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Sparse graphs and an augmentation problem

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Abstract

For two integers $k > 0$ and ℓ, a graph $G = (V, E)$ is called (k, ℓ)-tight if $|E| = k|V| - \ell$ and $|E(X)| \leq k|X| - \ell$ for all $X \subseteq V$ for which $k|X| - \ell \geq 0$. G is called (k, ℓ)-redundant if $G - e$ has a spanning (k, ℓ)-tight subgraph for all $e \in E$. We consider the following augmentation problem. Given a graph $G = (V, E)$ that has a (k, ℓ)-tight spanning subgraph, find a graph $H = (V, F)$ with minimum number of edges, such that $G + H$ is (k, ℓ)-redundant.

In this paper, we give a polynomial algorithm and a min-max theorem for this augmentation problem when the input is (k, ℓ)-tight. For general inputs, we give a polynomial algorithm when $k \geq \ell$ and show the NP-hardness of the problem when $k < \ell$. Since (k, ℓ)-tight graphs play an important role in rigidity theory, these algorithms can be used to make several types of rigid frameworks redundantly rigid by adding a smallest set of new bars.

1 Introduction

Let k be a positive integer and ℓ be an integer such that $\ell < 2k$. A graph $G = (V, E)$ is called **(k, ℓ)-sparse** if $i_G(X) \leq k|X| - \ell$ for all $X \subseteq V$ for which $k|X| - \ell \geq 0$, where **$i_G(X)$** denotes the number of edges of G induced by $X \subseteq V$. A (k, ℓ)-sparse graph is called **(k, ℓ)-tight** if $|E| = k|V| - \ell$. A graph G is called **(k, ℓ)-rigid** if G has a (k, ℓ)-tight spanning subgraph. We will call an edge e of G a **(k, ℓ)-redundant edge** if $G - e$ is (k, ℓ)-rigid. A graph G is called a **(k, ℓ)-redundant graph** if each edge of G is (k, ℓ)-redundant. We consider the following augmentation problem that we call the **general (augmentation) problem**.

Problem. *Let k and ℓ be integers with $k \geq 0$ and $\ell < 2k$ and let $G = (V, E)$ be a (k, ℓ)-rigid graph. Find a graph $H = (V, F)$ on the same vertex set with minimum number of edges, such that $G + H = (V, E \cup F)$ is (k, ℓ)-redundant.*

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We call the special case of this problem, where the input graph G is (k, ℓ)-tight, the **reduced (augmentation) problem**. In this paper, we give a min-max theorem for the reduced problem which leads to an algorithm of $O(n^2)$ running time on a graph with n vertices for fixed k and ℓ. We also show how the algorithm that solves the reduced problem can be extended to solve the general problem in the same running time when $\ell \leq k$. In contrast, we note that the general problem is NP-hard whenever $\ell > k$.

Sparsity properties are important in rigidity theory as they can be used in the characterization of many rigidity classes. For example, the generically rigid graphs in $\mathbb{R}^2$ are the $(2, 3)$-rigid graphs by the fundamental theorem of Pollaczek-Geiringer [19] and Laman [14]. The 'body-bar graph' induced by a given graph G is generically rigid in $\mathbb{R}^d$ if and only if G contains $\binom{d+1}{2}$ edge-disjoint spanning trees by Tay's theorem [20], that is, when G is $\left(\binom{d+1}{2}, \binom{d+1}{2}\right)$-rigid by Nash-Williams' [18] famous result.

Besides the effect of redundancy, redundant rigidity is an important concept in rigidity theory since variants of Hendrickson's result [9] show that redundant rigidity is often a necessary condition of 'global rigidity' which plays a crucial role in many applications [1, 22, 23]. Furthermore, in some cases, for example for 'body-bar graphs' (see [3]), redundant rigidity is also a sufficient condition of global rigidity. It is thus natural to ask how many new edges are needed to make a rigid graph redundantly rigid.

There are special pairs of k and ℓ for which these problems were already investigated. For example, García and Tejel [6] showed that the general problem is NP-hard for $(2, 3)$-rigid graphs but can be solved in polynomial time for minimally rigid graphs, that is, when G is $(2, 3)$-tight. In contrast to this result, our method is based on a new, well-posed min-max theorem for the reduced problem. One ca observe that the general problem for $(1, 1)$-rigid graphs is the well-studied 2-edge-connectivity augmentation problem. We also note that Frank and T. Király [5] gave a polynomial algorithm to augment a graph to a (k, h)-tree-connected graph using polyhedral techniques. (A graph $G = (V, E)$ is called $\boldsymbol{(k, h)}$**-tree-connected** if $G - E'$ contains k edge-disjoint spanning trees for every $E' \subseteq E$ with $|E'| = h$.) The famous result of Nash-Williams [18] states that the graphs that can be partitioned to k edge-disjoint spanning trees are exactly the (k, k)-tight graphs. Therefore, the algorithm of Frank and Király with parameters $k \in \mathbb{Z}_+$ and $h = 1$, can be used to give a polynomial algorithm for the general problem when $\ell = k$. The algorithm, that will be presented here, is a rather simple and more efficient solution for this problem, however, it does not deal with the case of $h \geq 2$ and also the (k, k)-rigidity of the input is needed.

To obtain the solution for the general problem, we need more general concepts. Let ℓ be an integer, and let $m : V \to \mathbb{Z}_+$ be a function where $\mathbb{Z}_+$ denotes the set of non-negative integers. For $X \subseteq V$, let $\widetilde{m}(X) := \sum_{v \in X} m(v)$. A graph $G = (V, E)$ is called $\boldsymbol{(\underline{m}, \ell)}$**-sparse** if $i_G(X) \leq \widetilde{m}(X) - \ell$ holds for every $X \subseteq V$ for which $\widetilde{m}(X) - \ell \geq 0$, (that is, if $i_G(X) \leq (\widetilde{m}(X) - \ell)_+ := \max\{\widetilde{m}(X) - \ell, 0\}$ holds for every $X \subseteq V$). An $(\underline{m}, \ell)$-sparse graph is called $\boldsymbol{(\underline{m}, \ell)}$**-tight** if $|E| = \widetilde{m}(V) - \ell$. Throughout this paper, we make a slightly stronger assumption on m and ℓ, as follows.

(A0) $\ell \in \mathbb{Z}$ and $m : V \to \mathbb{Z}_+$ such that $m(u) + m(v) \geq \ell$ holds for each $u, v \in V$ and equality is only allowed when $m(u) = m(v) = \ell = 0$.

This assumption ensures that $(\widetilde{m}(X) - \ell)_+ = \widetilde{m}(X) - \ell$ holds for every $X \subseteq V$ with $|X| \geq 2$ and every $(\underline{m}, \ell)$-tight graph on vertex set V with $|V| \geq 2$ must have at least one edge whenever $\widetilde{m}(V) > 0$. Observe that each subgraph of an $(\underline{m}, \ell)$-sparse graph is $(\underline{m}, \ell)$-sparse. For simplicity, we will call a set $X \subseteq V$ **$(\underline{m}, \ell)$-tight in G** if the **induced subgraph $G[X]$** of X in G is $(\underline{m}, \ell)$-tight.

A graph G is called **$(\underline{m}, \ell)$-rigid** if G has an $(\underline{m}, \ell)$-tight spanning subgraph. We call an edge of G, e an **$(\underline{m}, \ell)$-redundant edge** if $G - e$ is $(\underline{m}, \ell)$-rigid. A graph G is called an **$(\underline{m}, \ell)$-redundant graph** if each edge of G is $(\underline{m}, \ell)$-redundant. Note that when $m \equiv k$, an $(\underline{m}, \ell)$-sparse/tight/rigid/redundant graph is (k, ℓ)-sparse/tight/rigid/redundant, respectively.

To solve the general problem, we shall use the polynomial algorithm for the reduced problem for $(\underline{m}, \ell)$-tight graphs. Our method will also give a polynomial time solution for the general problem whenever $m(v) \geq \ell$ holds for every $v \in V$.

We conclude the introduction by listing some notation used throughout this paper. All the graphs in this paper are multigraphs, that is, we allow parallel edges and loops. Given a graph $G = (V, E)$, $\boldsymbol{d_G(v)}$ denotes the number of edges incident to a vertex $v \in V$, $\boldsymbol{d_G(X, Y)}$ denotes the number of edges between $X - Y$ and $Y - X$ for $X, Y \subseteq V$, and $\boldsymbol{d_G(X)} := d_G(X, V - X)$. Note that our definition implies that $d_G(v) \neq d_G(\{v\})$ if there exist loop edges on v. Also note that, in the usual definition of the degree, loop edges count twice for the degree of a vertex, however, we only count them once. We use $\boldsymbol{N_G(X)}$ to denote the neighbour set of $X \subseteq V$, that is, $N_G(X) = \{v \in V - X : d_G(v, X) \geq 1\}$. For $X \subseteq V$, $\boldsymbol{G[X]}$ denotes the subgraph of G induced by X and $\boldsymbol{G/X}$ denotes the graph arising from G by contracting X into a single vertex. If G_1 and G_2 are graphs, then $G_1 \subseteq G_2$ will denote that G_1 is a subgraph of G_2. For a graph G and for a positive integer c, cG will denote the graph that arises from G by replacing each edge e of G by c parallel copies of e. If $\mathcal{C}$ is a set family, then we say that a set U **covers** $\mathcal{C}$ if $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$. Given a digraph $D = (V, A)$, let $\boldsymbol{\varrho(v)}$ and $\boldsymbol{\varrho(X)}$ denote the **in-degree** of a vertex $v \in V$ and a set $X \subseteq V$, respectively. When it is clear from the context, we omit the subscript G or D from several notations.

2 Preliminaries

In this section, we list some basic properties of $(\underline{m}, \ell)$-sparse graphs. We sketch their proofs for completeness. See [4, 16, 21] for more details. It follows from the definition that an $(\underline{m}, \ell)$-tight subgraph of an $(\underline{m}, \ell)$-sparse graph is always an induced subgraph. Therefore, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ both are $(\underline{m}, \ell)$-tight subgraphs of an $(\underline{m}, \ell)$-sparse graph G, then $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is also an induced subgraph of G. The following statement will be proved using standard submodular techniques.

Lemma 2.1. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-sparse graph, and let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be $(\underline{m}, \ell)$-tight subgraphs of G. If $\widetilde{m}(V_1 \cap V_2) \geq \ell$, then $T_1 \cup T_2$ is an $(\underline{m}, \ell)$-tight graph and there are no edges between $V_1 - V_2$ and $V_2 - V_1$. If $|V_1 \cap V_2| \geq 1$, then $T_1 \cap T_2$ is $(\underline{m}, \ell)$-tight as well.*

We note that, by (A0), the assumption on $\widetilde{m}(V_1 \cap V_2)$ always holds when $|V_1 \cap V_2| \geq 2$, and also when $\ell \leq 0$ (since $m \geq 0$).

Proof. As G_1 and G_2 are $(\underline{m}, \ell)$-tight,

$$\begin{aligned} i(V_1 \cup V_2) + i(V_1 \cap V_2) = i(V_1) + i(V_2) + d(V_1, V_2) \geq i(V_1) + i(V_2) \\ = \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell = \widetilde{m}(V_1 \cup V_2) - \ell + \widetilde{m}(V_1 \cap V_2) - \ell. \end{aligned} \tag{1}$$

Since $\widetilde{m}(V_1 \cap V_2) \geq \ell$, $(\widetilde{m}(V_1 \cap V_2) - \ell)_+ = \widetilde{m}(V_1 \cap V_2) - \ell$. Hence, as G is $(\underline{m}, \ell)$-sparse,

$$i(V_1 \cup V_2) + i(V_1 \cap V_2) \leq \widetilde{m}(V_1 \cup V_2) - \ell + \widetilde{m}(V_1 \cap V_2) - \ell \tag{2}$$

holds and equality must hold in both of (1) and (2). This is only possible if $T_1 \cup T_2$ is $(\underline{m}, \ell)$-tight and $d(V_1, V_2) = 0$. Furthermore, if $|V_1 \cap V_2| \geq 1$, then $T_1 \cap T_2$ must be $(\underline{m}, \ell)$-tight as well. □

It is useful to notice that G is connected in many cases.

Lemma 2.2. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph. If $\ell > 0$ and $m(v) > 0$ for every $v \in V$, then G is connected.*

Proof. We may assume that $|V| \geq 2$ since otherwise the statement of the lemma is obvious. If $\ell > 0$ and $m(v) > 0$, then v cannot be an isolated vertex since otherwise $\widetilde{m}(V) - \ell = |E| = i(V - v) + i(v) \leq (\widetilde{m}(V - v) - \ell)_+ + (m(v) - \ell)_+ < \widetilde{m}(V - v) + m(v) - \ell = \widetilde{m}(V) - \ell$ holds by (A0), a contradiction. Suppose now that G is not connected, that is, there exists a partition $\{V_1, V_2\}$ of V such that $E(V_1) \cup E(V_2) = E$. By $(\underline{m}, \ell)$-sparsity, $|E| = |E_1| + |E_2| \leq \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell = \widetilde{m}(V) - 2\ell = |E| - \ell$, contradicting $\ell > 0$. □

It is known that the edge sets of the $(\underline{m}, \ell)$-sparse subgraphs of a given graph form a matroid, called the **$(\underline{m}, \ell)$-sparsity matroid** or **count matroid** (see [4, 16, 21]). A circuit of this matroid is called an **$(\underline{m}, \ell)$-circuit**. Here, the notion of $(\underline{m}, \ell)$-circuit will be used for graphs and not only for edge sets. It follows from matroid theory (see details in [4]) that for an $(\underline{m}, \ell)$-sparse graph $G = (V, E)$ and $i, j \in V$ for which $G + ij$ is not $(\underline{m}, \ell)$-sparse, $G + ij$ contains a unique $(\underline{m}, \ell)$-circuit $C^G_{(\underline{m},\ell)}(ij)$. In this case $\boldsymbol{T^G_{(\underline{m},\ell)}(ij)} := C^G_{(\underline{m},\ell)}(ij) - ij$ is $(\underline{m}, \ell)$-tight. For every edge e' of $C^G_{(\underline{m},\ell)}(ij)$, $G' = G + ij - e'$ is also $(\underline{m}, \ell)$-sparse and the unique $(\underline{m}, \ell)$-circuit of $G' + e'$ is again $C^G_{(\underline{m},\ell)}(ij)$. Moreover, if $e'' \notin E(C^G_{(\underline{m},\ell)}(ij))$, then, $G' + ij - e''$ is not $(\underline{m}, \ell)$-sparse. The main property of $T^G_{(\underline{m},\ell)}(ij)$ is the following. It will be used several times in this paper.

Lemma 2.3. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-sparse graph and $i, j \in V$. Assume that $G + ij$ is not $(\underline{m}, \ell)$-sparse. If $G' = (V', E')$ is an $(\underline{m}, \ell)$-tight subgraph of G with $i, j \in V'$, then $T^G_{(\underline{m},\ell)}(ij) \subseteq G'$. Hence $T^G_{(\underline{m},\ell)}(ij) = \bigcap\{T_h : T_h$ is an $(\underline{m}, \ell)$-tight subgraph of G spanning i and $j\}$.* □

From now on, let $\boldsymbol{E^G_{(\underline{m},\ell)}(ij)} := E(T^G_{(\underline{m},\ell)}(ij))$, that is, the edge set of $T^G_{(\underline{m},\ell)}(ij)$, and $\boldsymbol{V^G_{(\underline{m},\ell)}(ij)} := V(T^G_{(\underline{m},\ell)}(ij))$, that is, the vertex set of $T(ij)$. We call an $(\underline{m}, \ell)$-tight subgraph G' of G **generated** if there are vertices i, j such that $T^G_{(\underline{m},\ell)}(ij) = G'$.

