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Decreasing Minimization on M-convex Sets: Algorithms and Applications

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Abstract

This paper is concerned with algorithms and applications of decreasing minimization on an M-convex set, which is the set of integral elements of an integral base-polyhedron. Based on a recent characterization of decreasingly minimal (dec-min) elements, we develop a strongly polynomial algorithm for computing a dec-min element of an M-convex set. The matroidal feature of the set of dec-min elements makes it possible to compute a minimum cost dec-min element, as well. Our second goal is to exhibit various applications in matroid and network optimization, resource allocation, and (hyper)graph orientation. We extend earlier results on semi-matchings to a large degree by developing a structural description of dec-min in-degree bounded orientations of a graph. This characterization gives rise to a strongly polynomial algorithm for finding a minimum cost dec-min orientation.

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Running head: Decreasing Minimization: Algorithms and Applications

1 Introduction

This paper is concerned with algorithms and applications of decreasing minimization on an M-convex set, which is the set of integral elements of an integral base-polyhedron. An element of a set of vectors, in general, is called decreasingly minimal (dec-min) if its largest component is as small as possible, within this, its second largest component is as small as possible, and so on. Decreasing minimization means the problem of finding a dec-min

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element of a given set of vectors (or even a cheapest dec-min element with respect to a given linear cost-function). When the given set of vectors consists of integral vectors, this problem is also referred to as discrete decreasing minimization.

In the companion paper [16], the present authors have investigated the structural aspects of the discrete decreasing minimization on an M-convex set. Among others, the dec-min elements are characterized as those admitting no local improvement. As dual objects to dec-min elements, the notions of canonical chain, canonical partition of the ground-set, and essential value sequence are defined, and the structure of the set of all dec-min elements is described in terms of these dual objects. We emphasize that the role of these dual objects is not merely to help us fully understand the problem from its dual side. Beyond this, the dual characterization reveals the fundamental feature of the primal problem that the set of dec-min elements itself forms an M-convex set, and, in fact, a rather special one arising from a matroid by translation. In addition, these dual objects are inherent in computing a dec-min element in strongly polynomial time and indispensable for efficient computation of a minimum weight dec-min element, as well.

The first goal of this paper is to develop, on the basis of the above-mentioned structural characterizations, a strongly polynomial algorithm for computing a dec-min element as well as the canonical chain of a given M-convex set. The second goal is to exhibit several applications. For example, we prove a conjecture of Borradaile et al. [4] on dec-min strongly connected orientations of undirected graphs. Our general approach makes it possible to solve algorithmically even the minimum edge-cost dec-min orientation problem when upper and lower bounds are imposed on the in-degrees and the orientation is expected to be k-edge-connected (or even (k, ℓ)-edge-connected). These orientation results form the basis of a major generalization of the so-called semi-matching problem initiated by Ladner et al. [21], which had been motivated by a resource allocation problem. Our approach is the first one that provides a strongly polynomial algorithm for the capacitated case, as well.

An algorithmic solution to a discrete counterpart of Megiddo's lexicographic flow problem [31, 32] is also developed. Yet another application of the structural results of [16] gives rise to an extension of a result of Levin and Onn [30] on finding k bases of a matroid on a ground-set S with n elements such that the degree-vector of the hypergraph formed by these k bases is decreasingly minimal. Our approach generalizes this problem to the case when one has k distinct matroids on S.

The paper is organized as follows. Algorithms for computing a dec-min element and the canonical chain are given in Section 2. In Section 3, various kinds of applications are shown, including those to matroids, network flows, arborescences, and connectivity augmentations. Sections 4, 5, and 6 are devoted to detailed account of applications to graph orientation problems.

1.1 Notation and terminology

We continue to use notation and terminology introduced in [16], while some additional ones are given here. Two subsets X and Y of a finite S are **intersecting** if $X \cap Y \neq \emptyset$ and **crossing** if none of $X - Y$, $Y - X$, $X \cap Y$, and $S - (X \cup Y)$ is empty. Let b be a set-function for which $b(X) = +\infty$ is allowed but $b(X) = -\infty$ is not. The submodular inequality for subsets

$X, Y \subseteq S$ is defined by

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \tag{1.1}$$

We say that b is (fully) **submodular** if this inequality holds for every pair of subsets $X, Y \subseteq S$ with finite b-values. When the submodular inequality is required only for intersecting (crossing) pairs of subsets, we say that b is **intersecting** (**crossing**) **submodular**. A set-function p is called (fully, intersecting, crossing) **supermodular** if $-p$ is (fully, intersecting, crossing) submodular.

For a (fully) submodular integer-valued set-function b on S for which $b(\emptyset) = 0$ and $b(S)$ is finite, the **base-polyhedron** B is defined by

$$B = B(b) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S),\ \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subset S\}, \tag{1.2}$$

which is a (possibly unbounded) integral polyhedron in $\mathbf{R}^S$. A (fully) supermodular integer-valued set-function p with $p(\emptyset) = 0$ and $p(S)$ finite also defines an integral base-polyhedron by

$$B = B'(p) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = p(S),\ \widetilde{x}(Z) \geq p(Z) \text{ for every } Z \subset S\}. \tag{1.3}$$

In discrete convex analysis [34, 35], the set of integral elements of an integral base-polyhedron is called an **M-convex set**. For any integral polyhedron B we use the notation $\ddot{B}$ for the set of integral element of B, that is,

$$\ddot{B} := B \cap \mathbf{Z}^S, \tag{1.4}$$

where $\ddot{B}$ may be pronounced 'dotted B.'

When an (intersecting) submodular function b and an (intersecting) supermodular function p meet the cross-inequality

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \tag{1.5}$$

for every (intersecting) pair $X, Y \subseteq S$, the polyhedron Q defined by

$$Q = Q(p, b) := \{x : p(Z) \leq \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subseteq S\} \tag{1.6}$$

is called a **generalized polymatroid** (**g-polymatroid**, for short). A base-polyhedron is a special g-polymatroid (where $p(S) = b(S)$) and every g-polymatroid arises from a base-polyhedron by projecting it along a single axis.

In applications it is important that weaker set-functions may also define base-polyhedra and g-polymatroids. For example, if p is an integer-valued crossing supermodular function, then $B'(p)$ is still an integral base-polyhedron, which may, however, be empty. To prove theorems on base-polyhedra, it is much easier to work with base-polyhedra defined by fully sub- or supermodular functions. On the other hand, in applications, base-polyhedra are often defined with a crossing sub- or supermodular (or even weaker) function. (We shall use this fact frequently in Sections 5 and 6.)

We assume that graphs or digraphs have no loops but parallel edges are allowed. For a digraph $D = (V, A)$, the **in-degree** of a node v is the number of arcs of D with head v. The in-degree $\varrho_D(Z) = \varrho(Z)$ of a subset $Z \subseteq V$ denotes the number of edges (= arcs) entering Z, where an arc uv is said to enter Z if its head v is in Z while its tail u is in $V - Z$. The **out-degree** $\delta_D(Z) = \delta(Z)$ is the number of arcs leaving Z, that is $\delta(Z) = \varrho(V - Z)$. The number of edges of a directed or undirected graph H induced by $Z \subseteq V$ is denoted by $i(Z) = i_H(Z)$. In an undirected graph $G = (V, E)$, the **degree** $d(Z) = d_G(Z)$ of a subset $Z \subseteq V$ denotes the number of edges connecting Z and $V - Z$ while $e(Z) = e_G(Z)$ denotes the number of edges with one or two end-nodes in Z. Clearly, $e(Z) = d(Z) + i(Z)$.

2 Algorithms

In this section, we consider algorithmic aspects of decreasing minimization over an M-convex set. In particular, we show how to compute efficiently a decreasingly minimal element along with its canonical chain and partition.

First we recall fundamental characterizations of a dec-min element of an M-convex set.

Theorem 2.1 ([16, Theorem 3.3])**.** *For an element m of an M-convex set $\overset{....}{B} = \overset{....}{B}'(p)$, the following four conditions are pairwise equivalent.*

(A) *There is no 1-tightening step for m.*

(B) *There is a chain $(\emptyset \subset)\ C_1 \subset C_2 \subset \cdots \subset C_\ell\ (= S)$ such that each C_i is an m-top and m-tight set (with respect to p) and m is near-uniform on each $S_i := C_i - C_{i-1}$ $(i = 1, 2, \ldots, \ell)$, where $C_0 := \emptyset$.*

(C1) *m is decreasingly minimal in $\overset{....}{B}$.*

(C2) *m is increasingly maximal in $\overset{....}{B}$.* ∎

An integral base-polyhedron B can be given in the form $B(b)$ in (1.2) with a (fully) submodular function b or in the form $B'(p)$ in (1.3) with a (fully) supermodular function p. Here b and p are complementary functions (that is, $p(X) = b(S) - b(S - X)$) and hence an algorithm described for one of them can easily be transformed to work on the other. In the present description, we use supermodular functions.

There is a one-to-one correspondence between B and p but, for an intersecting or crossing supermodular function p, $B'(p)$ is also a (possibly empty) base-polyhedron which is integral if p is integer-valued. For obtaining and proving results for B (or for $\overset{....}{B}$), it is much easier to work with a fully supermodular p while in applications base-polyhedra often arise from intersecting or crossing (or even weaker) supermodular functions. Therefore in describing and analysing algorithms, we must consider these weaker functions as well.

Remark 2.1. One of the most fundamental algorithms of discrete optimization is for minimizing a submodular function, that is, for finding a subset Z of S for which $b(Z) = \min\{b(X) : X \subseteq S\}$. There are strongly polynomial algorithms for this problem (for example, Schrijver [40] and Iwata et al. [26] are the first, while Orlin [37] is one of the fastest), and we shall refer to such an algorithm as a **submod-minimizer** subroutine. The complexity of Orlin's algorithm [37], for example, is $O(n^6)$ (where $n = |S|$) and the algorithm calls $O(n^5)$ times a routine which evaluates the submodular function in question. (An evaluation routine outputs the value $b(X)$ for any input subset $X \subseteq S$). This complexity bound is definitely attractive from a theoretical point of view but in concrete applications it is always a challenge to develop faster algorithms for the special case. Naturally, submodular function minimization and supermodular function maximization are equivalent. ∎

2.1 The basic algorithm for computing a dec-min element

Our first goal is to describe a natural approach—the **basic algorithm**—for finding a decreasingly minimal element of an M-convex set $\overset{....}{B}$. The basic algorithm is polynomial in $n + |p(S)|$, and hence it is polynomial in n when $|p(S)|$ is small in the sense that it can be bounded by a polynomial of n. This is the case, for example, in an application when we are interested in strongly connected decreasingly minimal (=egalitarian) orientations. In the general case, where typical applications arise by defining p with a 'large' capacity function, a (more complex) strongly polynomial-time algorithm will be described in Section 2.4.

In order to find a dec-min element of an M-convex set $\overset{....}{B}$, we assume that a subroutine is available to

$$\text{compute an integral element of } B. \tag{2.1}$$

When $B = B'(p)$ and p is fully supermodular, a variant of Edmonds' polymatroid greedy algorithm finds an integral member of B. (Namely, take any ordering $s_1, \dots, s_n$ of S, and define $m(s_1) := p(s_1)$ and, for $i = 2, \dots, n$, $m(s_i) = p(Z_i) - p(Z_{i-1})$ where $Z_i = \{s_1, s_2, \dots, s_i\}$. Edmonds [6] proved that vector m is indeed in B). This algorithm needs only a subroutine to evaluate $p(Z_i)$ for $i = 1, \dots, n$. For an intersecting supermodular function p, Frank and Tardos [18] described an algorithm which needs n applications of a submod-minimizer routine. For crossing supermodular p, a more complex algorithm is given in [18] which terminates after at most n^2 applications of a submod-minimizer. Note that the latter problem of finding an integral element of a base-polyhedron $B'(p)$ defined by a crossing supermodular function p covers such non-trivial problems as the one of finding a degree-constrained k-edge-connected orientation of an undirected graph, a problem solved first in [10].

Suppose now that an initial integral member m of B is available. The algorithm needs a subroutine to

$$\text{decide for } m \in \overset{....}{B} \text{ and for } s, t \in S \text{ if } m' := m + \chi_s - \chi_t \text{ belongs to } B. \tag{2.2}$$

Observe that Subroutine (2.2) is certainly available if we can

$$\text{decide for any } m' \in \mathbf{Z}^S \text{ whether or not } m' \text{ belongs to } B, \tag{2.3}$$

though applying this more general subroutine is clearly slower than a direct algorithm to realize (2.2).

Note that $m' = m + \chi_s - \chi_t$ is in B precisely if there is no m-tight $t\overline{s}$-set (with respect to p), and this is true even if B is defined by a crossing supermodular function p. Subroutine (2.2) can be carried out by a single application of a submod-minimizer.

