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Abstract

The two main concepts of Rigidity Theory are rigidity, where the framework has no continuous deformation, and global rigidity where the given distance set determines the locations of the points up to isometry. We consider the following augmentation problem. Given a minimally rigid graph $G = (V, E)$ in $\mathbb{R}^2$, find a minimum cardinality edge set F such that the graph $G' = (V, E + F)$ is globally rigid in $\mathbb{R}^2$. We provide a min-max theorem and an $O(|V|^3)$ time algorithm for this problem.

1 Introduction

A d-dimensional framework is a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map of the vertices to the d-dimensional space. We call (G, p) a *realization* of G. Two realizations of G, say (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ for every $uv \in E$. Two realizations are *congruent*, if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for every vertex pair $u, v \in V$, or in other words, when (G, p) is isometric to (G, q). We say that the framework (G, p) is *globally rigid*, if each of its equivalent realizations is also congruent, that is, the edge lengths of the framework uniquely determine its realization up to isometry of $\mathbb{R}^d$. This concept of global rigidity plays an important role in rigidity theory and network localization problems [2, 4, 10].

For example, given some sensors in the plane with known distances between some of them, one may consider the following question. At least how many sensor-localizations do we need to measure exactly to be able to reconstruct the exact location of each sensor? This is the so-called *global rigidity pinning* (or anchoring) problem. Sometimes measuring the exact sensor-locations are too expensive or even impossible. Instead, one may ask at least how many new distances need to be measured so that the distances uniquely determine the positions of the sensors (up to isometry). This problem is called the *global rigidity augmentation* problem. (We note that reconstructing the

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position of the sensors is a challenging task, even if they are uniquely determined by the framework, see [1, 13, 18]. In this paper we do not address this problem.)

Determining whether a given framework is globally rigid is NP-hard [17]. The analysis gets more tractable, if we consider *generic frameworks* where the set of coordinates of the points is algebraically independent over the rationals. In this case, the global rigidity of a framework depends only on the underlying graph G (see [4] and [7]). We call a graph G **globally rigid** in $\mathbb{R}^d$ if each (or equivalently some) of its generic realization is globally rigid. The characterization of globally rigid graphs is known for $d = 1, 2$ [9] and is a major open problem of rigidity theory for $d \geq 3$.

For generic frameworks, Fekete and Jordán [5] gave a constant factor approximation for the global rigidity pinning problem. In Section 5 we show how this result can be applied to give a constant factor approximation for the global rigidity augmentation problem for generic frameworks. Observe, that this latter augmentation problem can be modelled as follows.

Problem 1.1. Given a graph $G = (V, E)$, find an edge set F of minimum cardinality on the same vertex set, such that $G + F = (V, E + F)$ is globally rigid in $\mathbb{R}^2$.

While the above mentioned result provides only a constant factor approximation algorithm for Problem 1.1, we shall solve a specific case of it optimally. When the input graph is minimally rigid we give a min-max theorem and also an $O(|V|^3)$ time exact algorithm for it. From this result, it follows that the globally rigid pinning problem also can be solved optimally for minimally rigid graphs (see Section 5). These results may be used to develop an optimal solution of Problem 1.1 and the global rigidity pinning problem.

2 Preliminaries

2.1 Rigidity in $\mathbb{R}^2$

In this subsection we collect the basic definitions and results from rigidity theory that we shall use, including the formal definition of Problem 1.1. There are several equivalent approaches to graph rigidity, for our purpose, a combinatorial one is the most practical. For a detailed introduction to rigidity theory including the equivalency of our approach, the reader is referred to [12].

A graph $G = (V, E)$ is called **sparse** if $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$, where $\boldsymbol{i(X)}$ denotes the number of edges induced by X. A graph $G = (V, E)$ is called **tight** if it is sparse and $|E| = 2|V| - 3$.

The fundamental results of Pollaczek-Geiringer [16] and Laman [15] assert that the generic realization of a graph G in $\mathbb{R}^2$ is minimally rigid if and only if G is tight. Hence we call a graph **rigid** (in $\mathbb{R}^2$) if it contains a spanning tight subgraph.

Tight graphs have some well known properties. For example, any subgraph of a sparse graph is always sparse. If $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ both are tight subgraphs of a tight graph G, then $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is an induced subgraph of G. Moreover, with standard submodular techniques one can prove the following (see [12]).

Lemma 2.1. *Let $G = (V, E)$ be a tight graph, and let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be tight subgraphs of G. If $|V_1 \cap V_2| \geq 2$, then $T_1 \cup T_2$ is a tight subgraphs of G.* □

The edge sets of the tight graphs on vertex set V correspond to the bases of the so-called 2-dimensional rigidity matroid defined on the edge set of the complete graph on V. In particular, if G is tight and $e = ij$ is a new edge, then $G + e$ has a unique (matroid) circuit, the *fundamental circuit* of e with respect to G, denoted by $C(ij)$ or $C(e)$. This circuit contains e. $(V(C(e)), E(C(e)) - e)$ forms a tight subgraph of G, that we call $\boldsymbol{T(e)}$ or $T(ij)$. For the sake of convenience, we do not distinguish a graph from its edge set, that is, $T(e) = E(C(e)) - e$. The next lemma easily follows from basic matroid properties.

Lemma 2.2. *Let $G = (V, E)$ be a tight graph and let $e = ij$ be an edge for some $i, j \in V$. If G' is a tight subgraph of G with $\{i, j\} \subseteq V(G')$, then $T(ij)$ is a subgraph of G'. Thus $T(ij)$ is equal to the intersection of all tight subgraphs T_h of G with $\{i, j\} \subseteq V(T_h)$.* □

An edge e of a rigid graph G is called **redundant** if $G - e$ is rigid. A graph is **redundantly rigid** if all of its edges are redundant.

A graph $G = (V, E)$ is called **k-connected** if $|V| > k$ and $G - X$ is connected for any vertex set $X \subset V$ of cardinality at most $k - 1$. Connectivity has several connections to rigidity. An often used folklore result is the following.

Proposition 2.3. *If $G = (V, E)$ is a rigid graph for which $|V| \geq 3$, then G is 2-connected.* □

Regarding to our problem, the following characterization of global rigidity in $\mathbb{R}^2$, due to Jackson and Jordán, is the most important.

