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Sparse graphs and an augmentation problem

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Abstract

For two integers $k > 0$ and ℓ, a graph $G = (V, E)$ is called (k, ℓ)-tight if $|E| = k|V| - \ell$ and $i_G(X) \leq k|X| - \ell$ for each $X \subseteq V$ that induces at least one edge. G is called (k, ℓ)-redundant if $G - e$ has a spanning (k, ℓ)-tight subgraph for all $e \in E$. We consider the following augmentation problem. Given a graph $G = (V, E)$ that has a (k, ℓ)-tight spanning subgraph, find a graph $H = (V, F)$ with minimum number of edges, such that $G + H$ is (k, ℓ)-redundant.

In this paper, we give a polynomial algorithm and a min-max theorem for this augmentation problem when the input is (k, ℓ)-tight. For general inputs, we give a polynomial algorithm when $k \geq \ell$ and show the NP-hardness of the problem when $k < \ell$. Since (k, ℓ)-tight graphs play an important role in rigidity theory, these algorithms can be used to make several types of rigid frameworks redundantly rigid by adding a smallest set of new bars.

1 Introduction

Let k be a positive integer and ℓ be an integer such that $\ell < 2k$. A (multi)graph $G = (V, E)$ is called $\boldsymbol{(k, \ell)}$**-sparse** if $i_G(X) \leq k|X| - \ell$ for all $X \subseteq V$ for which $i_G(X) > 0$, where $\boldsymbol{i_G(X)}$ denotes the number of edges of G induced by $X \subseteq V$. A (k, ℓ)-sparse graph is called $\boldsymbol{(k, \ell)}$**-tight** if $|E| = k|V| - \ell$. A graph G is called $\boldsymbol{(k, \ell)}$**-rigid** if G has a (k, ℓ)-tight spanning subgraph. We will call an edge e of G a $\boldsymbol{(k, \ell)}$**-redundant edge** if $G - e$ is (k, ℓ)-rigid. A graph G is called a $\boldsymbol{(k, \ell)}$**-redundant graph** if each edge of G is (k, ℓ)-redundant. We consider the following augmentation problem that we call the **general (augmentation) problem**.

Problem. *Let k and ℓ be integers with $k \geq 0$ and $\ell < 2k$ and let $G = (V, E)$ be a (k, ℓ)-rigid graph. Find a graph $H = (V, F)$ on the same vertex set with minimum number of edges, such that $G + H = (V, E \cup F)$ is (k, ℓ)-redundant.*

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We call the special case of this problem, where the input graph G is (k, ℓ)-tight, the **reduced (augmentation) problem**. In this paper, we give a min-max theorem and an $O(n^2)$ running time algorithm on a graph with n vertices for fixed k and ℓ for the reduced problem. We also show how this algorithm can be extended to solve the general problem in the same running time when $\ell \leq k$. In contrast, we show that the general problem is NP-hard whenever $\ell > k$.

Sparsity properties are important in rigidity theory as they can be used in the characterization of many rigidity classes. For example, the generically rigid graphs in $\mathbb{R}^2$ are exactly the $(2, 3)$-rigid graphs by the fundamental theorem of Pollaczek-Geiringer [24] and Laman [18]. The 'body-bar graph' induced by a given graph G is generically rigid in $\mathbb{R}^d$ if and only if G is $\left(\binom{d+1}{2}, \binom{d+1}{2}\right)$-rigid by Tay's theorem [25]. Note that G is (k, k)-rigid if and only if G contains k edge-disjoint spanning trees by the fundamental result of Nash-Williams [22].

Besides the effect of redundancy, redundant rigidity is an important concept in rigidity theory since variants of Hendrickson's result [10] show that redundant rigidity is often a necessary condition of 'global rigidity' which plays a crucial role in many applications [1, 27, 28]. Furthermore, in some cases, for example for 'body-bar graphs' (see [4]), redundant rigidity is also a sufficient condition of global rigidity. It is thus natural to ask how many new edges are needed to make a rigid graph redundantly rigid.

There are special pairs of k and ℓ for which these problems were already investigated. The general problem for $(1, 1)$-rigid graphs is the well-studied 2-edge-connectivity augmentation problem solved by Eswaran and Tarjan [5]. The general problem for $k = \ell$ was solved by Frank and T. Király [8] who who gave a polynomial algorithm to augment a graph to an h-times (k, k)-redundant graph using polyhedral techniques (where h-times (k, k)-redundant means that it remains (k, k)-rigid after deleting any set of its edges of cardinality h). The algorithm that will be presented here, is a more efficient solution for this problem, however, it does not deal with the case of $h \geq 2$ and also the (k, k)-rigidity of the input is needed. García and Tejel [9] showed that the general problem is NP-hard for $(2, 3)$-rigid graphs but can be solved in polynomial time for $(2, 3)$-tight graphs. We use similar techniques to [9], however, our method is based on a new min-max theorem for the reduced problem.

1.1 Main ideas and results

It is known that the edge sets of the (k, ℓ)-sparse subgraphs of a given graph form a matroid, called the $(\boldsymbol{k}, \boldsymbol{\ell})$**-sparsity matroid** or **count matroid** (see [20] and [26, Appendix A]). This matroid is the well-known 2-dimensional rigidity matroid when $k = 2$ and $\ell = 3$, and the graphic matroid when $k = \ell = 1$. When we add an edge e to a (k, ℓ)-sparse graph G, the edges in the fundamental circuit of e (which is, in fact, the minimal (k, ℓ)-tight subgraph of G containing both ends of e) in the (k, ℓ)-sparsity matroid become (k, ℓ)-redundant. By extending a result of García and Tejel [9, Lemma 4] we show in Section 2 that if we add more than one edges, still exactly the edges of the fundamental circuits of the augmenting edges in G become (k, ℓ)-redundant.

Let $G = (V, E)$ be a (k, ℓ)-tight graph. We call a set of vertices $\emptyset \neq C \subsetneq V$ **(k, ℓ)-co-tight** in G if the complement of C induces a (k, ℓ)-tight subgraph of G. It can be seen easily by using the above mentioned matroid properties that, if an edge set F augments G to a (k, ℓ)-redundant graph, then $V(F)$ must intersect all (k, ℓ)-co-tight sets in G. Hence, if $\mathcal{C}$ is a family of disjoint (k, ℓ)-co-tight sets in G, then at least $\left\lceil \frac{|\mathcal{C}|}{2} \right\rceil$ edges are needed to augment G to an (m, ℓ)-redundant graph. This proves the min $\geq$ max part of our following min-max theorem.

Theorem 1.1. *Let $k > 0$ and ℓ be two integers such that $\ell < 2k$ and let $G = (V, E)$ be a (k, ℓ)-tight graph on at least $\max(4, k^2+2)$ vertices. If there exists any (k, ℓ)-co-tight set in G, then*

$$\min\{|F| : H = (V, F) \text{ is a graph for which } G + H \text{ is } (k, \ell)\text{-redundant}\}$$
$$= \max\left\{ \left\lceil \frac{|\mathcal{C}|}{2} \right\rceil : \mathcal{C} \text{ is a family of disjoint } (k, \ell)\text{-co-tight sets in } G \right\}.$$

Otherwise, $G + uv$ is (m, ℓ)-redundant for every pair $u, v \in V$.

Sketch of the proof of the min $\leq$ max *part.* First we show in Lemma 5.5 that the inclusion-wise minimal (k, ℓ)-co-tight sets, that we call **(k, ℓ)-MCT sets**, are pairwise disjoint or $G + uv$ is (k, ℓ)-redundant for some pair $u, v \in V$. This proof is the most technical part of our paper. We note that for $(2, 3)$-tight graphs this statement was proved by Jordán [15, Theorem 3.9.13].

We can assume thus that there exists at least three disjoint (k, ℓ)-MCT sets in G. Let us take a transversal X of the family of (k, ℓ)-MCT sets. (A set X is called a **transversal** of a family $\mathcal{S}$ if $|X \cap S| = 1$ for each $S \in \mathcal{S}$.) We show in Lemma 5.8 that any connected graph (in particular, a star $K_{1,|X|-1}$) on X augments G to a (k, ℓ)-redundant graph. Finally, with Lemma 5.9 we reduce the cardinality of this edge set to the optimum. We note that the reduction that we use here is similar to the reduction which is used by García and Tejel [9] for the (2,3)-tight case. The connections are shown by Lemma 5.10. □

It is easy to provide a polynomial algorithm for the reduced problem, however, with some further work we will reduce its running time to $O(n^2)$ on graphs on n vertices (for fixed k and ℓ) in Section 6.

To obtain the solution for the general problem by using the reduced problem when $k \geq \ell$, we need more general concepts. The idea of our method comes from Jackson and Jordán [11] who proved that the (k, k)-redundant edges of a (k, k)-rigid graph $\bar{G}$ form induced subgraphs of $\bar{G}$ with disjoint vertex sets. If we contract these subgraphs into single vertices one can show that the resulting graph is (k, k)-tight for which we can use the algorithm for the reduced problem. In order to extend this idea for $k > \ell$, we need the following generalization of sparsity. Suppose throughout this paper that

(A0) $\ell \in \mathbb{Z}$ and $m : V \to \mathbb{Z}_+$ such that $m(u) + m(v) \geq \ell$ holds for each $u, v \in V$ and equality is only allowed when $m(u) = m(v) = \ell = 0$.

A graph $G = (V, E)$ is called **(m, ℓ)-sparse** if $i_G(X) \leq m(X) - \ell$ holds for every $X \subseteq V$ for which $m(X) - \ell \geq 0$ (and $i_G(X) = 0$ otherwise), where $m(X) := \sum_{v \in X} m(v)$. Note that our assumption (A0) implies that $m(X) - \ell \geq 0$ holds whenever $X \subseteq V$ with $|X| \geq 2$, furthermore, $m(X) - \ell > 0$ also holds for such set X except when $X = \{u, v\}$ and $m(u) = m(v) = \ell = 0$ hold. Observe that each subgraph of an (m, ℓ)-sparse graph is (m, ℓ)-sparse. Note that, when $m \equiv k$, an (m, ℓ)-sparse graph is (k, ℓ)-sparse. **(m, ℓ)-tight/rigid/redundant** graphs and **(m, ℓ)-co-tight/MCT** sets can be defined by extending the definitions of (k, ℓ)-tight/rigid/redundant graphs and (k, ℓ)-co-tight/MCT sets similarly. For simplicity, we call a set $X \subseteq V$ **(m, ℓ)-tight** (**(k, ℓ)-tight**, respectively) **in G** if the **induced subgraph $G[X]$** of X in G is (m, ℓ)-tight ((k, ℓ)-tight, respectively). When the pair (m, ℓ) ((k, ℓ), respectively) is clear from the context we may omit the prefix (m, ℓ) ((k, ℓ), respectively) from the notions above.

When we contract the (k, ℓ)-redundant subgraphs of a given (k, ℓ)-rigid graph for $k > \ell$, the resulting graph becomes (m, ℓ)-tight for some function m and constant ℓ. Hence, for our solution of the general problem in Section 3, we need an algorithm for the reduced problem on (m, ℓ)-tight graphs. Thus we shall solve the reduced problem also for (m, ℓ)-tight graphs.

The idea of our NP-hardness result in Section 7 also comes from the work of García and Tejel [9]. We first show that the colored version of the reduced problem (when we want to augment a subset of the edges of our (k, ℓ)-tight graph to (k, ℓ)-redundant) is NP-hard for all $k > 1$ and $\ell < 2k$. Next we use this result and some gadgets to show the NP-hardness of the general problem for $k < \ell < 2k$.

Notations

We conclude the introduction by listing some notation used throughout this paper. All the graphs in this paper are multigraphs, that is, we allow parallel edges and loops. For simplicity, we do not distinguish graphs and their edge sets when it is clear from the context. Given a graph $G = (V, E)$, $\boldsymbol{d_G(v)}$ denotes the number of edges incident to a vertex $v \in V$, $\boldsymbol{d_G(X, Y)}$ denotes the number of edges between $X - Y$ and $Y - X$ for $X, Y \subseteq V$, $\boldsymbol{d_G(X)} := d_G(X, V - X)$, and $\boldsymbol{e_G(X)} := i_G(X) + d_G(X)$. Note that our definition implies that $d_G(v) \neq d_G(\{v\})$ if there exist loop edges on v. Also note that, in the usual definition of the degree, loop edges count twice for the degree of a vertex, however, we only count them once. We use $\boldsymbol{N_G(X)}$ to denote the neighbor set of $X \subseteq V$, that is, $N_G(X) = \{v \in V - X : d_G(v, X) \geq 1\}$. For $X \subseteq V$, $\boldsymbol{G[X]}$ denotes the subgraph of G induced by X and $\boldsymbol{G/X}$ denotes the graph arising from G by contracting X into a single vertex. If G_1 and G_2 are graphs, then $G_1 \subseteq G_2$ will denote that G_1 is a subgraph of G_2. If $\mathcal{C}$ is a set family, then we say that a set U **covers** $\mathcal{C}$ if $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$. Given a digraph $D = (V, A)$, let $\boldsymbol{\varrho(v)}$ and $\boldsymbol{\varrho(X)}$ denote the **in-degree** of a vertex $v \in V$ and a set $X \subseteq V$, respectively. When it is clear from the context, we omit the subscript G or D from several notations.

2 Preliminaries

In this section, we list some basic properties of (m,ℓ)-sparse graphs. We sketch their proofs for completeness. It follows from the definition that an (m,ℓ)-tight subgraph of an (m,ℓ)-sparse graph is always an induced subgraph. Therefore, if $T_1=(V_1,E_1)$ and $T_2=(V_2,E_2)$ both are (m,ℓ)-tight subgraphs of an (m,ℓ)-sparse graph G, then $G_1\cap G_2=(V_1\cap V_2,E_1\cap E_2)$ is also an induced subgraph of G. Moreover, with standard submodular techniques we can prove the following.

Lemma 2.1. *Let $G=(V,E)$ be an (m,ℓ)-sparse graph, and let $T_1=(V_1,E_1)$ and $T_2=(V_2,E_2)$ be (m,ℓ)-tight subgraphs of G. If $\widetilde{m}(V_1\cap V_2)\geq \ell$, then $T_1\cup T_2$ is an (m,ℓ)-tight graph and there are no edges between V_1-V_2 and V_2-V_1. If $|V_1\cap V_2|\geq 1$, then $T_1\cap T_2$ is (m,ℓ)-tight as well.*

We note that, by (A0), the assumption on $m(V_1\cap V_2)$ always holds when $|V_1\cap V_2|\geq 2$, and also when $\ell\leq 0$ (since $m\geq 0$).