As long as possible, apply the 1-tightening step. Recall that a 1-tightening step replaces m by $m' := m + \chi_s - \chi_t$ where s and t are elements of S for which $m(t) \geq m(s) + 2$ and m' belongs to $\overset{....}{B}$. By Theorem 2.1, when no more 1-tightening step is available, the current m is a decreasingly minimal member of $\overset{....}{B}$ and the algorithm terminates. In order to estimate the number of 1-tightening steps, observe that a single 1-tightening step decreases the square-sum of the components. Since the largest square-sum of an arbitrary integral vector z with $\widetilde{z}(S) = p(S)$ is $p(S)^2$ and $\widetilde{z}(S) = p(S)$ holds for all members z of $\overset{....}{B}$, we conclude that the number of 1-tightening steps is at most $p(S)^2$. Therefore if $|p(S)|$ is bounded by a

Given a dec-min element m of $\ddot{B}$, the following procedure computes the canonical chain $\mathcal{C}^* = \{C_1, C_2, \dots, C_q\}$ and the canonical partition $\mathcal{P}^* = \{S_1, S_2, \dots, S_q\}$ of S along with the essential value-sequence $\beta_1 > \beta_2 > \cdots > \beta_q$ belonging to $\ddot{B}$.

1. Let β_1 denote the largest value of m. Let $C_1 := \bigcup\{T_m(u) : m(u) = \beta_1\}$, $S_1 := C_1$, and $i := 2$.

2. In the general step $i \geq 2$, the pairwise disjoint non-empty sets $S_1, S_2, \dots, S_{i-1}$ and a chain $C_1 \subset C_2 \subset \cdots \subset C_{i-1}$ have already been computed along with the essential values $\beta_1 > \beta_2 > \cdots > \beta_{i-1}$. If $C_{i-1} = S$, set $q := i - 1$ and stop. Otherwise, let

$$\begin{aligned} \beta_i &:= \max\{m(s) : s \in S - C_{i-1}\}, \\ C_i &:= \bigcup\{T_m(u) : m(u) \geq \beta_i\}, \\ S_i &:= C_i - C_{i-1}, \end{aligned}$$

and go to the next step for $i := i + 1$. ■

polynomial of n, then the basic algorithm to compute a dec-min element of $\ddot{B}$ is strongly polynomial.

The basic algorithm above is efficient when $|p(S)|$ is 'small' (that is, $|p(S)|$ is bounded by a power of n), but it is not strongly polynomial when $|p(S)|$ is 'large'. We postpone, till Section 2.4, the description of a strongly polynomial algorithm for computing a dec-min element of an M-convex set $\ddot{B}$ defined by a general p. In the next section we show how the canonical chain as well as the essential value-sequence can be computed, once a dec-min element m is available. It is emphasized that these dual objects are indispensable and must be computed when we are interested in identifying the set of all dec-min elements of $\ddot{B}$ or in finding a minimum weight dec-min element (cf., [16, Section 5.3]).

2.2 Computing the essential value-sequence and the canonical chain

In this section we show an algorithm to compute the essential value-sequence and the canonical chain, when we are given a dec-min element m of an M-convex set.

Let $B = B'(p)$ be again an integral base-polyhedron whose unique (fully) supermodular bounding function is p. In the algorithm, we assume that we can compute the smallest m-tight set $T_m(u) = T_m(u; p)$ containing a given element $u \in S$. Here m-tightness is with respect to p, that is, a set X is m-tight if $\widetilde{m}(X) = p(X)$. It is fundamental, however, to emphasize that $T_m(u)$ can be computed even in the case when p is not explicitly available and B is defined by a weaker function, for example, by a crossing supermodular function. Namely, we have $T_m(u) = \{s : m + \chi_s - \chi_u \in B\}$ and hence $T_m(u)$ is indeed computable by at most n applications of routine (2.2).

Corollary 5.4 from [16] states that the sequence $\beta_1, \beta_2, \dots, \beta_q$ provided by this algorithm is indeed the essential value-sequence belonging to $\ddot{B}$, and similarly the chain $C_1 \subset C_2 \subset \cdots \subset C_q$ is the canonical chain while the partition $\{S_1, S_2, \dots, S_q\}$ is the canonical partition.

We emphasize that Algorithm 2.2 to compute the essential value-sequence and the canonical chain is strongly polynomial for arbitrary p (independently of the magnitude of $|p(S)|$), provided that a dec-min element m of $\overset{....}{B}$ is already available as well as Oracle (2.2).

It is in order here to emphasize the significance of this algorithm for computing these dual objects. By Theorem 2.2 below, Algorithm 2.2 enables us to computationally capture the set of all dec-min elements. Concisely, the matroid associated with dec-min elements, as in Theorem 2.3 below, can be identified by this algorithm. Recall that a **matroidal M-convex set** is the translation of the incidence vectors of bases of a matroid by an integral vector.

Theorem 2.2 ([16, Corollary 5.2]). *An element m of an M-convex set $\overset{....}{B}$ is decreasingly minimal if and only if each C_i is m-tight (with respect to p) and $\beta_i - 1 \leq m(s) \leq \beta_i$ holds for each $s \in S_i$ $(i = 1, \ldots, q)$.* ■

Theorem 2.3 ([16, Theorem 5.7]). *The set of dec-min elements of an M-convex set $\overset{....}{B}$ is a matroidal M-convex set.* ■

We shall we use Theorem 2.2 in Sections 4.3 and 5.1, and Theorem 2.3 in Sections 4.5 and 5.2.

Adaptation to the intersection with a box Algorithm 2.2 can be adapted to the case when we have specific upper and lower bounds on the members of $\overset{....}{B} = \overset{....}{B'}(p)$. Let $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{+\infty\}$ be bounding functions with $f \leq g$ and let $T(f, g)$ denote the box defined by f and g. It is a basic fact on integral base-polyhedra that the intersection $B^{\square} := B \cap T(f, g)$ is also a (possibly empty) integral base-polyhedron. Assume that $B^{\square}$ is non-empty.

Let m be an element of $\overset{....}{B^{\square}}$ $(= B^{\square} \cap \mathbf{Z}^S)$. Let $T_m(u)$ denote the smallest m-tight set containing u with respect to p, and let $T^{\square}_m(u)$ be the smallest m-tight set containing u with respect to $p^{\square}$.

Claim 2.4.

$$T^{\square}_m(u) = \begin{cases} \{u\} & \text{if} \quad m(u) = f(u), \\ T_m(u) - \{v : m(v) = g(v)\} & \text{if} \quad m(u) > f(u). \end{cases}$$

Proof. We have $T^{\square}_m(u) = \{s : m - \chi_u + \chi_s \in B^{\square}\}$. Since $B^{\square} = B \cap T(f, g)$, we have $m - \chi_u + \chi_s \in B^{\square}$ if and only if (i) $m - \chi_u + \chi_s \in B$ and (ii) $m - \chi_u + \chi_s \in T(f, g)$ hold. For $s \neq u$, (i) holds if and only if $s \in T_m(u)$, and (ii) holds if and only if $m(u) > f(u)$ and $m(s) < g(s)$. Hence follows the claim. ■

The claim implies that Algorithm 2.2 can be adapted easily to compute the canonical chain and partition belonging to $\overset{....}{B^{\square}}$ along with its essential value-sequence.

Our next goal is to describe a strongly polynomial algorithm to compute a dec-min element of $\overset{....}{B}$ in the general case when no restriction is imposed on the magnitude of $|p(S)|$. To this end, we need an algorithm to maximize $\lceil p(X)/|X| \rceil$, which is given in Section 2.3. The strongly polynomial algorithm for computing a dec-min element is described in Section 2.4.

2.3 Maximizing $\lceil p(X)/|X|\rceil$ with the Newton–Dinkelbach algorithm

In this section we describe a variant of the Newton–Dinkelbach (ND) algorithm to compute the maximum of $\lceil p(X)/|X|\rceil$. We assume that p is an integer-valued set-function on a ground-set S with $n \geq 1$ elements, $p(\emptyset) = 0$, $p(S)$ is finite ($p(X)$ may be $-\infty$ for some X but never $+\infty$).

An excellent overview by Radzik [38] analyses the ND-algorithm concerning (among others) this problem and describes a strongly polynomial algorithm. We present a variant of the ND-algorithm whose specific feature is that it works throughout with integers $\lceil p(X)/|X|\rceil$. This has the advantage that the proof is simpler than the original one working with the fractions $p(X)/|X|$.

The algorithm works if a subroutine is available to

$$\text{find a subset } \; X \subseteq S \; \text{ maximizing } \; p(X) - \mu|X| \; \text{ for any fixed integer } \mu. \tag{2.4}$$

This routine will actually be needed only for special values of μ when $\mu = \lceil p(X)/\ell\rceil$ (where $X \subseteq S$ and $1 \leq \ell \leq n$). We do not have to assume that p is supermodular and the only requirement for the ND-algorithm is that Subroutine (2.4) be available. Via a submod-minimizer this is certainly the case when p happens to be supermodular (cf., Remark 2.1).

In several applications, the requested general purpose submod-minimizer can be superseded by a direct and more efficient algorithm such as the one for network flows or for matroid partition. Subroutine (2.4) is also available in the more general case (needed in applications) when p is only crossing supermodular. Indeed, for a given ordered pair of elements $s, t \in S$, the restriction of p on the family of $s\bar{t}$-sets is fully supermodular, and therefore we can apply a submod-minimizer to each of the $n(n-1)$ ordered pairs (s,t) to get the requested maximum of $p(X) - \mu|X|$.

We call a value μ **good** if $\mu|X| \geq p(X)$ for every $X \subseteq S$. A value that is not good is called **bad**. Clearly, a sufficiently large μ is good. Our goal is to compute the minimum $\mu_{\min}$ of the good integers. This number is nothing but the maximum of $\lceil p(X)/|X|\rceil$ over non-empty subsets of S.

Let $\mu_0 := \lceil p(S)/|S|\rceil - 1$. This (possibly negative) number is bad and the algorithm starts with μ_0. Let

$$X_0 \in \arg\max\{p(X) - \mu_0|X| : \; X \subseteq S\},$$

that is, X_0 is a set maximizing the function $p(X)-\mu_0|X|$. Note that the badness of μ_0 implies that $p(X_0) > 0$.

The procedure determines one by one a series of pairs (μ_j, X_j) for subscripts $j = 1, 2, \ldots$ where each integer μ_j is a tentative candidate for μ while X_j is a non-empty subset of S. Suppose that the pair (μ_{j-1}, X_{j-1}) has already been determined for a subscript $j \geq 1$. Let μ_j be the smallest integer for which $\mu_j|X_{j-1}| \geq p(X_{j-1})$, that is,

$$\mu_j := \left\lceil \frac{p(X_{j-1})}{|X_{j-1}|} \right\rceil.$$

If μ_j is bad, that is, if there is a set $X \subseteq S$ with $p(X) - \mu_j|X| > 0$, then let

$$X_j \in \arg\max\{p(X) - \mu_j|X| : \; X \subseteq S\},$$

that is, X_j is a set maximizing the function $p(X) - \mu_j|X|$. (If there are more than one maximizing set, we can take any). Since μ_j is bad, we have $X_j \neq \emptyset$ and $p(X_j) - \mu_j|X_j| > 0$.

Claim 2.5. *If μ_j is bad for some subscript $j \geq 0$, then $\mu_j < \mu_{j+1}$.*

Proof. The badness of μ_j means that $p(X_j) - \mu_j|X_j| > 0$, from which

$$\mu_{j+1} = \left\lceil \frac{p(X_j)}{|X_j|} \right\rceil = \left\lceil \frac{p(X_j) - \mu_j|X_j|}{|X_j|} \right\rceil + \mu_j \ > \ \mu_j. \qquad \blacksquare$$

Since there is a good μ and the sequence μ_j is strictly monotone increasing by Claim 2.5, there will be a first subscript $h \geq 1$ for which μ_h is good. The algorithm terminates by outputting this μ_h (and in this case X_h is not computed).

Theorem 2.6. *If h is the first subscript during the run of the algorithm for which μ_h is good, then $\mu_{\min} = \mu_h$ (that is, μ_h is the requested smallest good μ-value) and $h \leq n$.*

Proof. Since μ_h is good and μ_h is the smallest integer for which $\mu_h|X_{h-1}| \geq p(X_{h-1})$, the set X_{h-1} certifies that no good integer μ can exist which is smaller than μ_h, that is, $\mu_{\min} = \mu_h$.

Claim 2.7. *If μ_j is bad for some subscript $j \geq 1$, then $|X_{j-1}| > |X_j|$.*

Proof. As μ_j $(= \lceil p(X_{j-1})/|X_{j-1}|\rceil)$ is bad, we obtain that

$$\begin{aligned} p(X_j) - \mu_j|X_j| > 0 &= p(X_{j-1}) - \frac{p(X_{j-1})}{|X_{j-1}|}|X_{j-1}| \\ &\geq p(X_{j-1}) - \left\lceil \frac{p(X_{j-1})}{|X_{j-1}|} \right\rceil |X_{j-1}| = p(X_{j-1}) - \mu_j|X_{j-1}|, \end{aligned}$$

from which we get

$$p(X_j) - \mu_j|X_j| > p(X_{j-1}) - \mu_j|X_{j-1}|. \tag{2.5}$$

Since X_{j-1} maximizes $p(X) - \mu_{j-1}|X|$, we have

$$p(X_{j-1}) - \mu_{j-1}|X_{j-1}| \geq p(X_j) - \mu_{j-1}|X_j|. \tag{2.6}$$

By adding up (2.5) and (2.6), we obtain

$$(\mu_j - \mu_{j-1})|X_{j-1}| > (\mu_j - \mu_{j-1})|X_j|.$$

As μ_j is bad, so is μ_{j-1}, and hence, by applying Claim 2.5 to $j - 1$ in place of j, we obtain that $\mu_j > \mu_{j-1}$, from which we arrive at $|X_{j-1}| > |X_j|$, as required. ■

Claim 2.7 implies that $n \geq |X_0| > |X_1| > \cdots > |X_{h-1}| \geq 1$, from which $h \leq n$ follows. ■■

2.4 Computing a dec-min element in strongly polynomial time

In order to compute a dec-min element of an M-convex set $\overset{....}{B} = \overset{....}{B}'(p)$, our first task is to compute the smallest integer β_1 for which $\overset{....}{B}$ has an element with largest component β_1. Theorem 4.1 of [16] asserts that $\beta_1 = \max\{\lceil p(X)/|X|\rceil : \emptyset \neq X \subseteq S\}$. By applying the ND-algorithm described in Section 2.3, we can compute β_1 in strongly polynomial time. Note that, by Theorem 2.6, the algorithm terminates after at most n applications of Subroutine (2.4).

