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Abstract

The authors of this note gave a combinatorial characterization of globally rigid generic body-hinge frameworks in [6]. One step of the proof of this result used a specific property of the so-called vertex-splitting operation in graphs. This property, however, has not yet been verified in its full generality. Here we complete our proof by showing a different argument for this step.

1 Introduction

We start by stating the main result of [6] and refer the reader to our paper for a general introduction and basic definitions in rigidity theory. Our result is about body-hinge graphs. Given a multigraph H, the (d-dimensional) *body-hinge graph* induced by H, denoted by G_H, is obtained from H as follows. Each vertex $v \in V(H)$ corresponds to a complete graph $B(v)$ on $(d-1)d_H(v)+d+1$ vertices in G_H, in which $d+1$ vertices form the *core* $C(v)$ of the body $B(v)$ and the remaining vertices are partitioned into sets of $d-1$ vertices so that each set is assigned to one edge incident with v. Here $d_H(v)$ denotes the degree of v in H. For each edge $e = uv$ of H the bodies $B(u)$ and $B(v)$ share the $d-1$ vertices assigned to e in these bodies. This set of $d-1$ vertices, assigned to e, is the *hinge set* corresponding to e, denoted by $H(e)$. The cores of the bodies are pairwise disjoint. There are no other vertices or edges in G_H.

For a multigraph H we use kH to denote the graph obtained from H by replacing every edge e by k parallel copies of e. We say that H is *m-tree-connected* if it contains m edge-disjoint spanning trees. It is *highly m-tree-connected* if $H - e$ is m-tree-connected for every $e \in E(H)$. The global rigidity of graphs G_H possessing this body-hinge structure is characterized as follows.

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Theorem 1. *[6, Theorem 4.5] Let $H = (V, E)$ be a multigraph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ is highly $\binom{d+1}{2}$-tree-connected.*

In the final step of the inductive proof of the "if" direction of Theorem 1 (the second last sentence of the proof, [6, line 16, page 70]) we used a property of the vertex splitting operation to verify that a certain graph is globally rigid. This operation is defined as follows.

Let G be a graph, let $v_1 \in V$, let $v_1v_2, v_1v_3, ..., v_1v_d$ be $d-1$ designated edges incident with v_1, and let $v_1v_{d+1}, ..., v_1v_{d+k_1}$, and $v_1v_{d+k_1+1}, ..., v_1v_{d+k_1+k_2}$ be a bipartition of the remaining edges incident with v_1. The (d-dimensional) *vertex splitting* operation at v_1 removes the edges $v_1v_{d+1}, ..., v_1v_{d+k_1}$, adds a new vertex v_0, and adds the new edges $v_0v_1, v_0v_2, ..., v_0v_d, v_0v_{d+1}, ..., v_0v_{d+k_1}$. The new edge v_0v_1 is called the *bridging edge* in the resulting graph.

The following conjecture, which is a weaker version of a conjecture of Walter Whiteley posed in [1], see also [3, Conjecture 16], is still unsolved.

Conjecture 1. *Let G be globally rigid in $\mathbb{R}^d$ and let G' be obtained from G by a vertex splitting operation. If $G' - e$ is rigid in $\mathbb{R}^d$ for the bridging edge e, then G' is globally rigid in $\mathbb{R}^d$.*

We stated this conjecture as a result of Bob Connelly [2], called it Theorem 4.3, and used it in the proof. However, as it was pointed out by Bill Jackson [4], Connelly's result in [2] is different, and it does not imply the truth of the conjecture. To repair the proof we shall use the following statement, due to Jackson, which provides another way to show that a vertex-splitting operation preserves global rigidity. The proof is based on [3, Theorem 13]. See also [5, Section 5] for an application of this idea.

Lemma 1. *[4] Let G be a globally rigid graph in $\mathbb{R}^d$ and $v_1 \in V(G)$. Suppose that G' is obtained from G by a vertex splitting operation at v_1 and that G' has an infinitesimally rigid realization (G', p) in $\mathbb{R}^d$ with $p(v_0) = p(v_1)$. Then G' is globally rigid in $\mathbb{R}^d$.*

In the next section we define the graph that plays the role of G' in the proof of Theorem 1 and show that it satisfies the conditions of Lemma 1.

