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Vertex Splitting, Coincident Realisations and Global Rigidity of Braced Triangulations

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Abstract

We give a short proof of a result of Jordán and Tanigawa that a 4-connected graph which has a spanning planar triangulation as a proper subgraph is generically globally rigid in $\mathbb{R}^3$. Our proof is based on a new sufficient condition for the so called vertex splitting operation to preserve generic global rigidity in $\mathbb{R}^d$.

Keywords Bar-joint framework, global rigidity, vertex splitting, plane triangulation.

Mathematics Subject Classification 52C25, 05C10, 05C75

1 Introduction

We consider the problem of determining when a configuration consisting of a finite set of points in d-dimensional Euclidean space $\mathbb{R}^d$ is uniqely defined up to congruence by a given set of constraints which fix the distance between certain pairs of points. This problem was shown to be NP-hard for all $d \geq 1$ by Saxe [18], but becomes more tractable if we restrict our attention to generic configurations. Gortler, Healy and Thurston [9] showed that, for generic frameworks, uniqueness depends only on the underlying constraint graph. Graphs which give rise to uniquely realisable generic configurations in $\mathbb{R}^d$ are said to be *globally rigid in* $\mathbb{R}^d$. These graphs have been characterised for $d = 1, 2$, [13], but it is a major open problem in distance geometry to characterise globally rigid graphs when $d \geq 3$.

A recent result of Jordán and Tanigawa [17] characterises when graphs constructed from plane triangulations by adding some additional edges are globally rigid in $\mathbb{R}^3$.

Theorem 1. *Suppose that G is a graph which has a planar triangulation T as a spanning subgraph. Then G is globally rigid in $\mathbb{R}^3$ if and only if G is 4-connected and $G \neq T$.*

We will give a short proof of this result. The main tool in our inductive proof is the (3-dimensional version of) the following result which gives a sufficient condition for the so called vertex splitting operation to preserve global rigidity in $\mathbb{R}^d$.

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Theorem 2. *Let $G = (V, E)$ be a graph which is globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G' is obtained from G by a vertex splitting operation which splits v into two vertices v' and v'', and that G' has an infinitesimally rigid realisation in $\mathbb{R}^d$ in which v' and v'' are coincident. Then G' is generically globally rigid in $\mathbb{R}^d$.*

Theorem 2 may be of independent interest. It has aleady been used by Jordán, Kiraly and Tanigawa in [16] to repair a gap in the proof of their characterision of generic global rigidity for 'body-hinge frameworks' given in [15]. An analogous result to Theorem 2 was used in [12, 14] to obtain a characteriseation of generic global rigidity for 'cylindrical frameworks'. Theorem 2 is a special case of a conjecture of Whiteley, see [3, 4], that the vertex splitting operation preserves global rigidity in $\mathbb{R}^d$ if and only if both v' and v'' have degree at least $d + 1$ in G'.

2 Vertex splitting and coincident realisations

We will prove Theorem 2. We first define the terms appearing in the statement of this theorem. A *(d-dimensional) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a point configuration. The *rigidity map* for G is the map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ which maps a configuration $p \in \mathbb{R}^{d|V|}$ to the sequence of squared edge lengths $(\|p(u) - p(v)\|^2)_{uv \in E}$. The framework (G, p) is *gloablly rigid* if, for every framework (G, q) with $f_G(p) = f_G(q)$, we have p is congruent to q. It is *rigid* if it is globally rigid within some open neighbourhood of p and is *infinitesimally rigid* if the Jacobean matrix of the rigidity map of G has rank $\min\{d|V| - \binom{d+1}{2}, \binom{d}{2}\}$ at p. Gluck [6] showed that every infinitesimally rigid framework is rigid and that the two properties are equivalent when p is *generic* i.e. the coordinates of p are algebraically independent over $\mathbb{Q}$. We say that the graph G is *rigid*, respectively *globally rigid*, in $\mathbb{R}^d$ if some, or equivalently every, generic framework (G, p) in $\mathbb{R}^d$ is rigid, respectively globally rigid. We refer the reader to the survey article [20] for more information on rigid frameworks.