Given the value of β_1, a β_1-covered element m of $\overset{....}{B}$ can easily be computed with a greedy-type algorithm as follows. Since there is a β_1-covered member of B, the vector $(\beta_1, \beta_1, \dots, \beta_1)$ belongs to the so-called supermodular polyhedron $S'(p) := \{x : \widetilde{x}(X) \geq p(X)$ for every $X \subseteq S\}$. Consider the elements of S in an arbitrary order $\{s_1, \dots, s_n\}$. Let $m(s_1) := \min\{z : (z, \beta_1, \beta_1, \dots, \beta_1) \in S'(p)\}$. In the general step, if the components $m(s_1), \dots, m(s_{i-1})$ have already been determined, let

$$m(s_i) := \min\{z : (m(s_1), m(s_2), \dots, m(s_{i-1}), z, \beta_1, \beta_1, \dots, \beta_1) \in S'(p)\}. \tag{2.7}$$

This computation can be carried out by n applications of a subroutine for a submodular function minimization.

Given a β_1-covered integral element of B, our next goal is to obtain a pre-dec-min element of $\overset{....}{B}$. To this end, we apply 1-tightening steps. That is, as long as possible, we pick two elements s and t of S for which $m(t) = \beta_1$ and $m(s) \leq \beta_1 - 2$ such that there is no m-tight $t\overline{s}$-set, reduce $m(t)$ by 1 and increase $m(s)$ by 1. In this way, we obtain another integral element of B for which the largest component continues to be β_1 (as β_1 was chosen to be the smallest upper bound) but the number of β_1-valued components is strictly smaller. Therefore, after at most $|S| - 1$ such 1-tightening steps, we arrive at a vector for which no 1-tightening step (with $m(t) = \beta_1$ and $m(s) \leq \beta_1 - 2$) is possible anymore. Theorem 4.2 of [16] states that a β_1-covered element m of $\overset{....}{B}$ is pre-dec-min precisely if $m(s) \geq \beta_1 - 1$ for each $s \in S_1(m)$, where $S_1(m) = \cup\{T_m(t) : m(t) = \beta_1\}$. Hence the final vector the previous procedure is a pre-decreasingly minimal element of $\overset{....}{B}$. We use the same letter m to denote this pre-dec-min element.

Recall that $T_m(t)$ denoted the unique smallest tight set containing t when p is (fully) supermodular. But $T_m(t)$ can be described without explicitly referring to p since an element $s \in S$ belongs to $T_m(t)$ precisely if $m' := m - \chi_t + \chi_s$ is in B, and this is computable by subroutine (2.2). Therefore we can compute $S_1(m)$.

Theorem 4.4 of [16] states that $S_1(m)$ is the first member S_1 of the canonical partition associated with $\overset{....}{B}$. Let B_1 denote the restriction of the base-polyhedron B to S_1 and B'_1 the contraction of B by S_1. Theorem 4.6 of [16] states that, for $m_1 \in \mathbf{Z}^{S_1}$ and $m'_1 \in \mathbf{Z}^{S-S_1}$, (m_1, m'_1) is a dec-min element of $\overset{....}{B}$ precisely if m_1 is a dec-min element of $\overset{....}{B}_1$ and m'_1 is a dec-min element of $\overset{....}{B}'_1$. Let $m_1 := m|S_1$ for the pre-dec-min element m constructed above. Since m_1 is near-uniform on S_1, it is a dec-min element of $\overset{....}{B}_1$. Hence, if m'_1 is a dec-min element of $\overset{....}{B}'_1$, then (m_1, m'_1) is a dec-min element of $\overset{....}{B}$. Such a dec-min element m'_1 can be computed by applying iteratively the computation described above for computing m_1. In this way we can compute a dec-min element of $\overset{....}{B}$ in strongly polynomial time.

3 Applications

3.1 Background

There are two major sources of applicability of the structural results on decreasing minimization on an M-convex set. One of them relies on the fact that the class of integral base-polyhedra is closed under several operations. For example, a face of a base-polyhedron is also a base-polyhedron, and so is the intersection of an integral box with a base-polyhedron B. Also, the sum of integral base-polyhedra $B_1, \dots, B_k$ is a base-polyhedron B which has, in addition, the integer decomposition property meaning that any integral element of B can be obtained as the sum of k integral elements by taking one from each B_i. This latter property implies that the sum of M-convex sets is M-convex. We also mention the important operation of taking an aggregate of a base-polyhedron, to be introduced below in Section 3.2.

The other source of applicability is based on the fact that not only fully super- or submodular functions can define base-polyhedra but some weaker functions as well. For example, if p is an integer-valued crossing (in particular, intersecting) supermodular function with finite $p(S)$, then $B = B'(p)$ is a (possibly empty) integral base-polyhedron (and $\overset{....}{B}$ is an M-convex set). This fact will be exploited in solving dec-min orientation problems when both degree-constraints and edge-connectivity requirements must be fulfilled. In some cases even weaker set-functions can define base-polyhedra. This is why we can solve dec-min problems concerning edge- and node-connectivity augmentations of digraphs.

3.2 Matroids

Levin and Onn [30] solved algorithmically the following problem. Find k bases of a matroid M on a ground-set S such that the sum of their characteristic vectors be decreasingly minimal. Their approach, however, does not seem to work in the following natural extension. Suppose we are given k matroids $M_1, \dots, M_k$ on a common ground-set S, and our goal is to find a basis B_i of each matroid M_i in such a way that the vector $\sum[\chi_{B_i} : i = 1, \dots, k]$ is decreasingly minimal. Let B_Σ denote the sum of the base-polyhedra of the k matroids. By a theorem of Edmonds, the integral elements of B_Σ are exactly the vectors of form $\sum[\chi_{B_i} : i = 1, \dots, k]$ where B_i is a basis of M_i. Therefore the problem is to find a dec-min element of $\overset{....}{B}_\Sigma$. This can be found by the basic algorithm described in Section 2.1. Let us see how the requested subroutines are available in this special case. The algorithm starts with an arbitrary member m of $\overset{....}{B}_\Sigma$ which is obtained by taking a basis B_i from each matroid M_i, and these bases define $m := \sum_i \chi_{B_i}$.

To realize Subroutine (2.2), we mentioned that it suffices to realize Subroutine (2.3), which requires for a given integral vector m' with $\widetilde{m}'(S) = \sum_i r_i(S)$ to decide whether m' is in $\overset{....}{B}_\Sigma$ or not. But this can simply be done by Edmonds' matroid intersection algorithm. Namely, let $S_1, \dots, S_k$ be disjoint copies of S and M'_i an isomorphic copy of M_i on S_i. Let N_1 be the direct sum of matroids M'_i on ground-set $S' := S_1 \cup \cdots \cup S_k$. Let N_2 be a partition matroid on S' in which a subset Z is a basis if it contains exactly $m'(s)$ members of the k copies of s for each $s \in S$. Then m' is in $\overset{....}{B}_\Sigma$ precisely if N_1 and N_2 have a common basis.

In conclusion, with the help of Edmonds' matroid intersection algorithm, Subroutine (2.2) is available, and hence the basic algorithm can be applied. (Actually, the algorithm can be sped up by looking into the details of the matroid intersection algorithm for N_1 and N_2.)

Another natural problem concerns a single matroid M on a ground-set T. Suppose we are given a partition $\mathcal{P} = \{T_1, \dots, T_n\}$ of T and we consider the intersection vector $(|Z \cap T_1|, \dots, |Z \cap T_n|)$ assigned to a basis Z of M. The problem is to find a basis for which the intersection vector is decreasingly minimal.

To solve this problem, we recall an important construction of base-polyhedra, called the aggregate. Let T be a ground-set and B_T an integral base-polyhedron in $\mathbf{R}^T$. Let $\mathcal{P} = \{T_1, \dots, T_n\}$ be a partition of T into non-empty subsets and let $S = \{s_1, \dots, s_n\}$ be a set whose elements correspond to the members of $\mathcal{P}$. The aggregate B_S of B_T is defined as follows.

$$B_S := \{(y_1, \dots, y_n) : \text{there is an } x \in B_T \text{ with } y_i = \widetilde{x}(T_i)\ (i = 1, \dots, n)\}. \tag{3.1}$$

A basic theorem concerning base-polyhedra states that B_S is a base-polyhedron, moreover, for each integral member $(y_1, \dots, y_n)$ of B_S, the vector x in (3.1) can be chosen integer-valued. In other words,

$$\overset{....}{B_S} := \{(y_1, \dots, y_n) : \text{there is an } x \in \overset{....}{B_T} \text{ with } y_i = \widetilde{x}(T_i)\ (i = 1, \dots, n)\}. \tag{3.2}$$

We call $\overset{....}{B_S}$ the **aggregate** of $\overset{....}{B_T}$.

Returning to our matroid problem, let B_T denote the base-polyhedron of matroid M. Then the problem is nothing but finding a dec-min element of $\overset{....}{B_S}$.

We can apply the basic algorithm (concerning M-convex sets) for this special case since the requested subroutines are available through standard matroid algorithms. Namely, Subroutine (2.1) is available since for any basis Z of M, the intersection vector assigned to Z is nothing but an element of $\overset{....}{B_S}$.

To realize Subroutine (2.2), we mentioned that it suffices to realize Subroutine (2.3). Suppose we are given a vector $y \in \mathbf{Z}_+^S$ (Here y stands for m' in (2.3)). Suppose that $\widetilde{y}(S) = r(T)$ (where r is the rank-function of matroid M) and that $y(s_i) \leq |T_i|$ for $i = 1, \dots, n$.

Let $G = (S, T; E)$ denote a bipartite graph where $E = \{ts_i : t \in T_i, i = 1, \dots, n\}$. By this definition, the degree of every node in T is 1 and hence the elements of E correspond to the elements of M. Let M_1 be the matroid on E corresponding to M (on T). Let M_2 be a partition matroid on E in which a set $F \subseteq E$ is a basis if $d_F(s_i) = y(s_i)$. By this construction, the vector y is in $\overset{....}{B_S}$ precisely if the two matroids M_1 and M_2 have a common basis. This problem is again tractable by Edmonds' matroid intersection algorithm.

As a special case, we can find a spanning tree of a (connected) directed graph for which its in-degree-vector is decreasingly minimal. Since the family of unions of k disjoint bases of a matroid forms also a matroid, we can also compute k edge-disjoint spanning trees in a digraph whose union has a decreasingly minimal in-degree vector.

Another special case is when we want to find a spanning tree of a connected bipartite graph $G = (S, T; E)$ whose in-degree vector restricted to S is decreasingly minimal.

3.3 Flows

3.3.1 A base polyhedron associated with net-in-flows

Let $D = (V, A)$ be a digraph endowed with integer-valued bounding functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ for which $f \leq g$. We call a vector (or function) z on A **feasible** if $f \leq z \leq g$. The **net-in-flow** Ψ_z of z is a vector on V and defined by $\Psi_z(v) = \varrho_z(v) - \delta_z(v)$, where $\varrho_z(v) := \sum[z(uv) : uv \in A]$ and $\delta_z(v) := \sum[z(vu) : uv \in A]$. If m is the net-in-flow of a vector z, then we also say that z is an m-**flow**.

A variation of Hoffman's classic theorem on feasible circulations [24] is as follows.

Lemma 3.1. *An integral vector $m : V \to \mathbf{Z}$ is the net-in-flow of an integral feasible vector (or in other words, there is an integer-valued feasible m-flow) if and only if $\widetilde{m}(V) = 0$ and*

$$\varrho_f(Z) - \delta_g(Z) \leq \widetilde{m}(Z) \text{ holds whenever } Z \subseteq V, \tag{3.3}$$

where $\varrho_f(Z) := \sum[f(a) : a \in A \text{ and } a \text{ enters } Z]$ *and* $\delta_g(Z) := \sum[g(a) : a \in A \text{ and } a \text{ leaves } Z]$. ∎

Define a set-function p_{fg} on V by

$$p_{fg}(Z) := \varrho_f(Z) - \delta_g(Z).$$

Then p_{fg} is (fully) supermodular (see, e.g. Proposition 1.2.3 in [12]). Consider the base-polyhedron $B_{fg} := B'(p_{fg})$ and the M-convex set $\overset{....}{B}_{fg}$. By Lemma 3.1 the M-convex set $\overset{....}{B}_{fg}$ consists exactly of the net-in-flow integral vectors m.

