - vv_3 + v_3v_4)[X \cup Y \cup \{v\}], p^a)$ and $((G-vv_3+v_3v_4)[X\cup Y\cup\{v\}], p^b)$ are infinitesimally rigid since $G-vv_3+v_3v_4$ is Laman, $(G - vv_3 + v_3v_4, p^a)$ and $(G - vv_3 + v_3v_4, p^b)$ both are infinitesimally rigid, and the set $X \cup Y \cup \{v\}$ (for the tight sets X and Y of G defined above) is tight in $G - vv_3 + v_3v_4$. Note that $v_3 \notin X \cup Y$, hence $(G - vv_3 + v_3v_4)[X \cup Y \cup \{v\}] = (G - vv_3)[X \cup Y \cup \{v\}]$. Thus $((G - vv_3)[X \cup Y \cup \{v\}], p^a)$ and $((G - vv_3)[X \cup Y \cup \{v\}], p^b)$ both are also infinitesimally rigid.

Observe that $G - vv_3$ has only $2|V| - 4$ edges and hence neither $(G - vv_3, p^a)$ nor $(G-vv_3, p^b)$ is infinitesimally rigid. But, the infinitesimal rigidity of $(G-vv_3+v_3v_4, p)$ implies that the dimension of the space of the infinitesimal motions of $(G - vv_3, p^a)$ ($(G - vv_3, p^b)$, respectively) is 4. By adding a trivial infinitesimal motion to a non-trivial infinitesimal motion of $(G - vv_3, p^a)$, we can assume that the value of this non-trivial infinitesimal motion m is 0 on each element of $X\cup Y\cup\{v\}$. Note that $m(v_3) \neq 0$ since otherwise m is also a non-trivial infinitesimal motion of the infinitesimally rigid framework $(G - vv_3 + v_3v_4, p^a)$, a contradiction. Observe now that m is also a non-trivial infinitesimal motion for $(G - vv_3 + v_3v_4, p^b)$ for all such non-trivial infinitesimal motions m' of $(G - vv_3, p^a)$ (and of $(G - vv_3, p^b)$, respectively) $m(v_3)$ and $m'(v_3)$ are parallel since the dimension of space of the infinitesimal motions of $(G - vv_3, p^a)$ (and of $(G - vv_3, p^a)$, respectively) is 4. However, by the genericity of A and $a, b \neq p(v_3)$, $m(v_3) - m(v) = m(v_3)$ cannot be orthogonal to both $a - p(v_3)$ and $b - p(v_3)$ and hence at least one of (G, p^a) and (G, p^b) is infinitesimally rigid. $\square$

4 Rigid graphs with few locations

In this section we show that Theorem 1.6 can be extended to the class of graphs that can be embedded in some fixed closed surface. Later we use this generalization of Theorem 1.6 to prove Theorem 1.7. We refer to the book of Mohar and Thomassen [5, Chapter 3] for an introduction to the topic of graph embeddings in surfaces.

4.1 Graphs on surfaces

Note that in the proof of Theorem 1.6 we used planarity two times:

- In Lemma 2.3, we used the edge bound (which follows from Euler's formula) for planar graphs.
- In our reduction steps, we used planarity 'locally' to show that the reduced graphs are also F'-crossing (see Figures 1, 2 and 3).

Observe that if instead of planarity of G we only require that it can be embedded in a closed surface we also have the above properties since a closed surface has finite Euler characteristic (which is non-positive except for the sphere) and is locally homeomorphic to the plane [5]. Hence we get the following result with the same proof.

Theorem 4.1. *For every closed surface $\mathcal{C}$ with Euler characteristic $\chi_{\mathcal{C}} \leq 0$ there exists a constant $k_{\mathcal{C}} = O(\sqrt{-\chi_{\mathcal{C}}})$ such that for every graph $G = (V, E)$ which has an embedding into $\mathcal{C}$ and is rigid in $\mathbb{R}^2$ and for every set A of generic points in $\mathbb{R}^2$ with $|A| \geq k_{\mathcal{C}}$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

Proof (Sketch). Since the proof is just a copy of our proof for the planar case, we only show why $k_{\mathcal{C}} = O(\sqrt{-\chi_{\mathcal{C}}})$. In our proof for the planar case, we only used Euler's formula in the proof of Lemma 2.3. Similarly to the planar case, we say that $G' = (V, E \cup F)$ is ***F*-crossing on $\mathcal{C}$** for a closed surface $\mathcal{C}$ if $E \cap F = \emptyset$ and G' can be drawn with continuous curves on $\mathcal{C}$ such that only edges in F can cross each other and each edge in F can cross at most one other edge in F. Now Lemma 2.3 can be modified, as follows.