Proof. As G_1 and G_2 are (m,ℓ)-tight,

$$\begin{aligned} i(V_1\cup V_2)+i(V_1\cap V_2)&=i(V_1)+i(V_2)+d(V_1,V_2)\geq i(V_1)+i(V_2)\\ &=m(V_1)-\ell+m(V_2)-\ell=m(V_1\cup V_2)-\ell+m(V_1\cap V_2)-\ell. \end{aligned} \tag{1}$$

Since $\widetilde{m}(V_1\cap V_2)\geq\ell$, $(\widetilde{m}(V_1\cap V_2)-\ell)_+=\widetilde{m}(V_1\cap V_2)-\ell$. Hence, as G is an (m,ℓ)-sparse graph,

$$i(V_1\cup V_2)+i(V_1\cap V_2)\leq m(V_1\cup V_2)-\ell+m(V_1\cap V_2)-\ell \tag{2}$$

holds thus equity stands in (1) and (2). This is only possible if $T_1\cup T_2$ is (m,ℓ)-tight and $d(V_1,V_2)=0$. Furthermore, if $|V_1\cap V_2|\geq 1$, then $T_1\cap T_2$ must be (m,ℓ)-tight as well. □

It is useful to notice that (m,ℓ)-tight graphs have the following connectivity property.

Lemma 2.2. *Let $G=(V,E)$ be an (m,ℓ)-tight graph with $|V|\geq 3$ and let $v\in V$ with $0<m(v)<\ell$. Then no loop is incident with v and $d(v)\geq m(v)$.*

Proof. Note that v cannot induce any loop by the (m,ℓ)-sparsity condition. Hence the (m,ℓ)-tightness of G, the (m,ℓ)-sparsity of $V-v$ and (A0) imply that $m(V)-\ell=i(V)=d(v)+i(V-v)\leq d(v)+m(V-v)-\ell=m(V)-\ell-m(v)+d(v)$ thus $d(v)\geq m(v)$. Notice that this also means that v is connected to $V-v$. □

It is known that the edge sets of the (m,ℓ)-sparse subgraphs of a given graph form a matroid, called the **(m,ℓ)-sparsity matroid** or **count matroid**, which is a straightforward generalization of the (k,ℓ)-count matroid mentioned in the inroduction (see Frank [7, Section 13.5], Lorea [20], and Whiteley [26, Appendix A]). A circuit of this matroid is called an **(m,ℓ)-circuit**. It follows from matroid theory (see details in [7, Chapter 5]) that for an (m,ℓ)-sparse graph $G=(V,E)$ and $i,j\in V$ for which $G+ij$ is not (m,ℓ)-sparse, $G+ij$ contains a unique (m,ℓ)-circuit $C^G_{(m,\ell)}(ij)$. In this case, $\boldsymbol{T^G_{(m,\ell)}(ij)} := C^G_{(m,\ell)}(ij) - ij$ is (m,ℓ)-tight. (Note that $C^G_{(m,\ell)}(ij)$ consists of the single edge ij when $m(i) = m(j) = \ell = 0$, and hence $T^G_{(m,\ell)}(ij)$ consists of only two isolated vertices i and j in this case, however, this subgraph of G is (m,ℓ)-tight.) Let $\boldsymbol{V^G_{(m,\ell)}(ij)}$ denote the vertex set of $T^G_{(m,\ell)}(ij)$. (Recall that we do not distinguish graphs and their edge sets, hence $T^G_{(m,\ell)}(ij)$ may be used to denote its edge set.) For every edge e' of $C^G_{(m,\ell)}(ij)$, $G' = G + ij - e'$ is also (m,ℓ)-sparse and the unique (m,ℓ)-circuit of $G' + e'$ is again $C^G_{(m,\ell)}(ij)$. Moreover, if $e'' \notin C^G_{(m,\ell)}(ij)$, then, $G' + ij - e''$ is not (m,ℓ)-sparse. The main property of $T^G_{(m,\ell)}(ij)$ is the following. It will be used several times in this paper.

Lemma 2.3. *Let $G = (V,E)$ be an (m,ℓ)-sparse graph and let $i,j \in V$. Assume that $G + ij$ is not (m,ℓ)-sparse. If $G' = (V',E')$ is an (m,ℓ)-tight subgraph of G with $i,j \in V'$, then $T^G_{(m,\ell)}(ij) \subseteq G'$. Hence $T^G_{(m,\ell)}(ij) = \bigcap\{T_h : T_h$ is an (m,ℓ)-tight subgraph of G inducing i and $j\}$.* □

Let $\boldsymbol{R^G_{(m,\ell)}(i_1j_1,\dots,i_rj_r)}$ denote the subgraph induced by the (m,ℓ)-redundant edges of $G = (V,E)$ in $G \cup \{i_1j_1,\dots,i_rj_r\}$ where $i_1,\dots,i_r, j_1,\dots,j_r \in V$. For the sake of simplicity, when the graph G or (m,ℓ) is clear from the context, we will omit the superscript G or subscript (m,ℓ), respectively, from all of the above notation. Note that $R(ij) = T(ij)$ for any $i,j \in V$. The following lemma extends this simple fact by generalizing [9, Lemma 4] .

Lemma 2.4. *If G is an (m,ℓ)-tight graph, then*

$$R(i_1j_1,\dots,i_rj_r) = T(i_1j_1) \cup \cdots \cup T(i_rj_r).$$

Proof. Since $R(ij) = T(ij)$, $T(i_1j_1) \cup \cdots \cup T(i_rj_r) \subseteq R(i_1j_1,\dots,i_rj_r)$. For the other direction, let $e \in R(i_1j_1,\dots,i_rj_r)$ be an arbitrary edge. Now, $G - e$ is (m,ℓ)-sparse and $|E - e| = m(V) - \ell - 1$. $G \cup \{i_1j_1,\dots,i_rj_r\} - e$ is (m,ℓ)-rigid hence $E \cup \{i_1j_1,\dots,i_rj_r\} - e$ has a rank of $m(V) - \ell$ in the (m,ℓ)-sparsity matroid. Thus there is an edge f in $\{i_1j_1,\dots,i_rj_r\}$ for which $E - e + f$ is a basis of the (m,ℓ)-sparsity matroid. Since $E - e + f$ is independent in the (m,ℓ)-sparsity matroid, we must have $e \in T(f)$. □

As noted in the introduction, the main advantage of (m,ℓ)-tight graphs over (k,ℓ)-tight graphs is that an (m,ℓ)-tight graph remains (m',ℓ')-tight after contracting an (m,ℓ)-tight subgraph for some pair (m',ℓ'), as follows.

Lemma 2.5. *Let $G = (V,E)$ be an (m,ℓ)-tight graph. Suppose that $T \subsetneq V$ is an (m,ℓ)-tight set in G. Let t' be the new vertex that arises after shrinking T in G. Let $\ell' := \max(\ell, 0)$ and let $m' : V(G/T) \to \mathbb{Z}_+$ be a map such that $m'(v) = m(v)$ when $v \in V(G/T) \cap V$ and $m'(t') = \ell'$.*

(a) A set $S \subseteq V(G/T)$ containing t' induces an (m',ℓ')-tight subgraph of G/T if and only if $S - t' \cup T$ is (m,ℓ)-tight in G. In particular, G/T is (m',ℓ')-tight.

(b) A set $S \subseteq V(G/T) - t'$ is (m',ℓ')-tight in G/T if and only if either $\ell \geq 0$ and S is (m,ℓ)-tight in G, or $\ell < 0$ and $S \cup T$ is (m,ℓ)-tight in G such that no edge of G connects S and T.

Proof. (a) First we show that G/T is (m', ℓ')-sparse. Let $X \subseteq V(G/T)$. Assume first that $t' \in X$. Then $i_{G/T}(X) = i_G(X - t' \cup T) - i_G(T) \leq m(X - t' \cup T) - \ell - (m(T) - \ell) = m(X - t') = m'(X) - m'(t') = m'(X) - \ell'$ as T is (m, ℓ)-tight. If $t' \notin X$ and $\ell \geq 0$, then $i_{G/T}(X) = i_G(X) \leq \max(m(X) - \ell, 0) = \max(m'(X) - \ell', 0)$. If $t' \notin X$ and $\ell < 0$, then $i_{G/T}(X) = i_G(X) \leq i_G(X \cup T) - i_G(T) \leq m(X \cup T) - \ell - (m(T) - \ell) = m(X) = m'(X) - \ell'$. Hence G/T is (m', ℓ')-sparse.

Assume now $t' \in S \subseteq V(G/T)$. By the (m', ℓ')-sparsity of G/T, the tightness of S means that $i_{G/T}(S) = m'(S) - \ell' = m(S - t')$. Since $i_G(T) = m(T) - \ell$, replacing t' with T results $i_G(S - t' \cup T) = m(S - t' \cup T) - \ell$. The same argument stands for the other direction.

(b) First assume that $\ell \geq 0$ and $t' \notin S \subseteq V(G/T)$. By the (m', ℓ')-sparsity of G/T, the (m', ℓ')-tightness of S means that $i_{G/T}(S) = m'(S) - \ell' = m(S) - \ell$. On the other hand, the (m, ℓ)-tightness of S means that $i_G(S) = m(S) - \ell$. Since $i_G(S) = i_{G/T}(S)$, these two are equivalent.

Now if $\ell < 0$ and $t' \notin S \subseteq V(G/T)$. By the (m', ℓ')-sparsity of G/T, the (m', ℓ')-tightness of S means that $i_{G/T}(S) = m'(S) - \ell' = m(S) - 0$. Since T is (m, ℓ)-tight and $i_G(S) = i_{G/T}(S)$, $m(S) + m(T) - \ell \geq i_G(S \cup T) = i_G(S) + i_G(T) + d_G(S, T) \geq (m(S) - 0) + (m(T) - \ell) + 0$ follows and hence equality must hold throughout, that is, $i_G(S \cup T)$ is (m, ℓ)-tight and $d_G(S, T) = 0$. The proof of the other direction is similar. □

2.1 Algorithmic preliminaries

To give a polynomial algorithm for our (general or reduced) augmentation problem, one first needs an algorithm for testing the (m, ℓ)-sparsity of a graph. Such polynomial algorithm already exists for each pair of m and ℓ (see the works of Hendrickson and Jacobs [13, 14] and Berg and Jordán [2] for the case where $k = 2$ and $\ell = 3$, the paper of Lee and Streinu [19] for general k and $\ell \geq 0$, the book of Frank [7, Section 13.5.4] for the (m, ℓ) case, and the note of the first author [17] for the case of $\ell < 0$). All the above mentioned algorithms are based on in-degree constrained orientations (see [7, Lemma 13.5.9]).

We note that in the main applications of (k, ℓ)-sparse graphs k and ℓ are fixed constants thus we may assume the following condition.

(*) There exists a constant $c > 0$ such that $m(v) \leq c$ for every $v \in V$ and $|\ell| \leq c$.

We give the running time of our algorithms by assuming this condition. Observe that (*) implies that an (m, ℓ)-sparse graph on V has $O(|V|)$ edges. We shall use the algorithm provided by the following theorem (which can be constructed based on the algorithms in [7, 17, 19]) as a subroutine in our algorithms.

Theorem 2.6 (Based on [7, 17, 19])**.** *There exists an algorithm which decides in $O(|V|^2)$ time whether its input graph $G = (V, E)$ is (m, ℓ)-sparse. It has the following outputs:*

If G is (m, ℓ)-sparse, then it outputs this fact along with an orientation D of the edges in G minus a set $F' \subseteq E$ of at most $-\ell$ edges when $\ell < 0$. If G is also (m, ℓ)-tight, then it also outputs this fact.

If G is not (m,ℓ)-sparse, then it outputs a maximal (m,ℓ)-sparse subgraph $H = (V,F)$ of G along with an orientation D of the edges in H minus a set F' of $-\ell$ edges when $\ell < 0$. It also outputs the set R of edges in H which are (m,ℓ)-redundant in G.

Furthermore, if it returns that G is (m,ℓ)-sparse (including the case when G is (m,ℓ)-tight), then by only using the extra data in the output one can decide in $O(|V|)$ extra time whether $G+e$ is (m,ℓ)-sparse for any new edge e, and if the answer is no, then output the (m,ℓ)-tight subgraph $T(e)$ of G. □

We will also need the following observation.

Observation 2.7. *If D is the orientation in the output of the algorithm of Theorem 2.6 for checking the (m,ℓ)-sparsity of the graph G and T is an (m,ℓ)-tight set in G like in Lemma 2.5, then D/T can be an output of that algorithm when we check the (m',ℓ')-sparsity of G/T, that is, the in-degree of each vertex x in D/T is at most $m'(x)$.*

Proof. Since T is (m,ℓ)-tight, T induces $m(T)-\ell$ edges in G, that is, T induces $m(T)-\ell'$ arcs in D. Hence the in-degree of T is at most ℓ' in D which implies our statement. □

3 The reduction of the general problem

García and Tejel [9] showed that the general augmentation problem is NP-hard for $(2,3)$-rigid graphs by reducing it to the set-cover problem. Based on their method we will prove in Section 7 the NP-hardness of the general problem whenever $k < \ell$. In this section we show that, in any other case (that is, if $\ell \le k$), there exists an $O(|V|^2)$ time reduction from the general problem to the reduced problem that we shall solve in Sections 5 and 6. Moreover, we give our reduction for all (m,ℓ)-rigid graphs for which $m \ge \ell$. Throughout this section, $\boldsymbol{\bar{G} = (V, \bar{E})}$ denotes an (m,ℓ)-rigid graph and $\boldsymbol{G = (V, E)}$ denotes an (m,ℓ)-tight spanning subgraph of $\bar{G}$. Obviously, every edge in $\bar{E}-E$ is (m,ℓ)-redundant in $\bar{G}$. By Lemma 2.4, the (m,ℓ)-redundant edges of G in $\bar{G}$ are the edges of $R^G(\bar{E}-E) = \bigcup_{e\in\bar{E}-E} T^G(e)$.

As mentioned in the Introduction, the idea of the reduction method comes from a paper of Jackson and Jordán [11]. Since $m \le \ell$, the (m,ℓ)-redundant edges of G in $\bar{G}$ form some vertex disjoint (m,ℓ)-redundant induced subgraphs of G by Lemmas 2.1 and 2.4. (Note that we have only one such subgraph when $\ell < 0$.) By shrinking each of these subgraphs to a single vertex and by defining $\ell' := \max(\ell, 0)$ and m' to be ℓ' on each of the shrunken vertices and to be $m(v)$ on each non-shrunken vertex v, we get the shrunken graph $G' = (V', E')$ along with the map $m' : V' \to \mathbb{Z}_+$. The following statement follows by using Lemma 2.5 for each shrunken tight subgraph, sequentially.