By the algorithm described in Section 2, we can compute a decreasingly minimal element of $\overset{....}{B}_{fg}$ in strongly polynomial time. By relying on a strongly polynomial push-relabel algorithm, we can check whether or not (3.3) holds. If it does not, then the push-relabel algorithm can compute a set most violating (3.3) (that is, a maximizer of $\varrho_f(Z) - \delta_g(Z) - \widetilde{m}(Z)$) while if (3.3) does hold, then the push-relabel algorithm computes an integral valued feasible m-flow. Therefore the requested oracles in the general algorithm for computing a dec-min element are available through a network flow algorithm, and we do not have to rely on a general-purpose submodular function minimizing oracle.

For the sake of an application of this algorithm to capacitated dec-min orientations in Section 4.2, we remark that the algorithm can also be used to compute a dec-min element of the M-convex set obtained from $\overset{....}{B}_{fg}$ by translating it with a given integral vector.

3.3.2 Discrete version of Megiddo's flow problem

Megiddo [31], [32] considered the following problem. Let $D = (V, A)$ be a digraph endowed with a non-negative capacity function $g : A \to \mathbf{R}+$. Let S and T be two disjoint non-empty subsets of V. Megiddo described an algorithm to compute a feasible flow from S to T with maximum flow amount M for which the net-in-flow vector restricted on S is (in our terms) increasingly maximal. Here a feasible flow is a vector x on A for which $\Psi_x(v) \leq 0$ for $v \in S$, $\Psi_x(v) \geq 0$ for $v \in T$, and $\Psi_x(v) = 0$ for $v \in V - (S \cup T)$. The flow amount x is $\sum[\Psi_x(t) : t \in T]$.

We emphasize that Megiddo solved the continuous (fractional) case and did not consider the corresponding discrete (or integer-valued) flow problem. To our knowledge, this natural optimization problem has not been investigated so far.

To provide a solution, suppose that g is integer-valued. Let $f \equiv 0$ and consider the net-in-flow vectors belonging to feasible vectors. These form a base-polyhedron B_1 in $\mathbf{R}^V$. Let B_2 denote the base polyhedron obtained from B_1 by intersecting it with the box defined by $z(v) \leq 0$ for $v \in S$, $z(v) \geq 0$ for $v \in T$ and $z(v) = 0$ for $v \in V - (S \cup T)$.

The restriction of B_2 to S is a g-polymatroid Q in $\mathbf{R}^S$. And finally, we can consider the face of Q defined by $\widetilde{z}(S) = -M$. This is a base-polyhedron B_3 in $\mathbf{R}^S$, and the discrete version of Megiddo's flow problem is equivalent to finding an inc-max element of $\ddot{B}_3$. (Recall that an element of an M-convex set is dec-min precisely if it is inc-max.)

It can be shown that in this case again the general submodular function minimizing subroutine used in the algorithm to find a dec-min element of an M-convex set can be replaced by a max-flow min-cut algorithm.

A recent paper [17] addresses a more general problem to find an integral feasible flow that is dec-min on an arbitrarily specified edge set.

3.4 Further applications

3.4.1 Root-vectors of arborescences

A graph-example comes from packing arborescences. Let $D = (V, A)$ be a digraph and $k > 0$ an integer. We say that a non-negative integral vector $m : V \rightarrow \mathbf{Z}_+$ is a **root-vector** if there are k edge-disjoint spanning arborescences such that each node $v \in V$ is the root of $m(v)$ arborescences. Edmonds [7] classic result on disjoint arborescences implies that m is a root-vector if and only if $\widetilde{m}(V) = k$ and $\widetilde{m}(X) \geq k - \varrho(X)$ holds for every subset X with $\emptyset \subset X \subset V$. Define set-function p by $p(X) := k - \varrho(X)$ if $\emptyset \subset X \subseteq V$ and $p(\emptyset) := 0$. Then p is intersecting supermodular, so $B'(p)$ is an integral base-polyhedron. The intersection B of $B'(p)$ with the non-negative orthant is also a base-polyhedron, and the theorem of Edmonds is equivalent to stating that a vector m is a root-vector if and only if m is in $\ddot{B}$.

Therefore the general results on base-polyhedra can be specialized to obtain k disjoint spanning arborescences whose root-vector is decreasingly minimal.

3.4.2 Connectivity augmentations

Let $D = (V, A)$ be a directed graph and $k > 0$ an integer. We are interested in finding a so-called augmenting digraph $H = (V, F)$ of γ arcs for which $D + H$ is k-edge-connected or k-node-connected. In both cases, the in-degree vectors of the augmenting digraphs are the integral elements of an integral base-polyhedron [11], [13]. Obviously, the in-degree vectors of the augmented digraphs are the integral elements of an integral base-polyhedron.

Again, our results on general base-polyhedra can be specialized to find an augmenting digraph whose in-degree vector is decreasingly minimal.

4 Orientations of graphs

Let $G = (V, E)$ be an undirected graph. For $X \subseteq V$, let $i_G(X)$ denote the number of edges induced by X while $e_G(X)$ is the number of edges with at least one end-node in X. Then i_G is supermodular, e_G is submodular, and they are complementary functions, that is, $i_G(X) = e_G(V) - e_G(V - X)$. Let $B_G := B(e_G) = B'(i_G)$ denote the base-polyhedron defined by e_G or i_G.

We say that a function $m : V \rightarrow \mathbf{Z}$ is the **in-degree vector** of an orientation D of G if $\varrho_D(v) = m(v)$ for each node $v \in V$. An in-degree vector m obviously meets the equality $\widetilde{m}(V) = |E|$. The following basic result, sometimes called the Orientation lemma, is due to Hakimi [19].

Lemma 4.1 (Orientation lemma)**.** *Let $G = (V, E)$ be an undirected graph and $m : V \rightarrow \mathbf{Z}$ an integral vector for which $\widetilde{m}(V) = |E|$. Then G has an orientation with in-degree vector m if and only if*

$$\widetilde{m}(X) \leq e_G(X) \ \text{ for every subset } \ X \subseteq V, \tag{4.1}$$

which is equivalent to

$$\widetilde{m}(X) \geq i_G(X) \ \text{ for every subset } \ X \subseteq V. \tag{4.2}$$

■

This immediately implies the following claim.

Claim 4.2. *The in-degree vectors of orientations of G are precisely the integral elements of base-polyhedron B_G ($= B(e_G) = B'(i_G)$), that is, the set of in-degree vectors of orientations of G is the M-convex set $\overset{....}{B}_G$.* ■

The proof of Lemma 4.1 is algorithmic (see, e.g., Theorem 2.3.2 of [12]) and the orientation corresponding to a given m can be constructed easily.

4.1 Decreasingly minimal orientations

Due to Claim 4.2, we can apply the results on dec-min elements to the special base-polyhedron B_G. Borradaile et al. [4] called an orientation of G egalitarian if its in-degree vector is decreasingly minimal but we prefer the term **dec-min orientation** since an orientation with an increasingly maximal in-degree vector also has an intuitive egalitarian feeling. Such an orientation is called **inc-max**. Theorem 2.1 immediately implies the following.

Corollary 4.3. *An orientation of G is dec-min if and only if it is inc-max.* ■

Note that the term dec-min orientation is asymmetric in the sense that it refers to in-degree vectors. One could also aspire for finding an orientation whose out-degree vector is decreasingly minimal. But this problem is clearly equivalent to the in-degree version and hence in the present work we do not consider out-degree vectors, apart from a single exception in Section 4.5.

By Theorem 2.1, an element m of $\overset{....}{B}_G$ is decreasingly minimal if and only if there is no 1-tightening step for m. What is the meaning of a 1-tightening step in terms of orientations?

Claim 4.4. *Let D be an orientation of G with in-degree vector m. Let t and s be nodes of G. The vector $m' := m + \chi_s - \chi_t$ is in B_G if and only if D admits a dipath from s to t.*

Proof. $m' \in B_G$ holds precisely if there is no $t\overline{s}$-set X which is tight with respect to i_G, that is, $\widetilde{m}(X) = i_G(X)$. Since $\varrho(Y) + i_G(Y) = \sum[\varrho(v) : v \in Y] = \widetilde{m}(Y)$ holds for any set $Y \subseteq V$, the tightness of X is equivalent to requiring that $\varrho(X) = 0$. Therefore $m' \in B_G$ if and only if $\varrho(Y) > 0$ holds for every $t\overline{s}$-set Y, which is equivalent to the existence of a dipath of D from s to t. ∎

Recall that a 1-tightening step at a member m of B_G consists of replacing m by m' provided that $m(s) \geq m(t) + 2$ and $m' \in B_G$. By Claim 4.4, a 1-tightening step at a given orientation of G corresponds to reorienting an arbitrary dipath from a node s to node t for which $\varrho(s) \geq \varrho(t) + 2$. Therefore, Theorem 2.1 immediately implies the following basic theorem of Borradaile et al. [4].

Theorem 4.5 (Borradaile et al. [4])**.** *An orientation D of a graph $G = (V, E)$ is decreasingly minimal if and only if no dipath exists from a node s to a node t for which $\varrho(t) \geq \varrho(s) + 2$.* ∎

Note that this theorem also implies Corollary 4.3. It immediately gives rise to an algorithm for finding a dec-min orientation. Namely, we start with an arbitrary orientation of G. We call a dipath **feasible** if $\varrho(t) \geq \varrho(s) + 2$ holds for its starting node s and end-node t. The algorithm consists of reversing feasible dipaths as long as possible. Since the sum of the squares of in-degrees always drops when a feasible dipath is reversed, and originally this sum is at most $|E|^2$, the dipath-reversing procedure terminates after at most $|E|^2$ reversals. By Theorem 4.5, when no more feasible dipath exists, the current orientation is dec-min. The basic algorithm concerning general base-polyhedra in Section 2.1 is nothing but an extension of the algorithm of Borradaile et al.

It should be noted that they suggested to choose at every step the current feasible dipath in such a way that the in-degree of its end-node t is as high as possible, and they proved that the algorithm in this case terminates after at most $|E||V|$ dipath reversals.

Note that we obtained Corollary 4.3 as a special case of a result on M-convex sets but it is also a direct consequence of Theorem 4.5.

4.2 Capacitated orientation

Consider the following capacitated version of the basic dec-min orientation problem of Borradaile et al. [4]. Suppose that a positive integer $\ell(e)$ is assigned to each edge e of G. Denote by G^+ the graph arising from G by replacing each edge e of G with $\ell(e)$ parallel edges. Our goal is to find a dec-min orientation of G^+. In this case, an orientation of G^+ is described by telling that, among the $\ell(e)$ parallel edges connecting the end-nodes u and v of e how many are oriented toward v (implying that the rest of the $\ell(e)$ edges are oriented toward u). In principle, this problem can be solved by applying the algorithm described above to G^+, and this algorithm is satisfactory when ℓ is small in the sense that its largest value can be bounded by a power of $|E|$. The difficulty in the general case is that the algorithm will be polynomial only in the number of edges of G^+, that is, in $\widetilde{\ell}(E)$, and hence this algorithm is not polynomial in $|E|$.

We show how the algorithm in Section 3.3.1 can be used to solve the decreasingly minimal orientation problem in the capacitated case in strongly polynomial time. To this end, let $D = (V, A)$ be an arbitrary orientation of G serving as a reference orientation. Define a capacity function g on A by $g(\vec{e}) := \ell(e)$, where $\vec{e}$ denotes the arc of D obtained by orienting e.

We associate an orientation of G^+ with an integral vector $z : A \rightarrow \mathbf{Z}_+$ with $z \leq g$ as follows. For an arc uv of D, orient $z(uv)$ parallel copies of $e = uv \in E$ toward v and $g(uv) - z(uv)$ parallel copies toward u. Then the in-degree of a node v is $m_z(v) := \varrho_z(v) + \delta_{g-z}(v) = \varrho_z(v) - \delta_z(v) + \delta_g(v)$. Therefore our goal is to find an integral vector z on A for which $0 \leq z \leq g$ and the vector m_z on V is dec-min. Consider the set of net-in-flow vectors $\{(\Psi_z(v) : v \in V) : 0 \leq z \leq g\}$. In Section 3.3.1, we proved that this is a base-polyhedron B_1. Therefore the set of vectors $(m_z(v) : v \in V)$ is also a base-polyhedron B arising from B_1 by translating B_1 with the vector $(\delta_g(v) : v \in V)$.

As remarked at the end of Section 3.3.1, a dec-min element of $\ddot{B}$ can be computed in strongly polynomial time by relying on a push-relabel subroutine for network flows (and not using a general-purpose submodular function minimizer).

4.3 Canonical chain and essential value-sequence for orientations

In Section 2.2, we described Algorithm 2.2 for an arbitrary M-convex set $\ddot{B}$ that computes, from a given dec-min element m of $\ddot{B}$, the canonical chain and essential value-sequence belonging to $\ddot{B}$. That algorithm needed an oracle for computing the smallest m-tight set $T_m(u)$ containing u. Here we show how this general algorithm can be turned into a pure graph-algorithm in the special case of dec-min orientations.