Let $\boldsymbol{R^G_{(\underline{m},\ell)}(i_1j_1,\dots,i_rj_r) := (V^G_{(\underline{m},\ell)}(i_1j_1,\dots,i_rj_r), E^G_{(\underline{m},\ell)}(i_1j_1,\dots,i_rj_r))}$ denote the subgraph induced by the $(\underline{m},\ell)$-redundant edges of $G=(V,E)$ in $G\cup\{i_1j_1,\dots,i_rj_r\}$ for $i_1,\dots,i_r,\ j_1,\dots,j_r\in V$. That is, the edge $i_1j_1\notin R^G_{(\underline{m},\ell)}(i_1j_1)$. Note that $R^G_{(\underline{m},\ell)}(ij)=T^G_{(\underline{m},\ell)}(ij)$ for any $i,j\in V$. When the graph G or $(\underline{m},\ell)$ is clear from the context, then we omit the superscript G or subscript $(\underline{m},\ell)$, respectively, from all of the above notations. The following lemma is the first key of the solution of our augmentation problem.

Lemma 2.4. *Assume (A0). If G is an $(\underline{m},\ell)$-tight graph, then $R(i_1j_1,\dots,i_rj_r)=T(i_1j_1)\cup\dots\cup T(i_rj_r)$.*

Proof. Since $R(i_1j_1)=T(i_1j_1)$, $T(i_1j_1)\cup\dots\cup T(i_rj_r)\subseteq R(i_1j_1,\dots,i_rj_r)$. For the other direction, let $e\in E(i_1j_1,\dots,i_rj_r)$ be an arbitrary edge. Now, $G-e$ is $(\underline{m},\ell)$-sparse and $|E-e|=\widetilde{m}(V)-\ell-1$. $G\cup\{i_1j_1,\dots,i_rj_r\}-e$ is $(\underline{m},\ell)$-rigid hence $E\cup\{i_1j_1,\dots,i_rj_r\}-e$ has a rank of $\widetilde{m}(V)-\ell$ in the $(\underline{m},\ell)$-sparsity matroid. Thus there is an edge f in $\{i_1j_1,\dots,i_rj_r\}$ for which $E-e+f$ is a basis of the $(\underline{m},\ell)$-sparsity matroid. Since $E-e+f$ is independent in the $(\underline{m},\ell)$-sparsity matroid, we must have $e\in T(f)$. □

2.1 Algorithmic preliminaries

To give a polynomial algorithm for our (general or reduced) augmentation problem, one first needs an algorithm for testing the $(\underline{m},\ell)$-sparsity of a graph. Such a polynomial algorithm already exists for each pair of m and ℓ and it has several good properties that we shall use to obtain an efficient algorithm for our problem. We note that in the main applications of (k,ℓ)-sparse graphs k and ℓ are fixed constants which means in the general $(\underline{m},\ell)$-sparse case that

(*) there exists a constant $c>0$ such that $m(v)\le c$ for every $v\in V$ and $|\ell|\le c$.

Thus we will give the running time of our algorithms by assuming this condition. Observe that (*) implies that an $(\underline{m},\ell)$-sparse graph on V has $O(|V|)$ edges.

Hendrickson and Jacobs [11] (see also [12]) gave the first algorithm for testing $(2,3)$-sparsity which relies on in-degree constrained orientations (see the paper of Berg and Jordán [2] how this algorithm is connected to in-degree constrained orientations). This algorithm is called the "pebble game" algorithm. It is easy to extend this algorithm for testing (k,ℓ)-sparsity for other values of k and ℓ (and also for $(\underline{m},\ell)$-sparsity) if $\ell\ge 0$, see Lee and Streinu [15] for a full investigation of this algorithm for (k,ℓ)-sparsity and [4] for its extension for $(\underline{m},\ell)$-sparsity. Frank [4] gave the following extension of the orientation lemma of Hakimi [8] which can be used to modify the previous algorithm for the case where $\ell<0$ (see [13]).

Lemma 2.5 ([4, Lemma 13.5.9])**.** *Let $G=(V,E)$ be an undirected graph, $g:V\to\mathbb{Z}_+$ an upper-bound function, and $\gamma\ge 0$ an integer. It is possible to remove at most γ edges from G in such a way that the remaining graph G' has an orientation with in-degree function ϱ satisfying $\varrho(v)\le g(v)$ for every vertex v if and only if $\widetilde{g}(X)+\gamma\ge i_G(X)$ holds for every $X\subseteq V$ where $\widetilde{g}(X)=\sum_{v\in X}g(v)$.* □

We will use the algorithm provided by the following theorem (which can be constructed based on the algorithms in [4, 13, 15]) as a subroutine in our algorithms.

Theorem 2.6 (Based on [4, 13, 15])**.** *There exists an algorithm which decides in $O(|V|^2)$ time whether its input graph $G = (V, E)$ is $(\underline{m}, \ell)$-sparse. It has the following outputs:*

If G is $(\underline{m}, \ell)$-sparse, then it outputs this fact along with an orientation D of the edges in G minus a set F' of at most $-\ell$ edges when $\ell < 0$. If G is also $(\underline{m}, \ell)$-tight, then it also outputs this fact.

If G is not $(\underline{m}, \ell)$-sparse, then it outputs a maximal $(\underline{m}, \ell)$-sparse subgraph $H = (V, F)$ of G along with an orientation D of the edges in H minus a set F' of $-\ell$ edges when $\ell < 0$. It also outputs the set R of edges in H which are $(\underline{m}, \ell)$-redundant in G.

Furthermore, if it returns that G is $(\underline{m}, \ell)$-sparse (including the case when G is $(\underline{m}, \ell)$-tight), then by only using the extra data in the output one can decide in $O(|V|)$ extra time whether $G + e$ is $(\underline{m}, \ell)$-sparse for any new edge e, and if the answer is no, then output the generated $(\underline{m}, \ell)$-tight subgraph $T(e)$ of G. □

3 Preprocessing

It is easier to formulate our results by assuming the following conditions when $\ell > 0$.

(A1) There exists no $v \in V$ such that $m(v) = 0$ and v is an isolated vertex.

(A2) There exist $u, v \in V$ such that $V(uv) \neq \{u, v\}$.

(A3) There exists no $v \in V$ such that $V(uv) = \{u, v\}$ for all $u \in V - v$ and $V - v$ induces an $(\underline{m}, \ell)$-tight graph.

We note that these conditions automatically hold for (k, ℓ)-tight graphs with sufficiently many vertices and also for $(\underline{m}, \ell)$-tight graphs that arise from (k, ℓ)-rigid graphs in the algorithm of Section 6. In this section we show how to reduce the augmentation problem of any general $(\underline{m}, \ell)$-tight graph G satisfying (A0) to the augmentation problem of an $(\underline{m}, \ell)$-tight graph G' which satisfies the conditions (A0), (A1), (A2), and (A3). In some of our results we will also assume that $|V| \geq 4$. Note that it is straightforward to solve the augmentation problems for graphs on constant number of vertices.

Lemma 3.1. *Assume (A0), $\ell > 0$, and assume that $|V| \geq 4$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph such that (A1) does not hold, that is, there exists an isolated vertex $v \in V$ such that $m(v) = 0$. Let $\dot{H} = (V - v, F)$ be an optimal solution of the reduced augmentation problem on $G - v$. Then $H = (V, F)$ is an optimal solution of the reduced augmentation problem on G.*

Proof. As v is isolated, the edges in F generate every $e \in E$ thus H indeed augments G to redundantly rigid. We need to prove that for any graph $H' = (V, F')$ augmenting G to $(\underline{m}, \ell)$-redundant $|F'| \geq |F|$. For the sake of contradiction, let us suppose that

$|F'| < |F|$ and $d_{H'}(v)$ is as small as possible. There exists a vertex $u \in V - v$ such that $uv \in F'$ since otherwise $\dot{H}' = (V - v, F')$ also augments $G - v$ to $(\underline{m}, \ell)$-redundant, contradicting the optimality of $\dot{H}$. If V' is an $(\underline{m}, \ell)$-tight set in G and $v \in V'$, then $V' - v$ is also $(\underline{m}, \ell)$-tight. Thus for any $x \in V - v - u$, $V(ux) \cap (V - v) \supseteq V(uv) \cap (V - v)$ by Lemma 2.3. Thus, since $|V| \geq 4 > 2$, we can change the edge uv to ux in H', contradicting the minimality of $d_{H'}(v)$. □

Lemma 3.2. *Assume (A0), and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph such that (A2) does not hold, that is, $V(uv) = \{u, v\}$ for every $u, v \in V$. Then optimal solution of the reduced augmentation problem on G is the complete graph K_V on V.*

Proof. If (A2) does not hold, then there exists at least one edge between every pair $u, v \in V$ in G by (A0), the sparsity condition, and $\ell > 0$. By our assumption, $T(uv) = G[\{u, v\}]$ for every $u, v \in V$. Thus every added edge uv generates only the edges of G with endvertices u and v. Hence we need to add one edge between every vertex pair to make G $(\underline{m}, \ell)$-redundant. □

Lemma 3.3. *Assume (A0), and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph such that (A3) does not hold, that is, there exists a vertex $v \in V$ such that $V(uv) = \{u, v\}$ for every $u \in V - v$, and $V - v$ is an $(\underline{m}, \ell)$-tight set in G. Let $\dot{H} = (V - v, F)$ be an optimal solution of the reduced augmentation problem on $G - v$. Then $H = (V, F \cup \{uv : u \in V - v\})$ is an optimal solution of the reduced augmentation problem on G.*

Proof. As $\{u, v\}$ is an $(\underline{m}, \ell)$-tight set in G, there exists at least edge between u and v for every $u \in V - v$ by (A0), the sparsity condition, and $\ell > 0$. Adding the edge uv to G does not effect the redundancy of any edges other than the ones between u and v. On the other hand, by Lemma 2.3, no other edge can generate the edges between u and v since $V - v$ is an $(\underline{m}, \ell)$-tight set in G. Thus, for every optimal solution $H^* = (V, F^*)$ of the reduced augmentation problem on G, $\{uv : u \in V - v\} \subseteq F^*$. However, if we add the edges in $\{uv : u \in V - v\}$ to G, we only generate edges incident to v. Hence $\dot{H}^* = (V - v, F^* - (\{uv : u \in V - v\}$ must be an optimal solution of the reduced problem on the $(\underline{m}, \ell)$-tight graph $G - v$. □

It is easy to see that (A3) is equivalent to the following, when both $|V| \geq 3$ and (A0) hold.

(A3') There exists no $v \in V$ such that $V(uv) = \{u, v\}$ for all $u \in V - v$ and $d(v) = m(v)$.

It is easy to see by Theorem 2.6 that the proofs of the above lemmas give rise a polynomial algorithm to preprocess $G = (V, E)$. In section 5 we show that this algorithm needs $O(|V|^2)$ time.

4 The min-max theorem for the reduced problem

Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph. We call a set of vertices $\emptyset \neq C \subsetneq V$ **$(\underline{m}, \ell)$-co-tight** in G if $i(V - C) = \widetilde{m}(V - C) - \ell$, that is, the complement of C is a non-trivial $(\underline{m},\ell)$-tight set in G. Equivalently, C is $(\underline{m},\ell)$-co-tight if $|\widehat{E}(C)| = \widetilde{m}(C)$ where $\boldsymbol{\widehat{E}(C)}$ denotes the set of edges that are incident with at least one vertex in C. Let us denote $|\widehat{E}(C)|$ by $\boldsymbol{e(C)}$. Note that for every $X \subset V$ for which $\widetilde{m}(V-X) \geq \ell$, $e(X) \geq m(X)$ holds by the sparsity of $V - X$. For the sake of brevity let us denote the inclusion-wise minimal $(\underline{m},\ell)$-co-tight sets as **$(\underline{m},\ell)$-MCT** sets. We omit the $(\underline{m},\ell)$ prefix of the notions $(\underline{m},\ell)$-tight $(\underline{m},\ell)$-co-tight, and $(\underline{m},\ell)$-MCT when it is clear from the context.

Observation 4.1. *Assume (A0). Let $G = (V,E)$ be an $(\underline{m},\ell)$-tight graph and let $C \subset V$ be an $(\underline{m},\ell)$-co-tight set. If $G+H$ is $(\underline{m},\ell)$-redundant for an edge set H, then there exists an edge $uv \in H$ such that $u \in C$ or $v \in C$.*

Proof. For any edge e which is not incident with at least one vertex in C, $V(e) \subseteq V-C$ by Lemma 2.3. Hence our statement follows by Lemma 2.4. □

This observation immediately gives a lower bound for the cardinality of the optimal solution of the reduced problem.

Lemma 4.2. *Assume (A0). Let $G = (V,E)$ be an $(\underline{m},\ell)$-tight graph. The minimum number of edges that augment G to an $(\underline{m},\ell)$-redundant graph is at least the half of the maximal number of pairwise disjoint $(\underline{m},\ell)$-co-tight sets.* □

In this section we show that with some conditions the lower bound of Lemma 4.2 is tight. To prove this, we need some structural results. Remember, we say that a set U covers a set family $\mathcal{C}$, if $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$.

Lemma 4.3. *Assume (A0). Let $G = (V,E)$ be an $(\underline{m},\ell)$-tight graph and let $\mathcal{C}$ be the family of all $(\underline{m},\ell)$-MCT sets of G. Suppose that $U \subseteq V$ is a set that covers $\mathcal{C}$. If $V' \subseteq V$ such that $U \subseteq V'$ and V' induces an $(\underline{m},\ell)$-tight subgraph in G, then $V' = V$. In particular, if there are two vertices $u,v \in V$, for which $\{u,v\}$ covers $\mathcal{C}$, then $G+uv$ is $(\underline{m},\ell)$-redundant.*

Proof. Let us suppose that there exists a proper tight set $V' \subsetneq V$ in G for which $U \subseteq V'$. Then $V - V'$ is co-tight by definition and hence there exists an MCT $C \in \mathcal{C}$ such that $C \subseteq V - V'$. However, as $U \subset V'$, this contradicts the assumption that $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$. The second statement follows by the first one and the tightness of $V(uv) \ni u, v$. □

Now, we turn to prove that the MCT sets in G are pairwise disjoint except when there exists an edge e for which $G+e$ is redundant. First we prove that there are no intersecting MCT sets when $\ell \leq 0$.

Lemma 4.4. *Assume (A0) and $\ell \leq 0$. Let $G = (V,E)$ be an $(\underline{m},\ell)$-tight graph and let $\mathcal{C}$ be the family of all $(\underline{m},\ell)$-MCT sets of G. Then the members of $\mathcal{C}$ are pairwise disjoint, and either $|\mathcal{C}| \geq 3$ or there exists a pair $u,v \in V$ such that $E(uv) = E$.*

Proof. If $X, Y \in \mathcal{C}$ such that $X \cap Y \neq \emptyset$, then $X \cap Y$ is co-tight since the union $V - (X \cap Y)$ of the tight sets $V - X$ and $V - Y$ is tight by Lemma 2.1 and $\ell \leq 0$. However, this contradicts the minimality of X and Y. If $|\mathcal{C}| \leq 2$ then there exists a 2 element set $\{u,v\} \subset V$ covering $\mathcal{C}$ which implies $E(uv) = E$ by Lemma 4.3. □

Now we turn to prove the disjointness of MCT sets for the case when $\ell > 0$. In this case, we will need to assume that the conditions (A1), (A2), and (A3) hold for our graph G at some points of the proof. The following lemma shows that $|X \cup Y| \geq |V| - 1$ holds whenever X and Y are intersecting MCT sets.

Lemma 4.5. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph. If X and Y are two $(\underline{m}, \ell)$-MCT sets, such that $X \cap Y \neq \emptyset$, then $\widetilde{m}(V - (X \cup Y)) < \ell$. In particular, $|X \cup Y| \geq |V| - 1$.*

Proof. For the sake of contradiction, let us suppose that $\widetilde{m}(V - (X \cup Y)) \geq \ell$, which implies $e(X \cup Y) \geq \widetilde{m}(X \cup Y)$. As X and Y are MCT sets, $X \cap Y$ cannot be co-tight, that is, $e(X \cap Y) \geq \widetilde{m}(X \cap Y) + 1$. Hence $\widetilde{m}(X) + \widetilde{m}(Y) = e(X) + e(Y) = e(X \cap Y) + e(X \cup Y) + d(X - Y, Y - X) \geq \widetilde{m}(X \cup Y) + \widetilde{m}(X \cap Y) + 1 \geq \widetilde{m}(X) + \widetilde{m}(Y) + 1$, a contradiction. □

For a vertex $v \in V$ and a family $\mathcal{C}$, let $\boldsymbol{\mathcal{C}(v)} := \{C \in \mathcal{C} : v \notin C\}$. We use Lemma 4.5 to prove the following.