Proposition 3.1. *Let G be an (m,ℓ)-tight graph and let $G' = (V', E')$ and $m' : V' \to \mathbb{Z}_+$ arise from G as we defined above. Then G' is (m',ℓ')-tight. Furhermore, the pre-image of any (m',ℓ')-tight subgraph of G' is either (m,ℓ)-tight or, when $\ell < 0$, it gets (m,ℓ)-tight if we union it to the sole shrunken (m,ℓ)-tight subgraph of G. Moreover, the shrunken image of an (m,ℓ)-tight subgraph of G is (m',ℓ')-tight, and contains the only shrunken vertex when $\ell < 0$.*

Proof. The first two statements follow directly from Lemma 2.5. For the last statement, let T be an (m,ℓ)-tight subgraph of G. If we take the union of T with the shrunken T_i components whose vertex set is intersected by $V(T)$, we get another (m,ℓ)-tight subgraph of G by Lemma 2.1. (Note that T must intersect the single shrunken component T_1 when $\ell < 0$ since otherwise $T \cup T_1$ violates the (m,ℓ)-sparsity condition as $i_G(T \cup T_1) \geq i_G(T) \cup i_G(T_1) = m(T) - \ell + m(T_1) - \ell > m(T \cup T_1) - \ell$.) The shrunken image of this union, that coincides with the shrunken image of T, is (m',ℓ)-tight by Lemma 2.5. □

By Proposition 3.1, a covering of the edges of G that are not (m,ℓ)-redundant in $\bar{G}$ with (m,ℓ)-tight subgraphs gives a covering of G' with (m',ℓ')-tight subgraphs. Hence the minimum number of edges that we need to make G (m,ℓ)-redundant is at least the minimum number of edges that we need to make G' (m',ℓ')-redundant. The following statement shows that these two values are equal.

Proposition 3.2. *Let F' denote an edge set of minimum cardinality on V' for which $G' \cup F'$ is (m',ℓ')-redundant. Let F be an arbitrary pre-image of F', that is, we get F' from F by our shrinking procedure. Then $\bar{G} \cup F$ is (m,ℓ)-redundant.*

Proof. The shrunken image of $T^G_{(m,\ell)}(uv)$ is an (m',ℓ)-tight subgraph of G' that induces both of the images u' and v' of u and v by Proposition 3.1. Thus it is a supergraph of $T^{G'}_{(m',\ell')}(u'v')$ by Lemma 2.3. Since the image of each non-(m,ℓ)-redundant edge of $\bar{G}$ is in G' and the subgraphs $\{T^{G'}_{(m',\ell)}(u'v') : u'v' \in F'\}$ cover the edge set of G', the subgraphs $\{T^G_{(m,\ell)}(uv) : uv \in F\}$ cover every non-(m,ℓ)-redundant edge of $\bar{G}$. Hence $\bar{G} \cup F$ is (m,ℓ)-redundant by Lemma 2.4. □

With Proposition 3.2, we have reduced the problem of augmenting an (m,ℓ)-rigid graph to an (m,ℓ)-redundant graph to the problem of augmenting an (m',ℓ')-tight graph to an (m',ℓ')-redundant graph. It is easy to check that (A0) still holds after the reduction. Based on Theorem 2.6 the reduction can be done in $O(|V|^2)$ time. Note that when we get a solution for the arisen reduced problem, we can get back a solution to the original problem in linear time. This implies the following.

Theorem 3.3. *Let $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$ for which $m \geq \ell$ and (*) hold, and let $\bar{G} = (V, \bar{E})$ be an (m,ℓ)-rigid graph. In $O(|V|^2)$ time we can reduce the solution of the general augmentation problem on $\bar{G}$ to the solution of the reduced problem on an (m',ℓ')-tight graph on a vertex set of cardinality at most $|V|$.* □

Finally we note the following.

Remark 3.4. When we proceed the algorithm of Theorem 3.3 on a (k,ℓ)-rigid graph, then, for the resulting (m',ℓ')-tight graph, the function m' may only have two values: k and $\ell' = \max(\ell, 0)$.

4 Preprocessing

Our goal is to provide a solution for the reduced problem for all (m, ℓ)-tight graphs, however, it is easier to formulate our results by assuming the following conditions when $\ell > 0$.

(A) Assuming (A0) and (*) for m and ℓ, $G = (V, E)$ is an (m, ℓ)-tight graph on at least four vertices such that either $\ell \leq 0$ or all of the following three conditions hold.

- **(A1)** There exists no $v \in V$ such that $m(v) = 0$ and v is an isolated vertex.
- **(A2)** There exist $u, v \in V$ such that $V(uv) \neq \{u, v\}$.
- **(A3)** There exists no $v \in V$ such that $V(uv) = \{u, v\}$ for all $u \in V - v$ and $V - v$ induces an (m, ℓ)-tight graph.

We note that these conditions automatically hold for (k, ℓ)-tight graphs with sufficiently many vertices. Furthermore, using our conditions (A0) and (*) it is not hard to see that (A2) and (A3) hold for every (m, ℓ)-tight graph on at least $c^2 + 2$ vertices for which (A1) holds. Indeed, (A0) and $\ell > 0$ imply that, if $\{u, v\}$ is an (m, ℓ)-tight set in G, then it induces at least one edge except and hence the denial of (A2) (or (A3), respectively) contradicts the (m, ℓ)-sparsity of G (or the (m, ℓ)-tightness of $G[V - v]$ and G) when $m(u) = m(v) = \ell = 0$ does not hold for any two vertices u and v of G.

When G violates (A1), we may delete its isolated vertices with $m(v) = 0$ and solve the reduced problem for the arising graph. It is easy to check that this yields a solution of our original problem, too. Since it is straightforward to solve our augmentation problem on graphs on constant number of vertices and the above reduction can be done in $O(|V|^2)$ running time, the assumption of (A) does not restrict the usability of our results.

5 The min-max theorem for the reduced problem

In this section we prove Theorem 1.1 and its following extension for (m, ℓ)-tight graphs for which, throughout this whole section, **we assume (A)**, including that $G = (V, E)$ is an (m, ℓ)-tight graph on at least 4 vertices. The generalization of Theorem 1.1 for (m, ℓ)-tight graphs is the following.

Theorem 5.1. *If there exists any (m, ℓ)-co-tight set in G, then*

$$\begin{aligned}\min\{|F| : H = (V, F) \text{ is a graph for which } G + H \text{ is } (m,\ell)\text{-redundant}\}\\ = \max\left\{\left\lceil \frac{|\mathcal{C}|}{2} \right\rceil : \mathcal{C} \text{ is a family of disjoint } (m,\ell)\text{-co-tight sets}\right\}.\end{aligned}$$

Otherwise, $G + uv$ is (m, ℓ)-redundant for every pair $u, v \in V$.

Recall that we call a set of vertices $\emptyset \neq C \subsetneq V$ (m, ℓ)-co-tight in G if $i_G(V - C) = m(V - C) - \ell$, that is, if the complement of C is an (m, ℓ)-tight set in G. Equivalently (by $m(V) - \ell = |E| = e_G(C) + i_G(V - C)$), C is (m, ℓ)-co-tight if $e_G(C) = m(C)$. Note that for every $X \subset V$ for which $m(V - X) \geq \ell$, $e_G(X) \geq m(X)$ holds by the sparsity of $V - X$. A useful observation on the MCT sets is the following.

Lemma 5.2. *Assume that $\ell > 0$. Let C be an (m, ℓ)-MCT set and let $v \in C$. Then $m(v) \neq 0$.*

Proof. Suppose that $m(v) = 0$. When $|C| = 1$, the co-tightness of C implies that $d(v) = 0$, contradicting (A1). When $|C| \geq 2$, then $i(V - (C - v)) \geq i(V - C) = m(V - C) - \ell = m(V - (C - v)) - \ell$ follows by the co-tightness of C hence $V - (C - v)$ is also tight in G, contradicting the minimality of C. □

If an edge e is not incident with at least one vertex in a co-tight set C, then $V(e) \subseteq V - C$ by Lemma 2.3. Thus Lemma 2.4 implies the following simple observation

Observation 5.3. *Assume (A0). Let $G = (V, E)$ be an (m, ℓ)-tight graph and let $C \subset V$ be an (m, ℓ)-co-tight set. If $G + H$ is (m, ℓ)-redundant for $H = (V, F)$, then there exists an edge $uv \in F$ such that $u \in C$ or $v \in C$.* □

This observation immediately implies that min $\geq$ max in Theorem 5.1 since each co-tight set in $\mathcal{C}$ must contain an end-vertex of an edge of H.

Remember, we say that a set U covers a set family $\mathcal{C}$, if $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$. The following is another useful property of MCT sets.

Lemma 5.4. *Let $\mathcal{C}$ be the family of all (m, ℓ)-MCT sets of G. Suppose that $U \subseteq V$ is a set that covers $\mathcal{C}$. If $V' \subseteq V$ is a set of vertices such that $U \subseteq V'$ and V' induces an (m, ℓ)-tight subgraph in G, then $V' = V$. In particular, for two vertices $u, v \in V$, the set $\{u, v\}$ covers $\mathcal{C}$ if and only if $G + uv$ is (m, ℓ)-redundant.*

Proof. Let us suppose that there exists a proper tight set $V' \subsetneq V$ in G for which $U \subseteq V'$. Then $V - V'$ is co-tight by definition and hence there exists an MCT set $C \in \mathcal{C}$ such that $C \subseteq V - V'$. However, as $U \subset V'$, this contradicts the assumption that $|U \cap C| \geq 1$ for every $C \in \mathcal{C}$. The second statement follows by the first one and the tightness of $V(uv) \ni u, v$. □

Note that it is possible that there are no co-tight sets in a graph G. For example, $K_6 - e$ is $(3, 4)$-tight and there are no $(3, 4)$-co-tight sets in it. By Lemma 5.4, if there are no MCT sets in G then $G + uv$ is (m, ℓ)-redundant for any pair $u, v \in V$ which proves the last part of Theorem 5.1. Thus we may suppose that G contains at least one MCT set. We shall show that in this case min $\leq$ max also holds. This statement is obvious when the minimum is one since we assumed the existence of an MCT set in G. Hence we may assume that the minimum is at least two. We will show that, in this case, the maximum in Theorem 5.1 is obtained by the family of all (m, ℓ)-MCT sets in G that we denote by $\boldsymbol{\mathcal{C}^*}$ from now on. The following statement on the structure of MCT sets shows that the members of $\mathcal{C}^*$ are pairwise disjoint. It is an extension of a result of Jordán [15, Theorem 3.9.13] that states the same for $(2, 3)$-MCT sets.

Theorem 5.5. *The members of $\mathcal{C}^*$ are either pairwise disjoint and $|\mathcal{C}^*| \geq 3$ or there exists a pair $u, v \in V$ such that $T(uv) = G$, that is, $G + uv$ is (m, ℓ)-redundant.*

The proof of Theorem 5.5 requires more involved thoughts hence we prove it separately in Section 5.1. Now, to finish the proof of Theorem 5.1, we need to show that G can be augmented to an (m, ℓ)-redundant graph by using a set of $\left\lceil \frac{|\mathcal{C}^*|}{2} \right\rceil$ edges when at least two edges are needed for the augmentation. Our plan is to add these new edges between the members of $\mathcal{C}^*$. By Theorem 5.5, we may suppose from now on that all the MCT sets of G are pairwise disjoint and $|\mathcal{C}^*| \geq 3$. First, we show the following strengthening of the statement of Theorem 5.5.

Lemma 5.6. *Suppose that the members of $\mathcal{C}^*$ are pairwise disjoint and $|\mathcal{C}^*| \geq 3$. Let $C, C' \in \mathcal{C}^*$. Then $N(C) \cap C' = \emptyset$.*

Proof. Suppose first that $m(V - (C \cup C')) < \ell$ (and $\ell > 0$). In this case $|V - (C \cup C')| \leq 1$ holds by (A0). $V \neq C \cup C'$ since there exist at least three MCT sets. Therefore, $V - (C \cup C') = v$ for some $v \in V$ and v is a co-tight set on its own with $m(v) < \ell$. (A1) implies that $m(v) > 0$ also holds. By Lemma 2.2, there is no loop in G incident with v. Furthermore, since $\{v\}$ is co-tight in G, $d_G(v) = m(v)$. Note that $C \cup \{v\} = V - C'$ and $C' \cup \{v\} = V - C$ are tight sets in G each of which must induce at least one edge incident with v by Lemma 2.2. Since $V = C \cup C' \cup \{v\}$ and $|V| \geq 4$, at least one of C and C', say C, must contain at least 2 vertices. Hence $d_{G[C \cup \{v\}]}(v) \geq m(v)$ also follows by Lemma 2.2. But now the disjointness of C and C' implies that $d_G(v) > m(v)$, a contradiction.

On the other hand, if $m(V - (C \cup C')) \geq \ell$, then the statement follows by using Lemma 2.1 for $T_1 = G[V - C]$ and $T_2 = G[V - C']$. □

As we have noted before, we plan to connect the members of $\mathcal{C}^*$ by the new edges. Based on the above result, the following statement (which is illustrated by Figure 1) will provide a useful property of the arising redundant subgraphs.

Lemma 5.7. *Let C be an (m, ℓ)-MCT set in G and let $u \in C$ and $v \in V - (C \cup N(C))$ such that $m(v) \neq 0$. Then $C \cup N(C) \subset V(uv)$.*

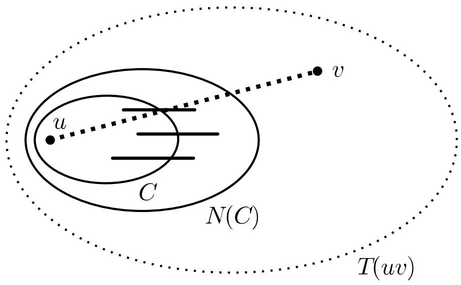

Figure 1: Illustration of Lemma 5.7

Proof. As $v \notin N(u)$ and $m(v) \neq 0$, condition (A0) implies that $\{u, v\}$ is not tight in G.

First we show that $C \subset V(uv)$. Note that $V - C$ is maximally tight that implies that either $V(uv) \cup (V - C) = V$ and hence $C \subset V(uv)$ or $V(uv) \cup (V - C)$ is not tight in G. In the latter case, $V(uv) \cap (V - C) = v$ and $m(v) < \ell$ both follow by Lemma 2.1. Thus, since v is not connected to C, the tightness of $T(uv)$ and the sparsity of G imply that $m(V(uv)) - \ell = i(V(uv)) = i(v) + i(V(uv) - v) \leq 0 + m(V(uv) - v) - \ell = m(V(uv)) - \ell - m(v)$ contradicting $m(v) > 0$.

Now the (m, ℓ)-tightness of $V(uv)$, the (m, ℓ)-sparsity of G, and the co-tightness of C imply that $m(V(uv)) - \ell = i_G(V(uv)) = i_G(V(uv) - C) + e_{T(uv)}(C) \leq i_G(V(uv) - C) + e_G(C) \leq m(V(uv) - C) - \ell + m(C)$. Hence equality must hold throughout, in particular, $e_{T(uv)}(C) = e_G(C)$ implying $N(C) \subset V(uv)$. □

Note that a transversal of $\mathcal{C}^*$ can be easily constructed by taking one arbitrary element of each member of $\mathcal{C}^*$ when the members of $\mathcal{C}^*$ are pairwise disjoint. Next we show that any connected graph on a transversal of $\mathcal{C}^*$ augments G to an (m, ℓ)-redundant graph, that is, $|\mathcal{C}^*| - 1$ edges are enough for the augmentation.