To this end, consider the special M-convex set, denoted by $\ddot{B}_G$, consisting of the in-degree vectors of the orientations of an undirected graph $G = (V, E)$. By the Orientation lemma, $B_G = B'(i_G)$ where $i_G(X)$ denotes the number of edges induced by X. Recall that i_G is a fully supermodular function. For an orientation D of G with in-degree vector m, the smallest m-tight set $T_m(t)$ (with respect to i_G) containing a node t will be denoted by $T_D(t)$.

Claim 4.6. *Let D be an arbitrary orientation of G with in-degree vector m.* (A) *A set $X \subseteq V$ is m-tight (with respect to i_G) if and only if $\varrho_D(X) = 0$.* (B) *The smallest m-tight set $T_D(t)$ containing a node t is the set of nodes from which t is reachable in D.*

Proof. We have

$$\varrho_D(X) + i_G(X) = \sum[\varrho_D(v) : v \in X] = \widetilde{m}(X) \geq i_G(X),$$

from which X is m-tight (that is, $\widetilde{m}(X) = i_G(X)$) precisely if $\varrho_D(X) = 0$, and Part (A) follows. Therefore the smallest m-tight set $T_D(t)$ containing t is the smallest set containing t with in-degree 0, and hence $T_D(t)$ is indeed the set of nodes from which t is reachable in D, as stated in Part (B). ■

By Claim 4.6, $T_D(t)$ is easily computable, and hence Algorithm 2.2 for general M-convex sets can easily be specialized to graph orientations. By applying Theorem 2.2 to $p := i_G$ and recalling from Claim 4.6 that C_i is m-tight, in the present case, precisely if $\varrho_D(C_i) = 0$, we obtain the following.

Theorem 4.7. *An orientation of D of G is dec-min if and only of $\varrho_D(C_i) = 0$ for each member C_i of the canonical chain and $\beta_i - 1 \leq \varrho_D(v) \leq \beta_i$ holds for every node $v \in S_i$ ($i = 1, \dots, q$).* ■

We remark that the members of the canonical partition computed by our algorithm for $\ddot{B}_G$ is exactly the non-empty members of the so-called density decomposition of G introduced by Borradaile et al. [5].

4.4 Cheapest dec-min orientations

It is indicated in [16] that, in decreasing minimization on an M-convex set in general, we can construct an algorithm to compute a cheapest dec-min element with respect to a given (linear) cost-function on the ground-set. In the special case of dec-min orientations, this means that if c is a cost-function on the node-set of $G = (V, E)$, then we have an algorithm to compute a dec-min orientation of G for which $\sum[c(v)\varrho(v) : v \in V]$ is minimum.

But the question remains: what happens if, instead of a cost-function on the node-set, we have a cost-function c on $\vec{E}_2$, where $\vec{E}_2$ arises from E by replacing each element $e = uv$ ($= vu$) of E by two oppositely oriented arcs uv and vu, and we are interested in finding a cheapest orientation with specified properties? (As an orientation of e consists of replacing e by one of the two arcs uv and vu and the cost of its orientation is, accordingly, $c(uv)$ or $c(vu)$. Therefore we can actually assume that $\min\{c(uv), c(vu)\} = 0$.)

It is important to remark that the standard minimum cost in-degree constrained orientation problem itself can be reduced with a well-known technique to a minimum cost feasible flow problem in a digraph with small integral capacities. This latter problem is tractable in strongly polynomial time via the classic min-cost flow algorithm of Ford and Fulkerson (that is, we do not need the more sophisticated min-cost flow algorithm of Tardos, which is strongly polynomial for an arbitrary capacity). Actually, we shall need a version of this minimum cost orientation problem when some of the edges are already oriented, and this slight extension is also tractable by network flows.

Theorem 4.7 implies that the problem of finding a cheapest dec-min orientation is equivalent to finding a cheapest in-degree constrained orientation by orienting edges connecting C_i and $V - C_i$ toward $V - C_i$ ($i = 1, \dots, q$). Here the in-degree constraints are given by $\beta_i - 1 \leq \varrho_D(v) \leq \beta_i$ for $v \in S_i$ ($i = 1, \dots, q$).

Note that Harada et al. [20] provided a direct algorithm for the minimum cost version of the so-called semi-matching problem, which problem includes the minimum cost dec-min orientation problem. For this link, see Section 5.4.

4.5 Orientation with dec-min in-degree vector and dec-min out-degree vector

We mentioned that dec-min and inc-max orientations always concern in-degree vectors. As an example to demonstrate the advantage of the general base-polyhedral view, we outline here one exception when in-degree vectors and out-degree vectors play a symmetric role. The problem is to characterize undirected graphs admitting an orientation which is both dec-min with respect to its in-degree vector and dec-min with respect to its out-degree vector.

For the present purposes, we let d_G denote the degree vector of G, that is, $d_G(v)$ is the number of edges incident to $v \in V$. (This notation differs from the standard set-function meaning of d_G.)

Let $B_{\rm in}$ denote the convex hull of the in-degree vectors of orientations of G, and $B_{\rm out}$ the convex hull of out-degree vectors of orientations of G. (Earlier $B_{\rm in}$ was denoted by B_G but now we have to deal with both out-degrees and in-degrees.) As before, $\overset{....}{B}_{\rm in}$ is the set of in-degree vectors of orientations of G, and $\overset{....}{B}_{\rm out}$ is the set of out-degree vectors of orientations of G. Let $\overset{....}{B}{}^{\bullet}_{\rm in}$ denote the set of dec-min in-degree vectors of orientations of G, and $\overset{....}{B}{}^{\bullet}_{\rm out}$ the set of dec-min out-degree vectors of orientations of G. By Theorem 2.3, both $\overset{....}{B}{}^{\bullet}_{\rm in}$ and $\overset{....}{B}{}^{\bullet}_{\rm out}$ are matroidal M-convex sets.

Note that the negative of a (matroidal) M-convex set is also a (matroidal) M-convex set, and the translation of a (matroidal) M-convex set by an integral vector is also a (matroidal) M-convex set. Therefore $d_G - \overset{....}{B}{}^{\bullet}_{\rm out}$ is a matroidal M-convex set. Clearly, a vector $m_{\rm in}$ is the in-degree vector of an orientation D of G precisely if $d_G - m_{\rm in}$ is the out-degree vector of D.

We are interested in finding an orientation whose in-degree vector is dec-min and whose out-degree vector is dec-min. This is equivalent to finding a member $m_{\rm in}$ of $\overset{....}{B}{}^{\bullet}_{\rm in}$ for which the vector $m_{\rm out} := d_G - m_{\rm in}$ is in the matroidal M-convex set $\overset{....}{B}{}^{\bullet}_{\rm out}$. But this latter is equivalent to requiring that $m_{\rm in}$ is in the M-convex set $d_G - \overset{....}{B}{}^{\bullet}_{\rm out}$. That is, the problem is equivalent to finding an element of the intersection of the matroidal M-convex sets $\overset{....}{B}{}^{\bullet}_{\rm in}$ and $d_G - \overset{....}{B}{}^{\bullet}_{\rm out}$. This latter problem can be solved by Edmonds matroid intersection algorithm [8].

5 In-degree constrained orientations of graphs

In this section we first describe an algorithm to find a dec-min in-degree constrained orientation. Second, we develop a complete description of the set of dec-min in-degree constrained orientations, which gives rise to an algorithm to compute a cheapest dec-min in-degree constrained orientation.

5.1 Computing a dec-min in-degree constrained orientation

Let $f : V \to \mathbf{Z} \cup \{-\infty\}$ be a lower bound function and $g : V \to \mathbf{Z} \cup \{+\infty\}$ an upper bound function for which $f \leq g$. We are interested in in-degree constrained orientations D of G, by which we mean that $f(v) \leq \varrho_D(v) \leq g(v)$ for every $v \in V$. Such an orientation is called (f, g)**-bounded**, and we assume that G has such an orientation. (By a well-known orientation theorem, such an orientation exists if and only if $i_G \leq \widetilde{g}$ and $\widetilde{f} \leq e_G$.

As before, let $\overset{....}{B}_G$ denote the M-convex set of the in-degree vectors of orientations of G, and let $\overset{....}{B}{}^{\square}_G$ denote the intersection of $\overset{....}{B}_G$ with the integral box $T(f, g)$. That is, $\overset{....}{B}{}^{\square}_G$ is the set of in-degree vectors of (f, g)-bounded orientations of G. Let D be an (f, g)-bounded orientation of G with in-degree vector m. We denote the smallest tight set containing a node t by $T^{\square}_D(t)$ $(= T^{\square}_m(t))$. By applying Claim 2.4 to $\overset{\cdots\cdot}{B}{}^{\square}_G$, we obtain that

$$T^{\square}_D(t) = \begin{cases} \{t\} & \text{if } \ \varrho_D(t) = f(t), \\ T_D(t) - \{s : \varrho_D(s) = g(s)\} & \text{if } \ \varrho_D(t) > f(t), \end{cases} \tag{5.1}$$

implying that, in case $\varrho_D(t) > f(t)$, the set $T^{\square}_D(t)$ consists of those nodes s from which t is reachable and for which $\varrho_D(s) < g(s)$.

Formula (5.1) implies for distinct nodes s and t that the vector $m' := m + \chi_s - \chi_t$ belongs to $\overset{\cdots\cdot}{B}{}^{\square}_G$ precisely if there is an st-dipath (i.e. a dipath from s to t) for which $\varrho_D(s) < g(s)$ and $\varrho_D(t) > f(t)$. We call such a dipath P of D **reversible**. Note that the dipath P' of D' obtained by reorienting P is reversible in D'.

If P is a reversible st-dipath of D for which $\varrho_D(t) \geq \varrho_D(s) + 2$, then the orientation D' is decreasingly smaller than D. We call such a dipath **improving**. Therefore, reorienting an improving st-dipath corresponds to a 1-tightening step. Hence Theorem 2.1 implies the following extension of Theorem 4.5.

Theorem 5.1. *An (f,g)-bounded orientation D of G is dec-min if and only if there is no improving dipath, that is, a dipath from a node s to a node t for which $\varrho_D(t) \geq \varrho_D(s) + 2$, $\varrho_D(s) < g(s)$, and $\varrho_D(t) > f(t)$.* ■

In Section 2.1 we have presented an algorithm that computes a dec-min element of an arbitrary M-convex set. By specializing it to $\overset{\cdots\cdot}{B}{}^{\square}_G$, we conclude that in order to construct a dec-min (f,g)-bounded orientation of G, one can start with an arbitrary (f,g)-bounded orientation, and then reorient (currently) improving dipaths one by one, as long as such a dipath exists. As we pointed out after Theorem 4.5, after at most $|E|^2$ improving dipath reorientations, the algorithm terminates with a dec-min (f,g)-bounded orientation of G.

Canonical chain and essential value-sequence for (f,g)-bounded orientations In Section 2.2, we indicated that Algorithm 2.2 can immediately be applied to compute the canonical chain, the canonical partition, and the essential value-sequence belonging to the intersection $\overset{\cdots\cdot}{B}{}^{\square}$ of an arbitrary M-convex set $\overset{\cdots\cdot}{B}$ with an integral box $T(f,g)$.

This algorithm needs only the original subroutine to compute $T_m(u)$ since, by Claim 2.4, $T^{\square}_m(u)$ is easily computable from $T_m(u)$. As we indicated above, in the special case of orientations, the corresponding sets $T_D(t)$ and $T^{\square}_D(t)$ are immediately computable from D. Therefore this extended algorithm can be used in the special case when we are interested in dec-min (f,g)-bounded orientations of $G = (V,E)$. The algorithm starts with a dec-min (f,g)-bounded orientation D of G and outputs the canonical chain $\mathcal{C}^{\square} = \{C^{\square}_1, \dots, C^{\square}_q\}$, the canonical partition $\mathcal{P}^{\square} = \{S^{\square}_1, \dots, S^{\square}_q\}$, and the essential value-sequence $\beta^{\square}_1 > \cdots > \beta^{\square}_q$. In view of Theorem 2.2, we also define bounding functions f^* and g^* as

$$\begin{aligned} f^*(v) &:= \beta^{\square}_i - 1 \text{ if } \ v \in S_i \qquad (i = 1, \dots, q), \\ g^*(v) &:= \beta^{\square}_i \qquad \text{if } \ v \in S_i \qquad (i = 1, \dots, q). \end{aligned}$$

We say that the small box

$$T^* := T(f^*, g^*) \tag{5.2}$$

belongs to $\overset{....}{B}{}^{\square}_{G}$. Clearly, $f \leq f^*$ and $g^* \leq g$, and hence $T(f^*, g^*) \subseteq T(f, g)$. In Section 5.2 below we assume that these data are available.

Remark 5.1. A special case of in-degree constrained orientations is when we have a prescribed subset T of V and a non-negative function $m_T : T \to \mathbf{Z}_+$ serving as an in-degree specification on T, and we are interested in orientations of G for which $\varrho(v) = m_T(v)$ holds for every $t \in T$. We call such an orientation T**-specified**. This notion will have applications in Section 5.4. ∎

5.2 Cheapest dec-min in-degree constrained orientations

We are given a cost-function c on the possible orientations of the edges of G and our goal is to find a cheapest dec-min (f, g)-bounded orientation of G. This will be done with the help of a purely graphical description of the set of all dec-min (f, g)-bounded orientations, which is given in Theorem 5.3.