Lemma 4.6. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph and let $\mathcal{C}$ be the family of all $(\underline{m}, \ell)$-MCT sets of G. Assume that there exists two $(\underline{m}, \ell)$-MCT sets $X, Y \in \mathcal{C}$ such that $X \cap Y \neq \emptyset$ and $X \cup Y = V - v$ for some $v \in V$. Then $\mathcal{C}(v)$ is a co-partition of $V - v$ with $|\mathcal{C}(v)| \geq 3$ or there exists a vertex $u \in V - v$ such that $E(uv) = E$.*

Proof. Assume that there exists no vertex $u \in V - v$ such that $E(uv) = E$. Then $\mathcal{C}(v) \neq \{X, Y\}$ since otherwise the set formed by a vertex $u \in X \cap Y$ and v would cover $\mathcal{C}$ contradicting Lemma 4.3 and $E(uv) \neq E$. Hence $|\mathcal{C}(v)| \geq 3$ as $X, Y \in \mathcal{C}(v)$. Let $Z \in \mathcal{C}(v) - \{X, Y\}$. According to Lemma 4.5, $X \cup Z$ and $Y \cup Z$ both must be equal to $V - v = X \cup Y$. Thus $Z \supseteq (X - Y) \cup (Y - X)$. This implies also that every two members of $\mathcal{C}(v)$ are intersecting and hence, for every three members W_1, W_2, and W_3 of $\mathcal{C}(v)$, $W_3 \supseteq (W_1 - W_2) \cup (W_2 - W_1)$ holds. Therefore, every vertex in $V - v$ is avoided by at most one member of $\mathcal{C}(v)$. If there exists a vertex u that is contained in every member of $\mathcal{C}(v)$, then $\{u, v\}$ covers $\mathcal{C}$ contradicting Lemma 4.3 and $E(uv) \neq E$. Therefore, every vertex in $V - v$ is avoided by exactly one member of $\mathcal{C}(v)$, that is, $\mathcal{C}(v)$ is a co-partition of $V - v$. □

For a vertex $v \in V$ and a set $W \subseteq V - v$, let $\boldsymbol{\widetilde{W}^v} := V - v - W$. The following two observations are useful.

Observation 4.7. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph and let $v \in V$ be a vertex for which the family $\mathcal{C}(v)$ of $(\underline{m}, \ell)$-MCT sets avoiding v is a co-partition of $V - v$. Let $W_1, W_2 \in \mathcal{C}(v)$ and let $w_1 \in \widetilde{W}_1^v$ and $w_2 \in \widetilde{W}_2^v$. Suppose that $V' \subseteq V$ is an $(\underline{m}, \ell)$-tight set in G with $v, w_1, w_2 \in V'$. Then $V' = V$.*

Proof. Since $\mathcal{C}(v)$ is a co-partition and every MCT set which is not in $\mathcal{C}(v)$ is covered by v, $\{v, w_1, w_2\}$ covers the family of all MCT sets. Hence our statement follows by Lemma 4.3. □

Observation 4.8. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph. and let $v \in V$ be a vertex for which the family $\mathcal{C}(v)$ of $(\underline{m}, \ell)$-MCT sets avoiding v is a co-partition of $V - v$ with $|\mathcal{C}(v)| \geq 3$. Then $m(v) < \ell$.*

Proof. For the sake of contradiction, assume that $\ell \leq m(v)$. As W is MCT, $\widetilde{W}^v \cup v = V - W$ is tight in G for every $W \in \mathcal{C}(v)$, furthermore, there is no proper tight set in G containing $\widetilde{W}^v \cup v$. However, Lemma 2.1 and $\ell \leq m(v)$ imply that $\widetilde{X}^v \cup \widetilde{Y}^v \cup v$ is tight for each $X, Y \in \mathcal{C}(v)$. Since $|\mathcal{C}(v)| \geq 3$, $\widetilde{X}^v \cup \widetilde{Y}^v \cup v \neq V$, contradicting the fact that there is no proper tight set in G containing $\widetilde{X}^v \cup v$. □

Using the above two observations we prove the following.

Lemma 4.9. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph, and let $v \in V$ be a vertex for which the family $\mathcal{C}(v)$ of $(\underline{m}, \ell)$-MCT sets avoiding v is a co-partition of $V - v$ with $|\mathcal{C}(v)| \geq 3$. Suppose that there exists a vertex $u \in V - v$ with $m(u) \leq m(v)$. Let $W_1, W_2 \in \mathcal{C}(v)$ and let $w_1 \in \widetilde{W}_1^v$ and $w_2 \in \widetilde{W}_2^v$. Suppose that V' is an $(\underline{m}, \ell)$-tight set in G with $w_1, w_2 \in V'$. Then either $V' = V$ or $V' = \{w_1, w_2\}$. In particular, either $V(w_1w_2) = V$ (and $E(w_1w_2) = E$) or $V(w_1w_2) = \{w_1, w_2\}$.*

Proof. Assume that $V' \subsetneq V$ is a proper tight set in G such that $w_1, w_2 \in V'$ (for example, $V(w_1w_2)$ is such a set if $V(w_1w_2) \neq V$). Since $V' \neq V$, Observation 4.7 implies that $v \notin V'$. Suppose that $|V' \cap \widetilde{Z}^v| \geq 2$ for some $Z \in \mathcal{C}(v)$. In this case $V' \cup (V - Z) = V' \cup (\widetilde{Z}^v \cup v)$ is a tight set in G by Lemma 2.1. Hence $V' \cup (V - Z) = V$ by Observation 4.7. Lemma 2.1 also states that $d(v, Z) = 0$. This implies that $d(v, \widetilde{W}^v) = 0$ for each $W \in \mathcal{C}(v) - \{Z\}$. Note that $m(v) < \ell$ by Observation 4.8. Hence there are no loops on v in G. Thus, as $d(v, \widetilde{W}^v) = 0$ for each $W \in \mathcal{C}(v) - \{Z\}$, $m(v) = 0$ follows from the tightness of $\widetilde{W}^v \cup v$. By our assumption in the lemma, there exists a vertex $u \in V$ such that $m(u) \leq m(v)$. Consequently, $0 + 0 = m(u) + m(v) \geq \ell$ holds by (A0) and this contradicts our assumption of $0 < \ell$. Therefore, $|V' \cap \widetilde{Z}^v| \leq 1$ for every $Z \in \mathcal{C}(v)$.

Let us consider the complement sets of the members of $\mathcal{C}(v)$. V' intersects at least two of them: $\widetilde{W}_1^v \cup v$ and $\widetilde{W}_2^v \cup v$.

Claim 4.10. *V' intersects exactly two maximal tight sets containing v, namely, $\widetilde{W}_1^v \cup v$ and $\widetilde{W}_2^v \cup v$.*

Proof. Let us denote the family of maximal tight sets containing v by $\mathcal{F} = \{\widetilde{Z}^v \cup v$ where $Z \in \mathcal{C}(v)\}$. Suppose that V' intersects t members of $\mathcal{F}$, say $V_1, \ldots, V_t$. Since $|V' \cap \widetilde{Z}^v| \leq 1$ for every $Z \in \mathcal{C}(v)$ and $v \notin V'$, $|V'| = t$. Suppose that $t \geq 3$. Let E' denote the set of edges induced by V'.

As V' is $(\underline{m}, \ell)$-tight in G, $\widetilde{m}(V') - \ell = |E'|$. Since every pair of vertices in V' induces an $(\underline{m}, \ell)$-sparse subgraph in G, $|E'| \leq (t-1)\widetilde{m}(V') - \binom{t}{2}\ell$. This results $\widetilde{m}(V') - \ell \leq (t-1)\widetilde{m}(V') - \frac{t(t-1)}{2}\ell$. Hence $\frac{t(t-1)-2}{2}\frac{1}{t-2}\ell \leq \widetilde{m}(V')$, and thus $\frac{t+1}{2}\ell \leq \widetilde{m}(V')$.

Let E^* denote the union of E' and the set of edges induced by $V_1, \ldots, V_t$. Clearly, $m(V') - \ell + \widetilde{m}(V_1) - \ell + \cdots + \widetilde{m}(V_t) - \ell \leq |E^*|$. By the sparsity condition, $|E^*| \leq \widetilde{m}(V_1 \cup \cdots \cup V_t) - \ell$. As $\widetilde{m}(V_1 \cup \cdots \cup V_t) = \widetilde{m}(V_1) + \cdots + \widetilde{m}(V_t) - (t-1)m(v)$, we get

$t\ell \geq \widetilde{m}(V') + (t-1)m(v) \geq \frac{t+1}{2}\ell + (t-1)m(v)$. Thus $m(v) \leq \ell\frac{2t-t-1}{2(t-1)} = \frac{\ell}{2}$. By our condition in the lemma, there exists a vertex $u \in V - v$, for which $m(u) \leq m(v) \leq \frac{\ell}{2}$. Since $m(v) < \ell$ (by Observation 4.8), $m(u) = m(v) = \ell = 0$ cannot hold. Hence $m(u) + m(v) > \ell$ must hold by (A0), contradicting $m(u) \leq m(v) \leq \frac{\ell}{2}$. Therefore, $t = 2$. □

This finishes the proof of the lemma since we have seen that $|V' \cap \widetilde{W_1}^v| \leq 1$, and $|V' \cap \widetilde{W_2}^v| \leq 1$. □

Based on Lemma 4.9, we can prove the following.

Lemma 4.11. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph for which (A2) holds. Suppose that there exists a vertex $v \in V$ for which the family $\mathcal{C}(v)$ of $(\underline{m}, \ell)$-MCT sets avoiding v is a co-partition of $V - v$ with $|\mathcal{C}(v)| \geq 3$. Then $m(v) < m(u)$ holds for every $u \in V - v$ or there exist two vertices $x, y \in V - v$ such that $E(xy) = E$.*

Proof. For the sake of contradiction, suppose that there exists a vertex $u \in V - v$ with $m(u) \leq m(v)$.

Suppose first that there is an MCT set $Z \in \mathcal{C}(v)$ for which $|\widetilde{Z}^v| \geq 2$. Let $z_1, z_2 \in \widetilde{Z}^v$ and let us take a vertex $x \in Z$ such that $m(x)$ is not the unique minimum of m. (Note that such x must exist as $|\mathcal{C}(v)| \geq 3$ and $\mathcal{C}(v)$ is a co-partition of $V - v$.) By Lemma 4.9, either $V(xz_i) = \{x, z_i\}$, or $V(xz_i) = V$ and $E(xz_i) = E$ for $i = 1, 2$. Hence we can assume $V(xz_i) = \{x, z_i\}$ for both $i = 1, 2$ since otherwise $E(xy) = E$ holds for $y = z_i$. This means that $\{z_1, x\}$ and $\{z_2, x\}$ are proper tight sets in G. Moreover, Lemma 4.9 also implies that $\{z_1, x\}$ and $\{z_2, x\}$ are the only proper tight sets in G containing z_1 and w, and z_2 and w, respectively. Then $\{z_1, x\}$ and $\{z_2, x\}$ are maximal proper tight sets in G and their complements are MCT sets. Therefore, there are at least two MCT sets avoiding x which are intersecting. Hence the family $\mathcal{C}(x)$ of MCT sets avoiding x is a co-partition of $V - x$ or there exists a vertex $y \in V - x$ such that $E(xy) = E$ by Lemma 4.6. Note that in the latter case $y \neq v$ since there exists an MCT set in the co-partition $\mathcal{C}(v)$ of $V - v$ avoiding x and v, hence in this case we are done. In the first case, note that z_1 and z_2 are avoided by different members of $\mathcal{C}(w)$ since $V - \{z_1, w\}, V - \{z_2, w\} \in \mathcal{C}(w)$. Thus Lemma 4.9 for w assures that the only proper tight set in G containing z_1 and z_2 is $\{z_1, z_2\}$ which contradicts the tightness of $\widetilde{Z}^v \cup v$.

Finally, suppose that, $|\widetilde{Z}^v| = 1$ for each co-tight set $Z \in \mathcal{C}(v)$. This implies that $V(uv) = \{u, v\}$ for every $u \in V - v$. By (A2), there must exist a pair $x, y \in V - v$ for which $V(xy) \neq \{x, y\}$. Lemma 4.9 implies now that $V(xy) = V$ and $E(xy) = E$. □

We can now improve the statement of Lemma 4.11 by assuming (A1) and (A3), as follows.

Lemma 4.12. *Assume (A0) and $\ell > 0$. Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph for which (A1), (A2), and (A3) holds. Suppose that there exists a vertex $v \in V$ for which the family $\mathcal{C}(v)$ of $(\underline{m}, \ell)$-MCT sets avoiding v is a co-partition of $V - v$ with $|\mathcal{C}(v)| \geq 3$. Then there exists a pair of vertices $x, y \in V - v$ such that $E(xy) = E$.*

Proof. By Lemma 4.11, we only need to prove the case where $m(v) < m(u)$ hold for every $u \in V - v$. Furthermore, in this case the following claim also follows by Lemma 4.11.

Claim 4.13. *Assume that there exist two intersecting MCT sets X and Y for which $X \cup Y = V - u$ for some $u \in V - v$. Then there exists a pair of vertices $x, y \in V - v$ for which $E(xy) = E$.*

Proof. Note that $m(v) < m(u)$ by our assumption and there exists no vertex $w \in V - v$ for which $E(vw) = E$ since $\mathcal{C}(v)$ is a co-partition. Hence our statement follows by Lemmas 4.6 and 4.11. □

We can also suppose that there exists an MCT set Z containing v since otherwise the family $\mathcal{C}$ of all MCT sets is equal to $\mathcal{C}(v)$ which is a co-partition; hence it could be covered by a set $\{x, y\} \subseteq V - v$ implying $E(xy) = E$ by Lemma 4.3. Then, for every $X \in \mathcal{C}(v)$ intersecting Z, $X \cup Z = V$ holds by Claim 4.13. Thus $Z \neq V$ cannot intersect any member of $\mathcal{C}(v)$ since $\mathcal{C}(v)$ is a co-partition. Hence Z must be the singleton v. Therefore, $V - v$ is tight and $d_G(v) = m(v)$.

As $V - v$ is tight, (A3) implies that there exists a vertex $u \in V - v$ such that $V(uv) \neq \{u, v\}$, that is, $\{u, v\}$ is not tight. Let W_1 be the member of the co-partition $\mathcal{C}(v)$ of $V - v$ which does not contain u. Since $\{u, v\}$ is not tight and W_1 is a co-tight set in G with $u, v \notin W_1$, $|V - W_1| \geq 3$. Now, by using the tightness of $V - W_1$ and the sparsity of $V - W_1 - v$ combined with (A0) and $|V - W_1| \geq 3$, we get $\widetilde{m}(V - W_1) - \ell = i(V - W_1) = i(V - W_1 - v) + d_{G[V-W_1]}(v) \leq \widetilde{m}(V - W_1 - v) - \ell + d_{G[V-W_1]}(v)$, that is, $m(v) \leq d_{G[V-W_1]}(v)$.

Let $W_2 \in \mathcal{C}(v) - \{W_1\}$. The tightness of $V - v$ and condition (A1) implies that $m(v) > 0$ and hence $m(u) > m(v) > 0$ for every $u \in V$. Thus Lemma 2.2, $\ell > 0$, and the tightness of $G[V - W_2]$ imply that $G[V - W_2]$ is connected, and hence $d_G(v, (V - W_2) - v) > 0$. However, now we have $m(v) = d_G(v) \geq d_{G[V-W_1]}(v) + d_G(v, (V - W_2) - v) > m(v)$, a contradiction. □

Now we are ready to prove the following structural result on the MCT sets.

Lemma 4.14. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Let $\mathcal{C}$ be the family of all $(\underline{m}, \ell)$-MCT sets of G. Suppose that there exists no pair $u, v \in V$ such that $E(uv) = E$. Then the members of $\mathcal{C}$ are pairwise disjoint and $|\mathcal{C}| \geq 3$.*

Proof. By Lemma 4.4, it is enough to prove the statement when $\ell > 0$ and hence (A1), (A2), and (A3) hold.

If $|\mathcal{C}| \leq 2$ then there exists a 2 element set $\{u, v\} \subset V$ covering $\mathcal{C}$ which implies $E(uv) = E$ by Lemma 4.3, a contradiction. Hence the second statement of the Lemma follows easily.