Lemma 5.8. *Suppose that there exists no pair $u, v \in V$ such that $T(uv) = G$. Let X be a transversal of $\mathcal{C}^*$ and let F be an edge set which induces a connected graph on $Y \subseteq X$. Then $R(F)$ is the minimal (m, ℓ)-tight subgraph inducing all elements of Y. In particular, if F is the edge set of a star $K_{1,|X|-1}$ on the vertex set X, then $G + F$ is (m, ℓ)-redundant.*

Proof. Recall that, $R(F)$ denotes the set of (m, ℓ)-redundant edges of G in $G+F$, and Lemma 2.4 claims that $R(F) = \bigcup_{f \in F} T(f)$. Let us use induction on $|F|$. If $F = \{yy'\}$, then $R(F) = T(yy')$ which is the minimal (m, ℓ)-tight subgraph of G containing both of y and y' by Lemma 2.3.

Let now $yy' \in F$ such that $F - yy'$ is connected. We may assume by switching the role of y and y' that $y \in V(R(F - yy'))$. Let $y'' \in V(R(F - yy')) - y$. By induction, $R(F - yy')$ is a tight subgraph of G which induces each elements of $V(R(F - yy'))$, in particular, it induces y and y''. Thus by Lemma 2.3, $T(yy'')$ is a subgraph of $R(F - yy')$.

Assume that the vertices $y, y', y'' \in Y$ are elements of the sets $C, C', C'' \in \mathcal{C}^*$, respectively. Lemmas 5.6 and 5.7 imply that $C \cup N(C) \subseteq V(yy')$ and $C \cup N(C) \subseteq V(yy'') \subseteq V(R(F - yy'))$. Note that $i_G(C) \leq m(C) - \ell$ by (m, ℓ)-sparsity and $e_G(C) = m(C)$ by the co-tightness of C in G. Hence $N(C) \neq \emptyset$ and $|C \cup N(C)| \geq 2$ holds when $\ell > 0$. Therefore, $m(V(yy') \cap V(R(F - yy'))) \geq m(C \cup N(C)) \geq \ell$ always holds, and Lemma 2.1 implies that $R(F - yy') \cup T(yy')$ is tight. This implies that $R(F)$ is tight since $R(F) = R(F - yy') \cup T(yy')$ follows by Lemma 2.4. Note that $Y \subseteq R(F)$ is obvious by $Y = V(F)$.

Let now T be the minimal tight subgraph of G which induces all elements of Y. Lemma 2.3 imply that $T(f) \subseteq T$ for each $f \in F$. Hence it follows by Lemma 2.4 that $R(F) = \bigcup_{f \in F} T(f) \subseteq T$, that is, $R(F) = T$.

Finally, observe that Lemma 5.4 and the tightness of $R(F)$ imply that, if F induces the whole set X (in particular, when it is a star on X), then $R(F)$ must be equal to G. □

The cardinality of the edge set provided by Lemma 5.8 can be decreased by iteratively using the following statement.

Lemma 5.9. *Suppose that the members of $\mathcal{C}^*$ are pairwise disjoint and $|\mathcal{C}^*| \geq 4$. Let y, x_1, x_2, x_3 be elements of distinct members of $\mathcal{C}^*$. Let $T^* = T(yx_1) \cup T(yx_2) \cup T(yx_3)$. Then $T^* = T(yx_1) \cup T(x_2x_3)$ or $T^* = T(yx_3) \cup T(x_1x_3)$ holds.*

Proof. Let us suppose that $T^* \neq T(yx_1) \cup T(x_2x_3)$. Thus there exists an edge e for which $e \in T^*$ and $e \notin T(yx_1) \cup T(x_2x_3)$.

Lemmas 2.4 and 5.8 imply that T^* is the minimal tight subgraph of G inducing all of y, x_1, x_2, and x_3. However, they similarly imply that this statement also holds for $T(yx_1) \cup T(x_2x_3) \cup T(yx_3)$ and $T(yx_1) \cup T(x_2x_3) \cup T(x_1x_2)$, that is, these two graphs both are equal to T^*. Since $e \in T^*$ and $e \notin T(yx_1) \cup T(x_2x_3)$, we get $e \in T(yx_3)$ and $e \in T(x_1x_2)$. (See Figure 2). Lemma 2.1 imply now that $T(yx_3) \cup T(x_1x_2)$ is a tight

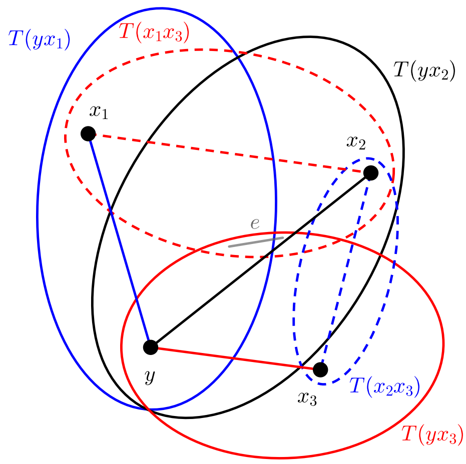

Figure 2: Illustration of Lemma 5.9

subgraph of G inducing all of y, x_1, x_2, and x_3, hence it must be equal to T^*. (We note that in this case $T^* = T(yx_2) \cup T(x_1x_3)$ also follows by a similar proof.) □

This finally allows us to finish the proof of Theorem 5.1.

Proof of Theorem 5.1. As mentioned before, we only need to prove that max $\geq$ min holds. This is obvious when min $= 1$. Hence we may assume that at least 2 edges are needed for the augmentation. In this case, the members of $\mathcal{C}^*$ are pairwise disjoint by Theorem 5.5 and we need to show that the augmentation can be done by using $\left\lceil \frac{|\mathcal{C}^*|}{2} \right\rceil$ edges. Let us denote a transversal set of the family of MCT sets by X. For an arbitrary $y \in X$, let the edge set $F_0 := \{yx : x \in X - y\}$, and let $H_0 = (V, F_0)$. By Lemma 5.8 $G + H_0$ is redundant. While there are at least three edges in H_i that are incident with y, we can decrease the number of edges in H_i and also the edges incident with y by Lemma 5.9 so that the arising graph $H_i = (V, F_i)$ still augments G to an (m, ℓ)-redundant graph. We can repeat this until the degree of y is at most two and the degree of every other vertex is at most one in the final graph $H = (V, F)$. Thus $|F| = \left\lceil \frac{|X|}{2} \right\rceil = \left\lceil \frac{|\mathcal{C}^*|}{2} \right\rceil$ and $G + H$ is (m, ℓ)-redundant. □

Before we turn to prove Theorem 5.5, we show the direct link between the results of this paper and the paper of García and Tejel [9]. We also use this result in the proof of the NP-hardness result in Section 7. Let us call an (m, ℓ)-tight subgraph G' of G **generated** if there are $u, v \in V$ such that $T(uv) = G'$.

Lemma 5.10. *Assume that there is no edge uv that augments G to an (m, ℓ)-redundant graph. Then $T(uv)$ is inclusion-wise maximal amongst all the generated subgraphs of G if and only if $u, v \in V$ are elements of two distinct (m, ℓ)-MCT sets. Moreover, two inclusion-wise maximal generated subgraphs $T(uv_1)$ and $T(uv_2)$ are equal if and only if v_1, v_2 are in the same (m, ℓ)-MCT set.*

Note that this result implies that 'classes of extreme vertices' defined in [9] are exactly the $(2, 3)$-MCT sets when no edge uv augments G to a $(2, 3)$-redundant graph

Proof. By Theorem 5.5, all the MCT sets of G are pairwise disjoint. Assume that $T(uv)$ is an inclusion-wise maximal generated subgraph of G for some $u, v \in V$. Let $\{x_1, \dots, x_t\}$ be a transversal of $\mathcal{C}^*$. By Lemma 5.8, $G = T(x_1x_2) \cup \dots \cup T(x_1x_t)$ hence we can assume that $u \in V(x_1x_2)$ and $v \in V(x_1x_2) \cup V(x_1x_3)$. On the other hand, $T(x_1x_2) \cup T(x_1x_3) = T(x_1x_2) \cup T(x_2x_3) = T(x_1x_3) \cup T(x_2x_3)$ by Lemmas 2.4 and 5.8. This means that one of the above three generated tight subgraphs, say $T(x_1x_2)$, must contain u and v both thus $V(uv) \subseteq V(x_1x_2)$ by Lemma 2.3. Since $T(uv)$ is inclusion-wise maximal equality must hold.

Let $C_1, C_2 \in \mathcal{C}^*$ such that $x_1 \in C_1$ and $x_2 \in C_2$. Suppose that $\{u, v\} \cap C_1 = \emptyset$. Note that $V - C_1$ is a tight set in G and hence $V(uv)$ is disjoint from C_1, in particular $x_1 \notin V(uv)$. However, this contradicts $V(uv) = V(x_1x_2)$ by Lemma 2.3. Therefore, with the same argument for C_2, either $u \in C_1$ and $v \in C_2$, or $u \in C_2$ and $v \in C_1$ must hold.

It is clear that taking any element x of C_1 and any element y of C_2 the generated tight set $T(xy)$ will be the same by Lemmas 5.7, 5.6 and 2.3. Finally, we need to show that $T(y_iy_j)$ is an inclusion-wise maximal generated subgraph of G for every $y_i \in C_i$, $y_j \in C_j$, and $1 \leq i < j \leq t$. This follows by the fact that, for a member $C \in \mathcal{C}$, $C \subseteq T(y_iy_j)$ if and only if $C = C_i$ or $C = C_j$ by Lemma 2.3 and the tightness of $V - C$. □

5.1 Proof of Theorem 5.5

In this subsection we prove Theorem 5.5. Recall that we assume (A) and that $G = (V, E)$ is an (m, ℓ)-tight graph while $\mathcal{C}^*$ is the family of all (m, ℓ)-MCT sets of G.

Lemma 5.11. *If X and Y are two (m, ℓ)-MCT sets in G, such that $X \cap Y \neq \emptyset$, then $m(V - (X \cup Y)) < \ell$. In particular, $|X \cup Y| \geq |V| - 1$.*

Proof. For the sake of contradiction, let us suppose that $m(V-(X\cup Y))\geq \ell$. $V-X$ and $V-Y$ are tight sets in G by the co-tightness of X and Y. $(V-X)\cap(V-Y)=V-(X\cup Y)$ hence $V-(X\cap Y)$ is also tight in G by Lemma 2.1. Thus $X\cap Y$ is co-tight, contradicting the minimality of X and Y. □

Note that Lemma 5.11 implies that all the MCT sets of G are pairwise disjoint when $\ell\leq 0$. Hence we only need to prove Theorem 5.5 for the case where $\ell>0$. In this case, we have two possibilities for intersecting MCT sets: their union may either equal to V or be of cardinality $|V|-1$. As we will see, the latter possibility (which does not arise if $m(v)\leq\ell$ holds for all $v\in V$) is more difficult.

For a vertex $v\in V$, let $\boldsymbol{\mathcal{C}^*(v)}:=\{C\in\mathcal{C}^*: v\notin C\}$. We use Lemma 5.11 to prove the following.

Lemma 5.12. *Suppose that $\ell>0$. Assume that there exists two (m,ℓ)-MCT sets $X,Y\in\mathcal{C}^*$ such that $X\cap Y\neq\emptyset$ and $X\cup Y=V-v$ for some $v\in V$. Then $\mathcal{C}^*(v)$ is a co-partition of $V-v$ with $|\mathcal{C}^*(v)|\geq 3$ or there exists a vertex $u\in V-v$ such that $T(uv)=G$.*

Proof. Assume that there exists no vertex $u\in V-v$ such that $T(uv)=G$. Then $\mathcal{C}^*(v)\neq\{X,Y\}$ since otherwise the set formed by a vertex $u\in X\cap Y$ and v would cover $\mathcal{C}^*$ contradicting Lemma 5.4 and $T(uv)\neq G$. Hence $|\mathcal{C}^*(v)|\geq 3$ as $X,Y\in\mathcal{C}^*(v)$. Let $Z\in\mathcal{C}^*(v)-\{X,Y\}$. As Z must intersect X or Y, $X\cup Z$ or $Y\cup Z$ is equal to $V-v=X\cup Y$ according to Lemma 5.11. However, then both $X\cup Z$ and $Y\cup Z$ are equal to $V-v$ by the same reasoning. Thus $Z\supseteq(X-Y)\cup(Y-X)$. This implies also that every two members of $\mathcal{C}^*(v)$ are intersecting and hence, for every three members W_1, W_2, and W_3 of $\mathcal{C}^*(v)$, $W_3\supseteq(W_1-W_2)\cup(W_2-W_1)$ holds. Therefore, every vertex in $V-v$ is avoided by at most one member of $\mathcal{C}^*(v)$. If there exists a vertex u that is contained in every member of $\mathcal{C}^*(v)$, then $\{u,v\}$ covers $\mathcal{C}^*$ contradicting Lemma 5.4 and $T(uv)\neq G$. Therefore, every vertex in $V-v$ is avoided by exactly one member of $\mathcal{C}^*(v)$, that is, $\mathcal{C}^*(v)$ is a co-partition of $V-v$. □

For a vertex $v\in V$ and a set $W\subseteq V-v$, let $\boldsymbol{\widetilde{W}^v}:=V-v-W$. This is the point where our proof gets more involved than that of [15, Theorem 3.9.13]. This is because the following lemma is quite complicated in the case of $\ell>\frac{3}{2}k$. Otherwise we could state it in an even stronger form and the proof would come by a slightly different version of Lemma 2.1 easily.

Lemma 5.13. *Suppose that $\ell>0$. Let $v\in V$ be a vertex for which the family $\mathcal{C}^*(v)$ is a co-partition of $V-v$ with $|\mathcal{C}^*(v)|\geq 3$. Suppose that there exists a vertex $u\in V-v$ with $m(u)\leq m(v)$. Let $W_1,W_2\in\mathcal{C}^*(v)$ and let $w_1\in\widetilde{W}_1^v$ and $w_2\in\widetilde{W}_2^v$. Suppose that V' is an (m,ℓ)-tight set in G with $w_1,w_2\in V'$. Then either $V'=V$ or $V'=\{w_1,w_2\}$. In particular, either $V(w_1w_2)=V$ (and $T(w_1w_2)=G$) or $V(w_1w_2)=\{w_1,w_2\}$.*

Proof. Assume that $V'\subsetneq V$ is a proper tight set in G such that $w_1,w_2\in V'$ (for example, $V(w_1w_2)$ is such a set if $V(w_1w_2)\neq V$). Note that $\{v,w_1,w_2\}$ covers $\mathcal{C}^*$ hence $V'\neq V$ implies that $v\notin V'$ by Lemma 5.4.