Lemma 4.2. *Let $G = (V, E)$ be a Laman graph on n vertices, let $\mathcal{C}$ be a closed surface with Euler characteristic $\chi_{\mathcal{C}}$, and let $G' = (V, E \cup F)$ be F-crossing on $\mathcal{C}$ and simple. Then it has less than $n/3$ vertices of F-degree more than $12 - \frac{36}{n}(\chi_{\mathcal{C}} - 1)$.*

Proof. Since G is Laman, $|E| = 2n-3$. Like in the planar case, there exists a partition of F into to sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ can be embedded into $\mathcal{C}$. As G'_i can be embedded into $\mathcal{C}$ that has Euler characteristic $\chi_{\mathcal{C}}$, $|E \cup F_i| = n + n^*_i - \chi_{\mathcal{C}}$ for $i = 1, 2$ where n^*_i is the number of faces of G'_i embedded into $\mathcal{C}$. Since $n^*_i \leq \frac{2}{3}|E \cup F_i|$ follows by the simplicity of G_i, we get $|E \cup F_i| \leq 3n - 3\chi_{\mathcal{C}}$ for $i = 1, 2$. Hence $|F_1| \leq n - 3\chi_{\mathcal{C}} + 3$ and $|F_2| \leq n - 3\chi_{\mathcal{C}} + 3$ and thus $|F| \leq 2n - 6\chi_{\mathcal{C}} + 6$.

For a constant $c \in \mathbb{R}^+$ let $n'_{>c}$ denote the number of vertices in G' of F-degree more than c. Now, $cn'_{>c} < 2|F| \leq 4n - 12\chi_{\mathcal{C}} + 12$. To prove $n'_{>c} < n/3$ we need $4n - 12\chi_{\mathcal{C}} + 12 \leq cn/3$ and hence $12 + \frac{36}{n}(1 - \chi_{\mathcal{C}}) \leq c$. □

Now, similarly to the planar case, it is enough to prove Theorem 4.1 for Laman graphs and we prove a bit stronger result, as follows.

Theorem 4.3. *Let $\mathcal{C}$ be a closed surface with Euler characteristic $\chi_{\mathcal{C}} \leq 0$, let $G = (V, E)$ be a Laman graph and let us assume that $G' = (V, E \cup F)$ is an F-crossing graph on $\mathcal{C}$. There exist constants $c, c' \geq 1$ for which if A is a set of generic points in the plane with $|A| = c\sqrt{-\chi_{\mathcal{C}}} + c'$, then there exists a map $p : V \to A$ such that the framework (G, p) is infinitesimally rigid in the plane and $p(u) \neq p(v)$ holds for every edge $uv \in E \cup F$.*

Proof (Sketch). When $|V| \leq k_{\mathcal{C}}$, then the statement is obvious, hence we can assume that $|V| > c\sqrt{-\chi_{\mathcal{C}}} + c' \geq \sqrt{-\chi_{\mathcal{C}}} + 1$. By Lemmas 2.1 and 4.2 there exists a vertex $v \in V$ with $d_G(v) \leq 4$ and $d_F(v) \leq \left\lfloor 12 + \frac{36(1-\chi_{\mathcal{C}})}{\sqrt{-\chi_{\mathcal{C}}}+1} \right\rfloor$ since we only need to use the previous formula when $n > c\sqrt{-\chi_{\mathcal{C}}} + c' \geq \sqrt{-\chi_{\mathcal{C}}} + 1$. Now, by following the proof of Theorem 3.1, one can get that $|A| \geq \left\lfloor 27 + \frac{36(1-\chi_{\mathcal{C}})}{\sqrt{-\chi_{\mathcal{C}}}+1} \right\rfloor$ suffices. □

This finishes the proof of Theorem 4.1. □

4.2 Genus of Laman graphs

Next we show that the Laman graphs on n vertices have genus less than n, that is, they can be embedded in an orientable surface which has genus less than n.