For the sake of contradiction, let us suppose that there exists $X, Y \in \mathcal{C}$ such that $X \cap Y \neq \emptyset$. By Lemma 4.5, $|X \cup Y| \geq |V| - 1$ holds. According to Lemmas 4.6 and 4.12, $X \cup Y = V$ must hold since otherwise there would exist a pair $u, v \in V$ such that $E(uv) = E$ contradicting our assumption. By the minimality of MCT sets, every member of $\mathcal{C} - \{X, Y\}$ must contain at least one element of both of $V - Y = X - Y$ and $V - X = Y - X$. Hence each member of $W \in \mathcal{C} - \{X\}$ intersects X. Thus, again by Lemmas 4.6 and 4.12, $X \cup W = V$ for every $W \in \mathcal{C} - \{X\}$, that is, $V - X \subset W$. Let us take $u \in X$ and $v \in V - X$. Then $\{u, v\}$ covers $\mathcal{C}$ and hence $E(uv) = E$ by Lemma 4.3, a contradiction. □

Note that it can be checked in polynomial time whether $G + uv$ is $(\underline{m}, \ell)$-redundant for some pair $u, v \in V$. The naïve algorithm (which can be constructed from the algorithm of Theorem 2.6) has $O(|V|^3)$ running time. In Section 5, we give a rather complex algorithm deciding this problem based on the above proof that has $O(|V|^2)$ running time.

From now on, we deal with the case where all the MCT sets of G are disjoint. We prove that the disjointness of the MCT sets also imply that no edge of G connects two different MCT sets. We first claim the following.

Lemma 4.15. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph and let X and Y be disjoint $(\underline{m}, \ell)$-co-tight sets in G such that $\widetilde{m}(V - (X \cup Y)) \geq \ell$. Then $\widehat{E}(X) \cap \widehat{E}(Y) = \emptyset$.*

Proof. By $(\underline{m}, \ell)$-sparsity,

$$i(V - (X \cup Y)) \leq \widetilde{m}(V - (X \cup Y)) - \ell = \widetilde{m}(V) - \widetilde{m}(X) - \widetilde{m}(Y) - \ell$$

On the other hand, by the co-tightness of X and Y,

$$i(V - (X \cup Y)) = \widetilde{m}(V) - \ell - |\widehat{E}(X) \cup \widehat{E}(Y)| = \widetilde{m}(V) - \widetilde{m}(X) - \widetilde{m}(Y) - \ell + |\widehat{E}(X) \cap \widehat{E}(Y)|.$$

Hence the first inequality implies

$$\widetilde{m}(V) - \widetilde{m}(X) - \widetilde{m}(Y) - \ell + |\widehat{E}(X) \cap \widehat{E}(Y)| \leq \widetilde{m}(V) - \widetilde{m}(X) - \widetilde{m}(Y) - \ell$$

which can only hold when $\widehat{E}(X) \cap \widehat{E}(Y) = \emptyset$. □

This leads to the following lemma.

Lemma 4.16. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Suppose that there exists no pair $u, v \in V$ such that $E(uv) = E$. If X and Y are $(\underline{m}, \ell)$-MCT sets in G, then $\widehat{E}(X) \cap \widehat{E}(Y) = \emptyset$.*

Proof. By Lemma 4.14, we may suppose that the MCT sets in G are pairwise disjoint and there exist at least three of them.

If $\widetilde{m}(V - (X \cup Y)) \geq \ell$, then the statement follows by Lemma 4.15. Hence we may suppose that $\widetilde{m}(V - (X \cup Y)) < \ell$ (and hence $\ell > 0$). In this case $|V - (X \cup Y)| \leq 1$ holds by (A0). $V \neq X \cup Y$ since there exist at least three MCT sets. Therefore, $V - (X \cup Y) = v$ for some $v \in V$. Then v is a co-tight set on its own as there are at least 3 disjoint MCT sets in G. If both $\widetilde{m}(X) \geq \ell$ and $\widetilde{m}(Y) \geq \ell$ hold (which follows by (A0) when both of them contain at least two vertices), then we can use Lemma

4.15 on v and Y, and on v and X, respectively, to conclude that v is separated from X and Y, respectively, and thus v is isolated. Hence $m(v) = 0$ follows by Lemma 2.2 and $\ell > 0$, contradicting (A1).

If, say, Y contain only one vertex u, then $|X| \geq 2$ by $|V| \geq 4$. Hence Lemma 4.15 asserts for u and v that there is no edge between u and v. Since X is a co-tight set, $\{u, v\}$ is a tight set in G. As u and v are co-tight singleton sets, $m(u) \neq 0$ and $m(v) \neq 0$ by (A1). Hence the tight set $\{u, v\}$ must induce at least one edge in G by (A0), contradicting the non-existence of any edge between u and v. □

Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a family of sets on the ground set V. A set $X = \{x_1, \ldots, x_t\} \subseteq V$ is called a **transversal** of $\mathcal{C}$ if $x_i \in C_i$ for each $i \in \{1, \ldots, t\}$. We use now Lemma 4.16 to prove the following.

Lemma 4.17. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Suppose that there exists no pair $u, v \in V$ such that $E(uv) = E$. Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be the family of MCT sets in G. Let $\{x_1, \ldots, x_t\}$ be a transversal of $\mathcal{C}$ where $x_i \in C_i$ for $i \in \{1, \ldots, t\}$. Then $V(x_i x_j) \supseteq C_i \cup N(C_i) \cup C_j \cup N(C_j)$ for each $i, j \in \{1, \ldots, t\}$ with $i \neq j$.*

Proof. Note that the members of $\mathcal{C}$ are pairwise disjoint by Lemma 4.14, moreover, $C_i \cap (C_j \cup N(C_j)) = \emptyset$ by Lemma 4.16. We claim the following.

Claim 4.18. $\widetilde{m}((V - C_i) \cap V(x_i x_j)) \geq \ell$.

Proof. If $\ell \leq 0$ then the statement is obvious by $m \geq 0$. Hence we may suppose that $\ell > 0$. If $m(v) > 0$ for every $v \in V$, then $T(x_i x_j)$ is connected by Lemma 2.2, and hence $|V - C_i \cap V(x_i x_j)| \geq 2$ as no edge of G connects $x_j \in C_j$ to C_i. Thus our statement follows by (A0) in this case.

Assume now that $m(v) = 0$ for a $v \in V$. By (A1), v cannot be a co-tight set hence $|C_q| \geq 2$ for any C_q where $v \in C_q \in \mathcal{C}$. It follows that $C_q - v$ is co-tight in this case since $e(C_q - v) \leq e(C_q) = \widetilde{m}(C_q) = \widetilde{m}(C_q - v) \leq e(C_q - v)$ by the co-tightness of C_q and $m(v) = 0$. However, this contradicts the minimality of C_q. Hence $v \notin C_q$ for any $q \in \{1, \ldots, t\}$, in particular, $v \neq x_j$. Thus $m(x_j) \geq \ell$ by (A0). Therefore, $\widetilde{m}((V - C_i) \cap V(x_i x_j)) \geq m(x_j) \geq \ell$ follows also in this case. □

Since both $V - C_i$ and $V(x_i x_j)$ are tight sets in G, $(V - C_i) \cup V(x_i x_j)$ is tight as well by Claim 4.18 and Lemma 2.1. This implies that $C_i - V(x_i x_j)$ is co-tight or empty. As C_i is an MCT set, $C_i - V(x_i x_j)$ must be empty, that is, $C_i \subset V(x_i x_j)$. Suppose now that there is at least one edge $e \in \widehat{E}(C_i)$ which leaves $V(x_i x_j)$. This assumption along with the tightness of $V(x_i x_j)$ and the co-tightness of C_i implies

$$
\begin{aligned}
i(V(x_i x_j) - C_i) &\geq i(V(x_i x_j)) - (e(C_i) - 1) = \\
= \widetilde{m}(V(x_i x_j)) - \ell - (\widetilde{m}(C_i) - 1) &= \widetilde{m}(V(x_i x_j) - C_i) - \ell + 1
\end{aligned}
$$

which contradicts the sparsity condition on $V(x_i x_j) - C_i$ by Claim 4.18. Therefore, we can conclude that $N(C_i) \subset V(x_i x_j)$.

Note that $C_j \cup N(C_j) \subseteq V(x_i x_j)$ follows from the same proof by changing the indices i and j. □

Based on Lemmas 2.1 and 4.17 we can construct a (not yet optimal) edge set, that augments G to an $(\underline{m},\ell)$-redundant graph, as follows.

Lemma 4.19. *Assume (A0). Let $G=(V,E)$ be an $(\underline{m},\ell)$-tight graph on at least 4 vertices such that $\ell \le 0$ or (A1), (A2) and (A3) hold. Suppose that there exists no pair $u,v \in V$ such that $E(uv)=E$. Let $\mathcal{C}=\{C_1,\dots,C_t\}$ be the family of $(\underline{m},\ell)$-MCT sets in G. Let $\{x_1,\dots,x_t\}$ be a transversal of $\mathcal{C}$. If H is a connected edge set on the vertices $\{x_1,\dots,x_t\}$, then $G+H$ is $(\underline{m},\ell)$-redundant.*

Proof. Lemma 4.16 claims that the edges of H are disjoint from that of G. Choose three vertices from $\{x_1,\dots,x_t\}$, say $x_i, x_j, x_{j'}$.

Claim 4.20. *$T(x_ix_j)\cup T(x_ix_{j'})$ is tight.*

Proof. If $\ell > 0$, then we can use Lemmas 4.17 and 2.1 to deduce our statement. If $\ell \le 0$, then the statement follows directly by Lemma 2.1. □

By Lemma 4.14 we know that $|\{x_1,\dots,x_t\}| > 2$ and thus $|H| > 1$. Choose an arbitrary edge $e \in H$ and let $H^* = \{e\}$. Recall that $R(H*)$ denotes the graph spanned by the $(\underline{m},\ell)$-redundant edges of G in $G+H^*$. Let us add edges from H to H^*, such that H^* always spans a connected graph. As H spans a connected graph, such an order exists. The order of addition of the edges does not affect the set of the generated edges according to Lemma 2.4. We claim the following.

Claim 4.21. *During this process $R(H^*)$ is an $(\underline{m},\ell)$-tight subgraph of G.*

Proof. Let us use induction on $|H^*|$. If $|H^*|=1$, $R(H^*)=T(e)$ by Lemma 2.4. Suppose now that $R(H^*)$ is proved to be $(\underline{m},\ell)$-tight, and we want to add $e'=x_ix_j$ to H^*. $R(H^*+e')=R(H^*)\cup T(e')$ by Lemma 2.4. If both the vertices of e' are spanned by $R(H^*)$, then $T(e')\subseteq R(H^*)$ by the tightness of $R(H^*)$ and Lemma 2.3, and thus the statement still holds for H^*+e'. Otherwise, only one endvertex of e', say, x_j is spanned by H^*. As x_j is spanned by H^*, there exists an index $j' \in \{1,\dots,t\}$ such that $x_jx_{j'} \in H^*$. By Lemma 2.4, $T(x_jx_{j'})$ is a subgraph of $R(H^*)$. Hence Claim 4.20 and Lemma 2.1 assures us that $R(H^*)\cup T(e)$ is also tight. □

The above claim implies now that $R(H)$ is an $(\underline{m},\ell)$-tight subgraph of G that spans each of $x_1,\dots,x_t$. Since $\{x_1,\dots,x_t\}$ is a transversal of $\mathcal{C}$, it covers $\mathcal{C}$. Hence $R(H)=G$ follows by Lemma 4.3. This finishes the proof of Lemma 4.19. □

The cardinality of the edge set provided by Lemma 4.19 can be decreased by using the following statement.

Lemma 4.22. *Assume (A0). Let $G=(V,E)$ be an $(\underline{m},\ell)$-tight graph on at least 4 vertices such that $\ell \le 0$ or (A1), (A2) and (A3) hold. Suppose that there exists no pair $u,v \in V$ such that $E(uv)=E$. Let $\mathcal{C}=\{C_1,\dots,C_t\}$ be the family of at least 4 $(\underline{m},\ell)$-MCT sets in G. Let $\{x_1,\dots,x_t\}$ be a transversal of $\mathcal{C}$ and let x_i,x_j,x_k,y be distinct vertices from this transversal set. Let $T^*=T(yx_i)\cup T(yx_j)\cup T(yx_k)$. Then $T^*=T(yx_i)\cup T(x_jx_k)$ or $T^*=T(yx_j)\cup T(x_ix_k)$ holds.*

Proof. Let $T^* = (V^*, E^*)$. Let us suppose that $T^* \neq T(yx_i) \cup T(x_jx_k)$. Thus there exists an edge e, for which $e \in E^*$ and $e \notin E(yx_i) \cup E(x_jx_k)$.

Claim 4.23. *The $(\underline{m}, \ell)$-MCT sets of T^* are exactly the four $(\underline{m}, \ell)$-MCT sets of G containing y, x_i, x_j, and x_k, respectively.*

Proof. Let first C be an MCT set of G containing x_i, x_j, x_k, or y. As $\widetilde{m}(C) = e_G(C) \geq e_{T^*}(C)$ we can conclude that C is also co-tight in T^*. Note that $N(C) \subset V^*$ by Lemma 4.17, hence $k|C'| < e_G(C') = e_{T^*}(C')$ for each $\emptyset \neq C' \subsetneq C$. Thus C is MCT in T^*. Therefore, the four $(\underline{m}, \ell)$-MCT sets of G containing y, x_i, x_j, and x_k, respectively, are MCT sets in T^*. On the other hand, $T^* = T^G(yx_i) \cup T^G(yx_j) \cup T^G(yx_k) = T^{T^*}(yx_i) \cup T^{T^*}(yx_j) \cup T^{T^*}(yx_k)$. Hence there are no other MCT sets in T^* by Observation 4.1. □

By Lemma 4.19, $T^* = T(yx_i) \cup T(x_jx_k) \cup T(yx_j)$ (and also $T^* = T(yx_i) \cup T(x_jx_k) \cup T(x_ix_k)$). As a consequence, $e \in E(yx_j)$ and also $e \in E(x_ix_k)$. Hence $T(yx_j) \cup T(x_ix_k)$ is tight by Lemma 2.1, and thus $T(yx_i) \subseteq T(yx_j) \cup T(x_ix_k)$ by Lemma 2.3. Therefore, $T^* = T(yx_i) \cup T(x_jx_k) \cup T(yx_j) = T(yx_j) \cup T(x_ix_k)$. (We note that in this case $T^* = T(yx_k) \cup T(x_jx_k)$ also follows by a similar proof.) □

This finally allows us to prove that the lower bound in Lemma 4.2 can be reached in certain cases.

Theorem 4.24. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Then*

$$\min\{|F| : H = (V, F) \text{ is a graph for which } G + H \text{ is } (\underline{m}, \ell)\text{-redundant}\} =$$
$$= \max(1, \max\{\lceil \tfrac{|\mathcal{C}|}{2} \rceil : \mathcal{C} \text{ is a family of disjoint co-tight sets}\}).$$

Proof. Since G is not $(\underline{m}, \ell)$-redundant, one edge is always needed. Hence max $\leq$ min always holds by Lemma 4.2. Therefore, we only need to show a graph H for which equality holds.

If G cannot be augmented to an $(\underline{m}, \ell)$-redundant graph with only one edge, then the MCT sets are disjoint by Lemma 4.14. Let us denote a transversal set of the family of MCT sets by X. For an arbitrary $y \in X$, let $F' := \{yx : x \in X - y\}$, and let $H_0 = (V, F_0)$. By Lemma 4.19 $G + H_0$ is redundant. While there are at least three edges in H_i that are incident with y, we can decrease the number of edges in H_i and also the edges incident with y by Lemma 4.22 so that the arising graph $H_i = (V, F_i)$ still augments G to an $(\underline{m}, \ell)$-redundant graph. We can repeat this until the degree of y is at most two and the degree of every other vertex is at most one in the final graph $H = (V, F)$. Thus $|F| = \lceil \frac{|X|}{2} \rceil$ and $G + H$ is $(\underline{m}, \ell)$-redundant. □

We conclude this section by proving one more lemma that shows a direct link between the results of this paper and the paper of García and Tejel [7]. We will also use this result in the proof of the NP-hardness result in Section 7.

