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On the global rigidity of tensegrity graphs

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Abstract

A tensegrity graph is a graph with edges labeled as bars, cables and struts. A realization of a tensegrity graph T is a pair (T, p) , where p maps the vertices of T into \mathbb{R}^d for some $d \geq 1$. The realization is globally rigid if any realization (T, q) in \mathbb{R}^d in which the bars have the same length and the cables and struts are not longer and not shorter, respectively, is an isometric image of (T, p) . A tensegrity graph is weakly globally rigid in \mathbb{R}^d if it has a generic globally rigid realization in \mathbb{R}^d , and strongly globally rigid in \mathbb{R}^d if every generic realization in \mathbb{R}^d is globally rigid.

In this paper we give a necessary condition for weak global rigidity in \mathbb{R}^d and prove that in the $d = 1$ case the same condition is also sufficient. In particular, our results imply that a tensegrity graph has a generic globally rigid realization in \mathbb{R}^1 if and only if it has a generic universally rigid realization in \mathbb{R}^1 . We also show that recognizing strongly globally rigid tensegrity graphs in \mathbb{R}^d is co-NP-hard for all $d \geq 1$.

1 Introduction

A *tensegrity graph* $T = (V, B, C, S)$ is a simple graph with vertex set V and labelled edge set $E = B \cup C \cup S$, where B , C and S represent, respectively, rigid bars, inextensible cables and incompressible struts. We shall call the elements of E the *members* of T . A *tensegrity framework in \mathbb{R}^d* , or simply a *tensegrity*, is a pair (T, p) , where T is a tensegrity graph and $p : V \rightarrow \mathbb{R}^d$ is an embedding of its vertices into Euclidean space. We also say that (T, p) is a *realization* of T in \mathbb{R}^d . Tensegrity frameworks can be used to model various physical and biological systems (see e.g. [5], [8]).

Of particular interest is the study of the rigidity properties of tensegrities. Intuitively, we say that a tensegrity is *rigid* if it cannot be continuously deformed so that the length of the bars remains constant and the length of the cables (struts, respectively) does not increase (decrease, resp.). A tensegrity is *globally rigid* if the pairwise distances of the vertices is uniquely determined under these length constraints. We shall give precise definitions in the next section. One approach to the study of the rigidity and global rigidity of tensegrities is to focus on the underlying tensegrity graph and explore the

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extent to which the combinatorial structure of the graph determines the rigidity of its realizations. This has been especially successful in the case of *bar frameworks*, where every member is a bar¹. In this case it has been shown that for any fixed dimension d , either all realizations in \mathbb{R}^d in sufficiently general position are rigid (globally rigid, respectively), or none of them are (see [1], [2], [6]). The most frequently used notion of general position in this setting is that of a *generic* framework, one in which the set of all vertex coordinates is algebraically independent. Thus we can speak of (*generically*) *rigid* and (*generically*) *globally rigid* graphs in \mathbb{R}^d , for which every generic realization in \mathbb{R}^d is rigid (globally rigid, respectively). A combinatorial characterization of these graphs is known for $d = 1, 2$, and is a major open problem for $d \geq 3$.

For general tensegrity graphs, the situation is different: it may happen that some generic realizations are rigid, while others are not. However, we can still explore the connection between the structure of the tensegrity graph and the rigidity of its generic realizations. For example, we may ask which tensegrity graphs have a generic rigid realization in \mathbb{R}^d ; we shall call these graphs *weakly rigid* in \mathbb{R}^d . We may also require every generic realization in \mathbb{R}^d to be rigid; in this case we shall say that the tensegrity graph is *strongly rigid* in \mathbb{R}^d . Weak and strong global rigidity may be defined analogously.

These notions are not very well understood. In fact, the only general results known are concerning weak and strong rigidity in \mathbb{R}^1 . Recki and Shai [10] gave a polynomial-time checkable combinatorial characterization of weak rigidity in \mathbb{R}^1 . In the case of strong rigidity in \mathbb{R}^1 , Jackson, Jordán and Király gave a combinatorial characterization in terms of the so-called alternating cycle property but showed that recognizing these graphs is co-NP-complete [9].