Theorem 2.4 ([9]). *A graph $G = (V, E)$ is globally rigid in $\mathbb{R}^2$ if and only if it is redundantly rigid and 3-connected or if G is the complete graph (when $|V| \leq 3$).* □

The problem we solve in this paper is the following.

Problem 2.5. Given a tight graph $G = (V, E)$ with $|V| > 3$, find a graph $H = (V, F)$ with a minimum cardinality edge set F, such that $G + H$ is redundantly rigid and 3-connected.

Note, that if G has at most 3 vertices then G is tight if and only if it is globally rigid by Theorem 2.4, hence the solution of Problem 2.5 is obvious. Thus we may suppose in what follows that G contains at least 4 vertices.

2.2 Redundant augmentation and co-rigid sets

Let us first investigate the problem of augmenting a tight graph $G = (V, E)$ to a redundantly rigid graph by a minimum number of edges. This problem was considered and solved before by García and Tejel [6]. A generalization of this augmentation problem to so called (k, ℓ)-tight graphs appears in a work by the authors of this paper [14]. We use some ideas from both of these works.

If we add the edges $e_1, \dots, e_k$ to G, we augment some edges of G to redundant. Let us denote the set of these edges by $R(e_1, \dots, e_k)$. Note that $R(e_1) = T(e_1)$. The following statement of García and Tejel generalizes this simple fact.

Lemma 2.6 ([6, Lemma 4])**.** *Let $G = (V, E)$ be a tight graph. Then $R(e_1, \dots, e_k) = T(e_1) \cup \dots \cup T(e_k)$ for arbitrary edges $e_1, \dots, e_k$.* □

Following [12] we define the co-rigid sets of a tight graph. Given a tight graph $G = (V, E)$, a set $C \subsetneq V$ is called **co-rigid** if $V - C$ induces a tight subgraph. This is equivalent to the following: C is co-rigid in G if $|C| \leq |V| - 2$ and $|\widehat{E}(C)| = 2|C|$ where $\boldsymbol{\widehat{E}(C)}$ denotes the set of edges in G for which at least one of its two endpoints is in C. Notice, that $|\widehat{E}(X)| = i(X) + d(X, V - X)$ and $|E| = |\widehat{E}(X)| + i(V - X)$ holds for every $X \subsetneq V$, where $\boldsymbol{d(X, Y)}$ denotes the number of edges between two disjoint sets $X, Y \subsetneq V$. Hence $|\widehat{E}(X)| \geq 2|X|$ for every $X \subsetneq V$ where $|X| \leq |V| - 2$ by $|E| = 2|V| - 3$ and the sparsity of $G - X$. As G is tight, one might use the term co-tight instead of co-rigid (following the terminology of [14]), however, in this paper we will use notation of [12]. Observe that every tight graph G on at least 4 vertices contains at least 2 co-rigid sets that do not contain each other as any edge forms a tight subgraph of G.

By Lemma 2.2, the following property follows easily:

Proposition 2.7. *Let C be a co-rigid set of a tight graph. If $\{u, v\} \cap C = \emptyset$, then $T(uv) \cap \widehat{E}(C) = \emptyset$.* □

Let us abbreviate the name of minimal co-rigid sets by **MCR** sets and let $\boldsymbol{\mathcal{C}^*}$ denote the family of all MCR sets of G. We shall use the following results on MCR sets.

Lemma 2.8 ([12, Lemma 3.9.12])**.** *Let C_1 and C_2 be two intersecting MCR sets of a tight graph $G = (V, E)$. Then $|C_1 \cup C_2| \geq |V| - 1$.* □

Lemma 2.9 ([12, Theorem 3.9.13])**.** *Let G be a tight graph. Then the members of $\mathcal{C}^*$ are pairwise disjoint or there are two vertices $v, w \in V$ such that $\{v, w\} \cap C \neq \emptyset$ for all $C \in \mathcal{C}^*$.* □

Lemma 2.10 ([14, Lemma 5.4])**.** *Let G be a tight graph and let $P \subset V$ be a set which intersects each member of $\mathcal{C}^*$. Suppose that $G' = (V', E')$ is a tight subgraph of G such that $P \subset V'$. Then $G' = G$.* □

Lemmas 2.9 and 2.10 imply that if there are at least two intersecting MCR sets, then there exists an edge e such that $T(e) = G$. If we consider the other case, then the MCR sets are disjoint. This motivates us to investigate the disjoint MCR sets. The following lemma slightly extends the statement of [14, Lemma 5.6].

Lemma 2.11. *Let $G = (V, E)$ be a tight graph and let C, K be two disjoint MCR sets of G. If $|V \setminus (C \cup K)| > 1$, then $\widehat{E}(C) \cap \widehat{E}(K) = \emptyset$.*

Proof. By counting the edges induced by $V \setminus (C \cup K)$, we get that

$$i(V\setminus(C\cup K)) \leq 2|V\setminus(C\cup K)|-3 = 2|V|-2|C|-2|K|-3 = 2|V|-|\widehat{E}(C)|-|\widehat{E}(K)|-3$$

where the first inequality comes from the sparsity of G and the property $|V\setminus(C\cup K)| > 1$, while the equalities hold because C and K are disjoint co-rigid sets.

Counting the same edges with their complements implies

$$i(V \setminus (C \cup K)) = 2|V| - 3 - |\widehat{E}(C) \cup \widehat{E}(K)| \geq 2|V| - 3 - |\widehat{E}(C)| - |\widehat{E}(K)|$$

and equality is only possible if $\widehat{E}(C) \cap \widehat{E}(K) = \emptyset$. □

Lemma 2.12 ([14, Lemma 5.7])**.** *Let $G = (V, E)$ be a tight graph on at least 4 vertices. Let C be an MCR set, $u \in C$ and $v \in V \setminus (C \cup N(C))$. Then $C \cup N(C) \subset V(uv)$.* □

Theorem 2.13 ([14, Theorem 1.1])**.** *Let G be a tight graph on at least 4 vertices. Then*

$$\min\{|F| : H = (V,F) \text{ is a graph for which } G + H \text{ is a redundantly rigid graph } \} =$$
$$= \max\left\{\left\lceil \frac{|\mathcal{C}|}{2} \right\rceil : \mathcal{C} \text{ is a family of disjoint co-rigid sets in } G\right\}.$$

□

If G is 3-connected graph, then Problem 2.5 is equivalent to augment G to a redundantly rigid graph. In this case, the previous theorem provides an optimal solution to Problem 2.5. Thus we may suppose in what follows, that G is not 3-connected hence we also need to augment it to a 3-connected graph.