We need the following result of Connelly and Whiteley [5] which shows that global rigidity is a stable property for infinitesimally rigid frameworks.

Lemma 3. *Suppose that (G, p) is an infinitesimally rigid, globally rigid framework on n vertices in $\mathbb{R}^d$. Then there exists an open neighbourhood N_p of p in $\mathbb{R}^{dn}$ such that (G, q) is infinitesimally rigid and globally rigid for all $q \in N_p$.*

Given a graph $G = (V, E)$ and $v \in V$ with neighbour set $N(v)$ the *(d-dimensional) vertex splitting operation* constructs a new graph G' by deleting v, adding two new vertices v' and v'' with $N(v') \cup N(v'') = N(v) \cup \{v', v''\}$ and $|N(v') \cap N(v'')| = d - 1$. Whiteley [19] showed that vertex splitting preserves generic rigidity in $\mathbb{R}^d$ and conjectured in [3, 4] that it will preserve generic global rigidity if and only if both v' and v'' have degree at least $d + 1$ in G'.

Proof of Theorem 2: Let (G, p) be a generic realisation of G in $\mathbb{R}^d$ and let (G', p') be the $v'v''$-coincident realisation of G' obtained by putting $p'(u) = p(u)$ for all

$u \in V - v$ and $p'(v') = p'(v'') = p(v)$. The genericity of p implies that the rank of the rigidity matrix of any $v'v''$-coincident realisation of G' will be maximised at (G', p') and hence (G', p') is infinitesimally rigid. The genericity of p also implies that (G, p) is globally rigid, and this in turn implies that (G', p') is globally rigid. We can now use Lemma 3 to deduce that (G', q) is globally rigid for any generic q sufficiently close to p'. Hence G' is globally rigid. •

3 Braced triangulations

A graph T is a *planar (near) triangulation* if it has a 2-cell embeding in the plane in which every (bounded) face has three edges on its boundary. A *braced planar triangulation* is a graph $G = (V, E \cup B)$ which is the union of a planar triangulation $T = (V, E)$ and a (possibly empty) set of additional edges B, which we refer to as the *bracing edges of* G. We say that G is a *braced plane triangulation* when G is given with a particular 2-cell embedding of T in the plane.

We will need the following notation and elementary results for a plane triangulation T. Every cycle C of T divides the plane into two open regions exactly one of which is bounded. We will refer to the bounded region as the *inside of* C and the unbounded region as the *outside of* C. We say that C is a *separating cycle* of T if both regions contain vertices of T. If S is a minimal vertex cut-set of T then S induces a separating cycle C. It follows that every plane triangulation is 3-connected and that a plane triangulation is 4-connected if and only if it contains no separating 3-cycles. Given an edge e of T which belongs to no separating 3-cycle of T, we can obtain a new plane triangulation T/e by contacting the edge e and its end-vertices to a single vertex (which is located at the same point as one of the two end-vertices of e), and replacing the multiple edges created by this contraction by single edges.

Given a braced plane triangulation $G = (T, B)$ and an edge e of T which belongs to no separating 3-cycle of T, we denote the braced plane triangulation obtained by contacting the edge e by $G/e = (T/e, B_e)$ where the set of bracing edges B_e is obtained from B by replacing any multiple edges in G/e by single edges (in particular any edge of B which becomes parallel to an edge of T/e is deleted). We say that B *crosses* a separating cycle C of T if at least one edge of B has one end-vertex inside C and one end-vertex outside C. Thus G is 4-connected if and only if B crosses every separating 3-cycle of T.

Our first result implies that every 4-connected braced planar triangulation $G = (T, B)$ can be reduced to a braced octahedron by recursively contracting edges of T. The special case when $B = \emptyset$, i.e. G is a 4-connected planar triangulation, was obtained by Hama and Nakamoto [10], see also Brinkman et al [1].