As a preparation, we derive the following claim as an immediate consequence of the structural result stated in Theorem 2.3. Let m be a dec-min element of an M-convex set $\overset{....}{B}$ on ground-set S. Suppose that $m' := m + \chi_s - \chi_t$ is in $\overset{....}{B}$ (that is, $s \in T_m(t)$). Since m is dec-min, $m(t) \leq m(s) + 1$. If $m(t) = m(s) + 1$, then m' and m are value-equivalent and hence m' is also a dec-min element of $\overset{....}{B}$. We say that m' is obtained from m by an **elementary step**.

Claim 5.2. *Any dec-min element of $\overset{....}{B}$ can be obtained from a given dec-min element m by a sequence of at most $|S|$ elementary steps.*

Proof. By Theorem 2.3, the set of dec-min elements of $\overset{....}{B}$ is a matroidal M-convex set in the sense that it can be obtained from a matroid M^* by translating the incidence vectors of the bases of M^* by the same integral vector Δ^*. A simple property of matroids is that any basis can be obtained from a given basis through a sequence of at most $|S|$ bases such that each member of the series can be obtained from the preceding one by taking out one element and adding a new one. The corresponding change in the translated vector is exactly an elementary step. ∎

For a subset $E_0 \subseteq E$ and for an orientation A_0 of E_0, we say that an orientation D of G is A_0**-extending** if every element e of E_0 is oriented in D in the same direction as in A_0.

Theorem 5.3. *Let $G = (V, E)$ be an undirected graph admitting an (f, g)-bounded orientation. Let $\overset{....}{B}{}^{\square}_{G}$ denote the M-convex set consisting of the in-degree vectors of (f, g)-bounded orientations of G, and let T^* be the small box, belonging to $\overset{....}{B}{}^{\square}_{G}$, as defined in* (5.2). *There are a subset E_0 of E and an orientation A_0 of E_0 such that an (f, g)-bounded orientation D of G is a dec-min (f, g)-bounded orientation if and only if D is an orientation of G extending A_0 and the in-degree vector of D belongs to T^*.*

Proof. Let D be a dec-min (f, g)-bounded orientation of G, and let m denote its in-degree vector. Consider the canonical chain $\mathcal{C}^{\square} = \{C_1^{\square}, \dots, C_q^{\square}\}$, the canonical partition $\mathcal{P}^{\square} = \{S_1^{\square}, \dots, S_q^{\square}\}$, and the essential value-sequence $\beta_1^{\square} > \cdots > \beta_q^{\square}$ belonging to $\overset{....}{B}{}^{\square}_{G}$.

For $i \in \{1, \dots, q\}$, define

$$F_i := \{v : v \in S_i^\square, f(v) = \beta_i^\square\}.$$

Since $f(v) \le m(v) \le \beta_i^\square$ holds for every element v of $S_i^\square$, we obtain that $f(v) = m(v) = \beta_i^\square$ for $v \in F_i$. Note that F_i does not depend on D.

Claim 5.4. *For every $h = 1, \dots, i$, there is no dipath P from a node $s \in V - C_i^\square$ with $m(s) < g(s)$ to a node $t \in S_h^\square$ with $\beta_h^\square > f(t)$.*

Proof. Suppose indirectly that there is such a dipath P. If $m(t) = \beta_h^\square$, then P would be an improving dipath which is impossible since D is dec-min (f, g)-bounded. Therefore $m(t) = \beta_h^\square - 1$. But a property of the canonical partition is that there is an element t' of $S_h^\square - F_h$ for which $m(t') = \beta_h^\square$ and $t \in T_D^\square(t')$. This means that t' is reachable from t in D, and therefore there is a dipath from s to t' in D which is improving, a contradiction again. ■

We are going to define a chain $\mathcal{Z}$ of subsets $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_q$ $(= \emptyset)$ of V with the help of D, and will show that this chain actually does not depend on D. Let

$$Z_i := \{t : t \text{ is reachable in } D \text{ from a node } \ s \in V - C_i^\square \ \text{ with } \varrho_D(s) < g(s)\}. \tag{5.3}$$

Note that $Z_{i-1} \supseteq Z_i$ follows from the definition, where equality holds precisely if $\varrho_D(s) = g(s)$ for each $s \in S_i$.

Lemma 5.5. *Every dec-min (f, g)-bounded orientation defines the same family $\mathcal{Z}$.*

Proof. By Claim 5.2, it suffices to prove that a single elementary step does not change $\mathcal{Z}$. An elementary step in $\ddddot{B}_G^\square$ corresponds to the reorientation of an st-dipath P in D where $s, t \in S_h^\square - F_h$, $m(t) = \beta_h^\square$ and $m(s) = \beta_h^\square - 1$ hold for some $h \in \{1, \dots, q\}$. We will show for $i \in \{1, \dots, q\}$ that the reorientation of P does not change Z_i.

If $h \le i$, then Claim 5.4 implies that $Z_i \cap S_h^\square \subseteq F_h$. Since $\delta_D(Z_i) = 0$, the dipath P is disjoint from Z_i, implying that reorienting P does not affect Z_i.

Suppose now that $h \ge i + 1$. Since reorienting P results in a dec-min (f, g)-bounded orientation D', we get that $m(s) + 1 \le g(s)$ and hence $s \in Z_i - C_i^\square$. Since $\delta_D(Z_i) = 0$, we obtain that $t \in Z_i - C_i^\square$. Since $\varrho_{D'}(t) = \varrho_D(t) - 1 < g(t)$ and the set of nodes reachable from s in D is equal to the set of nodes reachable from t in D', it follows that the reorientation of P does not change Z_i. ■

Let E_0 consist of those edges of G which connect Z_i with $V - Z_i$ for some $i = 1, \dots, q$. Let A_0 denote the orientation of E_0 obtained by orienting each edge connecting Z_i and $V - Z_i$ toward Z_i.

Lemma 5.6. *The subset $E_0 \subseteq E$ and its orientation A_0 meet the requirements in the theorem.*

Proof. Consider first an arbitrary dec-min (f, g)-bounded orientation D of G. Then $\delta_D(Z_i) = 0$ and hence D extends A_0. Moreover, by a basic property of the canonical chain, the in-degree vector of D belongs to T^*.

Conversely, let D be an orientation of G extending A_0 whose in-degree vector belongs to T^*, that is,

$$f^*(v) \le \varrho_D(v) \le g^*(v) \text{ for every } v \in V.$$

Then D is clearly (f, g)-bounded.

Claim 5.7. *There is no improving dipath in D.*

Proof. Suppose, indirectly, that P is an improving st-dipath, that is, a dipath from s to t such that $\varrho_D(t) \geq \varrho_D(s) + 2$, $\varrho_D(t) > f(t)$, and $\varrho_D(s) < g(s)$. Suppose that t is in $S_i^{\square}$ for some $i \in \{1, \dots, q\}$. If s is in $S_k^{\square}$ for some $k \in \{1, \dots, q\}$, then

$$\beta_k^{\square} - 1 \leq \varrho_D(s) \leq \varrho_D(t) - 2 \leq \beta_i^{\square} - 2,$$

that is, $\beta_k^{\square} < \beta_i^{\square}$, and hence $k > i$, implying that s is in $V - C_i^{\square}$. This and $\varrho_D(s) < g(s)$ imply that s is in Z_i. Furthermore, $\beta_i^{\square} \geq \varrho_D(t) > f(t)$ implies that t is not in F_i, and since $S_i^{\square} \cap Z_i \subseteq F_i$, we obtain that t is not in Z_i. On the other hand, we must have $t \in Z_i$, since there is a dipath from $s \in V - C_i^{\square}$ to t and $\varrho_D(s) < g(s)$. This is a contradiction. ■

By proving Claim 5.7, we have shown Lemma 5.6. Thus the proof of Theorem 5.3 is completed. ■■

Algorithm for computing a cheapest dec-min (f, g)-bounded orientation First we compute a dec-min (f, g)-bounded orientation D of G with the help of the algorithm outlined in Section 5.1. Second, by applying the algorithm described in the same section, we compute the canonical chain and partition belonging to $\ddddot{B}_G^{\square}$ along with the essential value-sequence. Once these data are available, the sets Z_i $(i = 1, \dots, q)$ defined in (5.3) are easily computable. Lemma 5.5 ensures that these sets Z_i do not depend on the starting dec-min (f, g)-bounded orientation D. Let E_0 be the union of the set of edges connecting some Z_i with $V - Z_i$, and define the orientation A_0 of E_0 by orienting each edge between Z_i and $V - Z_i$ toward Z_i.

Theorem 5.3 implies that, once E_0 and its orientation A_0 are available, the problem of computing a cheapest dec-min (f, g)-bounded orientation of G reduces to finding cheapest in-degree constrained (namely, (f^*, g^*)-bounded) orientation of a mixed graph. We indicated already in Section 4.4 that such a problem is easily solvable by the strongly polynomial min-cost flow algorithm of Ford and Fulkerson in a digraph with identically 1 capacities.

Remark 5.2. In Section 4.2 we have considered the capacitated dec-min orientation problem in the basic case where no in-degree constraints are imposed. With the technique presented there, we can cope with the capacitated, min-cost, in-degree constrained variants as well. Furthermore, the algorithms above can easily be extended, with a slight modification, to the case when one is interested in orientations of mixed graphs. ■

5.3 Dec-min (f, g)-bounded orientations minimizing the in-degree of T

One may consider (f, g)-bounded orientations of G when the additional requirement is imposed that the in-degree of a specified subset T of nodes be as small as possible. We shall show that these orientations of G can be described as (f', g')-bounded orientations of a mixed graph arising from G by orienting the edges between a certain subset X_T of nodes and its complement $V - X_T$ toward $V - X_T$.

It is more comfortable, however, to show the analogous statement for a general M-convex set $\ddot{B}'(p) \subseteq \mathbf{Z}^V$ defined by a (fully) supermodular function p for which $\ddot{B}^\square := \ddot{B}'(p) \cap T(f, g)$ is non-empty. (Here, instead of the usual S, we use V to denote the ground-set of the general M-convex set. We are back at the special case of graph orientations when $p = i_G$.) We assume that each of p, f, and g is finite-valued.

Let $p^\square$ denote the unique (fully) supermodular function defining $B^\square$. This function can be expressed with the help of p, f, and g, as follows (see, for example, Theorem 14.3.9 in [12]):

$$p^\square(Y) = \max\{p(X) + \widetilde{f}(Y - X) - \widetilde{g}(X - Y) : X \subseteq V\} \qquad (Y \subseteq V). \tag{5.4}$$

As $B^\square$ is defined by the supermodular function $p^\square$ (that is, $B^\square = B'(p^\square)$), we have

$$\min\{\widetilde{m}(T) : m \in \ddot{B}^\square\} = p^\square(T). \tag{5.5}$$

This implies that the set of elements of $\ddot{B}^\square$ minimizing $\widetilde{m}(T)$ is itself an M-convex set. Namely, it is the set of integral elements of the base-polyhedron arising from $B^\square$ by taking its face defined by $\{m \in B^\square : \widetilde{m}(T) = p^\square(T)\}$. The next theorem shows how this M-convex set can be described in terms of f, g, and p, without referring to $p^\square$.

Theorem 5.8. *There is a box $T(f', g') \subseteq T(f, g)$ and a subset $X_T \subseteq V$ such that an element $m \in \ddot{B}^\square$ minimizes $\widetilde{m}(T)$ if and only if $\widetilde{m}(X_T) = p(X_T)$ and $m \in \ddot{B} \cap T(f', g')$.*

Proof. Let X_T be a set maximizing the right-hand side of (5.4).

Claim 5.9. *An element $m \in \ddot{B}^\square$ is a minimizer of the left-hand side of* (5.5) *if and only if the following three optimality criteria hold:*

$$\begin{aligned} &\widetilde{m}(X_T) = p(X_T), \\ &v \in T - X_T \quad \text{implies} \quad m(v) = f(v), \\ &v \in X_T - T \quad \text{implies} \quad m(v) = g(v). \end{aligned}$$

Proof. For any $m \in \ddot{B}^\square$ and $X \subseteq V$, we have $\widetilde{m}(T) = \widetilde{m}(X) + \widetilde{m}(T - X) - \widetilde{m}(X - T) \geq p(X) + \widetilde{f}(T - X) - \widetilde{g}(X - T)$. Here we have equality if and only if $\widetilde{m}(X) = p(X)$, $\widetilde{m}(T - X) = \widetilde{f}(T - X)$, and $\widetilde{m}(X - T) = \widetilde{g}(X - T)$, implying the claim. ∎

Define f' and g' as follows:

$$f'(v) := \begin{cases} g(v) & \text{if } v \in X_T - T, \\ f(v) & \text{if } v \in V - (X_T - T), \end{cases} \tag{5.6}$$

$$g'(v) := \begin{cases} f(v) & \text{if } v \in T - X_T, \\ g(v) & \text{if } v \in V - (T - X_T). \end{cases} \tag{5.7}$$

The claim implies that $T(f', g')$ and X_T meet the requirement of the theorem. ∎∎

As the set of elements of $\ddot{B}^\square$ minimizing $\widetilde{m}(T)$ is itself an M-convex set, all the algorithms developed earlier can be applied once we are able to compute set X_T occurring in Theorem 5.8. (By definitions (5.6) and (5.7), X_T immediately determines f' and g').