Lemma 4.25. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Assume that there is no edge uv that augments G to an $(\underline{m}, \ell)$-redundant graph. Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be the family of*

$(\underline{m}, \ell)$-MCT sets of G. Then the edge uv generates a subgraph that is inclusion-wise maximal amongst the generated subgraphs if and only if u and v are elements of two disjoint $(\underline{m}, \ell)$-MCT sets. Moreover, two inclusion-wise maximal generated subgraphs $T(uv_1)$ and $T(uv_2)$ are equal if and only if v_1, v_2 are in the same $(\underline{m}, \ell)$-MCT set.

Proof. By Lemma 4.14, all the MCT sets of G are pairwise disjoint. Assume first that $T(uv)$ is an inclusion-wise maximal generated subgraph of G for some $u, v \in V$. Let $\{x_1, \ldots, x_t\}$ be a transversal of $\mathcal{C}$. By Lemma 4.19, $G = T(x_1x_2) \cup \ldots T(x_1x_t)$ hence we can assume that $u \in V(x_1x_2)$ and $v \in V(x_1x_2) \cup V(x_1x_3)$. On the other hand, $T(x_1x_2) \cup T(x_1x_3) = T(x_1x_2) \cup T(x_2x_3) = T(x_1x_3) \cup T(x_2x_3)$ by Claim 4.20 and Lemma 2.3. This means that one of the above three generated tight subgraphs, say $T(x_1x_2)$, must contain u and v both thus $V(uv) \subseteq V(x_1x_2)$ by Lemma 2.3. Since $T(uv)$ is inclusion-wise maximal equality must hold. Note that $V - C_1$ and $V - C_2$ are tight sets in G by the co-tightness of C_1 and C_2 and hence $V(uv)$ does not contain C_1 (or C_2, respectively) if $\{u, v\} \cap C_1 = \emptyset$ (or $\{u, v\} \cap C_2 = \emptyset$, respectively). On the other hand, $C_1 \cup C_2 \subseteq V(x_1x_2) = V(uv)$ by Lemma 4.17. Therefore, either $u \in C_1$ and $v \in C_2$, or $u \in C_2$ and $v \in C_1$ must hold.

It is also clear that taking any element x of C_1 and any element y of C_2 the generated tight set $T(xy)$ will be the same by Lemma 4.17 and Lemma 2.3. Finally, we need to show that $T(y_iy_j)$ is an inclusion-wise maximal generated subgraph of G for every $y_i \in C_i$, $y_j \in C_j$, and $1 \leq i < j \leq t$. This follows by the fact that, for a member $C \in \mathcal{C}$, $C \subseteq T(y_iy_j)$ if and only if $C =$ either C_i or C_j since otherwise y_i and y_j are elements of the tight set $V - C$ and hence $T(y_iy_j) \subseteq V - C$ by Lemma 2.3. □

5 The algorithm for the reduced problem

Recall from Section 2.1 that we assume (*), that is, there exists a constant $c > 0$ such that $m(v) \leq c$ for every $v \in V$ and $|\ell| \leq c$. All the running times are given with this assumption. Our goal is to show that the reduced augmentation problem can be solved in $O(|V|^2)$ time. We first show that the preprocessing can be done under this time bound.

Lemma 5.1. *Assume (A0), $\ell > 0$, and (*). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph with $|V| \geq 4$. We can determine whether G violates (A1), (A2) or (A3) and we can construct a graph G' such that either G' does not violate any of (A1), (A2) and (A3) or G' has less than three vertices, and from the optimal solution of the reduced augmentation problem on G' we can compute the optimal solution of the reduced augmentation problem on G. All these computations can be done in $O(|V|^2)$ running time.*

Proof. Note that $m(v) = 0$ can only hold for at most one vertex v by (A0) and $\ell > 0$. For checking (A1), it is sufficient to check if $m(v) = 0$ and v is isolated for every vertex $v \in V$, which obviously can be done in $O(|V|)$ time (since the neighborhood should only be checked for the at most one vertex v with $m(v) = 0$. By Lemma 3.1 an optimal solution $\dot{H}$ of the reduced augmentation problem on $G' = (V - v, E)$ is also an optimal solution of the original problem.

(A2) can be checked, as follows. For every vertex-pair $\{u, v\}$, we need to check if it induces a tight subgraph. This can be done by counting all the edges connecting u and v and the loops incident with u or v. This all can be done in constant time by $(\underline{m}, \ell)$-sparsity and (*). Thus Lemma 3.2 claims that we can build the optimal H set in $O(|V|^2)$ time.

Similarly to the previous case, we can find all the vertices $v \in V$ such that $V(uv) = \{u, v\}$ for every $u \in V - v$ in $O(|V|^2)$ time. Let us denote the set of these vertices by S. With an $O(|E|) = O(|V|)$ preprocessing we can tell the degree of any vertex in $O(1)$, thus we can decide if any vertex $v \in V$ (more precisely $v \in S$) violates (A3') (which is equivalent with (A3)) in $O(|V|)$ extra time. However, if there exists such a vertex v, and we consider the reduced augmentation problem on $G - v$, any other vertex u violating (A3) in the graph $G - v$ is already in S, as $V(uv) = \{u, v\}$ and $V(uw) = \{u, w\}$ for all $w \in V - v - u$ implies $V(ux) = \{u, x\}$ for every $x \in V - u$. Hence, if there exists such a $u \in V - v$, $u \in S$ thus we can find the next vertex contradicting (A3) in the graph $G - v$ in $O(|V|)$. Repeating this step while we can (if our graph still has at least four vertices), we can generate a $G' = (V', E')$ graph, that satisfies the (A3) condition in $(O|V|^2)$. By Lemma 3.3 we conclude that from any optimal H' solution of the reduced problem on G' we can get an optimal solution to the reduced problem of G in $O(|V|^2)$ simply adding all edges that has at least one endvertex in $V - V'$.

Note that, after checking (A3), we need to check whether the vertex with $m(v) = 0$ got isolated and if this is the case, then we should check (A2) and (A3) again (if our graph still has at least four vertices). However, since $m(v) = 0$ can only hold for at most one vertex v, after doing this we can stop. This implies that the total running time is $O(|V|^2)$. □

By Section 4, the MCT sets of G play an important role in giving a solution to the reduced problem. To give an efficient algorithm for finding MCT sets, we need the following simple statement.

Lemma 5.2. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph. Suppose that $T \subsetneq V$ is an $(\underline{m}, \ell)$-tight set in G. Let t' be the new vertex that arises after shrinking T in G. Let $m' : V(G/T) \to \mathbb{Z}_+$ be a map such that $m'(v) := m(v)$ when $v \in V(G/T) \cap V$ and $m'(t') := \max(\ell, 0)$ and let $\ell' = \max(\ell, 0)$. Then S is an $(\underline{m}', \ell')$-tight set in G/T such that $t' \in S$ if and only if $S - t' \cup T$ is $(\underline{m}, \ell)$-tight in G. Furthermore, if $\ell \geq 0$, then $S \subseteq V(G/T) - t'$ is an $(\underline{m}', \ell')$-tight set in G/T if and only if S is an $(\underline{m}, \ell)$-tight set in G.*

Proof. First we show that G/T is $(\underline{m}', \ell')$-sparse, that is, $i_{G/T}(X) \leq \widetilde{m}'(X) - \ell'$ holds for every $X \subseteq V(G/T)$. Let $X \subseteq V(G/T)$. Assume first that $t' \in X$. Then $i_{G/T}(X) = i_G(X - t' \cup T) - i_G(T) \leq \widetilde{m}(X - t' \cup T) - \ell - (\widetilde{m}(T) - \ell) = \widetilde{m}(X - t') = \widetilde{m}'(X) - m'(t') = \widetilde{m}'(X) - \ell'$ as T is $(\underline{m}, \ell)$-tight. If $t' \notin X$ and $\ell \geq 0$, then $i_{G/T}(X) = i_G(X) \leq \widetilde{m}(X) - \ell = \widetilde{m}'(X) - \ell'$. If $t' \notin X$ and $\ell < 0$, then $i_{G/T}(X) = i_G(X) \leq i_G(X \cup T) - i_G(T) \leq \widetilde{m}(X \cup T) - \ell - (\widetilde{m}(T) - \ell) = \widetilde{m}(X) = \widetilde{m}'(X) - \ell'$.

Assume now $t' \in S$. By the $(\underline{m}', \ell')$-sparsity of G/T, the tightness of S means that $i_{G/T}(S) = \widetilde{m}'(S) - \ell' = \widetilde{m}(S - t')$. Since $i_G(T) = \widetilde{m}(T) - \ell$, replacing t' with T results $i_G(S - t' \cup T) = \widetilde{m}(S - t' \cup T) - \ell$. The same argument stands for the other direction.

Finally assume that $\ell \geq 0$ and $t' \notin S$. By the $(\underline{m}', \ell')$-sparsity of G/T the $(\underline{m}', \ell')$-tightness of S means that $i_{G/T}(S) = \widetilde{m}'(S) - \ell' = \widetilde{m}(S) - \ell$. On the other hand, the $(\underline{m}, \ell)$-tightness of S means that $i_G(S) = \widetilde{m}(S) - \ell$. Since $i_G(S) = i_{G/T}(S)$, these two are equivalent. □

We will also need the following observation.

Observation 5.3. *If D is the orientation in the output of the algorithm of Theorem 2.6 for checking the $(\underline{m}, \ell)$-sparsity of the graph G in Lemma 5.2, then D/T can be an output of that algorithm when we check the $(\underline{m}', \ell')$-sparsity of G/T, that is, the in-degree of each vertex x in D/T is at most $m'(x)$.*

Proof. Since T is $(\underline{m}, \ell)$-tight, T induces $\widetilde{m}(T) - \ell$ edges in G, that is, T induces $\widetilde{m}(T) - \ell'$ arcs in D. Hence the in-degree of T is at most ℓ' in D which implies our statement. □

Lemma 5.2 gives the idea to use the following algorithm to find an MCT set.

Algorithm 5.4. Input: *$m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ such that (A0) and (*) hold, an $(\underline{m}, \ell)$-tight graph $G = (V, E)$ on at least 4 vertices, and two vertices $u, v \in V$.*
Output: *An $(\underline{m}, \ell)$-MCT set in G that does not contain u, v or an edge e such that $E(e) = E$.*

0. *Run the algorithm of Theorem 2.6 on G, let D be the output orientation.*
1. *Using the output of* Step *0, calculate $T := V^G_{(\underline{m},\ell)}(uv)$.*
2. ***If*** *$T = V$* ***then Output:*** *the edge uv,* ***STOP****.*
3. *Shrink T to t' according to Lemma 5.2, that is, $G' := G/T$, $D' := D/T$, $\ell' := \max(\ell, 0)$, $m'(u) := m(u)$ for each $u \in V(G') \cap V$, $m'(t') := \ell'$.*
4. *$v := t'$ (hence $m'(v) = \ell'$), $V^* := V(G') - v$.*
5. ***While*** *$V^* \neq \emptyset$,* ***do****:*
6. *Calculate $T' := V^{G'}_{(\underline{m}',\ell')}(uv)$ using D'.*
7. ***If*** *$V' = V(G')$,* ***then*** *$V^* := V^* - u$.*
8. ***Else****,*
Shrink T' to t', so $G' := G'/T', D' := D'/T'$.
9. *$v := t'$, $V^* := V^* \cap V(G') - v$.*
10. ***Output:*** *$V(G') - v$.*

Remark 5.5. If G is a graph such that the algorithm of Theorem 2.6 was already executed on G and we have D, then Step 0 may be omitted.

Lemma 5.6. *The output of Algorithm 5.4 is either an edge e for which $E(e) = E$ or an $(\underline{m}, \ell)$-MCT set of G not containing u and v. The running time of Algorithm 5.4 is $O(|V|^2)$.*

Proof. If the output is an edge, then $T^G_{(\underline{m},\ell)}(uv) = G$. For the other case we first prove the following.

Claim 5.7. *If $u \in V(G') - V^* - v$ in any state of the Algorithm, then $T^{G'}_{(\underline{m}',\ell')}(uv) = G'$.*

Proof. The statement is obvious when we delete u from V^*. In any later state the statement follows by Lemma 5.2. $\square$

Suppose now, that the output of Algorithm 5.4 is a vertex set. Applying Lemma 5.2 repeatedly, we can conclude that the original vertex set $U \subset V$ which is contracted during the algorithm (that is, for which $G' = G/U$), is an $(\underline{m}, \ell)$-tight set in G. Thus the output of Algorithm 5.4 is an $(\underline{m}, \ell)$-co-tight set.

Suppose that U is not an inclusion-wise maximal $(\underline{m}, \ell)$-tight set, that is, there exists a proper $(\underline{m}, \ell)$-tight set $T^* \supsetneq U$ in G. Take the image of T^* in the final G', denote it with $T^*_{G'}$. By Lemma 5.2, $T^*_{G'}$ is tight in G', and clearly $v \in T^*_{G'}$. Let $u \in T^*_{G'} - v$. Since $V^* = \emptyset$, $u \in V(G') - V^* - v$ also holds. Hence, by Lemma 2.3 and Claim 5.7 $V(G') \neq T^*_{G'} \supseteq V(T^{G'}_{(\underline{m}', \ell')}(uv)) = V(G')$, a contradiction.

Step 0 runs in $O(|V|^2)$ time by Theorem 2.6. After this, Step 1 needs $O(|V|)$ running time and every execution of the loop takes at most $O(|V|)$ time by Theorem 2.6 and Observation 5.3. Thus the total running time of the algorithm is $O(|V|^2)$. $\square$

We need to decide now whether there is any edge e for which $G + e$ is $(\underline{m}, \ell)$-redundant. The motivation for this lays in Lemma 4.14, as the stucture of the MCT sets are completely different in case if there exists such an edge, and in the case if there exists no. We noted after Lemma 4.14 how this can be decided by a naïve algorithm with running time $O(|V|^3)$. However, our goal is to give an algorithm that answers this question in $O(|V|^2)$ time. We start with the case when we have an MCT set consisting of a single vertex.

Lemma 5.8. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. If we are given an $(\underline{m}, \ell)$-MCT singleton set $C = \{v\}$, then we can check whether there exists an edge e such that $E(e) = E$ in $O(|V|^2)$ time.*

Proof. By Observation 4.1, if there is an edge e such that $E(e) = E$, then one of the endvertices of e must be v. Thus there are $O(|V|)$ possible edges. For an edge e, we need to decide whether $E(e) = E$ and this can be done in $O(|V|)$ by Theorem 2.6. $\square$

Now we give an algorithm that decides whether G can be augmented to an $(\underline{m}, \ell)$-redundant graph by using one edge.

Algorithm 5.9. Input: *$m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ such that (A0) and (*) hold, and an $(\underline{m}, \ell)$-tight graph $G = (V, E)$ on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold.*
Output: *If there exists an edge e such that $E(e) = E$, then e, otherwise a vertex v of an $(\underline{m}, \ell)$-MCT set.*

1. *Choose two vertices $u, v \in V$, such that $|V(uv)| > 2$. Also suppose that $m(v) \geq m(u)$.*
2. *Run Algorithm 5.4 with u and v. Result: edge e or MCT set Z.*
  If *the result is an edge e,* ***then Output*** *e,* ***STOP****.*

3. ***If*** $|Z| = 1$, ***then*** *check every edge from it by the algorithm of Lemma 5.8.*
If *this outputs an edge* e, ***then Output*** e, ***STOP***.
Else, ***Output*** *the single element* z *of* Z, ***STOP***.
4. *Let* $z \in Z$ *be such that* $m(z)$ *is not the unique minimum of* m.
5. *Run Algorithm 5.4 with* z *and* v. *Result: edge* e *or MCT set* C.
If *the result is an edge* e, ***then Output*** e, ***STOP***.
7. ***If*** $|C| = 1$, ***then*** *check every edge from it by the algorithm of Lemma 5.8.*
If *this outputs an edge* e, ***then Output*** e, ***STOP***.
Else, ***Output*** z, ***STOP***.
8. *Let* $c \in C - Z$.
9. *Run Algorithm 5.4 with* z *and* c. *Result: edge* e *or MCT set* S.
If *the result is an edge* e, ***then Output*** e, ***STOP***.
10. ***If*** $Z \cap C = \emptyset$, $C \cap S = \emptyset$, *and* $S \cap Z = \emptyset$, ***then Output:*** z.
11. ***Else***, *check each possible edge from* v *and* z, *it gives a suitable edge* e. ***Output*** e.