2 Infinitesimally rigid realizations with coincident vertices

The *skeleton* S_H of the body-hinge graph G_H induced by H is obtained from G_H by deleting the cores $C(v)$ for all $v \in V(H)$. We showed that G_H is globally rigid if and only if S_H is globally rigid [6, Lemma 3.1]. Suppose that H contains a vertex v of degree two with $N_H(v) = \{u, w\}$. Let $H(uv) = \{x_1, x_2, \ldots, x_{d-1}\}$ and $H(vw) = \{y_1, y_2, \ldots, y_{d-1}\}$ denote the hinge sets corresponding to edges uv, uw. Consider the skeleton S_H and define a new graph S_H^v as follows: if $d \geq 4$ then S_H^v is obtained from S_H by contracting the edges x_i, y_i for all $3 \leq i \leq d-1$. If $d = 3$ then $S_H^v = S_H$. We showed that S_H is globally rigid if and only if S_H^v is globally rigid [6, Lemma 4.4].

In S_H^v the bodies of u, v, w are modified with respect to S_H and the hinge sets of edges uv, uw are also changed. We shall use $B^v(a)$ and $H^v(e)$ to denote the bodies and hinges in S_H^v associated with the vertices and edges of H, respectively. The $d-3$ vertices obtained by the contractions are shared by $B^v(u)$, $B^v(w)$, and $B^v(v)$. Thus $B^v(v)$ induces a complete graph on $d+1$ vertices. We also introduce $H_v = H - v + uw$ in the proof and show that $(\binom{d+1}{2} - 1)H_v - 2(uw)$ is $\binom{d+1}{2}$-tree-connected, see the proof of [6, Claim 4.6].

Our goal in the proof of Theorem 1 is to prove that S_H^v (and hence also G_H) is globally rigid. Since S_H^v is obtained from a globally rigid graph by a vertex splitting operation with bridging edge x_1y_1 (as shown in the proof), the global rigidity of S_H^v will follow from the next lemma. The proof is similar to the proof of [6, Lemma 3.2].

Lemma 2. *Let $H = (V, E)$ be a multigraph and let G_H be its d-dimensional body-hinge graph induced by H for some $d \geq 3$. Suppose that v is a vertex of degree two in H and let S_H^v be graph defined above. Then S_H^v has an infinitesimally rigid realization (S_H^v, p) in $\mathbb{R}^d$ with $p(x_1) = p(y_1)$.*

Proof. Since $(\binom{d+1}{2} - 1)H_v - 2(uw)$ is $\binom{d+1}{2}$-tree-connected, it contains $\binom{d+1}{2}$ edge-disjoint spanning trees $T_{i,j}, 0 \leq i < j \leq d$. We shall define a configuration p of $V(S_H^v)$ by using these trees. By relabelling some trees, if necessary, we can assume that

$$T_{0,d}, T_{1,d}, T_{2,d} \text{ do not contain (a copy of) edge } uw \tag{1}$$

Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. It will be convenient to denote the origin of $\mathbb{R}^d$ by e_0. Note that $V(S_H^v)$ is the disjoint union of the hinge sets $H^v(f)$, for $f \in E(H-v)$, the four-tuple $\{x_1, y_1, x_2, y_2\}$, and the vertices $\{x_3 = y_3, \ldots, x_{d-1} = y_{d-1}\}$ (where the last set of $d-3$ vertices exists only if $d \geq 4$).

For each edge $f \in E(H-v)$ there is at least one tree $T_{k,l}$ which does not contain a copy of f. We fix such a tree and define the realization of the vertices in $H^v(f)$ in such a way that

$$\{p(x) \mid x \in H^v(f)\} = \{e_i \mid 0 \leq i \leq d, i \neq k, l\} \tag{2}$$

holds. For the remaining vertices, we define

$$\begin{aligned}
p(x_1) &= p(y_1) = e_0,\\
p(x_2) &= e_1,\\
p(y_2) &= e_2,\\
p(x_i) &= e_i \ (3 \leq i \leq d-1).
\end{aligned}$$

Observe that

$$p(B^v(a)) \text{ affinely spans } \mathbb{R}^d \text{ for all } a \in V(H) - v \tag{3}$$

since a is incident with an edge of $T_{i,j}$ in H_v for every $1 \leq i < j \leq d$.

We shall show that (S_H^v, p) is infinitesimally rigid in $\mathbb{R}^d$. Note that the existence (or removal) of the edge x_1y_1 makes no difference since its end-vertices are coincident. It is also useful to remark that the only body which contains x_1y_1 is $B^v(v)$.

Consider an infinitesimal motion $m : V(S_H^v) \to \mathbb{R}^d$ of (S_H^v, p). Since the bodies are complete subgraphs and (3) holds, for each $a \in V(H) - v = V(H_v)$ there exists a $d \times d$ skew-symmetric matrix S_a and a vector $t_a \in \mathbb{R}^d$ such that $m(x) = S_a p(x) + t_a$ for every $x \in B^v(a)$.