Lemma 4. *Let $G = (T, B)$ be a 4-connected braced plane triangulation on at least seven vertices and C be the bounding cycle of a face of T. Then $G/e = (T/e, B_e)$ is a 4-connected braced plane triangulation for at least one edge $e \in E(T) \setminus E(C)$. In addition, we may choose e such that $B_e \neq \emptyset$ whenever $B \neq \emptyset$.*

Proof: It suffices to show that we can find an edge $e \in E(T)\backslash E(C)$ with the properties that e is in no separating 3-cycle of T, every separating 3-cycle of T/e is crossed by B_e, and $B_e \neq \emptyset$ when $B \neq \emptyset$. We may assume without loss of generality that C is the bounding cycle of the outer face of T. Choose a 3-cycle C_1 in T as follows. If T has a separating 3-cycle then choose C_1 to be a separating 3-cycle of T such that the set of vertices inside C_1 is minimal with respect to inclusion. If T has no separating 3-cycles then put $C_1 = C$. Let T_1 be the plane triangulation induced in T by $V(C_1)$ and the vertices inside of C_1. The choice of C_1 implies that T_1 is either K_4 or is 4-connected.

We first consider the case when $T_1 = K_4$. Then G/e will be 4-connected for all edges $e \in E(T_1)\backslash E(C_1)$, since the set of separating 3-cycles of T/e is the set of all separating 3-cycles of T other than C_1 (and hence every separating 3-cycle of G/e will be crossed by B). The 4-connectivity of G implies that some edge $b \in B$ crosses C_1 so we must have $B \neq \emptyset$ in this case. Let $C_1 = v_1v_2v_3v_1$ and $b = uw$ where u is the unique vertex inside C_1. If $wv_i \notin E(T)$ for some $1 \leq i \leq 3$ then we may choose $e = uv_i$ to ensure that $B_e \neq \emptyset$. Hence we may assume that $wv_i \in E(T)$ for all $1 \leq i \leq 3$. Since G has more than five vertices, $C_1' = wv_iv_{i+1}w$ is a separating cycle of G for some $1 \leq i \leq 3$, reading subscripts modulo three. Hence some edge $b' \in B$ crosses C_1'. We may now choose $e = uv_j$ with $j \neq i, i+1$ to ensure that $B_e \neq \emptyset$.

We next consider the case when T_1 is 4-connected and has no separating cycles of length four. Then T_1 is 5-connected and T_1/e will be 4-connected for all $e \in E(T_1)$. Hence G/e is 4-connected for all $e \in E(T_1)$ which are not incident with $V(C_1)$, since T and T/e will have the same set of separating 3-cycles (and hence every separating 3-cycle of G/e will be crossed by B). In addition, if $B \neq \emptyset$, then we may ensure that $B_e \neq \emptyset$ by choosing an $e \in E(T_1 - C_1)$ which is not adjacent to some edge in B (this is possible since the 5-connectivity of T_1 gives us lots of choices for e).

It remains to consider the case when T_1 is 4-connected and has a separating cycle C_2 of length four. We may suppose that C_2 has been chosen such that the set of vertices inside C_2 is minimal with respect to inclusion. Let $C_2 = v_1v_2v_3v_4v_1$ and let T_2 be the plane near triangulation induced in T by $V(C_2)$ and the vertices inside of C_2. The choice of C_2 implies that T_2 is a wheel on five vertices or T_2 is 4-connected.

Consider the subcase when T_2 is 4-connected. Then $T_2 - C_2$ is connected, each vertex of C_2 is adjacent to at least two vertices of $T_2 - C_2$, and no vertex of $T_2 - C_2$ is adjacent to two non-adjacent vertices of C_2. Suppose G/e is not 4-connected for some edge e of $T_2 - E(C_2)$. Then some separating 3-cycle of T/e is not a separating 3-cycle of T, and hence e is contained in a separating 4-cycle C_3 of T. The minimality of C_2 implies that $C_3 \cap T_2$ is a path of length three joining two non-adjacent vertices of C_2, say v_1, v_3, and $v_1v_3 \in E(T) \setminus E(T_2)$. Planarity now implies that $v_2v_4 \notin E(T)$ and hence all edges e of $T_2 - E(C_2)$ for which G/e is not 4-connected must lie on a v_1v_3-path in $T_2 - E(C_2)$ of length three. This implies that G/e will be 4-connected for all edges of $T_2 - E(C_2)$ which are incident with v_2 or v_4. This gives us sufficiently many edges to choose from to ensure that $B_e \neq \emptyset$ when $B \neq \emptyset$.