Suppose first that $|V' \cap \widetilde{Z}^v| \geq 2$ for some $Z \in \mathcal{C}^*(v)$. In this case $V' \cup (V - Z) = V' \cup (\widetilde{Z}^v \cup v)$ is a tight set in G by Lemma 2.1. Hence $V' \cup (V - Z) = V$ by Lemma 5.4, again. Lemma 2.1 also states that $d(v, Z) = 0$. This implies that $d(v, \widetilde{W}^v) = 0$ for each $W \in \mathcal{C}^*(v) - \{Z\}$. Note that $m(v) < \ell$ by Lemma 5.11 (since any two members of $\mathcal{C}^*(v)$ are intersecting and their union avoids v). If $m(v) > 0$ then Lemma 2.2 on the tight graph $G[V - W]$ imply that $d(v, \widetilde{W}^v) > 0$ for each $W \in \mathcal{C}^*(v) - \{Z\}$, contradicting $d(v, Z) = 0$ as $V - W - v = \widetilde{W}^v \subset Z$. However, if $m(v) = 0$, then by Lemma 5.2 v is not in any MCT sets. Thus $\{w_1, w_2\}$ covers all MCT sets, meaning that $V(w_1 w_2) = V$ by Lemma 5.4.

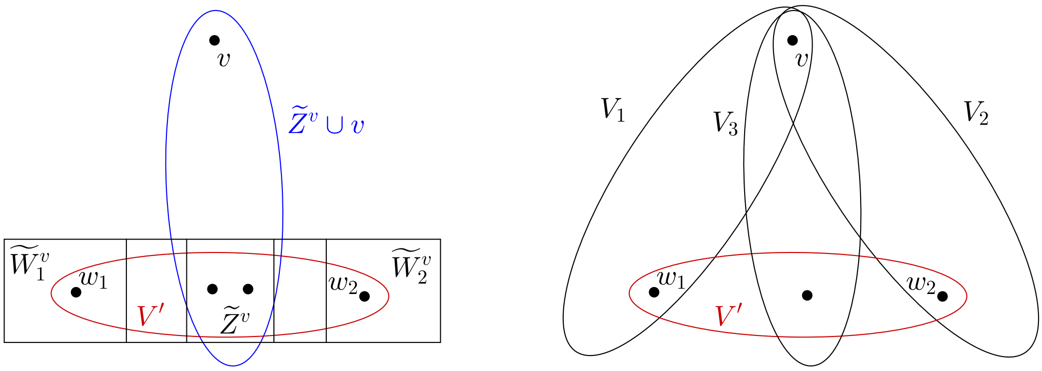

Figure 3: Illustrations of the proof of Lemma 5.13 and Claim 5.14

Now we might suppose that $|V' \cap \widetilde{Z}^v| \leq 1$ for every $Z \in \mathcal{C}^*(v)$. Let us consider the complement sets of the members of $\mathcal{C}^*(v)$. V' intersects at least two of them: $\widetilde{W}_1^v \cup v$ and $\widetilde{W}_2^v \cup v$. (See Figure 3 for an illustration.)

Claim 5.14. *V' intersects exactly two maximal tight sets containing v, namely, $\widetilde{W}_1^v \cup v$ and $\widetilde{W}_2^v \cup v$.*

Proof. Let us denote the family of maximal tight sets containing v by $\mathcal{F} = \{\widetilde{Z}^v \cup v$ where $Z \in \mathcal{C}^*(v)\}$. Suppose that V' intersects t members of $\mathcal{F}$, say $V_1, \ldots, V_t$. Since $|V' \cap \widetilde{Z}^v| \leq 1$ for every $Z \in \mathcal{C}^*(v)$ and $v \notin V'$, $|V'| = t$. Suppose that $t \geq 3$. Let E' denote the set of edges induced by V'.

As V' is (m, ℓ)-tight, $m(V') - \ell = |E'|$. Since every pair of vertices in V' induces an (m, ℓ)-sparse subgraph in G, $|E'| \leq (t-1)m(V') - \binom{t}{2}\ell$. This results $m(V') - \ell \leq (t-1)m(V') - \frac{t(t-1)}{2}\ell$. Hence $\frac{t(t-1)-2}{2}\frac{1}{t-2}\ell \leq m(V')$, and thus $\frac{t+1}{2}\ell \leq m(V')$.

Let E^* denote the union of E' and the set of edges induced by $V_1, \ldots, V_t$. Clearly, $m(V') - \ell + m(V_1) - \ell + \cdots + m(V_t) - \ell = |E^*|$. By the sparsity condition, $|E^*| \leq m(V_1 \cup \cdots \cup V_t) - \ell$. As $m(V_1 \cup \cdots \cup V_t) = m(V_1) + \cdots + m(V_t) - (t-1)m(v)$, we get $t\ell \geq m(V') + (t-1)m(v) \geq \frac{t+1}{2}\ell + (t-1)m(v)$. Thus $m(v) \leq \ell\frac{2t-t-1}{2(t-1)} = \frac{\ell}{2}$. By our condition in the lemma, there exists a vertex $u \in V - v$, for which $m(u) \leq m(v) \leq \frac{\ell}{2}$. Since $m(v) < \ell$ (by Lemma 5.11), $m(u) = m(v) = \ell = 0$ cannot hold. Hence $m(u) + m(v) > \ell$ must hold by (A0), contradicting $m(u) \leq m(v) \leq \frac{\ell}{2}$. Therefore, $t = 2$. ☐

This finishes the proof of the lemma. □

Based on Lemma 5.13, using assumption (A2) one can prove the following.

Lemma 5.15. *Suppose that $\ell > 0$. Let $v \in V$ be a vertex for which the family $\mathcal{C}^*(v)$ is a co-partition of $V - v$ with $|\mathcal{C}^*(v)| \geq 3$. Then $m(v) < m(u)$ holds for every $u \in V - v$ or there exist two vertices $x, y \in V - v$ such that $T(xy) = G$.*

Proof. Suppose that there exists a vertex $u \in V - v$ with $m(u) \leq m(v)$.

Suppose first that there is an MCT set $Z \in \mathcal{C}^*(v)$ for which $|\widetilde{Z}^v| \geq 2$. Let $z_1, z_2 \in \widetilde{Z}^v$ and let us take a vertex $x \in Z$ such that $m(x)$ is not the unique minimum of m. (Note that such x must exist as $|\mathcal{C}^*(v)| \geq 3$ and $\mathcal{C}^*(v)$ is a co-partition of $V - v$.) By Lemma 5.13, a tight set containing both of x and z_i is $\{x, z_i\}$ or V for $i = 1, 2$. Hence, for $i = 1, 2$, $\{x, z_i\}$ is a maximal proper (that is, $\neq V$) tight set in G or $T(xy) = G$ holds for $y = z_i$. Hence we may assume that $\{z_1, x\}$ and $\{z_2, x\}$ are maximal proper tight sets in G and their complements are MCT sets. Therefore, there exist at least two MCT sets avoiding x which are intersecting. Hence $\mathcal{C}^*(x)$ is a co-partition of $V - x$ or there exists a vertex $y \in V - x$ such that $T(xy) = G$ by Lemma 5.12. Note that in the latter case $y \neq v$ since there exists an MCT set in the co-partition $\mathcal{C}^*(v)$ of $V - v$ avoiding x and v, hence in this case we are done. In the first case, note that z_1 and z_2 are avoided by different members of $\mathcal{C}^*(w)$ since $V - \{z_1, w\}, V - \{z_2, w\} \in \mathcal{C}^*(w)$. Thus Lemma 5.13 for w assures that any tight set in G containing z_1 and z_2 is $\{z_1, z_2\}$ or V, however, this contradicts the tightness of $\widetilde{Z}^v \cup v$.

Finally, suppose that, $|\widetilde{Z}^v| = 1$ for each co-tight set $Z \in \mathcal{C}^*(v)$. This implies that $V(uv) = \{u, v\}$ for every $u \in V - v$. By (A2), there must exist a pair $x, y \in V - v$ for which $V(xy) \neq \{x, y\}$. Lemma 5.13 implies now that $V(xy) = V$ and $T(xy) = G$. □

Note that Lemma 5.11 implies that $m(v) < \ell$ when $\mathcal{C}^*(v)$ is a co-partition of $V - v$ with $|\mathcal{C}^*(v)| \geq 3$. Hence there always exists a vertex u with $m(u) \leq m(v)$ when G is a (k, ℓ)-tight graph or it arises from a (k, ℓ)-rigid graph with the algorithm of Theorem 3.3 (see Remark 3.4), that is, Lemma 5.15 implies automatically that $T(xy) = G$ for some $x, y \in V - v$. For completeness we show the extension of Lemma 5.15 that proves the same statement without using this condition on $m(v)$.

Lemma 5.16. *Suppose that $\ell > 0$. Let $v \in V$ be a vertex for which the family $\mathcal{C}^*(v)$ is a co-partition of $V - v$ with $|\mathcal{C}^*(v)| \geq 3$. Then there exists a pair of vertices $x, y \in V - v$ such that $T(xy) = G$.*

Proof. By Lemma 5.15, we only need to prove the case where $m(v) < m(u)$ holds for every $u \in V - v$. Furthermore, in this case the following claim also follows by Lemma 5.12 and 5.15. (Note that there exists no vertex $w \in V - v$ for which $T(vw) = G$ since $\mathcal{C}^*(v)$ is a co-partition.)

Claim 5.17. *Assume that there exist two intersecting MCT sets X and Y for which $X \cup Y = V - u$ for some $u \in V - v$. Then there exists a pair of vertices $x, y \in V - v$ for which $T(xy) = G$.* □

We may suppose that there exists an MCT set Z containing v since otherwise the family $\mathcal{C}^*$ is equal to $\mathcal{C}^*(v)$ which is a co-partition; hence it could be covered by a set $\{x, y\} \subseteq V - v$ implying $T(xy) = G$ by Lemma 5.4. Then, for every $X \in \mathcal{C}^*(v)$ intersecting Z, $X \cup Z = V$ holds by Claim 5.17. Thus $Z \neq V$ cannot intersect any member of $\mathcal{C}^*(v)$ since $\mathcal{C}^*(v)$ is a co-partition. Hence Z must be the singleton v. Therefore, $V - v$ is tight and $d_G(v) = m(v)$.

As $V - v$ is tight, (A3) implies that there exists a vertex $u \in V - v$ such that $V(uv) \neq \{u, v\}$, that is, $\{u, v\}$ is not tight. Let W_1 be the member of the co-partition $\mathcal{C}^*(v)$ of $V - v$ which does not contain this u. Since $\{u, v\}$ is not tight and W_1 is a minimal co-tight set in G with $u, v \notin W_1$, $|V - W_1| \geq 3$. Note that Lemma 5.11 implies that $m(v) < \ell$ and Lemma 5.2 implies that $m(v) > 0$. Hence Lemma 2.2 implies that $d_G(v, \widetilde{W_1}^v) = d_{G[V-W_1]}(v) \geq m(v)$. Let W_2 be another member of $\mathcal{C}^*(v)$. Now the tightness of $V - W_2$ in G and Lemma 2.2 implies that $d_G(v, \widetilde{W_2}^v) = d_{G[V-W_2]}(v) > 0$. However, since $\mathcal{C}^*(v)$ is a co-partition of $V - v$, $\widetilde{W_1}^v$ and $\widetilde{W_2}^v$ are disjoint, hence $m(v) = d_G(v) \geq d_G(v, \widetilde{W_1}^v) + d_G(v, \widetilde{W_2}^v) > m(v)$, a contradiction. □

Now we are ready to finish the prove Theorem 5.5.

Proof of Theorem 5.5. We have already seen that Lemma 5.11 implies the disjointness of the (m, ℓ)-MCT sets of G when $\ell \leq 0$. Hence it is enough to prove the statement when $\ell > 0$.

Let us suppose that there exists $X, Y \in \mathcal{C}^*$ such that $X \cap Y \neq \emptyset$. By Lemma 5.11, $|X \cup Y| \geq |V| - 1$ holds. According to Lemmas 5.12 and 5.16, either $X \cup Y = V$ holds or there exists a pair $u, v \in V$ such that $T(uv) = G$. By the minimality of MCT sets, every member of $\mathcal{C}^* - \{X, Y\}$ must contain at least one element of both of $V - Y = X - Y$ and $V - X = Y - X$. Hence each member $W \in \mathcal{C} - \{X\}$ intersects X. Thus, again by Lemmas 5.11, 5.12, and 5.16, $X \cup W = V$ for every $W \in \mathcal{C} - \{X\}$, that is, $V - X \subset W$. Let us take $u \in X$ and $v \in V - X$. Then $\{u, v\}$ covers $\mathcal{C}$ and hence $T(uv) = G$ by Lemma 5.4. □

Note that it can be checked in polynomial time whether $G + uv$ is (m, ℓ)-redundant for some pair $u, v \in V$. The naïve algorithm (which can be constructed from the algorithm of Theorem 2.6) has $O(|V|^3)$ running time. In Section 6, we give a rather complex algorithm deciding this problem based on the above proof that has $O(|V|^2)$ running time.

6 Algorithmic aspects

Our goal in this section is to show that the reduced augmentation problem can be solved in $O(|V|^2)$ time (under the assumption (*)).

By Section 5, the MCT sets of G play an important role in the solution of the reduced problem. Lemma 2.5 gives the idea to use the following algorithm to find an MCT set.

Algorithm 6.1. Input: *A graph $G = (V, E)$ along with $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$ such that (A) and (*) hold, and two vertices $u, v \in V$.*
Output: *An (m, ℓ)-MCT set in G that does not contain u, v or the edge uv when $T(uv) = G$.*

0. *Run the algorithm of Theorem 2.6 on G, let D be the output orientation.*
1. *Using the output of* Step *0, calculate $T := V^G_{(m,\ell)}(uv)$.*
2. ***If*** *$T = V$* ***then Output:*** *the edge uv,* ***STOP****.*
3. *Shrink T to t' according to Lemma 2.5, that is, $G' := G/T$, $D' := D/T$, $\ell' := \max(\ell, 0)$, $m'(u) := m(u)$ for each $u \in V(G') \cap V$, $m'(t') := \ell'$.*
4. *$v := t'$ (hence $m'(v) = \ell'$), $V^* := V(G') - v$.*
5. ***While*** *$V^* \neq \emptyset$,* ***do****:*
6. *Calculate $T' := V^{G'}_{(m',\ell')}(uv)$ using D'.*
7. ***If*** *$V' = V(G')$,* ***then*** *$V^* := V^* - u$.*
8. ***Else****,*
Shrink T' to t', so $G' := G'/T', D' := D'/T'$.
9. *$v := t'$, $V^* := V^* \cap V(G') - v$.*
10. ***Output:*** *$V(G') - v$.*

Lemma 6.2. *The output of Algorithm 6.1 is either the edge uv when $T(uv) = G$ or an (m, ℓ)-MCT set of G not containing u and v. The running time of Algorithm 6.1 is $O(|V|^2)$.*

Proof. If the output is the edge uv, then the algorithm returns this in Step 2 because $T^G_{(m,\ell)}(uv) = G$. For the other case we first prove the following.