Lemma 5.10. *Algorithm 5.9 decides whether there exists one edge* e *such that* $E(e) = E$ *and returns it. If there is no such edge, then it returns an element of an* $(\underline{m}, \ell)$*-MCT set. The algorithm runs in* $O(|V|^2)$ *time.*

For the proof of this lemma we shall use some notations from Section 4.

Proof. Observe that, whenever the algorithm returns a vertex, the output is a vertex $z \in Z$ where Z is an MCT set by Lemma 5.6.

By Lemma 5.6, if Algorithm 5.9 returns an edge e after any execution of Algorithm 5.4, then $E(e) = E$ holds thus we can stop. By Lemma 5.8, if it outputs an edge e in Step 3 or 7, then $E(e) = E$ holds thus we can stop. If the Algorithm returns z after Step 10, then it is easy to see that G cannot be augmented to a redundant graph with only one edge by Observation 4.1. Also if its output is given in Step 3 or 7, then this is correct due to Lemma 5.8 and Observation 4.1. Thus we only need to prove that if Algorithm 5.9 reaches Step 11, then one edge from v or z indeed augments G to an $(\underline{m}, \ell)$-redundant graph.

Assume that we reached Step 11. Then neither $m(v)$ nor $m(z)$ is the unique minimum of m and $|C|, |Z| \geq 2$. Also notice that in this case $\ell > 0$ by Lemma 4.4. Let $\mathcal{C}$ denote the family of MCT sets in G.

Suppose first that S and C are intersecting. $S, C \in \mathcal{C}(z)$ holds by their construction. By Lemmas 4.5 and 4.6, there exists a vertex $u \in V - z$ such that $E(uz) = E$ or $\mathcal{C}(z)$ form a co-partition. Since we have chosen $u, v \in V$ in Step 1 such that $|V(uv)| > 2$, $|V - Z| > 2$ holds. However, $V - Z$ is a tight set containing c (which is in $C - S$) and v (from $V - Z - C$). Hence, if $\mathcal{C}(z)$ form a co-partition, then the tightness of $V - Z$ contradicts Lemma 4.9. Therefore, in this case there exists a vertex $u \in V - z$ such that $E(uz) = E$.

Suppose now that $S \cap C = \emptyset$. If Z and C are intersecting, then $Z, C \in \mathcal{C}(v)$ and Lemmas 4.5 and 4.6 imply that there exists a vertex $u \in V - v$ such that $E(uv) = E$ or $\mathcal{C}(v)$ form a co-partition. However, if $\mathcal{C}(v)$ form a co-partition, then, by Lemma 4.9, either $E(zc) = E$ or $V - S = \{z, c\}$, contradicting the assumptions that $|C| \neq 1$

and $S \cap C = \emptyset$. Therefore, in this case there exists a vertex $u \in V - v$ such that $E(uv) = E$.

If neither S and C nor Z and C are intersecting, then S intersects Z and hence S contains every vertex in $C - c$ by Lemma 4.5, $Z \cap C = \emptyset$, and $c \notin S$. Thus S intersects C by $|C| > 1$, a contradiction. This proves the correctness of Algorithm 5.9.

When checking whether condition (A2) holds, we can store an appropriate vertices u and v to begin with in $O(|V|^2)$ time (see in Lemma 5.1). As Algorithm 5.4 and checking the size of the generated tight set for each possible new edge from a vertex runs in $O(|V|^2)$ time by Lemma 5.6 and Theorem 2.6 (see also the proof of Lemma 5.8), the running time of Algorithm 5.9 is $O(|V|^2)$. □

Now, we focus our attention to the case where there is no edge e that augments G to a redundant graph. We aim to find a minimal transversal set of the MCT sets of G starting from a vertex v that is in a MCT set. From this point our algorithm will be similar to the algorithm of García and Tejel [6], however, the proof of its correctness is simpler.

Algorithm 5.11. INPUT: *$m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}$ such that (A0) and (*) hold, and an $(\underline{m}, \ell)$-tight graph $G = (V, E)$ on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold, and there exists no edge e that augments G to an $(\underline{m}, \ell)$-redundant graph, and a vertex $v \in V$ from an $(\underline{m}, \ell)$-MCT set.*
OUTPUT: *A transversal system of the MCT sets of G: $X = \{v, x_2, \ldots, x_t\}$.*

- ***0.*** *Run the algorithm of Theorem 2.6 on G.*
- ***1.*** *Initialize $X = \emptyset$. All vertices are unmarked. Mark v.*
- ***2.*** *Explore all vertices $j \in V$:*
  If *j is unmarked,* ***then***
  - *Calculate $T(vj)$ by using the output of* STEP *0;*
  - *Mark all unmarked vertices in $V(vj)$;*
  - *$X := (S - V(vj)) + j$.*
- ***3.*** ***Output:*** *$X + v$.*

Lemma 5.12. *Algorithm 5.11 finds a transversal system of the MCT sets that contains v in $O(|V|^2)$ time.*

Proof. By Lemma 4.14, the MCT sets of G are pairvise disjoint and there are at least 3 of them. Let C be a MCT set such that $v \notin C$. Let us investigate the first moment in STEP 2 when the j is from C. By Lemma 4.17, all the vertices from C get marked now. Note that only such edges can generate C that have at least one vertex in C. Hence each vertex in C were unmarked before this step and j will not be deleted from X after this step. Thus from each MCT set not containing v we add exactly one vertex to X. Let $X' \subseteq X$ consist of the elements of X which are elements of some MCT set not containing v. By Lemma 4.19, $\bigcup\{T(xv)$ where $x \in X'\} = G$. This implies that $X' = X$ by the construction of X.

STEP 0 needs $O(|V|^2)$ running time and after this computing $T(vj)$ can be done in $O(|V|)$ time by Theorem 2.6. We need to compute $T(vj)$ for some $j \in V$ all together $O(|V|)$ times. Therefore, the total running time of the algorithm is $O(|V|^2)$. □

Finally, by using Lemma 4.22 and the above algorithms, we can give the following algorithm to find an optimal solution of the reduced problem in $O(|V|^2)$ time.

Algorithm 5.13. INPUT: *$m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}$ such that (A0) and (*) hold, and an $(\underline{m}, \ell)$-tight graph $G = (V, E)$ on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold.*
OUTPUT: *A minimum cardinality edge set F that augments G to an $(\underline{m}, \ell)$-redundant graph.*

1. *Run Algorithm 5.9. Result: edge e or vertex y.*
  If *the result is an edge e,* ***then Output*** *$\{e\}$,* ***STOP****.*
2. *Generate a transversal system X of the $(\underline{m}, \ell)$-MCT stets of G by using Algorithm 5.11 with y.*
3. *$F := \{vy | v \in X - y\}$.*
4. *Calculate $T(f)$ for each $f \in F$ by using the output of* STEP *0 of Algorithm 5.11.*
5. ***While*** *$d_F(y) \geq 3$,* ***do***
  Choose three neighbours of y in F, say x_i, x_j, x_k.
  Calculate $T(x_j x_k)$ by using the output of STEP *0 of Algorithm 5.11.*
  If *$T(yx_i) \cup T(x_j x_k) = T(yx_i) \cup T(yx_j) \cup T(yx_k)$,* ***then***
    $F := F - \{yx_j, yx_k\} + \{x_j x_k\}$.
  Else*,*
    $F := F - \{yx_i, yx_k\} + \{x_i x_k\}$.
6. ***Output:*** *F.*

Lemma 5.14. *Assume (A0). Let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight graph on at least 4 vertices such that $\ell \leq 0$ or (A1), (A2) and (A3) hold. Then Algorithm 5.13 finds a minimum cardinality edge set that augments G to an $(\underline{m}, \ell)$-redundant graph in $O(|V|^2)$.*

Proof. If the algorithm outputs $\{e\}$ in STEP 1, then $G + e$ is $(\underline{m}, \ell)$-redundant by Lemma 5.10. Otherwise, there is no edge such that $G + e$ is $(\underline{m}, \ell)$-redundant, and STEP 1 gives a vertex z from an MCT set of G. In this case, let $\mathcal{C}$ be the family of all MCT sets of G. By Lemma 4.14, all the sets in $\mathcal{C}$ are pairwise disjoint. Lemma 5.12 implies that the X that we get in STEP 2 is a transversal of $\mathcal{C}$ containing z. By Lemma 4.19, the set F defined in STEP 3 augments G to an $(\underline{m}, \ell)$-redundant graph. Lemma 4.22 implies that $G + F$ remains $(\underline{m}, \ell)$-redundant while we run STEP 5. Finally, Theorem 4.24 implies that the output of the algorithm is optimal.

By Lemma 5.10, STEP 1 can be done in $O(|V|^2)$ time. STEP 2 can be done in $O(|V|^2)$ time by Lemma 5.12. By Theorem 2.6, STEP 4 needs $O(|V|^2)$ time. Also by Theorem 2.6 and by $|E| = O(|V|)$, each loop in STEP 5 can be executed in $O(|V|)$ time thus the running time of STEP 4 is $O(|V|^2)$. Therefore, the running time of Algorithm 5.13 is $O(|V|^2)$. □

It follows by Lemma 5.1 that we do not need to assume that (A1), (A2) and (A3) hold for G. It is also straightforward to solve the reduced problem on graphs with less than 4 vertices in constant time. These imply the following.

Theorem 5.15. *Assume (A0) and (*). Let* $G = (V, E)$ *be an* $(\underline{m}, \ell)$*-tight graph. There exists an algorithm that gives an optimal solution for the reduced augmentation problem in* $O(|V|^2)$ *time.* □

6 The algorithm for the general problem for $m \geq \ell$

Let us consider now our general augmentation problem. García and Tejel [6] showed that this problem is NP-hard for $(2, 3)$-rigid graphs. In this section, we prove that when $m \geq \ell$ holds the general problem is solvable by a polynomial time algorithm. In Section 7, we will show that the NP-hardness result of García and Tejel [6] can also be extended for all pairs of k and ℓ with $k < \ell$, that is, for all pairs of k and ℓ for which the solution given in this section does not work.

Throughout this section, $\bar{G} = (V, \bar{E})$ will denote an $(\underline{m}, \ell)$-rigid graph and $G = (V, E)$ will denote an $(\underline{m}, \ell)$-tight spanning subgraph of $\bar{G}$. Obviously, every edge in $\bar{E} - E$ is $(\underline{m}, \ell)$-redundant in $\bar{G}$. By Lemma 2.4, the $(\underline{m}, \ell)$-redundant edges of G in $\bar{G}$ are the edges of $R^G(\bar{E} - E) = \bigcup_{uv \in \bar{E} - E} T^G(uv)$. As we already have solved the augmentation problem for $(\underline{m}, \ell)$-tight graphs, we assume that $\bar{E} - E \neq \emptyset$.

The idea of our method comes from Jackson and Jordán [10] who proved that the (k, k)-redundant edges of a (k, k)-rigid (that is, a $(k, 0)$-tree-connected) graph $\bar{G}$ form induced subgraphs of $\bar{G}$ with disjoint vertex sets.

We divide the problem into two cases depending on whether $\ell < 0$ or $\ell \geq 0$.

The case of $\ell \geq 0$

First, let us consider the case where $m \geq \ell \geq 0$. In this case, the $(\underline{m}, \ell)$-redundant edges of G in $\bar{G}$ form some vertex disjoint $(\underline{m}, \ell)$-redundant induced subgraphs of G by Lemmas 2.1 Lemma 2.4. By shrinking each of these subgraphs to a single vertex and by defining m' to be ℓ on each of the shrunken vertices and to be $m(v)$ on each non-shrunken vertex v, we get the shrunken graph $G' = (V', E')$. The following statement follows by Lemma 5.2.

Proposition 6.1. *Assume (A0). Let* G *be an* $(\underline{m}, \ell)$*-tight graph and let* $G' = (V', E')$ *and* $m' : V' \to \mathbb{Z}_+$ *arise from* G *as we defined above. Then* G' *is* $(\underline{m}', \ell)$*-tight. Moreover, the pre-image of any* $(\underline{m}', \ell)$*-tight subgraph of* G' *is* $(\underline{m}, \ell)$*-tight and the shrunken image of an* $(\underline{m}, \ell)$*-tight subgraph of* G *is* $(\underline{m}', \ell)$*-tight.*

Proof. The first two statement follows directly by Lemma 5.2. For the last statement, let T be an $(\underline{m}, \ell)$-tight subgraph of G. If we take the union of T with the shrunken T_i components whose vertex set is intersected by $V(T)$, we get another $(\underline{m}, \ell)$-tight subgraph of G by Lemma 2.1. The shrunken image of this union, that coincides with the shrunken image of T, is $(\underline{m}', \ell)$-tight by Lemma 5.2. □

By Proposition 6.1, a covering of G with $(\underline{m}, \ell)$-tight subgraphs gives a covering of G' with $(\underline{m}', \ell)$-tight subgraphs. Hence the minimum number of edges that we need to make G $(\underline{m}, \ell)$-redundant is at least the minimum number of edges that we need to make G' $(\underline{m}', \ell)$-redundant. The following statement shows that these two values are equal.

Proposition 6.2. *Let F' denote an edge set of minimum cardinality on V' for which $G' \cup F'$ is $(\underline{m}', \ell)$-redundant. Let F be an arbitrary pre-image of F', that is, we get F' from F by shrinking the $(\underline{m}, \ell)$-tight subgraphs of redundant edges of G. Then $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundant.*

Proof. The shrunken image of $T^{G}_{(\underline{m},\ell)}(uv)$ is an $(\underline{m}', \ell)$-tight subgraph of G' that spans both of the images u' and v' of u and v by Proposition 6.1. Thus it is a supergraph of $T^{G'}_{(\underline{m}',\ell)}(u'v')$ by Lemma 2.3. Since the image of each non-$(\underline{m}, \ell)$-redundant edge of $\bar{G}$ is in G' and the subgraphs $\{T^{G'}_{(\underline{m}',\ell)}(u'v') : u'v' \in F'\}$ cover the edge set of G', the subgraphs $\{T^{G}_{(\underline{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\underline{m}, \ell)$-redundant edge of $\bar{G}$. Hence $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundant by Lemma 2.4. □

The case of $\ell < 0$

Next, we consider the case where $\ell < 0$. In this case, the $(\underline{m}, \ell)$-redundant edges of G in $\bar{G}$ form an $(\underline{m}, \ell)$-tight induced subgraph T of G by Lemmas 2.1 and 2.4. (Note that an $(\underline{m}, \ell)$-tight set X in G must intersect $V(T)$ in this case since otherwise $X \cup V(T)$ induces at least $\widetilde{m}(X) - \ell + \widetilde{m}(V(T)) - \ell > \widetilde{m}(X \cup V(T)) - \ell$ edges.) By shrinking T to a single vertex and by defining m' to be 0 on the shrunken vertex and to be $m(v)$ on each non-shrunken vertex v, we get the shrunken graph $G' = (V', E')$ which is $(\underline{m}', 0)$-tight by Lemma 5.2. Like in Proposition 6.1, the image of an $(\underline{m}, \ell)$-tight subgraph of G (which must intersect T) is $(\underline{m}', 0)$-tight (as if we take its union with the T, we get another $(\underline{m}, \ell)$-tight subgraph of G by Lemma 2.1). Moreover, by a similar argument, the union of the pre-image of any $(\underline{m}', 0)$-tight subgraph of G' and T is $(\underline{m}, \ell)$-tight. Therefore, a covering of G with $(\underline{m}, \ell)$-tight subgraphs gives a covering of G' with $(\underline{m}', 0)$-tight subgraphs. Hence the minimum number of edges that we need to make G $(\underline{m}, \ell)$-redundant is at least the minimum number of edges that we need to make G' $(\underline{m}', 0)$-redundant. The following statement shows that these two values are equal.