2.3 3-connected augmentation

By Proposition 2.3, every tight graph is 2-connected and thus we need to augment a 2-connected graph to a 3-connected graph. There exists several methods to deal with this particular problem, even linear time algorithms [8]. However, we also need to augment G to a redundantly rigid graph hence we stick to a more simple approach following the ideas of [11]. Recall, that we suppose that G is not 3-connected.

Let us call $v_1, v_2 \in V$ a **cut-pair** of G, if $G - \{v_1, v_2\}$ is not connected.

Lemma 2.14 ([9])**.** *Let G be a rigid graph in $\mathbb{R}^2$. If there are two disjoint cut-pairs of G, say v_1, v_2 and u_1, u_2 then u_1 and u_2 are in the same component of $G - \{v_1, v_2\}$.* □

If u, v is a cut-pair in G, then let $\boldsymbol{b_{(u,v)}}$ denote the number of components of $G - \{u, v\}$. Let $\boldsymbol{b(G)}$ denote the maximum value of $b_{(u,v)}$ over all cut-pairs u, v of G. If there are no cut-pairs in G, let $b(G) := 1$. Clearly, any edge set F that augments G to a 3-connected graph needs to span a connected graph on the components of $b_{u,v}$ for every cut-pair u, v. Thus $|F| \geq b(G) - 1$.

Let $\boldsymbol{N(X)}$ denote the neighbor set of $X \subseteq V$, that is, $N(X) := \{v \in V - X :$ there exists an edge uv such that $u \in X\}$. A set $P \subsetneq V$ is called a **3-fragment** if

$|N(P)| = 2$ and $P \cup N(P) \neq V$. Increasing the connectivity of G is equivalent to increasing the number of neighbors of each 3-fragment of G. The maximum number of pairwise disjoint 3-fragments is denoted by $t(G)$. Let us call an inclusion-wise minimal 3-fragment a **3-end**. As every 3-fragment contains at least one 3-end, $t(G)$ is equal to the number of pairwise disjoint 3-ends. However, by Lemma 2.14 this latter value equals to the number of 3-ends for rigid graphs. Thus any edge set that augments G to a 3-connected graph must intersect all 3-ends. Hence the following lower bound can be given for the cardinality of an edge set that augments a rigid graph to a 3-connected graph.

Lemma 2.15. *Given a rigid graph G. The minimum number of edges that augments G to a 3-connected graph is at least* $\max\left\{b(G) - 1, \left\lceil \frac{t(G)}{2} \right\rceil\right\}$. □

In fact, the above two quantities are equal (see [11]).

3 Min-max theorem

In this section we shall merge the results on redundantly rigid augmentation and 3-connected augmentation to a new min-max theorem for globally rigid augmentation by mixing the statements of Theorem 2.13 and Lemma 2.15, as follows.

Theorem 3.1. *Let $G = (V, E)$ be a tight graph on at least 4 vertices. Then*

$$\min\{|F| : H = (V, F) \text{ is a graph for which } G + H \text{ is globally rigid }\} =$$
$$= \max\left\{b(G) - 1, \max\left\{\left\lceil \frac{|\mathcal{A}|}{2} \right\rceil : \mathcal{A} \text{ is a family of disjoint co-rigid sets and 3-fragments}\right\}\right\}$$

Recall that a graph on at least 4 vertices is globally rigid if and only if it is 3-connected and redundantly rigid by Theorem 2.4. Notice that, if G is 3-connected, then Theorem 3.1 follows directly by Theorem 2.13. Hence from now on, we assume that G is not 3-connected. The min $\geq$ max implication in Theorem 3.1 is obvious by Proposition 2.7 and Lemma 2.15. To prove the min $\leq$ max part, let us consider the family of all MCR sets and 3-ends of a tight graph G. Let us call the inclusion-wise minimal elements of this family the **atoms** of G . Let us denote the family of atoms by $\boldsymbol{\mathcal{A}^*}$. Similar to the proof of Theorem 2.13 in [14], we shall show that the atoms are pairwise disjoint and there exists a set of $\max\left\{b(G) - 1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$ edges that augments G to a globally rigid graph. Hence we first need to prove the following.

Lemma 3.2. *Let $G = (V, E)$ be a tight graph which is not 3-connected. Then the atoms of G are pairwise disjoint.*

To prove Lemma 3.2, we need the following two statements.

Observation 3.3. Suppose that C is a co-rigid set in G, and $C' \subsetneq C$ such that $d(C', C - C') = 0$. Then C' is also co-rigid.

Proof. As $d(C', C - C') = 0$, $|\widehat{E}(C)| = |\widehat{E}(C')| + |\widehat{E}(V - C')|$. Thus if $|\widehat{E}(C')| \geq 2|C'| + 1$, then $|\widehat{E}(C - C')| \leq 2|C - C'| - 1$, a contradiction. □

Lemma 3.4. *Suppose that $G = (V, E)$ is a tight graph. Let $a \in A \in \mathcal{A}^*$ be a vertex from an atom of G. Then there is no $v \in V$ such that a, v forms a cut-pair.*

Proof. If G is 3-connected, then the statement is obvious. If G is not 3-connected and A is a 3-end, then the statement holds by Lemma 2.14.

Now let A be an MCR set. Then $G[V - A]$ is tight and thus 2-connected by Proposition 2.3. Suppose that a, v forms a cut-pair for $a \in A$ and $v \in V$. Then $V - A$ intersects only one component of $G - \{a, v\}$, otherwise v would be a cut-vertex in the tight graph $G[V - A]$, contradicting its 2-connectivity. Thus A contains at least one component of $G - \{a, v\}$ (which also contains a 3-end), contradicting the minimality of A. □

Proof of Lemma 3.2. Let us recall that $\mathcal{C}^*$ denotes the family of MCR sets of G and let $\mathcal{L}^*$ denote the family of 3-ends of G. As we saw before, the members of $\mathcal{L}^*$ are pairwise disjoint.