It remains to consider the subcase when T_2 is a wheel on five vertices. Let u be the unique vertex of $T_2 - C_2$. Suppose that some vertex w of $T_1 - T_2$ is adjacent to all vertices of C_2 in T_1. Then the subgraph T_3 of T_1 obtained by adding w and all edges between w and C_2 to T_2 is isomorphic to the octahedron. Since T_1 is 4-connected and

$T_3 \subset T_1$ we must have $T_1 = T_3$. Since T has at least seven vertices, C_1 is a separating 3-cycle of T (this situation is illustrated in Figure 1). Since G is 4-connected, some edge $b \in B$ crosses C_1. Relabeling u, v_2, v_3 if necessary, we may suppose that b is incident to u. Let $e = v_2v_3$. Since T_1 is isomorphic to the icosahedron, $C_2 = v_1v_2v_3v_4v_1$ is the unique separating 4-cycle of T which contains e and hence C_2/e is the only separating 3-cycle of T/e which is not a separating 3-cycle of T. Since b crosses C_2/e in G/e, G/e is 4-connected.

Hence we may suppose that no vertex of $T_1 - T_2$ is adjacent to all vertices of C_2 in T_1. By symmetry and planarity, we may assume that v_1 and v_3 do not have a common neighbour in $T_1 - T_2$. Choose $e \in \{uv_1, uv_3\}$. Then e is not contained in a separating 4-cycle of T so G/e is 4-connected. Furthermore, if $B \neq \emptyset$, then we will have $B_e \neq \emptyset$ for either $e = uv_1$ or $e = uv_3$. •

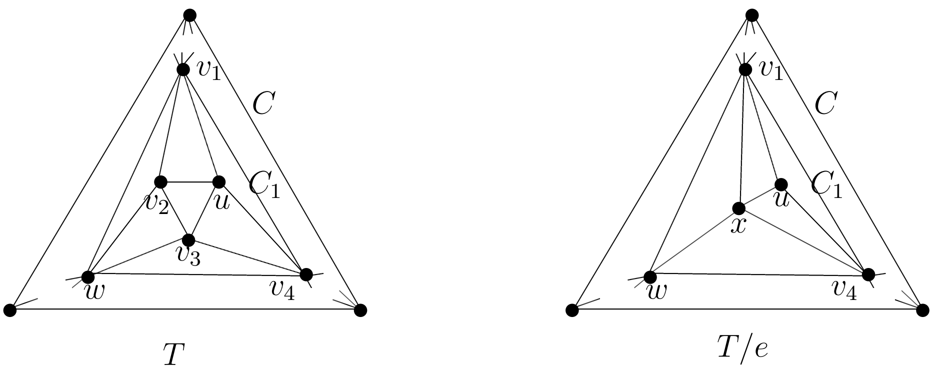

Figure 1: The plane triangulations T and T/e in the case when T_2 is the wheel on five vertices and T_1 is the octahedron. The edge $e = v_2v_3$ is contracted to a new vertex x to form T/e.

We are particularly interested in braced triangulations with at least one bracing edge. For such triangulations we can prove a slightly stronger result.

Corollary 5. *Let $G = (T, B)$ be a 4-connected braced plane triangulation on at least six vertices with $B \neq \emptyset$ and C be the bounding cycle of a face of T. Then $G/e = (T/e, B_e)$ is a 4-connected braced plane triangulation with $B_e \neq \emptyset$ for at least one edge $e \in E(T) \setminus E(C)$.*

Proof: The corollary follows immediately from Lemma 4 if G has at least seven vertices so we may assume that $|V(G)| = 6$. If T has a separating triangle C then one component of $G - C$ is a single vertex and we may proceed as in the case $T_1 = K_4$ of the proof of Lemma 4. On the other hand, if T has no separating triangle then T is isomorphic to the octahedron and $(T/e, B_e) = (K_5 - f, \{f\})$ for any $f \in B$ and any edge e of T which is not adjacent to f. •

We next use Corollary 5 to obtain a result on infinitesimally rigid realisations of 4-connected braced triangultions in $\mathbb{R}^3$ in which two adjacent vertices are coincident.