Proposition 6.3. *Let F' denote an edge set of minimum cardinality on V' for which $G' \cup F'$ is $(\underline{m}', 0)$-redundant. Let F be an arbitrary pre-image of F', that is, we get F' from F by shrinking T. Then $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundant.*

Proof. As we have seen, the shrunken image of $T^{G}_{(\underline{m},\ell)}(uv)$ is an $(\underline{m}', 0)$-tight subgraph of G' that spans the image u' and v' of u and v both. Thus it is a supergraph of $T^{G'}_{(\underline{m}',0)}(u'v')$ by Lemma 2.3. Since the image of each non-$(\underline{m}, \ell)$-redundant edge of $\bar{G}$ is in G' and the subgraphs $\{T^{G'}_{(\underline{m}',0)}(u'v') : u'v' \in F'\}$ cover the edge set of G', the subgraphs $\{T^{G}_{(\underline{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\underline{m}, \ell)$-redundant edge of $\bar{G}$. Hence $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundant. □

With Propositions 6.2 and 6.3, we have reduced the problem of augmenting an $(\underline{m}, \ell)$-rigid graph to an $(\underline{m}, \ell)$-redundant graph to the problem of augmenting an

$(\underline{m}', \max(\ell, 0))$-tight graph to an $(\underline{m}', \max(\ell, 0))$-redundant graph that we can solve in $O(|V|^2)$ time by Theorem 5.15 as (A0) holds obviously. Note that the contraction that we used can be done in $O(|V|^2)$ time by Theorem 2.6. This implies the following theorem.

Theorem 6.4. *There exists an $O(|V|^2)$ time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+,\ \ell \in \mathbb{Z}$ for which $m \geq \ell$ and (*) hold, and of an $(\underline{m}, \ell)$-rigid graph $\bar{G} = (V, \bar{E})$, such that $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundant.* □

7 Complexity results

In this section we prove that the general augmentation problem is NP-hard when $\ell > k$. Moreover, our method also implies that there exists no polynomial constant factor approximation algorithm for this problem if P≠NP.

García and Tejel showed in [6] that the general augmentation problem is NP-hard when $k = 2$ and $\ell = 3$. Our construction is based on their idea. We first recall this idea by showing the NP-hardness of a more general problem. As we will see, this result immediately implies the NP-hardness of the general augmentation problem when $\ell = 2k - 1$, however, some more work is needed to show the NP-hardness for other values of k and ℓ.

7.1 The Colored Tight Augmentation Problem

García and Tejel [6] introduced the following problem, called the **Colored Tight Augmentation** problem or **CTA** problem, for $k = 2$ and $\ell = 3$.

Problem. *Let k and ℓ be integers with $k \geq 0$ and $\ell < 2k$ and let $G = (V, E)$ be a (k, ℓ)-tight graph such that the edges in E are colored by red or black. Find a graph $H = (V, F)$ on the same vertex set with minimum number of edges, such that each black edge in $G + H = (V, E \cup F)$ is (k, ℓ)-redundant.*

It is easy to see that, for $\ell = 2k - 1$ this is problem equivalent to the general problem. Indeed, given an instance of the CTA problem, we can get an instance of the general augmentation problem by adding a parallel copy of each red edge. Clearly, if e is an edge of G, then $E^G_{(k,2k-1)}(e) = \{e\}$. Hence, in the above graph, all red edges are redundant while all black edges are not by Lemma 2.4. On the other hand, given an instance of the general augmentation problem, one can get an instance of the CTA problem by taking a spanning (k, ℓ)-tight subgraph and coloring each redundant edge to red and each other edge to black.

However, the above equivalency does not hold for general (k, ℓ). We shall prove that the CTA problem in NP-hard for every (k, ℓ) where $k > 1$, that is, also for $\ell \leq k$ in which case the general augmentation problem is solvable in polynomial time by Theorem 6.4. Furthermore, we shall also show that it does not have a polynomial time constant factor approximation unless $P = NP$. Our construction is a straightforward generalization of that of García and Tejel [6].

Theorem 7.1. *Let $k > 1$ and $\ell < 2k$ be two integers. Then the CTA problem is NP-hard on 2-colored (k, ℓ)-tight graphs, moreover, there exists no polynomial time constant factor approximation algorithm for it if P≠NP.*

Proof. We reduce the **set cover** problem to the CTA problem. Given a ground set $X = \{x_1, \ldots, x_n\}$ and a family $\mathcal{S} = \{S_1, \ldots, S_m\}$ of its subsets, a solution of the set cover problem is a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that $\bigcup \mathcal{S}' = X$ and $|\mathcal{S}|$ is minimal. It is well-known that the set cover problem is NP-hard, and cannot be approximated in constant factor unless P=NP (see [17]).

Given a family $\mathcal{S}$ on ground set X such that $|X| = n \geq 2k+1$ and no two members of $\mathcal{S}$ cover X, we construct our graph in the following way. Let us take a (k, ℓ)-tight graph G_0 on a copy X' of X. (It is easy to check that such graph exists when $|X| \geq 2k + 1$.) Let us take another copy X'' of X and connect the copies x' and x'' of each $x \in X$ by an edge e_x. These n edges will be the only black edges in our final graph; every other edge will be red. Add new edges between X' and X'' until $d(x''_i) = k$ holds for every $i \in \{1, \ldots, n\}$. Let us call the graph we got from G_0 by $G_1 = (V, E)$.

The following observation follows easily from the definition of the MCT sets and Lemma 4.14.

Lemma 7.2. *Let k and ℓ be two integers with $\ell < 2k$. Let $G = (V, E)$ be a (k, ℓ)-tight graph, and let $G' = (V', E')$ be a connected graph such that $i_{G'}(V \cap V') = 0$, the graph $G^* = (V \cup V', E \cup E')$ is (k, ℓ)-tight, and $V' - V$ forms a (k, ℓ)-MCT set in G^*. Suppose that there are at least two disjoint (k, ℓ)-MCT sets X and Y in G which do not intersect $V' \cap V$. Then the (k, ℓ)-MCT sets of G^* are exactly $V' - V$ and those (k, ℓ)-MCT sets of G that do not intersect $V' \cap V$.*

Proof. Observe that $e_{G'}(Z) = e_G(Z)$ holds for each $Z \subseteq V - V'$ and $e_{G'}(Z) > e_G(Z)$ holds for each $Z \subseteq V$ for which $Z \cap V' \neq \emptyset$. This implies that those MCT sets of G that do not intersect $V' \cap V$ are MCT in G' since for such a set Z $k|Z| = e_G(Z) = e_{G'}(Z)$ holds. Hence G' has at least 3 disjoint MCT sets: X, Y and $V' - V$. Thus, by Lemma 4.14, the MCT sets of G' are pairwise disjoint.

Assume now that Z is an MCT set in G' other than X, Y and $V' - V$. Then $Z \subseteq V - (X \cup Y)$ and hence $|Z| \leq |V| - 2$. This implies that $e_G(Z) \geq k|Z|$. Therefore, as Z is co-tight in G', $k|Z| \leq e_G(Z) \leq e_{G'}(Z) = k|Z|$ holds. Hence equality must hold throughout, that is Z is co-tight in G and Z does not intersect $V' \cap V$. □

From this we get the following claim.

Claim 7.3. *G_1 is a (k, ℓ)-tight graph, where $\{x''_1\}, \ldots, \{x''_n\}$ all are (k, ℓ)-MCT sets and these are the only (k, ℓ)-MCT sets of G_1.*

Proof. If we add a new v vertex with $d(v) = k$ to a (k, ℓ)-tight graph G', then $G' + v$ is (k, ℓ)-tight, as no set S with $v \in S$ violates the sparsity conditions. Hence G_1 is (k, ℓ)-tight, as we get it from a (k, ℓ)-tight graph G_0.

$d(x''_i) = k$ for every x''_i thus x''_i is a co-tight set by itself and hence it is also MCT. By Lemma 7.2, there can be no other MCT sets. □

Now for every $S \in \mathcal{S}$ we make the following extension on G_1. Let S'' denote the copy of S in X''. We start with $V_S = \emptyset$ and $E_S = \emptyset$. We first choose k vertices from S'' (if $S'' \geq k$) and add a new vertex v with a (red) edge to these k vertices. Also add v to V_S. Later, when V_S is not empty, we take the last vertex v that is added to V_S and $k-1$ other vertices from S'' which were not used before and add a new vertex of degree k neighbouring these k vertices. We proceed the above addition until there are vertices in S'' that were not used in such a step. In the last step, there may be less than $k-1$ (or k in the first step) such vertices in S''. In this case, we take the rest of the neighbours from X'.

Let us denote the graph that we get from G_1 after running the above procedure for each $S \in \mathcal{S}$ by $G^* = (V^*, E^*)$. Let $S \in \mathcal{S}$ an arbitrary fixed set. One can observe that $d(v) = k$ holds for every $v \in V_S$ in the moment when it is added to V_S, however, in the subsequent step this is increased to $k+1$, except for the last vertex added to V_S that we call m_S. By Lemma 7.2, this shows that the only MCT sets in G^* are the sets $M_S := \{m_S\}$ for all $S \in \mathcal{S}$. By the construction, it is also easy to see that V_S is co-tight in G^* for every $S \in \mathcal{S}$.

By Lemma 4.25, we may assume that the optimal solution of the CTA problem consists of edges between MCT sets. Hence to finish our proof we only need to prove the following statement.

Claim 7.4. *Let $S_i, S_j \in \mathcal{S}$ and let $e = m_{S_i}m_{S_j}$ be an edge connecting the two (k,ℓ)-MCT sets M_{S_i} and M_{S_j} of G^*. Then a black edge e_x is contained in $E^{G^*}(e)$ if and only if $x \in S_i \cup S_j$.*

Proof. Let $S \in \mathcal{S} - \{S_i, S_j\}$. We have seen that V_S is co-tight in G^*. Hence $V^{G^*}(e) \cap V_S = \emptyset$ by Lemma 2.3. This implies that $V^{G^*}(e) \subseteq X' \cup X'' \cup V_{S_i} \cup V_{S_j}$. Hence $T^{G^*}(e) = T^{G^*[X' \cup X'' \cup V_{S_i} \cup V_{S_j}]}(e)$ by the (k,ℓ)-tightness of $G^*[X' \cup X'' \cup V_{S_i} \cup V_{S_j}]$ and Lemma 2.3. Now observe that our construction, Lemma 7.2, and Claim 7.3 imply that the copy $x'' \in X''$ of $x \in X$ is co-tight in $G^*[X' \cup X'' \cup V_{S_i} \cup V_{S_j}]$ if $x \notin S_i \cup S_j$. Thus, again by Lemma 2.3, $V^{G^*}(e) \subseteq X' \cup S_i'' \cup S_j'' \cup V_{S_i} \cup V_{S_j}$ where S_i'' and S_j'' are the copies of S_i and S_j in X'', respectively. Therefore, if $x \notin S_i \cup S_j$ then $e_x \notin E^{G^*}(e)$.

On the other hand, e connects two MCT sets M_{S_i} and M_{S_j}. Hence, by Lemma 4.17, $M_{S_i} \cup M_{S_j} \cup N_{G^*}(M_{S_i}) \cup N_{G^*}(M_{S_j}) \subseteq V^{G^*}(e)$. Let $m^1_{S_i}$ and $m^1_{S_j}$ the two vertices that we added to V_{S_i} and V_{S_j} before m_{S_i} and m_{S_j}, respectively. Since m_{S_i} and m_{S_j} are neighbours of these vertices, $m^1_{S_i}, m^1_{S_j} \in V^{G^*}(e)$. Hence $V^{G^*}(m^1_{S_i}m^1_{S_j}) \subseteq V^{G^*}(e)$ by Lemma 2.3. Now, $G^* - (M_{S_i} \cup M_{S_j})$ is also (k,ℓ)-tight by the co-tightness of M_{S_i} and M_{S_j} hence $T^{G^*}(m^1_{S_i}m^1_{S_j}) = T^{G^*-(M_{S_i} \cup M_{S_j})}(m^1_{S_i}m^1_{S_j})$. Observe that $m^1_{S_i}$ and $m^1_{S_j}$ form singleton MCT sets in $G^* - (M_{S_i} \cup M_{S_j})$, hence Lemma 4.17 implies that $\{m^1_{S_i}, m^1_{S_j}\} \cup N_{G^*-(M_{S_i} \cup M_{S_j})}(\{m^1_{S_i}, m^1_{S_j}\}) \subseteq V^{G^*-(M_{S_i} \cup M_{S_j})}(m^1_{S_i}m^1_{S_j}) \subseteq V^{G^*}(e)$. By repeating this argument, it follows that $V_{S_i} \cup V_{S_j} \cup N(V_{S_i}) \cup N(V_{S_j}) \subseteq V^{G^*}(e)$.

Let S'' be the copy of $S_i \cup S_j$ in X''. By the construction of G^*, $S'' \subseteq N(V_{S_i}) \cup N(V_{S_j})$ and hence $S'' \subseteq V^{G^*}(e)$. Let $x'', y'' \in S''$ be the copy of $x, y \in S_i \cup S_j$. By Lemma 2.3 and by $x'', y'' \in V^{G^*}(e)$, it follows that $V^{G^*}(x''y'') \subseteq V^{G^*}(e)$. Since x'' and y'' are also vertices of the (k,ℓ)-tight subgraph G_1, $V^{G^*}(x''y'') = V^{G_1}(x''y'')$ by Lemma 2.3 again. Since $\{x''\}$ and $\{y''\}$ are MCT sets of G_1 by Claim 7.3, $\{x'', y''\} \cup N_{G_1}(\{x'', y''\}) \subseteq$

$V^{G_1}(x''y'')$ by Lemma 4.17. This implies that $e_x, e_y \in E^{G_1}(x''y'') \subseteq E^{G^*}(e)$, as we claimed. □

From the above claim one can see that any (not necessarily optimal) solution of the CTA problem on G^* of cardinality q gives a (not necessarily optimal) solution of the set cover problem that uses at most $2q$ sets and every (not necessarily optimal) solution of the set cover problem with cardinality q gives a (not necessarily optimal) solution of the CTA problem with cardinality $\lceil \frac{q}{2} \rceil$. However, there is no constant factor approximation of the set cover problem unless P=NP by [17]. This also holds for the version of the set cover problem we used (that is, when $|X| \geq 2k+1$ and there are no two sets that cover the ground set). Therefore, there is no constant factor approximation of the CTA problem unless P=NP. This finishes the proof of Theorem 7.1. □

7.2 The General Augentation Problem for $\ell > k$

As we have seen in the beginning of the previous subsection, the NP-hardness of the CTA problem implies the NP-hardness of the general augmentation problem when $\ell = 2k - 1$. This is because, in this single case, we can make some edges of a (k, ℓ)-tight graph redundant (or in other worlds "red") by adding a parallel copy of them. However, this fact is not true when $\ell \neq 2k - 1$. Although, we can guarantee a similar property when our (k, ℓ)-tight graph G has the form of $(2k - \ell)H$ for some simple graph H, that is, each pair of adjacent vertices induces $2k - \ell$ parallel edges. Note that with the above construction we can only construct such a graph when $2k - \ell$ is a divisor of k (that is, the degree of a vertex that forms an MCT set). However, by Lemma 7.2, we can also use extensions with MCT sets that are other than a single vertex. The following two lemmas shows that this extension is possible.

Lemma 7.5. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Then there exists a graph $H = (V' + w, E')$, called a (k, ℓ)-**gadget**, on $2k - \ell + 1$ vertices that has no loops, $|E'| = k$, $d_H(w) \geq 2$, $(2k - \ell)H[V]$ is (k, ℓ)-sparse, and $e_{(2k-\ell)H}(X) > k|X|$ holds for every $X \subsetneq V'$.*

Proof. Let $V' = \{v_1, \ldots, v_{2k-\ell}\}$ and let $v_{2k-\ell+1} := v_1$. Let us form a cycle $E'_C = \{v_i v_{i+1} : i \in \{1, \ldots, 2k - \ell\}\}$ on V'. This consists of $(2k - \ell)$ edges and it is easy to check that $(V', (2k - \ell)E'_C)$ is (k, ℓ)-sparse. Since $|E'| = k$, $|E' - E'_C| = \ell - k$. We will distribute these edges between V' and w as equal as possible.