Suppose that $C \in \mathcal{C}^*$ and $L \in \mathcal{L}^*$ such that $C, L \in \mathcal{A}^*$. By Lemma 3.4 $|C \cap N(L)| = 0$. However, by Observation 3.3 C is connected thus $C \cap L = \emptyset$ or $L = C$, as $C \subsetneq L$ contradicts the minimality of L.

Suppose now that there exist two distinct intersecting sets $C_1, C_2 \in \mathcal{C}^* \cap \mathcal{A}^*$. By Lemma 2.8 $|C_1 \cup C_2| \geq |V| - 1$ contradicting Lemma 3.4 as G is not 3-connected. □

Note that if G is 3-connected, Lemma 3.2 does not always hold (see Lemma 2.9).

Now, we turn to prove that there exists a set of $\max\left\{b(G) - 1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$ edges that augments G to a globally rigid graph. A set X is called a **transversal** of a family $\mathcal{S}$ if $|X \cap S| = 1$ for each $S \in \mathcal{S}$ and $|X| = |\mathcal{S}|$. Let P be a transversal of $\mathcal{A}^*$. As the members of $\mathcal{A}^*$ are pairwise disjoint if G is not 3-connected by Lemma 3.2, choosing one arbitrary vertex from every $A \in \mathcal{A}^*$ obtains a transversal. Observe that P is a minimum cardinality vertex set that intersects all MCR sets and 3-ends, and consequently all co-rigid sets and 3-fragments. Hence $|\mathcal{A}| \leq |P|$ holds for an arbitrary family $\mathcal{A}$ of disjoint co-rigid sets and 3-fragments.

We shall show now that a connected graph on a transversal P of $\mathcal{A}^*$ augments G to a globally rigid graph, that is, 3-connected and redundantly rigid. Later, we will reduce the number of edges needed for this augmentation to the optimum value.

Lemma 3.5. *Suppose that G is a tight graph which is not 3-connected. Let P be a transversal of $\mathcal{A}^*$. Then, for any connected graph $H = (P, F)$ on P, $G + H$ is 3-connected.*

Proof. G is 2-connected by Proposition 2.3. Also, P contains no member of any cut-pair by Lemma 3.4. If there exists a cut-pair in $G + H$, then in one of its components there is no vertex from P, but P intersected all 3-ends and this component is a 3-fragment which must contain a 3-end, a contradiction. □

To show that the above augmentation gives a redundantly rigid graph, we extend the ideas of the proof of Theorem 2.13 from [14].

Lemma 3.6. *Let $G = (V, E)$ be a tight graph which is not 3-connected and let $A, B \in \mathcal{A}^*$. Then $A \cap N(B) = \emptyset$.*

Proof. Since G is not 3-connected, $|V - (A \cup B)| \geq 2$ by Lemma 3.4. If B is a 3-end, then $N(B)$ is a cut-pair thus $A \cup N(B) = \emptyset$ by Lemma 3.4. If both are MCR sets, then the lemma follows by Lemma 2.11. □

Lemma 3.6 immediately implies the following.

Observation 3.7. The vertex set P induces no edge in G. □

Recall that $R(F)$ denotes the set of redundant edges of G in $G + F$. The following lemma and its proof is a direct extension of [14, Lemma 5.8].

Lemma 3.8. *Suppose that G is a tight graph which is not 3-connected. Let P be a transversal of $\mathcal{A}^*$ and let F be an edge set of a connected graph on $P' \subseteq P$. Then $R(F)$ is the minimal tight subgraph inducing all elements of P'. In particular, if F is the edge set of a star $K_{1,|P|-1}$ on the vertex set P, then $G + F$ is redundantly rigid.*

Proof. Recall that $R(F) = \bigcup_{f \in F} T(f)$ by Lemma 2.6. Let us use induction on $|F|$. If $F = \{ij\}$, then $R(F) = T(ij)$ which is the minimal tight subgraph of G containing both of i and j by Lemma 2.2.

Claim 3.9. *For each $p \in P$ there exists a set D_p such that $D_p \subset V(T(pq))$ with $|D_p| \geq 2$ for all $q \in P - p$.*

Proof. Let $A \in \mathcal{A}^*$ such that $p \in A$. We claim that $D_p := \{p\} \cup N(A)$ is a suitable set. By Lemma 2.2 $p \in V(T(pq))$, and by Proposition 2.3 $|D_p| \geq 2$. If A is a 3-end, then $D_p \subset T(pq)$ by Proposition 2.3 and Lemma 3.6. If A is an MCR set, then Lemmas 2.12 and 3.6 imply that $A \cup N(A) \subset V(T(pq))$ and thus $D_p \subset V(T(pq))$. □

Let $ij \in F$ such that $F - ij$ is connected. If $i, j \in V(R(F - ij))$, then $T(ij) \subseteq R(F - ij)$ by Lemma 2.2. We may assume (by possibly switching the role of i and j) that $i \in V(R(F - ij))$ and $j \notin V(R(F - ij))$. The connectivity of $F - ij$ implies that there exists an edge $ij' \in F - ij$. By induction, $R(F - ij)$ is a tight subgraph of G which induces each elements of $V(R(F - ij))$, particularly i. Note that $T(ij') \subseteq R(F - ij)$ by Lemma 2.6. Hence $D_i \subset V(T(ij')) \subseteq V(R(F - ij))$ and $D_i \subset V(T(ij))$ by Claim 3.9 thus we may use Lemmas 2.1 and 2.6 to conclude that $R(F) = R(F - ij) \cup T(ij)$ is tight.

Let now T be the minimal tight subgraph of G which induces all elements of P'. Lemma 2.2 imply that $T(f) \subseteq T$ for each $f \in F$. Hence it follows by Lemma 2.6 that $R(F) = \bigcup_{f \in F} T(f) \subseteq T$, that is, $R(F) = T$.