In this paper we consider the analogous problems in the case of global rigidity. The situation turns out to be similar to the case of weak and strong rigidity: we give a combinatorial characterization of weak global rigidity in \mathbb{R}^1 which can be checked in polynomial time (Theorem 3.5), while we show that recognizing strongly globally rigid tensegrity graphs in \mathbb{R}^d is co-NP-hard (Corollary 4.5). For the latter result, we introduce the *odd cycle property* for tensegrity graphs and show that it is a necessary condition for strong global rigidity in general and sufficient for some special families of tensegrity graphs.

The rest of the paper is laid out as follows. In Section 2 we recall the definitions and results that we use throughout the paper. In Section 3 we give a necessary condition for weak global rigidity in \mathbb{R}^d and show that it is also sufficient in \mathbb{R}^1 . Finally, in Section 4 we introduce and examine the odd cycle property and use it to show that recognizing strongly globally rigid tensegrity graphs is co-NP-hard in \mathbb{R}^d for all fixed $d \geq 1$.

¹In the case of bar frameworks we shall view the underlying tensegrity graph simply as a graph without any edge labeling.

2 Preliminaries

Let (T, p) and (T, q) be two d -dimensional realizations of the tensegrity graph $T = (V, B, C, S)$. We say that (T, p) *dominates* (T, q) if we have

$$\begin{aligned} \|p(u) - p(v)\| &= \|q(u) - q(v)\| && \text{for all } uv \in B, \\ \|p(u) - p(v)\| &\geq \|q(u) - q(v)\| && \text{for all } uv \in C, \\ \|p(u) - p(v)\| &\leq \|q(u) - q(v)\| && \text{for all } uv \in S. \end{aligned}$$

In this case we also say that (T, q) *satisfies the member constraints of* (T, p) , or, if every member of T is a bar, that (T, q) and (T, p) are *equivalent*. If we have

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \quad \text{for all } u, v \in V,$$

then we say that (T, p) and (T, q) are *congruent*.

We say that a tensegrity (T, p) is *rigid* if there is some $\varepsilon > 0$ such that any other realization (T, q) with $\|p(v) - q(v)\| < \varepsilon$ for all $v \in V$ that satisfies the member constraints of (T, p) is, in fact, congruent to it. It can be shown that a tensegrity framework is not rigid if and only if it is *flexible*: there is a continuous (indeed, an analytic) motion (T, p_t) , $0 \leq t \leq 1$ of frameworks satisfying the member constraints of $p = p_0$ such that p_t is not congruent to p for $t > 0$ (see [11]). We say that the d -dimensional tensegrity framework (T, p) is *globally rigid* if the only d -dimensional realizations satisfying its member constraints are the ones congruent to it. Finally, we may require that (T, p) remains globally rigid even when we consider it as a framework in \mathbb{R}^D for some $D \geq d$. If this holds for all $D \geq d$, then we say that (T, p) is *universally rigid*.

Again, we say that a tensegrity graph is *weakly rigid* (*weakly globally rigid*, respectively) in \mathbb{R}^d if it has a generic rigid (globally rigid, resp.) realization in \mathbb{R}^d . On the other hand, we say that a tensegrity graph is *strongly rigid* (*strongly globally rigid*, respectively) in \mathbb{R}^d if every generic realization of the graph in \mathbb{R}^d is rigid (globally rigid, resp.). As we mentioned before, for bar graphs, weak and strong rigidity are equivalent, as well as weak and strong global rigidity. In this case, we simply say that the graph is *rigid* in \mathbb{R}^d (*globally rigid in* \mathbb{R}^d , respectively).

Let $E = B \cup C \cup S$ denote the members of the tensegrity graph T . A *strict proper stress* of a tensegrity (T, p) is a mapping $\omega : E \rightarrow \mathbb{R}$ such that the value of ω is negative on cables and positive on struts and for every vertex $v \in V$ we have

$$\sum_{u:uv \in E} \omega(uv)(p(u) - p(v)) = 0.$$

We shall need the following result, which follows from Theorem 5.2 and Theorem 5.8 of [11].