Claim 6.3. *If $u \in V(G') - V^* - v$ in any state of the Algorithm, then $T^{G'}_{(m',\ell')}(uv) = G'$.*

Proof. The statement is obvious when we delete u from V^*. In any later state the statement follows by Lemma 2.5. □

Suppose now, that the output of Algorithm 6.1 is a vertex set. Applying Lemma 2.5 repeatedly, we can conclude that the original vertex set $U \subset V$ which is contracted during the algorithm (that is, for which $G' = G/U$), is an (m, ℓ)-tight set in G. Thus the output of Algorithm 6.1 is an (m, ℓ)-co-tight set.

Suppose that U is not an inclusion-wise maximal proper (m, ℓ)-tight set in G, that is, there exists a proper (m, ℓ)-tight set $T^* \supsetneq U$ in G. Take the image of T^* in the final G', denote it with $T^*_{G'}$. By Lemma 2.5, $T^*_{G'}$ is tight in G', and clearly $v \in T^*_{G'}$. Let $u \in T^*_{G'} - v$. Since $V^* = \emptyset$, $u \in V(G') - V^* - v$ also holds. Hence, by Lemma 2.3 and Claim 6.3 $V(G') \neq T^*_{G'} \supseteq V(T^{G'}_{(m',\ell')}(uv)) = V(G')$, a contradiction.

Step 0 runs in $O(|V|^2)$ time by Theorem 2.6. After this, Step 1 needs $O(|V|)$ running time and every execution of the loop takes at most $O(|V|)$ time by Theorem 2.6 and Observation 2.7. Thus the total running time of the algorithm is $O(|V|^2)$. □

We need to decide now whether there is any pair of vertices $u, v \in V$ for which $G + uv$ is (m, ℓ)-redundant. The motivation for this lays in Theorem 5.5, as we saw in Section 5 how the stucture of the MCT sets are completely different if there exists such an edge, or not. We noted in the end of Section 5.1 how this can be decided in running time $O(|V|^3)$. However, now we answer this question in $O(|V|^2)$ time. We start with the case when we have an MCT set consisting of a single vertex.

Lemma 6.4. *Assume (A) and (*). If we are given an (m, ℓ)-MCT singleton set $C = \{v\}$, then we can check whether there exists an edge xy such that $T(xy) = G$ and return it in $O(|V|^2)$ time.*

Proof. By Observation 5.3, if there exists a pair $x, y \in V$ such that $T(xy) = G$, then x or y must be v. Hence we need to check for each $x \in V - v$ whether $T(xv) = G$. This can be done in $O(|V|^2)$ total time by Theorem 2.6. □

Now we give the algorithm that decides whether G can be augmented to an (m, ℓ)-redundant graph by using one edge.

Algorithm 6.5. INPUT: *A graph $G = (V, E)$ along with $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$ such that (A) and (*) hold.*
OUTPUT: *If there exists an edge e such that $T(e) = G$, then e, otherwise a vertex v of an (m, ℓ)-MCT set.*

1. *Choose two vertices $u, v \in V$, such that $|V(uv)| > 2$. Also suppose that $m(v) \geq m(u)$.*
2. *Run Algorithm 6.1 with u and v. Result: edge e or MCT set C_{uv}.*
    If *the result is an edge e,* ***then Output*** *e,* ***STOP****.*
3. ***If*** *$|C_{uv}| = 1$,* ***then*** *run the algorithm of Lemma 6.4 with C_{uv}.*
    If *this outputs an edge e,* ***then Output*** *e,* ***STOP****.*
    Else*,* ***Output*** *the single element c_1 of C_{uv},* ***STOP****.*
4. *Let $c_1 \in C_{uv}$ be such that $m(c_1)$ is not the unique minimum of m.*
5. *Run Algorithm 6.1 with v, and c_1. Result: edge e or MCT set C_{vc_1}.*
    If *the result is an edge e,* ***then Output*** *e,* ***STOP****.*
7. ***If*** *$|C_{vc_1}| = 1$,* ***then*** *run the algorithm of Lemma 6.4 with C_{vc_1}.*
    If *this outputs an edge e,* ***then Output*** *e,* ***STOP****.*
    Else*,* ***Output*** *c_2,* ***STOP****.*
8. *Let $c_2 \in C_{vc_1} - C_{uv}$.*
9. *Run Algorithm 6.1 with c_1 and c_2. Result: edge e or MCT set $C_{c_1c_2}$.*
    If *the result is an edge e,* ***then Output*** *e,* ***STOP****.*
10. ***If*** *$C_{uv} \cap C_{vc_1} = \emptyset$, $C_{vc_1} \cap C_{c_1c_2} = \emptyset$, and $C_{c_1c_2} \cap C_{uv} = \emptyset$,* ***then Output:*** *$c_1$.*
11. ***Else****, check each possible edge from v and c_1, it gives a suitable edge e.* ***Output*** *e.*

Lemma 6.6. *Algorithm 6.5 decides whether there exists one edge e such that $T(e) = G$ and returns it. If there is no such edge, then it returns an element of an (m, ℓ)-MCT set. The algorithm runs in $O(|V|^2)$ time.*

For the proof of this lemma we shall use some notation from Section 5.

Proof. Observe that, whenever the algorithm returns a vertex, the output is a vertex $c_1 \in C_{uv}$ or $c_2 \in C_{vc_1}$ where C_{uv} and C_{vc_1} are MCT sets by Lemma 6.2.

By Lemma 6.2, if Algorithm 6.5 returns an edge e after any execution of Algorithm 6.1, then $T(e) = G$ holds thus we can stop. By Lemma 6.4, if it outputs an edge e in Step 3 or 7, then $T(e) = G$ holds thus we can stop. If the Algorithm returns c_1 after Step 10, then it is easy to see that G cannot be augmented to a redundant graph with only one edge by Observation 5.3. Also if its output is given in Step 3 or 7, then this is correct due to Lemma 6.4 and Observation 5.3. Thus we only need to prove that if Algorithm 6.5 reaches Step 11, then one edge from v or c_1 augments G to an (m,ℓ)-redundant graph.

Assume that we reached Step 11. Then neither $m(v)$ nor $m(c_1)$ is the unique minimum of m and $|C_{uv}|, |C_{vc_1}| \geq 2$. Also notice that in this case $\ell > 0$ according to Lemma 5.11.

Suppose first that $C_{c_1c_2}$ and C_{vc_1} are intersecting. $C_{c_1c_2}, C_{vc_1} \in \mathcal{C}^*(c_1)$ holds by their construction. By Lemma 5.12, there exists a vertex $w \in V - c_1$ such that $T(wc_1) = G$, or $\mathcal{C}^*(c_1)$ forms a co-partition. Since we have chosen $u, v \in V$ in Step 1 such that $|V(uv)| > 2$, $|V - C_{uv}| > 2$ holds. However, $V - C_{uv}$ is a tight set containing v and c_2 (which is in $C_{vc_1} - C_{uv}$). Hence, if $\mathcal{C}^*(c_1)$ forms a co-partition, then the tightness of $V - C_{uv}$ contradicts Lemma 5.13. Therefore, in this case there exists a vertex $w \in V - c_1$ such that $T(wc_1) = G$.

Suppose now that $C_{c_1c_2} \cap C_{vc_1} = \emptyset$. If C_{uv} and C_{vc_1} are intersecting, then $C_{uv}, C_{vc_1} \in \mathcal{C}^*(v)$. Now Lemma 5.12 implies that there exists a vertex $w \in V - v$ such that $T(wv) = G$, or $\mathcal{C}^*(v)$ forms a co-partition. However, if $\mathcal{C}^*(v)$ forms a co-partition, then, by Lemma 5.13, either $T(c_1c_2) = G$ or $V - C_{c_1c_2} = \{c_1, c_2\}$, contradicting the assumptions that $|C_{vc_1}| \neq 1$ and $C_{c_1c_2} \cap C_{vc_1} = \emptyset$. Therefore, in this case there exists a vertex $w \in V - v$ such that $T(wv) = G$.

If neither $C_{c_1c_2}$ and C_{vc_1} nor C_{uv} and C_{vc_1} are intersecting, then $C_{c_1c_2}$ intersects C_{uv} and hence $C_{c_1c_2}$ contains every vertex in $C_{vc_1} - c_2$ by Lemma 5.11 since $C_{uv} \cap C_{vc_1} = \emptyset$ and $c_2 \notin C_{c_1c_2}$. Thus $C_{c_1c_2}$ intersects C_{vc_1} by $|C_{vc_1}| > 1$, a contradiction. This proves the correctness of Algorithm 6.5.

We can find appropriate vertices u and v for Step 1 by checking the tighness of each 2-element vertex set in $O(|V|^2)$ total time. As Algorithm 6.1 and checking the size of the generated tight set for each possible new edge from a vertex runs in $O(|V|^2)$ time by Lemma 6.2 and Theorem 2.6 like in the proof of Lemma 6.4, and we need to run the $O(|V|^2)$ time algorithm of Lemma 6.4 at most twice, the total running time of Algorithm 6.5 is $O(|V|^2)$. □

Now, we focus our attention to the case where there is no edge e that augments G to a redundant graph. From this point on, idea of our algorithm is a generalization of that of García and Tejel [9] because of Lemma 5.10. The following greedy algorithm finds transversal set of the MCT sets of G starting from a vertex v that is in an MCT set.

Algorithm 6.7. Input: *A graph $G = (V, E)$ along with $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$ such that (A) and (*) hold and there exists no edge e that augments G to an (m,ℓ)-redundant graph, and a vertex $v \in V$ from an (m,ℓ)-MCT set of G.*

Output: *A transversal of the MCT sets of G:* $X = \{v, x_2, \ldots, x_t\}$.

0. *Run the algorithm of Theorem 2.6 on* G.
1. *Initialize* $X = \emptyset$. *All vertices are unmarked. Mark* v.
2. *Explore all vertices* $j \in V$:
If j *is unmarked,* ***then*** *calculate* $T(vj)$ *by using the output of* Step *0 and Mark all unmarked vertices in* $V(vj)$. *Let* $X := (X - V(vj)) + j$.
3. ***Output:*** $X + v$.

Lemma 6.8. *Algorithm 6.7 finds a transversal system of the MCT sets that contains* v *in* $O(|V|^2)$ *time.*

Proof. It is easy to see that, in the end of the algorithm, the subgraphs $\{T(vx) : x \in X\}$ are inclusion-wise maximal amongst the subgraphs $\{T(vx) : x \in V - v\}$. However, since v is an element of an MCT set, we can use Lemma 5.10 to conclude that x must be also an element of another MCT set, moreover, $X \cup v$ is indeed a transversal of $\mathcal{C}^*$.

Step 0 needs $O(|V|^2)$ running time and after this computing $T(vj)$ can be done in $O(|V|)$ time by Theorem 2.6. We need to compute $T(vj)$ for some $j \in V$ all together $O(|V|)$ times. Therefore, the total running time of the algorithm is $O(|V|^2)$. □

Finally, by using Lemma 5.9 and the above algorithms, we can give the following algorithm to find an optimal solution of the reduced problem in $O(|V|^2)$ time.

Algorithm 6.9. Input: *A graph* $G = (V, E)$ *along with* $m : V \to \mathbb{Z}_+$ *and* $\ell \in \mathbb{Z}$ *such that (A) and (*) hold.*
Output: *A minimum cardinality edge set* F *that augments* G *to an* (m, ℓ)*-redundant graph.*

1. *Run Algorithm 6.5. Result: edge* e *or vertex* y.
If *the result is an edge* e, ***then Output*** $\{e\}$, ***STOP***.
2. *Generate a transversal system* X *of the* (m, ℓ)*-MCT stets of* G *by using Algorithm 6.7 with* y.
3. $F := \{vy | v \in X - y\}$.
4. *Calculate* $T(f)$ *for all* $f \in F$ *by using the output of* Step *0 of Algorithm 6.7.*
5. ***While*** $d_F(y) \geq 3$, ***do***
Choose three neighbors of y *in* F, *say* x_i, x_j, x_k.
Calculate $T(x_j x_k)$ *by using the output of* Step *0 of Algorithm 6.7.*
If $T(yx_i) \cup T(x_j x_k) = T(yx_i) \cup T(yx_j) \cup T(yx_k)$, ***then***
$F := F - \{yx_j, yx_k\} + \{x_j x_k\}$.
Else, $F := F - \{yx_i, yx_k\} + \{x_i x_k\}$.
6. ***Output:*** F.

Lemma 6.10. *Algorithm 6.9 finds a minimum cardinality edge set that augments* G *to an* (m, ℓ)*-redundant graph in* $O(|V|^2)$ *time.*

Proof. If the algorithm outputs $\{e\}$ in Step 1, then $G + e$ is (m, ℓ)-redundant by Lemma 6.6. Otherwise, there is no edge such that $G + e$ is (m, ℓ)-redundant, and

STEP 1 gives a vertex y from an MCT set of G. In this case, Lemma 6.8 implies that the set X that we get in STEP 2 is a transversal of $\mathcal{C}^*$ containing y. By Lemma 5.8, the set F defined in STEP 3 augments G to an (m, ℓ)-redundant graph. Lemma 5.9 implies that $G + F$ remains (m, ℓ)-redundant while we run STEP 5. Finally, Theorem 5.1 implies that the output of the algorithm is optimal.

By Lemmas 6.6 and 6.8, STEP 1 and STEP 2 can be done in $O(|V|^2)$ time. By Theorem 2.6, STEP 4 needs $O(|V|^2)$ time. Also by Theorem 2.6 and by $|E| = O(|V|)$, each loop in STEP 5 can be executed in $O(|V|)$ time thus the running time of STEP 4 is $O(|V|^2)$. Therefore, the running time of Algorithm 6.9 is $O(|V|^2)$. □

It follows by Section 4 that we do not really need to assume that (A) hold for G. This imply the following.

Theorem 6.11. *Assume (*). Let $G = (V, E)$ be an (m, ℓ)-tight graph. There exists an algorithm that gives an optimal solution for the reduced augmentation problem in $O(|V|^2)$ time.* □

Theorem 3.3 implies now the following.

Theorem 6.12. *There exists an $O(|V|^2)$ time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}$ for which $m \geq \ell$ and (*) hold, and of an (m, ℓ)-rigid graph $\bar{G} = (V, \bar{E})$, such that $\bar{G} \cup F$ is (m, ℓ)-redundant.* □

7 Complexity results

In this section we prove that the general augmentation problem is NP-hard when $\ell > k$. Moreover, our method also implies that there exists no polynomial constant factor approximation algorithm for this problem if P≠NP.

García and Tejel showed in [9] that the general augmentation problem is NP-hard when $k = 2$ and $\ell = 3$. Our construction is based on their idea. First we show the NP-hardness of the following problem, called the **Colored Tight Augmentation** problem or **CTA** problem.