Finally, if $P' = P$, then $P \subset V(R(F))$ thus $R(F) = G$ by Lemma 2.10 since P intersects every MCR set. □

Now we turn to show how the cardinality of the augmenting edge set provided by the above lemmas can be reduced to the optimum. By a direct extension of [14, Lemma 5.9] and its proof, we get the following.

Lemma 3.10. *Let $G = (V, E)$ be a tight graph which is not 3-connected and let P be a transversal of $\mathcal{A}^*$. Suppose that $x_1, x_2, x_3, y \in P$ are distinct vertices. Let $T^* = T(x_1y) \cup T(x_2y) \cup T(x_3y)$. Then $T^* = T(x_1y) \cup T(x_2x_3)$ or $T^* = T(x_2y) \cup T(x_1x_3)$ holds.*

Proof. Let $T^* = (V^*, E^*)$. By Lemmas 2.6 and 3.8 T^* is tight. Let us suppose that $T^* \neq T(x_1y) \cup T(x_2x_3)$. Thus there exists an edge e, for which $e \in E^*$ and $e \notin T(x_1y) \cup T(x_2x_3)$.

Lemmas 2.6 and 3.8 imply that T^* is the minimal tight subgraph of G inducing all of $x_1,\ x_2,\ x_3$ and y. However, they similarly imply that this statement also holds for $T(x_1y) \cup T(x_2x_3) \cup T(x_3y)$ and $T(x_1y) \cup T(x_2x_3) \cup T(x_1x_2)$, that is, these two graphs both are equal to T^*. Since $e \in T^*$ and $e \notin T(x_1y) \cup T(x_2x_3)$, we get $e \in T(x_3y)$ and $e \in T(x_1x_2)$.

Lemma 2.1 imply now that $T(x_3y) \cup T(x_1x_2)$ is a tight subgraph of G (and also of T^*) inducing all of $x_1,\ x_2,\ x_3$ and y, hence it must be equal to T^*. □

Observe that the operation in Lemma 3.10 allows us to reduce the cardinality of the edge set used for the augmentation by maintaining the property that it augments G to a redundantly rigid graph. However, we also need to maintain the 3-connectivity of $G + F$ to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. As we have seen at the beginning of this section, we only need to prove the min $\leq$ max part of Theorem 3.1 for the case where G is not 3-connected. In this case, the atoms of G are pairwise disjoint by Lemma 3.2 and a tree on a transversal P of $\mathcal{A}^*$ augments G to a globally rigid graph with $|\mathcal{A}^*| - 1$ edges by Lemmas 3.5 and 3.8. Note that as $\mathcal{A}^*$ consists of pairwise disjoint MCR sets and 3-ends of G, the maximum in Theorem 3.1 is at least $\max\left\{b(G) - 1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$, furthermore, this latter value equals to $|\mathcal{A}^*| - 1$ when $|\mathcal{A}^*| \leq 3$ completing our proof for this case.

To reduce the number of edges needed for the augmentation, we do the following procedure. Let us define a vertex set $N \subseteq P$. The set N stands for "not fixed" vertices while vertices in $P - N$ are the "fixed" vertices. We can **fix** an edge xy by removing x and y from N and adding xy to F.

We shall keep some properties during the whole procedure:

1. For an arbitrary star S_N on the vertex set N, $G + F + S_N$ is a redundantly rigid graph.
2. In every 3-end of $G + F$, there is at least one vertex from N
3. $\max\left\{b(G + F) - 1, \left\lceil \frac{|N|}{2} \right\rceil\right\} + |F| = \max\left\{b(G) - 1, \left\lceil \frac{|P|}{2} \right\rceil\right\}$.

Notice that Properties 1–3 hold for $N = P$ and $F = \emptyset$ by Lemmas 3.5 and 3.8.

Remark 3.11. Properties 1 and 2 ensure that $G + F + S_N$ is redundantly rigid and 3-connected, and thus globally rigid by Theorem 2.4.

Remark 3.12. Suppose that $|N| \geq 4$. If there are vertices $x_1, x_2, x_3 \in N$, such that neither fixing x_1x_3 nor fixing x_2x_3 violates Properties 2 and 3, then by Lemma 3.10 fixing one of these maintains Property 1.

By Remark 3.12 we always aim to find at least two possibilities to fix such that Property 2 holds. Also, if it can be done such that $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\}$ decreases by one, then we can maintain Properties 1–3. Roughly, we distinguish 4 different possibilities in each of which we find 3 vertices from N such that we can apply Remark 3.12 and hence we can fix one edge while maintaining Properties 1–3.

Lemma 3.13. *Let G be a tight graph which is not 3-connected such that $|\mathcal{A}^*| \geq 4$. Let P be a transversal on $\mathcal{A}^*$. Let $N \subseteq P$ be a vertex set and F be an edge set on P such that they satisfy Properties 1–3. If $|N| \geq \max\{4, b(G+F)+1\}$, then we can choose $x, y \in N$, such that for $N - \{x, y\}$ and $F + \{xy\}$ (that is, for fixing xy) Properties 1–3 also hold.*

Proof. We use the following method for the proof. This is the core of our algorithm which we will describe in Section 4.

1 If $b(G+F) - 1 \geq \left\lceil \frac{|N|}{2} \right\rceil$, **then**

2 **If** there is only one cut-pair (u, v) such that $b_{(u,v)}(G+F) = b(G+F)$, **then**

Choose x_1, x_2 from a component of $G+F-\{u, v\}$ that contains at least two vertices from N. Let $x_3 \in N$ be a vertex from a component of $G+F-\{u, v\}$ that does not contain x_1 and x_2.

3 **else**

Let (u_1, v_1) and (u_2, v_2) be two cut-pairs for which $b_{(u_1,v_1)}(G+F) = b(G+F) = b_{(u_2,v_2)}(G+F)$. Choose $x_1, x_2 \in N$ from two different components of $G+F-\{u_1, v_1\}$ that do not contain $\{u_2, v_2\}$. Choose $x_3 \in N$ from a component of $G+F-\{u_2, v_2\}$ that does not contain $\{u_1, v_1\}$.

4 else

5 **If** there is a cut-pair $\{u, v\}$ such that for one component of $G - \{u, v\}$, say K, $|N \cap K| \geq 2$ and $|(V-K) \cap N| \geq 2$, **then**

Choose x_1, x_2 from $N \cap K$ and choose $x_3 \in N$ from $(V-K) \cap N$.