Claim *Let $f = ab \in T_{i,j}$ be an edge for some $0 \le i < j \le d$ and $a, b \in V(H_v)$. Then there is an edge $xy \in E(S_H^v - x_1 y_1)$ for which $x \in B^v(a)$, $y \in B^v(b)$, and $\{p(x), p(y)\} = \{e_i, e_j\}$.*

Proof. First suppose that $ab \neq uw$. Then it follows from (2) that there is at least one vertex $x \in H^v(f)$ for which $p(x)$ is equal to either e_i or e_j. By (3), there is a vertex y in $B^v(b)$ with $\{p(x), p(y)\} = \{e_i, e_j\}$. As $H^v(f) \subseteq B^v(b)$, we also have $xy \in E(S_H^v - x_1 y_1)$.

Next suppose that $ab = uw$. Then (1) implies

$$(i, j) \notin \{(0, d), (1, d), (2, d)\}. \tag{4}$$

If $\{i, j\} \cap \{3, \dots, d-1\} \neq \emptyset$ (which may hold only if $d \ge 4$), then, since $B^v(u) \cap B^v(w) = \{x_3, \dots, x_{d-1}\}$ and (3) holds, there is a pair $x \in B^v(u) \cap B^v(w)$ and $y \in B^v(w)$ with $\{p(x), p(y)\} = \{e_i, e_j\}$. Now $x, y \in B^v(w)$, and hence we also have $xy \in E(S_H^v - x_1 y_1)$, as required. Finally, if $\{i, j\} \cap \{3, \dots, d-1\} = \emptyset$, then $(i, j) \in \{(0, 1), (0, 2), (1, 2)\}$ by (4). Recall that $S_H^v - x_1 y_1$ contains the edges $x_1 y_2$, $x_2 y_1$, and $x_2 y_2$, and $x_1, x_2 \in B^v(u)$ and $y_1, y_2 \in B^v(w)$ hold. Since $p(x_1) = p(y_1) = e_0, p(x_2) = e_1$ and $p(y_2) = e_2$, the desired edge exists. This completes the proof of the claim. □

Consider $f = ab \in T_{i,j}$ for some $0 \le i < j \le d$ and $a, b \in V(H_v)$, and a pair $x \in B^v(a)$, $y \in B^v(b)$ with $xy \in E(S_H^v - x_1 y_1)$ and $\{p(x), p(y)\} = \{e_i, e_j\}$. Such a pair exists by the Claim. We may assume that $p(x) = e_i$ and $p(y) = e_j$.

Since m is an infinitesimal motion of (S_H^v, p), we have

$$\langle p(x) - p(y), m(x) - m(y) \rangle = 0.$$

If $i \ge 1$, this gives

$$\begin{aligned} 0 &= \langle p(x) - p(y), m(x) - m(y) \rangle \\ &= \langle p(x) - p(y), S_a p(x) + t_a - S_b p(y) - t_b \rangle \\ &= \langle e_i - e_j, S_a e_i + t_a - S_b e_j - t_b \rangle \\ &= -e_j^\top S_a e_i - e_i^\top S_b e_j + \langle e_i - e_j, t_a - t_b \rangle. \end{aligned} \tag{5}$$

On the other hand, if $i = 0$, then by $e_0 = 0$ we also have

$$\langle e_j, t_a - t_b \rangle = 0 \qquad \text{for } 1 \le j \le d \text{ and } ab \in T_{0,j}.$$

This implies $t_a = t_b$ for each pair $a, b \in V(H_v)$, since $T_{0,j}$ spans $V(H_v)$ for each j. Therefore, by using the skew-symmetry of S_v and (5), we can deduce that

$$S_a[i, j] = e_i^\top S_a e_j = e_i^\top S_b e_j = S_b[i, j] \qquad \text{for } 1 \le i < j \le d \text{ and } ab \in T_{i,j}.$$

Again, since $T_{i,j}$ spans $V(H_v)$ for all i, j, this implies that $S_a = S_b$ for each pair $a, b \in V(H - v)$. Since every vertex in $S_H^v - x_1y_1$ belongs to at least one body $B^v(a)$ for some $a \in V(H_v)$, we conclude that there is a skew-symmetric matrix S and a vector $t \in \mathbb{R}^d$ such that $m(x) = Sp(x) + t$ for every $x \in V(S_H^v)$. In other words, m is a trivial infinitesimal motion. This proves that (S_H^v, p) is infinitesimally rigid in $\mathbb{R}^d$ and completes the proof. □

Thus in the modified proof of Theorem 1 the last three sentences [6, lines 15-18, page 70] are as follows: "Therefore by Theorem 4.1 S' is globally rigid. Since S_H^v is constructed from S' by a vertex splitting operation, we can apply Lemma 1 and Lemma 2 to conclude that S_H^v is globally rigid. This completes the proof."

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