Problem. *Let $G = (V, E)$ be a (k, ℓ)-tight graph such that the edges in E are colored to red or black. Find a graph $H = (V, F)$ on the same vertex set with minimum number of edges, such that each black edge of G is (k, ℓ)-redundantin $G + H = (V, E \cup F)$.*

The CTA problem is an extension of the general problem: given an instance of the general problem, one can get an instance of the CTA problem by taking a spanning (k, ℓ)-tight subgraph and coloring each redundant edge to red and each other edge to black. By extending the work of García and Tejel [9], we shall prove that the CTA problem in NP-hard for every (k, ℓ) where $k > 1$, that is, also for $\ell \leq k$ in which case the general augmentation problem is solvable in polynomial time by Theorem 6.12. On the other hand, García and Tejel [9] observed in the $(k, \ell) = (2, 3)$ case that the CTA problem is equivalent to the general augmentation problem: adding parallel edges to the red ones reduces the solution of the CTA problem to the solution of the general problem. (We note that the same construction works when $\ell = 2k - 1$.)

We shall use the **set cover** problem to show the NP-hardness of the CTA problem. Given a ground set X and a family $\mathcal{S}$ of its subsets, a solution of the set cover problem is to find a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that $\bigcup \mathcal{S}' = X$ and $|\mathcal{S}|$ is minimal. It is well-known that the set cover problem is NP-hard, and cannot be approximated in constant factor unless P=NP (see [21]). Since it is easy to solve the problem on a ground set X of constant cardinality or when two members of $\mathcal{S}$ cover X, the problem remains NP-hard if we assume that $|X| \geq c$ for a given constant c and and no two members of $\mathcal{S}$ cover X.

Our NP-hardness proofs for both the CTA problem and the general problem are based on the following statement.

Lemma 7.1. *Let $G = (V, E)$ be a (k, ℓ)-tight graph, and let $G' = (V', E')$ be a connected graph such that $i_{G'}(V \cap V') = 0$, the graph $G^* = (V \cup V', E \cup E')$ is (k, ℓ)-tight, and $V' - V$ forms a (k, ℓ)-MCT set in G^*. Suppose that there are at least two disjoint (k, ℓ)-MCT sets in G which do not intersect $V' \cap V$.*

(a) Then the (k, ℓ)-MCT sets of G^* are exactly $V' - V$ and those (k, ℓ)-MCT sets of G that do not intersect $V' \cap V$.

(b) Moreover, assuming that X, Y are (k, ℓ)-MCT sets of G with $X \cap V' \neq \emptyset$ and $Y = V' \neq \emptyset$, for each triple $v' \in V' - V$, $x \in X$, and $y \in Y$, $T^G(xy) \subset T^{G^*}(v'y)$ holds.

Proof. (a) Observe that $e_{G^*}(Z) = e_G(Z)$ holds for each $Z \subseteq V - V'$ by $i_{G'}(V \cap V') = 0$ and $e_{G^*}(Z) > e_G(Z)$ holds for each $Z \subseteq V$ for which $Z \cap V' \neq \emptyset$, as every $v \in V \cap V'$ has a neighbor in $V' - V$ by the connectivity of G'. This implies that those MCT sets of G that do not intersect $V' \cap V$ are MCT in G^*, since for such a set Z, $k|Z| = e_G(Z) = e_{G^*}(Z)$ holds. Hence G^* has at least 3 disjoint MCT sets: the two MCT sets of G which do not intersect V', and $V' - V$. Thus, by Theorem 5.5, the MCT sets of G^* are pairwise disjoint. However, then by Lemma 5.6 there is no $Z \neq V' - V$ MCT set in G^* which intersects V'.

(b) By Lemmas 5.7 and 5.6, $(V' - V) \cup N_{G^*}(V' - V) = V' \subset V^{G^*}(v'y)$. Hence $T^{G^*}(x'y) \subset T^{G^*}(v'y)$ holds for each $x' \in X \cap V'$ by Lemma 2.3. On the other hand, $T^{G^*}(x'y) = T^G(x'y)$ since $x', y \in V$ and $V' - V$ is co-tight in G^*. Furthermore, $T^G(x'y) = T^G(xy)$ follows by Lemma 5.10, completing our proof. □

Now, we are ready to show the NP-hardness of the CTA problem.

Theorem 7.2. *Let $k > 1$ and $\ell < 2k$ be two integers. Then the CTA problem is NP-hard on (k, ℓ)-tight graphs, moreover, there exists no polynomial time constant factor approximation algorithm for it if P≠NP.*

Proof. Given an instance of the set cover problem, a family $\mathcal{S}$ on ground set X such that $|X| \geq 2k + 1$ and no two members of $\mathcal{S}$ cover X, we construct our graph in the following way. Let us take a (k, ℓ)-tight graph G_0 on a copy X' of X. (It is easy to check that such graph exists when $|X| \geq 2k + 1$.) Let us take another copy X'' of X and connect the copies x' and x'' of each $x \in X$ by an edge e_x. These $|X|$ edges will be the only black edges in our final graph; every other edge will be red. Add new edges between X' and X'' until $d(x'') = k$ holds for every $x'' \in X''$. Let us denote the graph we got with these steps by G_1.

Claim 7.3. *The family of MCT sets in G_1 equals to the family formed by all one element subsets of X''. Furthermore, $T^{G_1}(x''y'')$ induces exactly two black edges e_x and e_y for each pair $x, y \in X$.*

Proof. The first statement follows directly by Lemma 7.1 (a). Observe that $\{x''\} \cup N(\{x''\}) \cup \{y''\} \cup N(\{y''\}) \subseteq V^{G_1}(x''y'')$ by Lemma 5.7. Hence the two black edges e_x and e_y are induced by $T^{G_1}(x''y'')$. On the other hand, the co-tightness of $\{z''\}$ in G_1 implies that $z'' \notin V^{G_1}(x''y'')$ for each $z \in X - \{x, y\}$. Hence $e_z = z'z''$ is not induced by $T^{G_1}(x''y'')$ for any $z \in X - \{x, y\}$. □

Now for every $S \in \mathcal{S}$ we make the following extension on G_1. Let S'' denote the copy of S in X''. We start with $V_S = \emptyset$. We first choose k vertices from S'' (if $|S''| \geq k$) and add a new vertex v with (red) edges to these k vertices. We also add v to V_S. Later, when V_S is not empty, we take the last vertex that is added to V_S, say v, and $k-1$ other vertices from S'' which were not used before and add a new vertex w of degree k connecting to these k vertices. We also add w to V_S. We proceed the above addition until there are vertices in S'' that were not used in such a step. In the last step, there may be less than $k-1$ such vertices in S'' (or less than k in the first step). In this case, we take the rest of the neighbors from X' (remember, $|X'| \geq 2k+1$). (See Figure 4.)

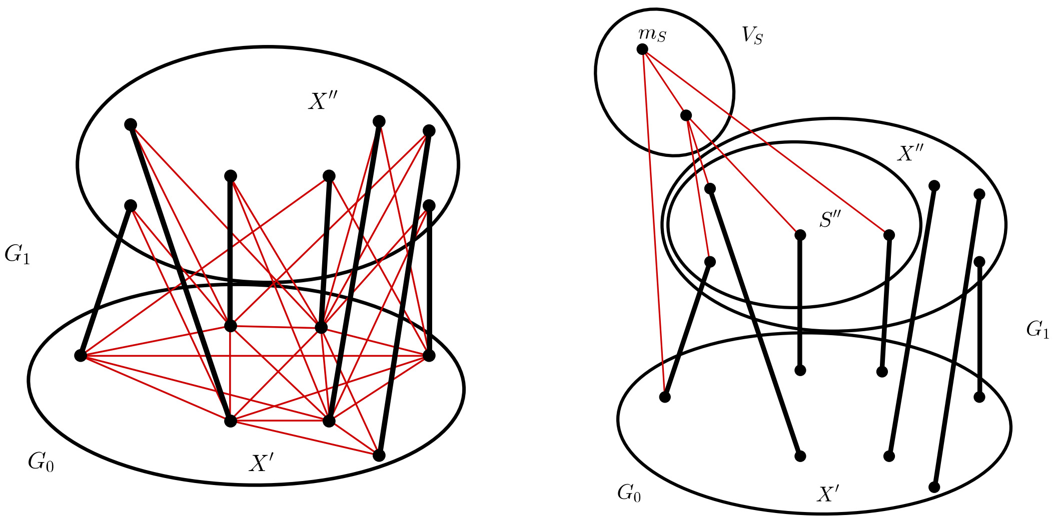

Figure 4: Constuction of G^* for $(k, \ell) = (3, 5)$. On the right side, the red edges of G_1 are hidden.

Let us denote the graph that we get from G_1 after running the above procedure for each $S \in \mathcal{S}$ by $G^* = (V^*, E^*)$. For an arbitrary $S \in \mathcal{S}$, $d(v) = k$ holds for every $v \in V_S$ in the moment when it is added to V_S, however, in the subsequent step this is increased to $k+1$, except for the last vertex added to V_S that we call m_S. By Lemma 7.1 (a), this shows that the only MCT sets in G^* are the singletons $M_S := \{m_S\}$ for all $S \in \mathcal{S}$. By the construction, it is also easy to see that V_S is co-tight in G^* for every $S \in \mathcal{S}$.

By Lemma 5.10, we may assume that the optimal solution of the CTA problem consists of edges between MCT sets. Hence to finish our proof we only need to prove the following statement.

Claim 7.4. *Let $S_1, S_2 \in \mathcal{S}$ and let $e = m_{S_1} m_{S_2}$ be an edge connecting the two (k,ℓ)-MCT sets M_{S_1} and M_{S_2} of G^*. Then a black edge e_x is contained in $T^{G^*}(e)$ if and only if $x \in S_i \cup S_j$.*

Proof. Since V_S is co-tight for each $S \in \mathcal{S}$, $T^{G^*}(e) \subseteq G^*[X' \cup X'' \cup V_{S_i} \cup V_{S_j}] =: G_{ij}$ by Lemma 2.3 and hence $T^{G^*}(e) = T^{G_{ij}}(e)$. Now, by using Lemma 7.1 (b) several times following the construction of V_{S_i} and V_{S_j}, we get that $T^{G_0}(x''y'') \subset T^{G_{ij}}(e)$ for each pair $x, y \in S_i \cup S_j$. Hence $T^{G^*}(e) = T^{G_{ij}}(e)$ induces the black edge e_x for each $x \in S_i \cup S_j$ by Claim 7.3. On the other hand, $z'' \notin V^{G_{ij}}(e)$ for $z \in X - (S_i \cup S_j)$ since $\{z''\}$ is co-tight in $G_i j$ by Lemma 7.1 (a). Hence the black edge e_z is not induced by $T^{G^*}(e) = T^{G_{ij}}(e)$. □

From the above claim, one can see that any (not necessarily optimal) solution of the CTA problem on G^* of cardinality q gives a (not necessarily optimal) solution of the set cover problem that uses at most $2q$ sets and every (not necessarily optimal) solution of the set cover problem with cardinality q gives a (not necessarily optimal) solution of the CTA problem with cardinality $\lceil \frac{q}{2} \rceil$. However, there is no constant factor approximation of the set cover problem unless P=NP by [21]. Therefore, there is no constant factor approximation of the CTA problem unless P=NP. This finishes the proof of Theorem 7.2. □

To show the NP hardness of the general problem for $k < \ell$, we shall modify the above NP-hardness proof in such a way that we usually add $2k - \ell$ parallel edges between two connected verices. To this end, we shall attach MCT sets – other than the singletons used in the above proof – by using Lemma 7.1. The following two lemmas assert that such extension is possible when $k < \ell$. For a set E and $c \in \mathbb{Z}_+$ let $\boldsymbol{cE}$ denote the multiset that arises by taking c copy of each element of E. For a graph $G = (V, E)$, $\boldsymbol{cG}$ denotes the graph (V, cE).

Lemma 7.5. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Then there exists a graph $H_{(k,\ell)} = (V_{(k,\ell)} + w, E_{(k,\ell)})$, called a (k,ℓ)-**gadget**, on $2k - \ell + 1$ vertices that has no loops, $|E_{(k,\ell)}| = k$, $d_{H_{(k,\ell)}}(w) \geq 2$, $(2k-\ell)H_{(k,\ell)}$ is (k,ℓ)-sparse, and $e_{(2k-\ell)H_{(k,\ell)}}(X) > k|X|$ holds for every $X \subsetneq V_{(k,\ell)}$.*

Proof. Let $V_{(k,\ell)} := \{v_1, \ldots, v_{2k-\ell}\}$ and let $v_{2k-\ell+1} := v_1$. Let us form a cycle $C_{(k,\ell)} = \{v_i v_{i+1} : i \in \{1, \ldots, 2k-\ell\}\}$ on $V_{(k,\ell)}$. This consists of $(2k-\ell)$ edges. Since $|E_{(k,\ell)}| = k$, we need to add $\ell - k$ edges to $C_{(k,\ell)}$ to construct $E_{(k,\ell)}$. We distribute these edges between $V_{(k,\ell)}$ and w as equal as possible. (See Figure 5 (a).)

Claim 7.6. *It is possible to define a set $E^w_{(k,\ell)}$ of $\ell - k$ edges between $V_{(k,\ell)}$ and w in such a way that, for every set X of consecutive vertices on the cycle $C_{(k,\ell)}$, $d_{H_{(k,\ell)}}(w, X) =$ either $\lfloor |X| \frac{\ell-k}{2k-\ell} \rfloor$ or $\lceil |X| \frac{\ell-k}{2k-\ell} \rceil$.*

Proof. Observe that it is enough to prove the statement when X is an interval on the cycle not containing $v_{2k-\ell}v_1$, since $d_{w,H_{(k,\ell)}}(X) + d_{w,H_{(k,\ell)}}(V_{(k,\ell)} - X) = k - \ell$ and $k - \ell - \lfloor |X|\frac{\ell-k}{2k-\ell} \rfloor = \lceil |V_{(k,\ell)} - X|\frac{\ell-k}{2k-\ell} \rceil$.

For the proof, we will use some results on **totally unimodular** (**TU** for short) matrices which are matrices with only 0 and ± 1 subdeterminants. For more details, we refer to [7, Section 4.2].