6 **else** (Notice that if $b(G+F) = 1$, then this is the only possible case.)

Choose $x_1, x_2, x_3 \in N$ arbitrarily.

7 If $G + F + S(N - \{x_1, x_3\}) + x_1x_3$ is redundantly rigid, **then**

$x := x_1, y := x_3$.

else

$x := x_2, y := x_3$.

First we prove that the above method is consistent, that is, we can execute each of its steps. As $|N| \geq b(G+F)+1$ and P contains no vertex from a cut-pair of G by Lemma 3.4, $|N| > b_{(u,v)}(G+F)$ for an arbitrary cut-pair $\{u,v\}$ hence there exists a component of $G+F-\{u,v\}$ that contains at least two vertices from N. This shows that we can choose vertices in STEPS **2** and **5** consistently. Meanwhile, in STEP **3** there are at least two components of $G+F-\{u_1,v_1\}$ that do not contain $\{u_2,v_2\}$ by $|N| \geq 4$ hence $b_{(u_1,v_1)}(G+F) \geq 3$.

Now let us show that the choice of x and y maintains Property 2.

Claim 3.14. *Suppose that there is a cut-pair $\{u,v\}$ such that for one component of $G-\{u,v\}$, say K, $x_1, x_2 \in N \cap K$ and $x_3, y \in (V-K) \cap N$. Then fixing either x_1x_3 or x_2x_3 maintains Property 2.*

Proof. Notice that the role of x_1 and x_2 is symmetric thus we might suppose that we fixed the edge x_1x_3. Suppose that we form a new 3-end L with it. Then necessarily $x_1, x_3 \in L$. If $x_2 \in L$ or $y \in L$, then Property 2 holds automatically. On the other hand, if none of them is in L, then there is a cut-pair in $K \cup \{u\}$ or in $K \cup \{v\}$ which separates x_1 from x_2. There is another cut-pair in $V-K$ (other than $\{u,v\}$) which separates x_3 from y. Both remain cut-pairs after fixing the edge x_1x_3. However, this contradicts the assumption that L is 3-end, as $|N(L)| = 2$ must hold for a 3-end. □

Notice, that the conditions of this claim hold in STEPS **2**, **3** and **5** thus with our choice of x_1, x_2, and x_3 Property 2 is maintained. If $G+F$ is already 3-connected, then Property 2 is obvious. Otherwise, in STEP **6**, every cut-pair cuts $G+F$ into two component one of which contains exactly one element from N by the condition of STEP **5**. For the sake of a contradiction, assume that $G+F+xy$ contains a 3-end L which contains no element of $N-\{x,y\}$. Let $N(L)=\{u,v\}$. Then $N \cap L = \{x,y\}$, $V-L-\{u,v\} \neq \emptyset$, and u,v is a cut pair of $G+F$. By our above condition, (u,v) cuts $G+F$ into two component one of which contains exactly one element from N. Hence exactly L and $V-L-\{u,v\}$ are these two components. Moreover, as $|L \cap N| = 2$, this implies $|N \cap (V-L-\{u,v\})| = 1$, contradicting $|N| \geq 4$.

Now we show that our method maintains Property 3. Fixing any edge decreases $\left\lceil \frac{|N|}{2} \right\rceil$ by one while increases F by one. After STEP **4** it is enough to keep Property 3 true as in this case $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\} > b(G+F)-1$. We need to show that if the condition in STEP **1** is true, then we also decrease $b(G+F)$. If $b(G+F)-1 \geq \left\lceil \frac{|N|}{2} \right\rceil$, then there can be at most two cut-pairs of $G+F$ satisfying $b_{(u,v)}(G+F) = b(G+F)$ by a simple calculation on the number of 3-ends (see [11]). If there is only one, the pair (u,v) chosen in STEP **2**, then we only need to decrease $b_{(u,v)}(G+F)$. Since x_1x_3 and x_2x_3 both connect two different components of $G+F-\{u,v\}$, $b_{(u,v)}(G+F)$ decreases by one after fixing any of them. If there are exactly two such cut-pairs, (u_1,v_1) and (u_2,v_2) chosen in STEP **3**, then we need to decrease $b_{(u_1,v_1)}(G+F)$ and $b_{(u_2,v_2)}(G+F)$ simultaneously. Again our choice of x_1x_3 and x_2x_3 guarantees this.

Therefore, by Remark 3.12 applied to STEP **7**, fixing xy maintains Properties 1–3. This completes the proof of Lemma 3.13. □

We apply Lemma 3.13 recursively until $|N| < \max\{4, b(G+F)+1\}$. To complete our proof, we need to show the following.

Claim 3.15. *If $|N| \leq \max\{3, b(G+F)\}$, then, for an arbitrary star S_N on N, $G+F+S_N$ forms a globally rigid graph for which $|F|+|S_N| = \max\left\{b(G)-1, \left\lceil \frac{|P|}{2} \right\rceil\right\}$.*

Proof. $G+F+S_N$ is globally rigid by Remark 3.11. By Property 3 it is enough to show that $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\} = |S_N| = |N|-1$. If $|N| = b(G+F)$, then $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\} = |N|-1$ as $\left\lceil \frac{|N|}{2} \right\rceil \leq |N|-1$. On the other hand, if $|N| < b(G+F)$, then $2 \leq |N| \leq 3$ thus $\left\lceil \frac{|N|}{2} \right\rceil = |N|-1$. □

Recall that $\mathcal{A}^*$ consists of pairwise disjoint MCR sets and 3-ends of G and hence the maximum in Theorem 3.1 is at least $\max\left\{b(G)-1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$. On the other hand, the above claim implies that G can be augmented to a globally rigid graph by an addition of an edge set of cardinality $\max\left\{b(G)-1, \left\lceil \frac{|P|}{2} \right\rceil\right\} = \max\left\{b(G)-1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$. This completes the proof of Theorem 3.1. □

4 Algorithmic aspects

It is easy to see that the proof of Theorem 3.1 provides an algorithm for Problem 2.5 when the input tight graph $G = (V, E)$ is not 3-connected. On the other hand, the algorithm of García and Tejel [6] or that of by the authors of this paper in [14] provides an algorithm for the case where G is 3-connected since in this case we only need a redundantly rigid augmentation of G. In this section we sketch how one can provide an $O(|V|^3)$ time algorithm for Theorem 3.1.