Theorem 6. *Let $G = (T, B)$ be a 4-connected braced planar triangulation with $B \neq \emptyset$ and $u, v \in V(G)$. Then G has an infinitesimally rigid realisation (G, p) in $\mathbb{R}^3$ with $p(u) = p(v)$.*

Proof: We use induction on $|V(G)|$. If $|V(G)| = 5$ then $G = K_5$ and it is straightforward to check that G has a infinitesimally rigid realisation (G, p) with $p(u) = p(v)$ for all $u, v \in V(G)$. Hence we may suppose that $|V(G)| \geq 6$. By Corollary 5, we can find an edge $f = xy \in E(T)$ with $\{x, y\} \neq \{u, v\}$ and such that $G/f = (T/f, B_f)$ is a 4-connected braced triangulation with $B_f \neq \emptyset$. We label the vertex obtained by contracting f as x, taking $x \in \{u, v\}$ if f is adacent to u or v. By induction G/f has an infinitesimally rigid realisation $(G/f, q)$ with $q(u) = q(v)$. We can now use the vertex-splitting result of Whiteley [19] to deduce that (G, p) is infinitesimally rigid for all p with $p(z) = q(z)$ for $z \in V(G/f)$ and $p(y)$ sufficiently close to $p(x)$. •

Proof of Theorem 1: Necessity follows from [11] (using the fact that if $G = T$ then G would not have enough edges to be redundantly rigid). We prove sufficiency by induction on $|V(G)|$. If $|V(G)| = 5$ then $G = K_5$ and G is globally rigid in $\mathbb{R}^3$. Hence we may suppose that $|V(G)| \geq 6$. By Lemma 4, we can find an edge $f = xy \in E(T)$ such that $G/f = (T/f, B_f)$ is a 4-connected braced triangulation with $B_f \neq \emptyset$. Then G/f is globally rigid by induction. Since G has an infinitesimally rigid xy-coincident realisation by Theorem 6, we can now use Theorem 2 to deduce that G is globally rigid. •

4 Closing Remarks

1. It follows from a result of Cauchy [2], that every graph which triangulates the plane is generically rigid in $\mathbb{R}^3$. Fogelsanger [8] extended this result to triangulations of an arbitrary surface. We conjecture that Theorem 1 can be extended in the same way.

Conjecture 7. *Let G be a graph which has a triangulation T of some surface S as a spanning subgraph. Then G is globally rigid if and only if G is 4-connected and, when S has genus zero, $G \neq T$.*

The conjecture is true for the special case when G itself is a triangulation of the projective plane or torus by [17, Theorem 10.3].

2. Let $G = (V, E)$ be a graph and $vv' \in E$. Fekete, Jordán and Kaszanitzky [7] showed that G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^2$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^2$ (where $G - vv'$ and G/vv' are obtained from G by, respectively, deleting and contracting the edge vv'). We conjecture that the same result holds in $\mathbb{R}^d$.

Conjecture 8. *Let $G = (V, E)$ be a graph and $vv' \in E$. Then G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^d$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^d$.*

The proof in [7] is based on a characterisation of independence in the '2-dimensional generic vv'-coincident rigidity matroid'. It is unlikely that a similar approach will work in $\mathbb{R}^d$ since it is notoriously difficult to characterise independence in the d-dimensional generic rigidity matroid for $d \geq 3$. But it is conceivable that there may be a geometric argument which uses the generic rigidity of $G - vv'$ and G/vv' to construct an infinitesimally rigid vv'-coincident realisation of G.

3. We can use the proof technique of Theorem 2 to show that Conjecture 8 would imply the following weak version of Whiteley's conjecture on vertex splitting.

Conjecture 9. *Let $H = (V, E)$ be a graph which is generically globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G is obtained from H by a d-dimensional vertex splitting operation which splits v into two new vertices v' and v''. If $G - v'v''$ is generically rigid in $\mathbb{R}^d$, then G is generically globally rigid in $\mathbb{R}^d$.*

Jordán, Király and Tanigawa [15, Theorem 4.3] state Conjecture 9 as a result of Connelly [4, Theorem 29] but this is not true - they are misquoting Connelly's theorem.

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