Let us take the incidence matrix A of all sets of consecutive vertices on the cycle not containing $v_{2k-\ell}v_1$. Then A is TU since it is an incidence matrix of subpaths of a path which is a network matrix and hence is TU (see [7, Corollary 4.2.6]). Let $x_0 = (\frac{\ell-k}{2k-\ell}, \dots, \frac{\ell-k}{2k-\ell})$ be a $2k - \ell$ dimensional vector. The rounding property of TU matrices [7, Lemma 4.3.4] asserts that there exists an integer vector q with $\lfloor x_0 \rfloor \le q \le \lceil x_0 \rceil$ and $\lfloor Ax_0 \rfloor \le Aq \le \lceil Ax_0 \rceil$. Let us add q_i edges between v_i and w to $E^w_{(k,\ell)}$. Now, the row corresponding to X in $\lfloor Ax_0 \rfloor \le Aq \le \lceil Ax_0 \rceil$ implies that $\lfloor |X|\frac{\ell-k}{2k-\ell} \rfloor \le e_{E^w_{(k,\ell)}}(X) \le \lceil |X|\frac{\ell-k}{2k-\ell} \rceil$ holds for each interval X on on the cycle not containing $v_{2k-\ell}v_1$, and $e_{E^w_{(k,\ell)}}(V_{(k,\ell)}) = \frac{\ell-k}{2k-\ell}(2k - \ell) = \ell - k$ as we wanted. $\square$

Clearly, it is enough to prove that $e_{(2k-\ell)H_{(k,\ell)}}(X) > k|X|$ holds for every $X \subsetneq V_{(k,\ell)}$ for which $H_{(k,\ell)}[X]$ is connected, that, is for every set X of consecutive vertices on the cycle such that $X \neq V_{(k,\ell)}$. Now $e_{(2k-\ell)H_{(k,\ell)}}(X) = e_{(2k-\ell)C_{(k,\ell)}}(X) + d_{(2k-\ell)E_{(k,\ell)}}(w, X) \ge (2k - \ell)|X| + (2k - \ell) + (2k - \ell)\lfloor |X|\frac{\ell-k}{2k-\ell} \rfloor > (2k - \ell)|X| + (2k - \ell)|X|\frac{\ell-k}{2k-\ell} = k|X|$.

To finish the proof we need to check whether $d_{H_{(k,\ell)}}(w) \ge 2$ holds. In our construction, $d_{H_{(k,\ell)}}(w) = \ell - k$ hence the only problem arises when $\ell = k + 1$. In this case we modify our construction, as follows. Assume that the only neighbor of w is v_1. Let us delete the edge v_1v_{k-1} and add the edge wv_{k-1}. It is easy to check that $e_{(k-1)H}(X) > k|X|$ still holds for every $X \subsetneq V'$. $\square$

Lemma 7.7. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Let $G = (V, E)$ be a (k, ℓ)-tight graph and let $G^* = (V \cup V_{(k,\ell)}, E \cup (2k - \ell)E_{(k,\ell)})$ be an extension of G such that all edges in $E_{(k,\ell)}$ are incident with at least one vertex in $V_{(k,\ell)}$, $E_{(k,\ell)}/V$ is isomorphic to a (k, ℓ)-gadget $H_{(k,\ell)}$ (with w representing the image of V), and $d_{E_{(k,\ell)}}(v) \le 1$ for each $v \in V$. Then G^* is (k, ℓ)-tight and $V_{(k,\ell)}$ is a (k, ℓ)-MCT set in G^*.*

We call the above extension of G a (k, ℓ)**-gadget extension** (see Figure 5 (b)). A (k, ℓ)-gadget extension plays roughly the same rule as the degree k vertex addition in the $\ell = 2k - 1$ case.

Proof. First observe that $e_{G^*}(X) > k|X|$ for all $X \subsetneq V_{(k,\ell)}$, furthermore, $e_{G^*}(V_{(k,\ell)}) = k|V_{(k,\ell)}|$ by Lemma 7.5. Hence if G^* is (k, ℓ)-tight, then $V_{(k,\ell)}$ is an MCT set in G^*.

Since $|E_{(k,\ell)}| = k$ and $|V_{(k,\ell)}| = 2k - \ell$ by Lemma 7.5, and since G is (k, ℓ)-tight, $|E \cup (2k - \ell)E_{(k,\ell)}| = k|V \cup V_{(k,\ell)}| - \ell$. Let $Y \subseteq V \cup V_{(k,\ell)}$ such that $|Y \cap V| \ge 2$. By the (k, ℓ)-sparsity of G and Lemma 7.5, $i_G(Y) = i_G(Y \cap V) + (2k - \ell)|E_{(k,\ell)}| - e_{G^*}(V_{(k,\ell)} - Y) \le k|Y \cap V| - \ell + k|V_{(k,\ell)}| - k|V_{(k,\ell)} - Y| = k|Y| - \ell$. Now let $Y \subseteq V \cup V_{(k,\ell)}$ such that $|Y \cap V| \le 1$. Then $G^*[Y - V]$ is (k, ℓ)-sparse by Lemma 7.5 and $G^*[Y]$ arises from $G^*[Y - V]$ by adding at most one vertex and $2k - \ell$ copy of at most one

(a) A $(10, 14)$-gadget.

(b) A $(10, 14)$-gadget-extension.

Figure 5: Illustration of the gadget-extension.

edge incident with this new vertex (where $2k - \ell < k$ by $k < \ell$). Hence $G^*[Y]$ is (k, ℓ)-sparse. Therefore, G^* is (k, ℓ)-tight. □

Now we are ready to prove the main result of this section.

Theorem 7.8. *Let k and ℓ be two positive integers such that $k < \ell < 2k$. Then the general augmentation problem is NP-hard on (k, ℓ)-rigid graphs, moreover, there exists no polynomial time constant factor approximation algorithm for it if P≠NP.*

Proof. Given a family $\mathcal{S}$ on ground set X such that $|X| \geq 2k+1$ and no two members of $\mathcal{S}$ cover X, we construct our 2-edge-colored (k, ℓ)-tight graph in the following way. Like in the proof of Theorem 7.2, let us take a (k, ℓ)-tight graph $G_0 = (X', E_0)$ on a copy X' of X. Let us perform $|X|$ (k, ℓ)-gadget extensions on G_0 with the (k, ℓ)-gadgets defined in Lemma 7.5. We perform the (k, ℓ)-gadget extensions in such a way that, in the step where we add $V^i_{(k,\ell)}$ for $i \in \{1, \dots, |X|\}$, we only connect $V^i_{(k,\ell)}$ to X' and one of the edges, say, e_{x_i} connecting $V^i_{(k,\ell)}$ and X' connects $V^i_{(k,\ell)}$ to the copy x'_i of x_i. The $(2k - \ell)$ parallel copies of the edges e_x for all $x \in X$ will be the only black edges in our final graph; every other edge will be red. Let us call the graph we got from G_0 by G_1. Lemmas 7.1 (a), 7.7, and 5.7 imply the following with a proof similar to that of Claim 7.3.

Claim 7.9. *G_1 is a (k, ℓ)-tight graph, where $V^i_{(k,\ell)}$ is a (k, ℓ)-MCT set for $i = 1, \dots, |X|$ and these are the only (k, ℓ)-MCT sets of G_1. Furthermore, $T^{G_1}(x''y'')$ induces exactly $2(2k - \ell)$ black edges (namely, the $2k - \ell$ copies of e_x and e_y) for each pair $x'' \in V^i_{(k,\ell)}$ and $y'' \in V^j_{(k,\ell)}$ for $1 \leq i < j \leq |X|$.* □

Now $G^* = (V^*, E^*)$ arises from G_1 like in the proof of Theorem 7.2, although, we use (k, ℓ)-gadget extension instead of degree k vertex addition. The only difference is that the MCT sets of G^* are not singletons, although, they are still the the set of vertices M_S added by the last (k, ℓ)-gadget extension for $S \in \mathcal{S}$. Thus, similarly to Theorem 7.2, there exists an optimal solution of the CTA problem on G^* which consists of edges between the sets M_S for $S \in S$. Now, by copying the proof of Claim 7.4, we get the following.

Claim 7.10. *Let $S_1, S_2 \in \mathcal{S}$ and let e be an edge connecting the two (k,ℓ)-MCT sets M_{S_1} and M_{S_2} of G^*. Then a black edge e_x is contained in $T^{G^*}(e)$ if and only if $x \in S_1 \cup S_2$.* □

Now the same argument that we used at the end of the proof of Theorem 7.2 shows that there is no constant factor approximation of the CTA problem on 2-edge-colored graphs G^* that arise by the above construction unless P=NP.

Let $\bar{G}$ arise from G^* by adding a new parallel copy to each red edge of G^*. It may mean more parallel new edges added between two vertices. It is thou obvious that all red edges of G^* are (k,ℓ)-redundant in $\bar{G}$. On the other hand, for an edge e of G_0, $T^{G^*}(e)$ is a subgraph of the (k,ℓ)-tight subgraph G_0 of G^* by Lemma 2.3; and for an edge $e = uv$ that we added with a (k,ℓ)-gadget extension $T^{G^*}(e) = G^*[\{u,v\}]$ since the set $\{u,v\}$ induces $2k-\ell$ parallel edges in G^* and hence it is (k,ℓ)-tight. Therefore, no black edge of G^* is (k,ℓ)-redundant in $\bar{G}$. This finishes the proof of Theorem 7.8. □

8 Concluding remarks

Simple graphs In several applications in rigidity theory (see [12, 23]), it must be assumed that all considered graphs are **simple**, that is, have no loops nor parallel edges. Hence only those redundant augmentations are appropriate that maintain this property, that is, we cannot add an edge parallel to an existing edge of the graph. (Recall that we never use two parallel edges to an optimal augmentation by Lemma 2.4.) We consider here a slight generalization of this problem that we call the **restricted general problem**: *Given an (m,ℓ)-rigid graph $G = (V,E)$ and $E' \subseteq E$, find a graph $H = (V,F)$ on the same vertex set with minimum number of edges, such that $G+H$ is (m,ℓ)-redundant and the edges in F are not parallel to the edges in E'.* We call the version where the input is (m,ℓ)-tight the **restricted reduced problem**. It is easy to see that our algorithm solves this problem for $(k,2k-1)$-tight inputs in $O(n^2)$ time (when a solution exists) since in this case two parallel edges form a circuit of the sparsity matroid. For general (m,ℓ), the problem is still solvable, as follows.

By using the same reduction as in Section 3, we can reduce the solution of any instance of the restricted general problem to the solution of the restricted reduced problem in $O(n^2)$ time when $m \geq \ell$. Hence we only need to solve the restricted reduced problem.

Run the algorithm of Theorem 6.11. First assume that the output is a single edge uv, that is, $T(uv) = G$ for a pair $u,v \in V$. If $uv \notin E'$, then uv is an optimal solution of the restricted reduced problem. Hence we may assume that $uv \in E' \subseteq E$. Let K_V denote the complete graph on V. Let $E^* = E(K_V) - E'$ the edges that we may use. Observe, that either $G + E^*$ is (m,ℓ)-redundant or the restricted reduced problem is not solvable. We may suppose that $G + E^*$ is (m,ℓ)-redundant, that is, $T(uv) = G = R(E^*) = \bigcup_{e \in E^*} T(e)$. Then, as $uv \in E$, $uv \in T(e)$ for an edge $e \in E^*$. Hence $G = T(uv) \subseteq T(e) = G$ by Lemma 2.3. Therefore, in this case we may calculate $T(e)$ for each edge $e \in E^*$ that takes $O(|V|^3)$ time by Theorem 2.6.

Next assume that the output consists of a set F of more than one edge and (A) holds for G. Then no edge in F is parallel to any edge $e \in E$ by Lemma 5.6 since the algorithm outputs edges connecting members of $\mathcal{C}^*$ in this case. Finally assume that the output consists of a set F of more than one edge and (A) does not hold for G. If (A2) or (A3) does nor hold for G, then G has less than $c^2 + 2$ vertices by Section 4 and hence the solution of the restricted problem is straightforward by checking all possibilities. When G violates (A1), we may delete its isolated vertices with $m(v) = 0$ like in Section 4. If (A) holds for the resulting graph then the output of our algorithm does not use any edge parallel to any edge $e \in E$ by Lemma 5.6 and we are done. Otherwise, the resulting graph again has less than $c^2 + 2$ vertices. Here we need to be careful since it is possible that the restricted augmentation is not solvable for the arising graph while it is solvable for G (for example by using some edges uv from a deleted vertex with $m(u) = 0$). However, observe that the set of edges in $T(u_1v)$ and $T(u_2v)$ is the same for $u_1, u_2, v \in V$ with $m(u_1) = m(u_2) = d(u_1) = d(u_2) = 0$, moreover, the edge set of $T(u_1u_2)$ is a subset of both. Hence, in this case, we may add back a single vertex u with $m(u) = d(u) = 0$ and solve the restricted reduced problem on the arising graph on less than $c^2 + 3$ vertices by checking all possibilities.

Adding multiple edges In some other applications, instead of adding non parallel edges, in contrast, we need to add several parallel copies of each edge in the augmentation, that is, we want to minimize $|F|$ such that $G + cF$ is (m, ℓ)-redundant. (For example, such a problem arises for $m \equiv k = \ell$ when one wants to augment a 'generically rigid body-hinge framework' to 'globally rigid' by [16].) Note that, if G is an (m, ℓ)-tight graph and F is a set of edges on the same vertex set, then $R(cF) = \bigcup_{f \in cF} T(f) = \bigcup_{f \in F} T(f) = R(F)$ by Lemma 2.4. Hence, for any (m, ℓ)-rigid graph G, $G + cF$ is (m, ℓ)-redundant if and only if $G + F$ is (m, ℓ)-redundant. Therefore, the above version of the augmentation problem is equivalent to the original one.

Further directions We considered our augmentation problem only for (k, ℓ)-rigid inputs, that is, when the input graph has a spanning (k, ℓ)-tight subgraph. It would be natural to consider the problem for general input graphs. This problem was solved for the case where $k = \ell$ by [5, 8], however, we leave this problem open for other values of k and ℓ. A specific instance of the previous open question is the case where the input is (k, ℓ)-sparse or more specifically contains no edges. Since there always exists a (k, ℓ)-circuit on a sufficiently large vertex set and such a graph is (k, ℓ)-redundant and has $k|V| - \ell + 1$ edges, this latter problem can be solved easily. However, if the edges are weighted, then even with a metric weight function the question becomes rather interesting. For example, the case of $k = \ell = 1$ is the **Traveling Salesman Problem**. It is well-known, that his problem is NP-hard, however, there exists a $\frac{3}{2}$-approximation algorithm to it by Christofides [3]. Recently, Jordán and the second author of this paper considered the $(k, \ell) = (2, 3)$ case of this problem and gave a 2-approximation algorithm to the metric weight case based on the algorithm of García and Tejel [9]. By generalizing their method and using the results of this paper, one can easily give similar approximation for other values of k and ℓ.

Another problem, which is also closely related to our augmentation problem, is a version of the **pinning problem** from rigidity theory. In this problem, our goal is to anchor a minimum set of joints in a bar-joint framework such that the resulting framework is rigid/globally rigid/redundantly rigid. In fact, in the generic case, pinning can be modeled by adding a complete graph on the anchored vertices to our graph. Fekete and Jordán [6] investigated the (2-dimensional) global rigidity version of this problem and gave a 3-approximation algorithm. We note that our results can be applied for the redundant rigidity version of this problem (with (k, ℓ)-tight or (k, ℓ)-rigid inputs) since it can be seen easily that each co-tight set must contain a pinned vertex. Specifically, one can obtain an optimal solution to the redundantly rigid pinning problem if the input is minimally rigid in $\mathbb{R}^2$, and one can show the NP-hardness of the problem when the input is rigid in $\mathbb{R}^2$.

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