Theorem 4.1. *Let $G = (V, E)$ be a tight graph. There exists an $O(|V|^3)$ time algorithm that finds a graph $H = (V, F)$ with a minimum cardinality edge set F for which $G + H$ is a globally rigid graph.*

Proof (sketch). Note that the tightness of G implies that $|E| = 2|V| - 3$. Hence the 3-connectivity of G and all cut-pairs and 3-fragments of G can be found in $O(|V|^3)$ time by searching for the connected components of $G - \{u, v\}$ with the BFS algorithm for each pair of vertices $u, v \in V$. (Note that Lemma 2.14 implies that a rigid graph on n vertices has $O(n)$ cut-pairs and 3-fragments.)

The algorithm of Berg and Jordán [3] checks the tightness of G in $O(|V|^2)$ time, moreover, after this it can be used to calculate $T(ij)$ for each pair of vertices $i, j \in V$ in linear time. This fact was used to show that the algorithms in [6] and [14] both provide an optimal redundantly rigid augmentation of G in $O(|V|^2)$ time which completes the proof when the input is 3-connected. Moreover, this fact also implies that each step of the algorithm given in the proof of Lemma 3.13 can be executed in linear time when all the 3-fragments and cut-pairs of $G + F$ are known. To output an optimal solution we need to run the steps of this algorithm $O(|V|)$ times, recursively. Hence to complete the proof we need to resolve two issues:

1. To start the algorithm we need a transversal of $\mathcal{A}^*$. (And this is also needed to solve the case where $|\mathcal{A}^*| \leq 3$.)
2. We need to update the 3-fragments and cut-pairs of $G+F$ after each edge-fix in $O(|V|^2)$ time (instead of recalculating it as in our initial step in $O(|V|^3)$ time).

Solution of Issue 1. To resolve Issue 1, we shall use some algorithms from [14]. First, we need to decide whether there exists a pair $x, y \in V$ such that $G+xy$ is redundantly rigid. We may do it by checking every vertex pair of G if $G+xy$ is redundantly rigid. By the properties of algorithm of [3] it can be done in $O(|V|^3)$ time. By Lemma 2.9, if no such pair exists, then the MCR sets are pairwise disjoint (and there are at least three of them by Theorem 2.13). In this case, we may use [14, Algorithm 6.1] (which has $O(|V|^2)$ running time) to find an MCR set of G. Next we take a vertex v of this MCR set in such a way that we take v from a 3-end of G if possible. This way v is contained by an atom of G. Next we run [14, Algorithm 6.9] (which outputs a transversal of the MCR sets of G in $O(|V|^2)$ time) in such a way that we first consider the vertices of G which are contained in any 3-end of G. This way the output of [14, Algorithm 6.9] consists of a transversal of the MCR sets of G with the extra property that the vertex representing an MCR set C will be always taken from a 3-end which is intersected by C if possible. Hence to get a transversal of $\mathcal{A}^*$ we only need to add one vertex from each 3-end which is not represented by the output of our algorithm.

If $G+xy$ is redundantly rigid, then each MCR set of G contains x or y by Proposition 2.7. We need to decide whether there exists any MCR set in G which is an atom. Since the atoms of a non-3-connected tight graph are pairwise disjoint by Lemma 3.2, there may only exist at most two such MCR set: one which contains x and one which contains y. To find these, we may use [14, Algorithm 6.1] in a special way. Since the role of x and y is similar, we show how to find an MCR set containing x which is a candidate to be an atom. We divide V into three sets, as follows. Let X be the 3-end of G which contains x (this means that $X = \emptyset$ when x is not contained by any 3-end). Let Y be the union of all 3-ends of G except X, and let $V' := V - X - Y$. Note that $Y \neq \emptyset$. We run [14, Algorithm 6.1] in such a way, that the initial vertex v is taken from Y, and we also take the initial vertex u from Y if possible and from V' if not. During the algorithm the vertex u is taken from $Y \cap V^*$ while that is non-empty, next from $V' \cap V^*$ while that is non-empty, and finally from $(X - x) \cap V'$. Thus we may choose x and y such that they intersect no 3-ends of G (in the case where $X = \emptyset$) if such MCR set exists. (Note that it is possible that $|Y| = 1$ and $V' = \emptyset$, however, in this case the single element of Y is a vertex of degree 2 which forms an MCR singleton hence it must be the vertex y since $\{x, y\}$ covers all MCR sets. Furthermore, G has exactly two 3-ends. Hence $\{x, y\}$ covers all 3-ends and MCR sets of G, that is, they form a transversal of the atoms of G.)

Solution of Issue 2. Throughout the algorithm, we maintain the set of cut-pairs of $G+F$ and, for each cut pair u, v we store the family of connected components of $G+F-\{u, v\}$. When we fix an edge xy, we need to check whether it connects two connected components of $G+F-\{u, v\}$ for each cut-pair u, v, and in such a case, we need to union the two maintained components (and delete u, v from the set of cut-pairs if only one component remains after the union). This update can be done in $O(|V|)$ time for each cut-pair which implies $O(|V|^2)$ total update time for each edge-fixing step. □

5 Concluding remarks

In this paper, we have solved Problem 1.1 in the case where the input is a tight graph. For general inputs, a constant factor approximation can be given, as follows.

Let us recall the global rigidity pinning problem. In this problem, the goal is to anchor a minimum set of points of a framework such that the resulting framework is globally rigid. In the generic case, pinning can be modelled by adding a complete graph on the anchored vertices to the graph (see [5]). Moreover, instead of a complete graph we can add any globally rigid graph on the anchored vertex set, for example the square graph of a cycle. (A square of a graph arises by connecting all pairs of vertices which has distance at most 2 in the original graph). Notice that the square graph of the cycle on the vertex set V consists of $2|V|$ edges. This way one can see that a constant approximation to the global rigidity pinning problem gives a constant approximation to the global rigidity augmentation problem and *vice versa*. Fekete and Jordán [5] investigated the global rigidity pinning problem and gave a constant approximation algorithm to it. This implies that there exists a polynomial time constant approximation algorithm to Problem 1.1 (and it has an approximation ratio at most 4 times more than that of the pinning problem).