Claim 7.6. *It is possible to define a set E'_w of $\ell - k$ edges between V' and w in such a way that, for every set X of consecutive vertices on the cycle E'_C, $e_{E'_w}(X) =$ either $\lfloor |X| \frac{\ell-k}{2k-\ell} \rfloor$ or $\lceil |X| \frac{\ell-k}{2k-\ell} \rceil$.*

Proof. Observe that it is enough to prove the statement when X is an interval on the cycle not containing $v_{2k-\ell}v_1$, since $e_{E'_w}(X) + e_{E'_w}(V' - X) = |E'| = k - \ell$ and $k - \ell - \lfloor |X| \frac{\ell-k}{2k-\ell} \rfloor = \lceil |V' - X| \frac{\ell-k}{2k-\ell} \rceil$.

For the proof, we will use some results on **totally unimodular** (**TU** for short) matrices which are matrices with only 0 and ± 1 subdeterminants. For more details, we refer to [4, Section 4.2]. Our proof is based on the following statement.

Lemma 7.7 ([4, Lemma 4.3.4]). *Let A be a totally unimodular matrix and x_0 a vector of appropriate dimension. Then there exists an integral vector q for which $\lfloor x_0 \rfloor \leq q \leq \lceil x_0 \rceil$ and $\lfloor Ax_0 \rfloor \leq Aq \leq \lceil Ax_0 \rceil$.* □

We use the above lemma for the incidence matrix of all sets of consecutive vertices on the cycle not containing $v_{2k-\ell}v_1$. This matrix is TU since it is an incidence matrix of subpaths of a path which is a network matrix and hence is TU (see [4, Corollary 4.2.6]). Let $x_0 = (\frac{\ell-k}{2k-\ell}, \ldots, \frac{\ell-k}{2k-\ell})$ be a $2k-\ell$ dimensional vector and let q be its rounding according to Lemma 7.7. Let us add q_i edges between v_i and w to E'_w. Now, by Lemma 7.7, $\lfloor |X| \frac{\ell-k}{2k-\ell} \rfloor \leq e_{E'_w}(X) \leq \lceil |X| \frac{\ell-k}{2k-\ell} \rceil$ holds for each interval X on on the cycle not containing $v_{2k-\ell}v_1$, and $e_{E'_w}(V') = \frac{\ell-k}{2k-\ell}(2k-\ell) = \ell - k$ as we wanted. □

It is easy to see that it is enough to prove that $e_{(2k-\ell)H}(X) > k|X|$ holds for every $X \subsetneq V'$ for which $H[X]$ is connected, that, is for every set X of consecutive vertices on the cycle E'_C such that $X \neq V'$. Now $e_{(2k-\ell)H}(X) = e_{(2k-\ell)E'_C}(X) + e_{(2k-\ell)E'_w}(X) \geq (2k-\ell)|X| + (2k-\ell) + (2k-\ell)\lfloor |X| \frac{\ell-k}{2k-\ell} \rfloor > (2k-\ell)|X| + (2k-\ell)|X| \frac{\ell-k}{2k-\ell} = k|X|$.

To finish the proof we need to check whether $d_H(w) \geq 2$ holds. In our construction, $d_H(w) = \ell - k$ hence the only problem arises when $\ell = k+1$. In this case we modify our construction, as follows. Assume that the neighbour of w is v_1. Let us delete the edge $v_1 v_{k-1}$ and add the edge wv_{k-1}. It is easy to check that $e_{(k-1)H}(X) > k|X|$ still holds for every $X \subsetneq V'$. □

Lemma 7.8. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Let $G = (V, E)$ be a graph and let $G' = (V \cup V', E \cup (2k-\ell)E')$ be an extension of G such that all edges in E' are incident with at least one vertex in V', E'/V is isomorphic to a (k,ℓ)-gadget, and $d_{E'}(v) \leq 1$ for each $v \in V$. Assume that G is (k,ℓ)-tight. Then G' is (k,ℓ)-tight and V' is a (k,ℓ)-MCT set in G'.*

We call the above extension of G a (k,ℓ)-**gadget extension**.

Proof. First observe that $e_{G'}(X) > k|X|$ for all $X \subsetneq V'$, furthermore, $e_{G'}(V') = k|V'|$ by Lemma 7.5. Hence if G' is (k,ℓ)-tight, then V' is an MCT set in G'.

Since $|E'| = k$ and $|V'| = 2k - \ell$ by Lemma 7.5, and since G is (k,ℓ)-tight, $|E \cup (2k-\ell)E'| = k|V \cup V'| - \ell$. Let $Y \subseteq V \cup V'$ such that $|Y \cap V| \geq 2$. By the (k,ℓ)-sparsity of G and Lemma 7.5, $i_{G'}(Y) = i_G(Y \cap V) + (2k-\ell)|E'| - e_{G'}(V' - Y) \leq k|Y \cap V| - \ell + k|V'| - k|V' - Y| = k|Y| - \ell$. Now let $Y \subseteq V \cup V'$ such that $|Y \cap V| \leq 1$. Then $G'[Y - V]$ is (k,ℓ)-sparse by Lemma 7.5 and $G'[Y]$ arises from $G'[Y - V]$ by adding at most one vertex and $2k - \ell$ copy of at most one edge incident with this new vertex (where $2k - \ell < k$ by $k < \ell$). Hence $G'[Y]$ is (k,ℓ)-sparse. Therefore, G' is (k,ℓ)-tight. □

Now we are ready to prove the main result of this section.

Theorem 7.9. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Then the general augmentation problem is NP-hard on (k,ℓ)-rigid graphs, moreover, there exists no polynomial time constant factor approximation algorithm for it if P≠NP.*

Proof. The proof has two main steps. First we show how to modify the proof of Theorem 7.1 in such a way that all edges except the edges of the starting (k,ℓ)-tight graph G_0 have $2k-\ell$ parallel copies in the graph. Next we show how to extend this graph to a (k,ℓ)-rigid graph where only the black edges are not (k,ℓ)-redundant.

Given a family $\mathcal{S} = \{S_1,\ldots,S_m\}$ on ground set $X = \{x_1,\ldots,x_n\}$ such that $|X| = n \geq 2k+1$ and no two members of $\mathcal{S}$ cover X, we construct our 2-edge-colored (k,ℓ)-tight graph in the following way. Like in the proof of Theorem 7.1, let us take a (k,ℓ)-tight graph $G_0 = (X', E_0)$ on a copy X' of X. Let us perform n (k,ℓ)-gadget extensions on G_0 with the (k,ℓ)-gadgets defined in Lemma 7.5 in the following way. Add n copies $V'_1,\ldots,V'_n$ of the vertex set $V' = \{v_1,\ldots,v_{2k-\ell}\}$ to the vertex set of our graph. These are the vertex sets added during the extensions. We perform the (k,ℓ)-gadget extensions in such a way that, in the step where we add V'_i for $i \in \{1,\ldots,n\}$, we only connect V'_i to X' and one of the edges, say, e_{x_i} connecting V'_i and X' connects V'_i to the copy x'_i of x_i. The $(2k-\ell)$ parallel copies of the edges $e_{x_1},\ldots,e_{x_n}$ will be the only black edges in our final graph; every other edge will be red. Let us call the graph we got from G_0 by $G_1 = (V,E)$. Lemmas 7.2 and 7.8 imply the following with a proof similar to that of Claim 7.3.

Claim 7.10. *G_1 is a (k,ℓ)-tight graph, where each of $V'_1,\ldots,V'_n$ is a MCT set and they are the only MCT sets of G_1.* □

Now $G^* = (V^*, E^*)$ arises from G_1 like in the proof of Theorem 7.1, although, we use (k,ℓ)-gadget extension instead of degree k vertex addition. We use the notations V_S analogously to that of in Theorem 7.1. Let M_S be the set of vertices added by the last (k,ℓ)-gadget extension for $S \in \mathcal{S}$. Lemmas 7.2 and 7.8 imply that the only MCT sets in G^* are the sets M_S for all $S \in \mathcal{S}$. By the construction, it is also easy to see that V_S is co-tight in G^* for $S \in \mathcal{S}$.

By Lemma 4.25, we may assume that the optimal solution of the CTA problem on G^* consists of edges between MCT sets. Finally, the proof of the following statement is similar to that of Claim 7.4 and follows by Lemmas 2.3, 4.17, 7.8 and Claim 7.10.

Claim 7.11. *Let $S_i, S_j \in \mathcal{S}$ and let e be an edge connecting the two (k,ℓ)-MCT sets M_{S_i} and M_{S_j} of G. Then a black edge e_x is contained in $E^{G^*}(e)$ if and only if $x \in S_i \cup S_j$.* □

From the above claim one can see that any (not necessarily optimal) solution of the CTA problem on G^* of cardinality q gives a (not necessarily optimal) solution of the set cover problem that uses at most $2q$ sets and every (not necessarily optimal) solution of the set cover problem with cardinality q gives a (not necessarily optimal) solution of the CTA problem with cardinality $\lceil \frac{q}{2} \rceil$. However, there is no constant factor approximation of the set cover problem unless P=NP by [17]. Therefore, there is no constant factor approximation of the CTA problem on 2-edge-colored graphs G^* that arise by the above construction unless P=NP.

To finish the proof, we need to define a (k,ℓ)-rigid graph $\bar{G} = (V^*, \bar{E})$ for the 2-edge-colored graph graph $G^* = (V^*, E^*)$ that arose by the above construction in such a way that G^* is a subgraph of $\bar{G}$ and exactly the black edges of G^* are not (k,ℓ)-redundant in $\bar{G}$. Let $\bar{G}$ arise from G^* by adding a new parallel copy to each red edge of G^*. Lemma 2.4 states that $R^{G^*}(\bar{E} - E^*) = \bigcup_{e \in \bar{E} - E^*} T^{G^*}(e)$. Hence it is obvious that all red edges of G^* are (k, ℓ)-redundant in $\bar{G}$. On the other hand, for an edge e of G_0, $T^{G^*}(e)$ is a subgraph of the (k, ℓ)-tight graph G_0 by Lemma 2.3; and for an edge $e = uv$ that we added with a (k, ℓ)-gadget extension $T^{G^*}(e) = G^*[\{u, v\}]$ since the set $\{u, v\}$ induces $2k - \ell$ parallel edges in G^* and hence it is (k, ℓ)-tight. Therefore, no black edge of G^* is (k, ℓ)-redundant in $\bar{G}$. This finishes the proof of Theorem 7.9. □

8 Concluding remarks

Further extensions

Assume (A0) and let G be a graph such that cG is $(\underline{m}, \ell)$-tight. $R(i_1j_1, \ldots, i_cj_c) = T(i_1j_1) \cup \cdots \cup T(i_cj_c)$ by Lemma 2.4 hence $R(ij, \ldots, ij) = T(ij) \cup \cdots \cup T(ij) = T(ij)$. Thus if we get G' by adding some edges to G, then we get the same $(\underline{m}, \ell)$-redundant edges in cG' as if we add just one (and not c) copy of these edges to cG. Hence our algorithms can be used to prove the following corollaries.

Corollary 8.1. *There is an $O(|V|^2)$ time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with (A0) and of a graph $G = (V, E)$ for which cG is $(\underline{m}, \ell)$-tight, such that $c(G \cup F)$ is $(\underline{m}, \ell)$-redundant. Furthermore, $cG \cup F$ is also $(\underline{m}, \ell)$-redundant.* □

Corollary 8.2. *There is an $O(|V|^2)$ time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with (A0) and $m \geq \ell$ and of a graph $G = (V, E)$ for which cG is $(\underline{m}, \ell)$-rigid, such that $c(G \cup F)$ is $(\underline{m}, \ell)$-redundant. Furthermore, $cG \cup F$ is also $(\underline{m}, \ell)$-redundant.* □

Augmenting to a simple redundant graph

In several applications in rigidity theory, it must be assumed that all considered graphs are **simple**, that is, have no loops nor parallel edges. Hence only those redundant augmentations are appropriate that maintain this property, that is, we cannot add an edge parallel to an existing edge of the graph. Note that it follows by Lemma 4.16 and how our algorithm works that, assuming (A1), (A2), and (A3), if the output of our algorithm consists of more than one edge, then the augmented graph will be simple if the input was also simple.

On the other hand we can also get an optimal simple solution when our algorithm in Section 5 returns a single edge. It is easy to see that the problem can be solved on constant number of vertices. If the simple $(\underline{m}, \ell)$-tight graph $G = (V, E)$ has more than $c^2 + c$ vertices (where c comes from our assumption (*)), then G has some non-adjacent vertices. It is also easy to see that the complete graph on V is $(\underline{m}, \ell)$-redundant in this case. Hence, by Lemma 2.4, $G = \bigcup\{T(uv) : u, v \in V, uv \notin E\}$. Assume that $G = T(e)$ for some edge e, that is, Algorithm 5.13 returns a single edge. If e joins non-adjacent vertices of G, we are done. Otherwise, $e \in \bigcup\{E(uv) : u, v \in V, uv \notin E\}$ by our observation above and hence $e \in E(uv)$ for some non-adjacent vertices $u, v \in V$.

Lemma 2.3 implies now that $T(e)$ is a subgraph of $T(uv)$. Therefore, $G = T(e) = T(uv)$ and the edge uv can be found in $O(|V|^3)$ time by calculating $T(uv)$ for each pair of non-adjacent vertices. We leave the question open whether this time bound can be reduced to $O(|V|^2)$.

Note that by Lemma 3.1 we still can get a solution when (A1) does not hold. However, we have seen in Section 3 that we always need to add parallel edges when (A2) or (A3) does not hold and in this case there is no graph H for which $G + H$ is simple and $(\underline{m}, \ell)$-redundant. This implies the following corollary.

Corollary 8.3. *Assume (A0) and (*). Let $G = (V, E)$ be a simple $(\underline{m}, \ell)$-tight. There exists an algorithm that gives an optimal solution H for the reduced augmentation problem for which $G + H$ is simple if such solution exists. The running time of the algorithm is $O(|V|^3)$ when the output H has only one edge and $O(|V|^2)$ when the output H has more than one edge.* □

Finally, observe that the above method also works when we want to give a solution H to the general augmentation problem for $m \geq \ell$ in such a way that $G+H$ is simple. Indeed, as we have just seen, the solution is automatically simple when H has at least two edges. On the other hand, let $G = (V, E)$ be an $(\underline{m}, \ell)$-tight subgraph of our $(\underline{m}, \ell)$-rigid simple graph $\bar{G} = (V, \bar{E})$. If H has only one edge e, then $G = T^G(e) \cup \bigcup\{T^G(e') : e' \in \bar{E} - E\}$ by Lemma 2.4. Assuming again that $|V| > c^2 + c$, the complete graph on V is $(\underline{m}, \ell)$-redundantand hence $G = \bigcup\{T(uv) : u, v \in V, uv \notin \bar{E}\} \cup \bigcup\{T(e') : e' \in \bar{E} - E\}$. Hence $e \in \bigcup\{E^G(uv) : u, v \in V, uv \notin E\} \cup \bigcup\{E^G(e') : e' \in \bar{E} - E\}$. If $e \in E^G(e')$ for an edge $e' \in \bar{E} - E$, then $T^G(e)$ is the subgraph of $T^G(e')$ by Lemma 2.3 and hence $T^G(e') = T^G(e) = G$, that is, $\bar{G}$ is already $(\underline{m}, \ell)$-redundant. On the other hand, if $e \in E^G(uv)$ for some non-adjacent vertices $u, v \in V$, then $T^G(e)$ is the subgraph of $T^G(uv)$ by Lemma 2.3 and hence $T^G(uv) = T^G(e) = G$, that is, uv augments $\bar{G}$ to an $(\underline{m}, \ell)$-redundant graph. Hence if the algorithm of Theorem 6.4 returns a set F containing more than one edge, then $G + F$ is simple or there is no simple solution because (A2) or (A3) fails; and otherwise, one needs to check each non-adjacent pair of vertices $u, v \in V$ whether $G + uv$ is $(\underline{m}, \ell)$-redundant. This implies the following.

Corollary 8.4. *There exists an algorithm that obtains a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}$ for which $m \geq \ell$ and (*) hold, and of a simple $(\underline{m}, \ell)$-rigid graph $\bar{G} = (V, \bar{E})$, such that $\bar{G} \cup F$ is $(\underline{m}, \ell)$-redundantand simple if such solution exists. The running time of the algorithm is $O(|V|^3)$ when $|F| = 1$ and $O(|V|^2)$ when $|F| = 0$ or $|F| \geq 2$.* □

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