The following straightforward algorithm computes an element $m \in \ddot{B}^{\square}$ minimizing the left-hand side of (5.5) and a subset X_T maximizing the right-hand side of (5.4). Start with an arbitrary element $m \in \ddot{B}^{\square}$. By an improving step we mean the change of m to $m' := m+\chi_s-\chi_t$ for some elements $s \in V-T, t \in T$ for which $m(s) < g(s)$, $m(t) > f(t)$, and $s \in T_m(t)$, where $T_m(t)$ is the smallest m-tight set (with respect to p) containing t. Clearly, $m' \in \ddot{B}^{\square}$, and $\widetilde{m}'(T) = \widetilde{m}(T) - 1$. The algorithm applies improving steps as long as possible. When no more improving step exists, the set $X_T := \cup(T_m(t) : t \in T, m(t) > f(t))$ meets the three optimality criteria. The algorithm is polynomial if $|p(X)|$ is bounded by a polynomial of $|V|$.

By applying Theorem 5.8 to the special case of $p = i_G$, we obtain the following.

Corollary 5.10. *Let $G = (V, E)$ be a graph admitting an (f, g)-bounded orientation. There is a box $T(f', g') \subseteq T(f, g)$ and a subset $X_T \subseteq V$ such that an (f, g)-bounded orientation of G minimizes the in-degree of T if and only if D is an (f', g')-bounded orientation for which $\varrho_D(X_T) = 0$.* ∎

In this case, the algorithm above to compute X_T starts with an (f, g)-bounded orientation D of G, whose in-degree vector is denoted by m. As long as there is an st-dipath P with $s \in V - T, t \in T, m(s) < g(s)$, and $m(t) > f(t)$, reorient P. When no such a dipath exists anymore, the set X_T of nodes from which a node $t \in T$ with $m(t) > f(t)$ is reachable in D, along with the bounding functions f' and g' defined in (5.6) and in (5.7), meet the requirement in the corollary.

Minimum cost version It follows that, in order to compute a minimum cost dec-min (f, g)-bounded orientation for which the in-degree of T is minimum, we can apply the algorithm described in Section 5.2 for the mixed graph obtained from G by orienting each edge between X_T and $V - X_T$ toward $V - X_T$.

Remark 5.3. Instead of a single subset T of V, we may consider a chain $\mathcal{T}$ of subsets $T_1 \subset T_2 \subset \cdots \subset T_h$ of V. Then $\mathcal{T}$ defines a face $B^{\square}_{\rm face}$ of the base-polyhedron $B^{\square}$. Namely, an element m of $B^{\square}$ belongs to $B^{\square}_{\rm face}$ precisely if $\widetilde{m}(T_i) = p^{\square}(T_i)$ for each $i \in \{1, \dots, h\}$. This implies that the integral elements of $B^{\square}_{\rm face}$ simultaneously minimize $\widetilde{m}(T_i)$ for each $i \in \{1, \dots, h\}$ (over the elements of $\ddot{B}^{\square}$). Therefore, we can consider (f, g)-bounded orientations of G with the additional requirement that each of the in-degrees of $T_1, T_2, \dots, T_h$ is (simultaneously) minimum. Corollary 5.10 can be extended to this case, implying that we have an algorithm to compute a minimum cost dec-min (f, g)-bounded orientation of G that simultaneously minimizes the in-degree of each member of the chain $\{T_1, T_2, \dots, T_h\}$. ∎

5.4 Application in resource allocation: semi-matchings

For a general M-convex set $\ddot{B}$, it is shown in [16, Section 6] that for an element m of $\ddot{B}$ the following properties are equivalent: (A) m is dec-min, (B) the square-sum of the components is minimum, (C) the difference-sum of the components of m is minimum. Therefore the corresponding equivalences hold in the special case of in-degree constrained (in particular, T-specified) orientations of undirected graphs.

As an application of this equivalence, we show first how a result of Harvey et al. [21] concerning a resource allocation problem follows immediately. They introduced the notion of a semi-matching of a simple bipartite graph $G = (S, T; E)$ as a subset F of edges for which $d_F(t) = 1$ holds for every node $t \in T$, and solved the problem of finding a semi-matching F for which $\sum[d_F(s)(d_F(s)+1) : s \in S]$ is minimum. The problem was motivated by practical applications in the area of resource allocation in computer science. Note that

$$\sum[d_F(s)(d_F(s)+1) : s \in S] = \sum[d_F(s)^2 : s \in S] + \sum[d_F(s) : s \in S]$$
$$= \sum[d_F(s)^2 : s \in S] + |F| = \sum[d_F(s)^2 : s \in S] + |T|,$$

and therefore the problem of Harvey et al. is equivalent to finding a semi-matching F of G that minimizes the square-sum of degrees in S.

By orienting each edge in F toward S and each edge in $E - F$ toward T, a semi-matching can be identified with the set of arcs directed toward S in an orientation of $G = (S, T; E)$ in which the out-degree of every node $t \in T$ is 1 (that is, $\varrho(t) = d_G(t) - 1$), and $d_F(s) = \varrho(s)$ for each $s \in S$. Since $\varrho(t)$ for $t \in T$ is the same in these orientations, it follows that the total sum of $\varrho(v)^2$ over $S \cup T$ is minimized precisely if $\sum[\varrho(s)^2 : s \in S] = \sum[d_F(s)^2 : s \in S]$ is minimized. Therefore the semi-matching problem of Harvey et al. is nothing but a special dec-min T-specified orientation problem. Note that not only semi-matching problems can be managed with graph orientations, but conversely, an orientation of a graph $G = (V, E)$ can also be interpreted as a semi-matching of the bipartite graph obtained from G by subdividing each edge by a new node. This implies, for example, that the algorithm of Harvey et al. to compute a semi-matching minimizing $\sum[d_F(v)^2 : v \in S]$ is able to compute an orientation of a graph G for which $\sum[\varrho(v)^2 : v \in S]$ is minimum. Furthermore, an orientation of a hypergraph means that we assign an element of each hyper-edge Z to Z as its head. In this sense, semi-matchings of bipartite graphs and orientations of hypergraphs are exactly the same. Several graph orientation results have been extended to hypergraph orientation, for an overview, see, e.g. [12].

Bokal et al. [3] extended the results to subgraphs of G meeting a more general degree-specification on T when, rather than the identically 1 function, one imposes an arbitrary degree-specification m_T on T satisfying $0 \leq m_T(t) \leq d_G(t)$ $(t \in T)$. The same orientation approach applies in this more general setting. We may call a subset F of edges an **m_T-semi-matching** if $d_F(t) = m_T(t)$ for each $t \in T$. The extended resource allocation problem is to find an m_T-semi-matching F that minimizes $\sum[d_F(s)^2 : s \in S]$. This is equivalent to finding a T-specified orientation of G for which the square-sum of the in-degrees is minimum and the in-degree specification in $t \in T$ is $m'_T(t) := d_G(t) - m_T(t)$. Therefore this extended resource allocation problem is equivalent to finding a dec-min T-specified orientation of G.

The same orientation approach, when applied to in-degree constrained orientations, allows us to extend the m_T-semi-matching problem when we have upper and lower bounds imposed on the nodes in S. This may be a natural requirement in practical applications where the elements of S correspond to available resources (e.g. computers), the elements of T correspond to users, and we are interested in a fair (= dec-min = square-sum minimizer) distribution (=m_T-semi-matchings) of the resources when the load (or burden) of each resource is requested to meet a specified upper and/or lower bound. Note that in the resource allocation framework, the degree $d_F(s)$ of node $s \in S$ may be interpreted as the burden of s, and hence a difference-sum minimizer semi-matching minimizes the total sum of burden-differences.

Katrenič and Semanišin [29] investigated the problem of finding a dec-min 'maximum (f, g)-semi-matching' problem where there is a lower-bound function f_T on T and an upper bound function g_S on S (in the present notation) and one is interested in maximum cardinality subgraphs of G meeting these bounds. They describe an algorithm to compute a dec-min subgraph of this type. With the help of the orientation model discussed in Section 5.3 (where, besides the in-degree bounds on the nodes, the in-degree of a specified subset T was requested to be minimum), we have a strongly polynomial algorithm for an extension of the model of [29] when there may be upper and lower bounds on both S and T. Actually, even the minimum cost version of this problem was solved in Section 5.3.

In another variation, we also have degree bounds (f_S, g_S) on S and (f_T, g_T) on T, but we impose an arbitrary positive integer γ for the cardinality of F. We consider degree-constrained subgraphs $(S, T; F)$ of G for which $|F| = \gamma$, and want to find such a subgraph for which $\sum[d_F(s)^2 : s \in S]$ is minimum. (Notice the asymmetric role of S and T.) This is equivalent to finding an in-degree constrained orientation D of G for which $\varrho_D(S) = \gamma$ and $\sum[\varrho_D(s)^2 : s \in S]$ is minimum. Here the corresponding in-degree bound (f, g) on S is the given (f_S, g_S) while (f, g) on T is defined for $t \in T$ by

$$f(t) := d_G(t) - g_T(t) \quad \text{and} \quad g(t) := d_G(t) - f_T(t).$$

Let B denote the base-polyhedron spanned by the in-degree vectors of the degree-constrained orientations of G. Then the restriction of B to S is a g-polymatroid Q. By intersecting Q with the hyperplane $\{x : \widetilde{x}(S) = \gamma\}$, we obtain an integral base-polyhedron B_S in $\mathbf{R}^S$, and then the elements of $\ddot{\ddot{B}}_S$ are exactly the in-degree vectors of the requested orientations restricted to S. That is, the elements of $\ddot{\ddot{B}}_S$ are the restriction of the degree-vectors of the requested subgraphs of G to S. Since B_S is a base-polyhedron, a dec-min element of $\ddot{\ddot{B}}_S$ will be a solution to our minimum degree-square sum problem.

We briefly indicate that a capacitated version of the semi-matching problem can also be formulated as a dec-min in-degree constrained and capacitated orientation problem (cf., Section 4.2 and Remark 5.2). Let $G = (S, T; E)$ be again a bipartite graph, γ a positive integer, and f_V and g_V integer-valued bounding functions on $V := S \cup T$ for which $f_V \leq g_V$. In addition, an integer-valued capacity function g_E is also given on the edge-set E, and we are interested in finding a non-negative integral vector $z : E \to \mathbf{Z}_+$ for which $\widetilde{z}(E) = \gamma$, $z \leq g_E$ and $f_V(v) \leq d_z(v) \leq g_V(v)$ for every $v \in V$. (Here $d_z(v) := \sum[z(uv) : uv \in E]$.) We call such a vector **feasible**. The problem is to find a feasible vector z whose degree vector restricted to S (that is, the vector $(d_z(s) : s \in S)$ is decreasingly minimal.

By replacing each edge e with $g_E(e)$ parallel edges, it follows from the uncapacitated case above that the vectors $\{(d_z(s) : s \in S) : z$ is a feasible integral vector$\}$ form an M-convex set. In this case, however, the basic algorithm is not necessarily polynomial since the values of g_E may be large. Therefore we need the general strongly polynomial algorithm described in Section 2.4. In this case the general Subroutine (2.4) can be realized via max-flow min-cut computations.

Minimum cost dec-min semi-matchings Harada et al. [20] developed an algorithm to solve the minimum cost version of the original semi-matching problem of Harvey et al. [21]. As the dec-min in-degree bounded orientation problem covers all the extensions of semi-matching problems mentioned above, the minimum cost version of these extensions can also be solved with the strongly polynomial algorithms developed in Section 5.2 for minimum cost dec-min in-degree bounded orientations.

We close this section with some historical remarks. The problem of Harvey et al. is closely related to earlier investigations in the context of minimizing a separable convex function over (integral elements of) a base-polyhedron. For example, Federgruen and Groenevelt [9] provided a polynomial time algorithm in 1986. Hochbaum and Hong [23] in 1995 developed a strongly polynomial algorithm; their proof, however, included a technical gap, which was fixed by Moriguchi, Shioura, and Tsuchimura [33] in 2011. For an early book on resource allocation, see the one by Ibaraki and Katoh [25] while three more recent surveys are due to Katoh and Ibaraki [27] from 1998, to Hochbaum [22] from 2007, and to Katoh, Shioura, and Ibaraki [28] from 2013. Algorithmic aspects of minimum degree square-sum problems for general graphs were discussed by Apollonio and Sebő [1].

6 Orientations of graphs with edge-connectivity requirements

In this section, we investigate various edge-connectivity requirements for the orientations of G. The main motivation behind these investigations is a conjecture of Borradaile et al. [4] on decreasingly minimal strongly connected orientations. Our goal is to prove their conjecture in a more general form.

6.1 Strongly connected orientations

Suppose that G is 2-edge-connected, implying that it has a strong orientation by a theorem of Robbins [39]. We are interested in dec-min strong orientations, meaning that the in-degree vector is decreasingly minimal over the strong orientations of G. This problem of Borradaile et al. was motivated by a practical application concerning optimal interval routing schemes.

Analogously to Theorem 4.5, they described a natural way to improve a strong orientation D to another one whose in-degree vector is decreasingly smaller. Suppose that there are two nodes s and t for which $\varrho(t) \geq \varrho(s) + 2$ and there are two edge-disjoint dipaths from s to t in D. Then reorienting an arbitrary st-dipath of D results in another strongly connected orientation of D which is clearly decreasingly smaller than D.