For tight input graphs, we can solve the global rigidity pinning problem optimally as follows. It can be shown easily that we must pin at least one vertex from each atom. On the other hand, a complete graph on a transversal of $\mathcal{A}^*$ indeed augments G to a globally rigid graph as it contains also the optimal edge set given by Theorem 3.1. Thus one vertex from each atom pins the graph optimally. (When G is 3-connected, we may apply the method of [14, Section 8] directly.)

One may try to use the methods of this paper to obtain an optimal solution to Problem 1.1 when the input is a rigid graph. However, these methods do not yield to a solution directly since García and Tejel [6] pointed out that the redundant rigidity augmentation problem is NP-hard when the input is an arbitrary rigid graph. Moreover, their proof implies that no constant factor approximation can be given to the the redundant rigidity augmentation problem unless P=NP (see [14]). This observation opens a possibility that Problem 1.1 is not that hard since we have seen that there exists a contant factor approximation algorithm for this problem. Hence with some further ideas our method may be extended to solve Problem 1.1 for rigid inputs.

Finally, we note that the results of this paper can be generalized, as follows. A well-known generalization of sparse graphs are the so-called (k, ℓ)-sparse graphs where the sparsity condition $i(X) \leq 2|X| - 3$ is substituted by $i(X) \leq k|X| - \ell$ where k and ℓ are given integers. The definition of (k, ℓ)-tight graphs and (k, ℓ)-redundant edges is similar. It is well-known that (k, ℓ)-tight graphs are 2-connected when $\ell > k$ and connected when $\ell > 0$. The result of García and Tejel [6] was generalized by [14] to (k, ℓ)-tight graphs. Since the proof of Theorem 3.1 relied on the proof of Theorem 2.13 in [14], we can prove a similar result to obtain a minimum cardinality edge set which augments a (k,ℓ)-tight graph to a (k,ℓ)-redundant and 3-connected graph when $\ell > k$. Furthermore, due to the similarity of 3-connected augmentations of 2-connected graphs, and 2-connected augmentations of connected graphs [11], a similar result can be given to obtain a minimum cardinality edge set which augments a (k,ℓ)-tight graph to a (k,ℓ)-redundant and 2-connected graph when $0 < \ell \leq k$.

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References

[1] B.D.O. Anderson, I. Shames, G. Mao, and B. Fidan. Formal theory of noisy sensor network localization. *SIAM J. Discrete Math.*, 24:684–698, 01 2010.

[2] J. Aspnes, T. Eren, D.K. Goldenberg, A.S. Morse, W. Whiteley, Y.R. Yang, B.D.O. Anderson, and P.N. Belhumeur. A theory of network localization. *IEEE Transactions on Mobile Computing*, 5(12):1663–1678, Dec 2006.

[3] A.R. Berg and T. Jordán. Algorithms for graph rigidity and scene analysis. In *Algorithms - ESA 2003, 11th Annual European Symposium, Budapest, Hungary, September 16-19, 2003, Proceedings*, pages 78–89, 2003.

[4] R. Connelly. Generic global rigidity. *Discrete & Computational Geometry*, 33(4):549–63, 2005.

[5] Zs. Fekete and T. Jordán. Uniquely localizable networks with few anchors. In S.E. Nikoletseas and J.D.P. Rolim, editors, *Algorithmic Aspects of Wireless Sensor Networks*, pages 176–183, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[6] A. García and J. Tejel. Augmenting the rigidity of a graph in $\mathbb{R}^2$. *Algorithmica*, 59(2):145–168, 2011.

[7] S.J. Gortler, A.D. Healy, and D.P. Thurston. Characterizing generic global rigidity. *American Journal of Mathematics*, 132(4):897–939, 2010.

[8] T.S. Hsu and V. Ramachandran. A linear time algorithm for triconnectivity augmentation. *Annual Symposium on Foundations of Computer Science (Proceedings)*, pages 548–559, 1991.

[9] B. Jackson and T. Jordán. Connected rigidity matroids and unique realizations of graphs. *J. Comb. Theory, Ser. B*, 94:1–29, 2005.

[10] B. Jackson and T. Jordán. Graph theoretic techniques in the analysis of uniquely localizable sensor networks. In G. Mao and B. Fidan, editors, *Localization algorithms and strategies for wireless sensor networks*, pages 146–73. IGI Global, 2009.

[11] T. Jordán. On the optimal vertex-connectivity augmentation. *J. Comb. Theory, Ser. B*, 63:8–20, 1995.

[12] T. Jordán. Combinatorial rigidity: Graphs and matroids in the theory of rigid frameworks. In *Discrete Geometric Analysis*, volume 34 of *MSJ Memoirs*, pages 33–112. Mathematical Society of Japan, Japan, 2016.

[13] P. Kaewprapha, J. Li, and N. Puttarak. Network localization on unit disk graphs. In *2011 IEEE Global Telecommunications Conference - GLOBECOM 2011*, pages 1–5, 2011.

[14] Cs. Király and A. Mihálykó. Sparse graphs and an augmentation problem. Technical Report TR-2020-06, Egerváry Research Group, Budapest, 2020. `www.cs.elte.hu/egres`
An extended abstract appeared in Bienstock D., Zambelli G., editors, *Integer Programming and Combinatorial Optimization, IPCO 2020*, volume 12125 of *Lecture Notes in Computer Science*, pages 238–251, Springer, Cham, 2020.

[15] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engineering Mathematics*, 4:331–340, 1970.

[16] H. Pollaczek-Geiringer. Über die Gliederung ebener Fachwerke. *ZAMM - Journal of Applied Mathematics and Mechanics*, 7(1):58–72, 1927.

[17] J.B. Saxe. Embeddability of weighted graphs in k-space is strongly NP-hard. Technical report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[18] A. So and Y. Ye. Theory of semidefinite programming for sensor network localization. *Mathematical Programming*, 109:405–414, 01 2005.