Lemma 4.4. *Let $G = (V, E)$ be a Laman graph. Then G can be embedded in an orientable closed surface which has genus* $\max(n - 5, 0)$*.*

Proof. It is easy to check that each Laman graph on at most 5 vertices is planar. By Lemma 2.10, each Laman graph can be constructed by 0- and 1-extensions from the complete graph on 2 vertices. Observe that both of the 0- and 1-extensions increases the number of vertices by one. Furthermore, if G has an embedding in an orientable closed surface $\mathcal{C}$ and G' is its 0-extension, then it is easy to see that we can add a new handle to $\mathcal{C}$ (with ends close to the location of the two neighbors of the new vertex in G') in such a way that G' is embeddable into this new surface. Similarly, we can add a new handle to $\mathcal{C}$ (with ends close to the subdivided edge and the third neighbor of the new vertex in G') when G' is a 1-extension of G. □

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Again it is enough to prove the case when G is Laman and hence its genus is at most $\max(n - 5, 0)$ by Lemma 4.4. Note that it is well-known that an orientable closed surface with genus g has Euler characteristic $2 - 2g$ [5]. Hence our statement follows from Theorem 4.1 (or Theorem 1.6 when $g = 0$). □

5 Rigid realizations on few integer points

Fekete and Jordán [2] showed that one can construct an infinitesimally rigid realization of a graph $G = (V, E)$ with integer coordinates from $\{1, \dots, |V|\}$ by changing the coordinates one-by-one of an infinitesimal rigid realization of G, preserving infinitesimal rigidity. We prove Theorem 1.8 by showing that the coordinates of coincident vertices can be changed simultaneously.

Proof of Theorem 1.8. The statement is obvious when $|V| = 1$ hence we can assume $|V| \geq 2$. Let x be a map which maps each point $a \in A$ to a d-dimensional vector with variables $(x_{a,1}, \dots, x_{a,d})$. Let us consider the matrix $R(G, x \circ p)$. Since (G, p) is rigid, $R(G, p)$ has a $\left(d|V| - \binom{d+1}{2}\right) \times \left(d|V| - \binom{d+1}{2}\right)$ non-singular submatrix $M(G, p)$. Now $M(G, x \circ p)$ is a $\left(d|V| - \binom{d+1}{2}\right) \times \left(d|V| - \binom{d+1}{2}\right)$ submatrix of $R(G, x \circ p)$ whose determinant is a polynomial $P \not\equiv 0$ as the substitution of a_i into $x_{a,i}$ gives the determinant of $M(G, p)$ which is nonzero. Note that no graph on at least two vertices has infinitesimally rigid realization with one location hence at most $|V| - 1$ vertex have the same location in (G, p). Thus the variable $x_{a,i}$ is only included in at most $|V| - 1$ columns of $R(G, x \circ p)$ for each $i \in \{1, \dots, d\}$ and $a \in A$. Hence the degree of P is at most $|V| - 1$ in each of its variables. Thus P vanishes on at most $|V| - 1$ entries for each variable. Therefore, fixing $a_0 \in A$ we can choose a value $\varphi_1(a_0) \in \{1, \dots, |V|\}$ for $x_{a_0,1}$ such that $P|_{x_{a_0,1} = \varphi_1(a_0)} \not\equiv 0$. Next, we add values $\varphi_i(a) \in \{1, \dots, |V|\}$ sequentially for each $a \in A$ and $i \in \{1, \dots, d\}$ in such a way that finally we get a nonzero value for the constant polynomial $P|_{\{x_{a,i} = \varphi_i(a) : a \in A, i \in \{1, \dots, d\}\}}$. Therefore, the rigidity matrix $R(G, \varphi \circ p)$ has a $\left(d|V| - \binom{d+1}{2}\right) \times \left(d|V| - \binom{d+1}{2}\right)$ non singular submatrix $M(G, \varphi \circ p)$, that is $(G, \varphi \circ p)$ is rigid. Furthermore, $B = \varphi(A) \subseteq \{1, \dots, |V|\}^d$ and $|B| \leq |A|$. □

The next corollary follows from Theorems 1.7 and 1.8.

Corollary 5.1. *There exists a constant $c > 0$ such that, for every graph $G = (V, E)$ which is rigid in $\mathbb{R}^2$, there exists a set A of points in $\{1, \dots, |V|\}^2$ with $|A| \leq c\sqrt{|V|}$ such that there exists an infinitesimally rigid realization $p : V \to A$ of G.* □

We obtain similar corollaries by combining Theorem 1.8 with Theorem 1.4, Theorem 1.6, Theorem 2.5, or Theorem 4.1, respectiely.

Note that Corollary 5.1 states that every graph on n vertices which is rigid in the plane has an infinitesimally rigid realization with $O(\sqrt{n})$ integral locations with coordinates in $\{1, \dots, n\}$. By contrast, we note that Fekete and Jordán [2] proved that such a graph has an infinitesimally rigid realization with integral locations with coordinates in $\{1, \dots, \lceil \sqrt{n-1} \rceil + 9\}$ in such a way that the locations are pairwise different.

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