Borradaile et al. [4] conjectured the truth of the converse (and this conjecture was the starting point of our investigations). The next theorem states that the conjecture is true.

Theorem 6.1. *A strongly connected orientation D of $G = (V, E)$ is decreasingly minimal if and only if there are no two arc-disjoint st-dipaths in D for nodes s and t with $\varrho(t) \geq \varrho(s)+2$.*

Proof. Suppose first that there are nodes s and t with $\varrho(t) \geq \varrho(s) + 2$ such that there are two arc-disjoint st-dipaths of D. Let P be any st-dipath in D and let D' denote the digraph arising from D by reorienting P. Then D' is strongly connected, since if it had a node-set Z ($\emptyset \subset Z \subset V$) with no entering arcs, then Z must be a $t\overline{s}$-set and P enters Z exactly once. But then $0 = \varrho_{D'}(Z) = \varrho_D(Z) - 1 \geq 2 - 1 = 1$, a contradiction. Therefore D' is indeed strongly connected and its in-degree vector is decreasingly smaller than that of D.

To see the non-trivial part, define a set-function p_1 as follows:

$$p_1(X) := \begin{cases} 0 & \text{if} \quad X = \emptyset, \\ |E| & \text{if} \quad X = V, \\ i_G(X) + 1 & \text{if} \quad \emptyset \subset X \subset V. \end{cases} \tag{6.1}$$

Then p_1 is crossing supermodular and hence $B_1 := B'(p_1)$ is a base-polyhedron.

Claim 6.2. *An integral vector m is the in-degree vector of a strong orientation of G if and only if m is in $\overset{....}{B}_1$.*

Proof. If m is the in-degree vector of a strong orientation of G, then $\widetilde{m}(V) = |E| = p_1(V)$, $\widetilde{m}(\emptyset) = 0 = p_1(\emptyset)$, and

$$\widetilde{m}(Z) = \sum [\varrho(v) : v \in Z] = \varrho(Z) + i_G(Z) \geq 1 + i_G(Z) = p_1(Z)$$

for $\emptyset \subset Z \subset V$, that is, $m \in \overset{....}{B}_1$.

Conversely, let $m \in \overset{....}{B}_1$. Then $m \in B_G$ and hence by Claim 4.2, G has an orientation D with in-degree vector m. We claim that D is strongly connected. Indeed,

$$\varrho(Z) = \sum [\varrho(v) : v \in Z] - i_G(Z) = \widetilde{m}(Z) - i_G(Z) \geq p_1(Z) - i_G(Z) = 1$$

whenever $\emptyset \subset Z \subset V$. ■

Claim 6.3. *Let D be a strong orientation of G with in-degree vector m. Let t and s be nodes of G. The vector $m' := m + \chi_s - \chi_t$ is in B_1 if and only if D admits two arc-disjoint dipaths from s to t.*

Proof. $m' \in B_1$ holds precisely if there is no $t\overline{s}$-set X which is m-tight with respect to p_1, that is, $\widetilde{m}(X) = i_G(X) + 1$. Since $\varrho(Y) + i_G(Y) = \sum[\varrho(v) : v \in Y] = \widetilde{m}(Y)$ holds for any set $Y \subset V$, the tightness of X (that is, $\widetilde{m}(X) = i_G(X) + 1$) is equivalent to requiring that $\varrho(X) = 1$. Therefore $m' \in B_1$ if and only if $\varrho(Y) > 1$ holds for every $t\overline{s}$-set Y, which is, by Menger's theorem, equivalent to the existence of two arc-disjoint st-dipaths of D. ■

By Theorem 2.1, m is a dec-min element of $\overset{....}{B}_1$ if and only if there is no 1-tightening step for m. By Claim 6.3 this is just equivalent to the condition in the theorem that there are no two arc-disjoint st-dipaths in D for nodes s and t for which $\varrho(t) \geq \varrho(s) + 2$. ■■

An immediate consequence of Claim 6.2 and Theorem 2.1 is the following.

Corollary 6.4. *A strong orientation of G is dec-min if and only if it is inc-max.* ■

We indicated in Section 5.1 how in-degree constrained dec-min orientations can be managed due to the fact that the intersection of an integral base-polyhedron B with an integral box T is an integral base-polyhedron. The same approach works for degree-constrained strong orientations. For example, in this case dec-min and inc-max again coincide and one can formulate the in-degree constrained version of Theorem 6.1. In the next section, we overview more general cases.

6.2 A counterexample for mixed graphs

Although Robbins' theorem on strong orientability of undirected graphs easily extends to mixed graphs, as was pointed out by Boesch and Tindell [2], it is not true anymore that a decreasingly minimal strong orientation of a mixed graph is always increasingly maximal. Actually, one may consider two natural variants.

In the first one, decreasing minimality and increasing maximality concern the total in-degree of the directed graph obtained from the initial mixed graph after orienting its undirected edges. Let $V = \{a, b, c, d\}$. Let $E = \{ab, cd\}$ denote the set of undirected edges and let $A = \{ad, ad, ad, da, da, bc, bc, cb\}$ denote the set of directed edges of a mixed graph $M = (V, A + E)$. There are two strong orientations of M. In the first one, the orientations of the elements of E are ba and dc, in which case the total in-degree vector is $(3, 1, 3, 3)$. In the second one, the orientations of the elements of E are ab and cd, in which case the total in-degree vector is $(2, 2, 2, 4)$. Now $(3, 1, 3, 3)$ is dec-min while $(2, 2, 2, 4)$ is inc-max.

In the second variant, we are interested in the in-degree vector of the digraph obtained by orienting the originally undirected part E. For this version the counterexample is as follows. Let $V = \{a, b, c, d, x, y, u, v\}$. Let $E = \{ab, cd, au, au, av, av, dy, dy, bx, bx\}$ denote the set of undirected edges and let $A = \{ad, da, bc, cb\}$ denote the set of directed edges of a mixed graph $M = (V, A + E)$. The undirected part of M is denoted by $G = (V, E)$.

In any strong orientation of $M = (V, A + E)$, the orientations of the undirected parallel edge-pairs $\{au, au\}$, $\{av, av\}$, $\{dy, dy\}$, $\{bx, bx\}$ are oriented oppositely, and hence their contribution to the in-degrees (in the order of a, b, c, d, u, v, x, y) is $(2, 1, 0, 1, 1, 1, 1, 1)$.

Therefore there are essentially two distinct strong orientations of M. In the first one, the undirected edges ab, cd are oriented as ba, dc, while in the second one the undirected edges ab, cd are oriented as ab, cd. Hence the in-degree vector of the first strong orientation corresponding to the orientation of G (in the order of a, b, c, d, u, v, x, y) is $(3, 1, 1, 1, 1, 1, 1, 1)$. The in-degree vector of second strong orientation corresponding to the orientation of G is $(2, 2, 0, 2, 1, 1, 1, 1)$. The first vector is inc-max while the second vector is dec-min.

These examples give rise to the question: what is behind the phenomenon that while dec-min and inc-max coincide for strong orientations of undirected graphs, they differ for strong orientations of mixed graph? The explanation is, as we pointed out earlier, that for an M-convex set the two notions coincide and the set of in-degree vectors of strong orientations of an undirected graph is an M-convex set, while the corresponding set for a mixed graph is, in general, not an M-convex set. It is actually the intersection of two M-convex sets. An algorithm for computing a dec-min element of the intersection of two M-convex sets will be described elsewhere.

6.3 Higher edge-connectivity

An analogous approach works in a much more general setting. We say that a digraph covers a set-function h if $\varrho(X) \geq h(X)$ holds for every set $X \subseteq V$. The following result was proved in [10].

Theorem 6.5 ([10]). *Let h be a finite-valued, non-negative crossing supermodular function with $h(\emptyset) = h(V) = 0$. A graph $G = (V, E)$ has an orientation covering h if and only if*

$$e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V_i) \quad \text{and} \quad e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V - V_i)$$

hold for every partition $\mathcal{P} = \{V_1, \dots, V_q\}$ of V, where $e_{\mathcal{P}}$ denotes the number of edges connecting distinct parts of $\mathcal{P}$. ∎

This theorem easily implies the classic orientation result of Nash-Williams [36] stating that a graph G has a k-edge-connected orientation precisely if G is $2k$-edge-connected. Even more, call a digraph (k, ℓ)-edge-connected ($\ell \leq k$) (with respect to a root-node r_0) if $\varrho(X) \geq k$ whenever $\emptyset \subset X \subseteq V - r_0$ and $\varrho(X) \geq \ell$ whenever $r_0 \in X \subset V$. (By Menger's theorem, (k, ℓ)-edge-connectedness is equivalent to requiring that there are k arc-disjoint dipaths from r_0 to every node and there are ℓ arc-disjoint dipaths from every node to r_0.) Then Theorem 6.5 implies:

Theorem 6.6. *A graph $G = (V, E)$ has a (k, ℓ)-edge-connected orientation if and only if*

$$e_{\mathcal{P}} \geq k(q - 1) + \ell$$

holds for every q-partite partition $\mathcal{P}$ of V. ∎

Note that an even more general special case of Theorem 6.5 can be formulated to characterize graphs admitting in-degree constrained and (k, ℓ)-edge-connected orientations.

It is important to emphasize that however general Theorem 6.5 is, it does not say anything about strong orientations of mixed graphs. In particular, it does not imply the pretty but easily provable theorem of Boesch and Tindell [2]. The problem of finding decreasingly minimal in-degree constrained k-edge-connected orientation of mixed graphs can be solved as a special case of decreasing minimization over the intersection of two M-convex sets.

The next lemma shows why the set of in-degree vectors of orientations of G covering the set-function h appearing in Theorem 6.5 is an M-convex set, ensuring in this way the possibility of applying the results on decreasing minimization over M-convex sets to general graph orientation problems.

Lemma 6.7. *An orientation D of G covers h if and only if its in-degree vector m is in the base-polyhedron $B = B'(p)$, where $p := h + i_G$ is a crossing supermodular function.*

Proof. Suppose first that m is the in-degree vector of a digraph covering h. Then $h(X) \leq \varrho(X) = \widetilde{m}(X) - i_G(X)$ for $X \subset V$ and $h(V) = 0 = \varrho(V) = \widetilde{m}(V) - i_G(V)$, that is, m is indeed in B.

Conversely, suppose that $m \in B$. Since h is finite-valued and non-negative, we have $\widetilde{m}(X) \geq p(X) \geq i_G(X)$ for $X \subset V$ and $\widetilde{m}(V) = i_G(V)$ and hence, by the Orientation lemma, there is an orientation D of G with in-degree vector m. Moreover, this digraph D covers h since $\varrho_D(X) = \widetilde{m}(X) - i_G(X) \geq p(X) - i_G(X) = h(X)$ holds for $X \subset V$. ■

By Lemma 6.7, Theorem 2.1 can be applied again to the general orientation problem covering a non-negative and crossing supermodular set-function h in the same way as it was applied in the special case of strong orientation above, but we formulate the result only for the special case of in-degree constrained and k-edge-connected orientations.

Theorem 6.8. *Let $G = (V, E)$ be an undirected graph endowed with a lower bound function $f : V \to \mathbf{Z}$ and an upper bound function $g : V \to \mathbf{Z}$ with $f \leq g$. A k-edge-connected and in-degree constrained orientation D of G is decreasingly minimal if and only if there are no two nodes s and t for which $\varrho(t) \geq \varrho(s) + 2$, $\varrho(t) > f(t)$, $\varrho(s) < g(s)$, and there are $k + 1$ arc-disjoint st-dipaths.* ■

The theorem can be extended even further to in-degree constrained and (k, ℓ)-edge-connected orientations ($\ell \leq k$).

An extension We say that a digraph $D = (V, A)$ is k-edge-connected in a specified subset S of nodes if there are k-arc-disjoint dipaths in D from any node of S to any other node of S.

By relying on Lemma 6.7, one can derive the following.

Theorem 6.9. *Let $G = (V, E)$ be an undirected graph with a specified subset S of V. Let m_0 be an in-degree specification on $V - S$. The set of in-degree vectors of those orientations of G which are k-edge-connected in S and in-degree specified in $V - S$ is an M-convex set.* ■

By this theorem, we can determine a decreasingly minimal orientation among those which are k-edge-connected in S and in-degree specified in $V - S$. Even additional in-degree constraints can be imposed on the elements of S.

Hypergraph orientation Let $H = (V, \mathcal{E})$ be a hypergraph for which we assume that each hyperedge has at least 2 nodes. Orienting a hyperedge Z means that we designate an element z of Z as its head-node. A hyperedge Z with a designated head-node $z \in Z$ is a directed hyperedge denoted by (Z, z). Orienting a hypergraph means the operation of orienting each of its hyperedges. We say that a directed hyperedge (Z, z) enters a subset X of nodes if $z \in X$ and $Z - X \neq \emptyset$. A directed hypergraph is called k-edge-connected if the in-degree of every non-empty proper subset of nodes is at least k.

The following result was proved in [15] (see, also Theorem 2.22 in the survey paper [14]).

Theorem 6.10. *The set of in-degree vectors of k-edge-connected and in-degree constrained orientations of a hypergraph forms an M-convex set.* ■

Therefore we can apply the general results obtained for decreasing minimization over M-convex sets.

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