Lemma 2.1. *If a generic tensegrity framework is rigid, then it has a strict proper stress.*

We shall also need some notions from graph theory. Let $G = (V, E)$ be a graph. An *open ear decomposition* of G is a partition of the edge set of G into subsets C, P_1, \dots, P_l , where C is a cycle and for $1 \leq i \leq l$, P_i is a path with distinct endpoints that belong to $V(C \cup P_1, \dots, P_{i-1})$ while its internal vertices are disjoint from it. Such a path is called an *open ear*. It is known that a graph is 2-connected if and only if it has an open ear decomposition. For subsets $X, Y \subseteq V$ we use $E(X, Y)$ to denote the set of edges with one endpoint in X and the other in Y . For an integer $d \geq 1$ and subsets $X, Y \subseteq V$ with $|X|, |Y| \geq d + 1$, we say that the pair (X, Y) is a *d-separation* of G if $|X \cap Y| \leq d$, $X \cup Y = V$ and $E(X - X \cap Y, Y - X \cap Y) = \emptyset$. We emphasize that, in contrast to some authors, we do not require $|X \cap Y| = d$ in the preceding definition.

If (G, p) is a bar framework in \mathbb{R}^1 and (X, Y) is a 1-separation of G with $X \cap Y = \{v\}$, then we may obtain an equivalent, non-congruent framework (G, p') by reflecting Y through v , so that $p'(y) = 2 \cdot p(v) - p(y)$ for $y \in Y$ and $p'(x) = p(x)$ for $x \in X$. It is folklore that if G is connected and (G, p) is generic, then, up to congruence, each realization equivalent to (G, p) may be obtained via a series of such reflections. In particular, if G has at least 3 vertices, then it is globally rigid in \mathbb{R}^1 if and only if it is 2-connected.

This can be generalized to higher dimensions as follows. Let (G, p) be a bar framework in \mathbb{R}^d and (X, Y) a d -separation of G . Let $H \subseteq \mathbb{R}^d$ be an affine hyperplane containing the points $p(v), v \in X \cap Y$. Then we may obtain an equivalent framework (G, p') by reflecting the points $p(v), v \in Y$ through H . If neither $\{p(v), v \in X\}$ nor $\{p(v), v \in Y\}$ is contained in H (for example, if (G, p) is generic), then (G, p') and (G, p) are not congruent. In particular, if a graph on at least $d + 2$ vertices is globally rigid in \mathbb{R}^d then it is $(d + 1)$ -connected (see [7]). As a shorthand, we shall refer to this construction as “reflecting Y through H ”.

3 Existence of globally rigid realizations

In this section we examine the notion of weak global rigidity. We shall need the following observation regarding rigid tensegrities in \mathbb{R}^d . Recall that a point p of a convex set $C \subseteq \mathbb{R}^d$ is an *extreme point* of C if it can not be written as a convex combination of points in $C - p$.

Lemma 3.1. *Let (T, p) be a generic rigid tensegrity in \mathbb{R}^d . Then every vertex v for which $p(v)$ is an extreme point of the convex hull of $\{p(u), u \in V\}$ is incident to at least one non-strut, as well as at least one non-cable member.*

Proof. By Lemma 2.1, (T, p) has a strict proper stress ω . Rearranging the equilibrium condition gives

$$\sum_{u:uv \in E} \omega(uv)p(u) = \sum_{u:uv \in E} \omega(uv)p(v),$$

where E denotes the members of T . Now ω cannot be positive on all members incident to v , since then dividing this equation by $\sum_{u:uv \in E} \omega(uv)$ would give that $p(v)$ is a convex combination of $p(u), uv \in E$, contradicting the assumption that $p(v)$ is an

extreme point. For the same reason, ω cannot be negative on all members incident to v . This means that v is incident to at least one non-strut, as well as at least one non-cable member, as desired. \square

We note that Lemma 3.1 remains true without the genericity assumption on (T, p) (provided that p is injective), but the proof is slightly more involved in that case.

The following lemma gives simple necessary conditions for weak global rigidity in \mathbb{R}^d .

Lemma 3.2. *If a tensegrity graph $T = (V, B, C, S)$ has a generic globally rigid realization in \mathbb{R}^d , then either it is a complete graph with only bar members, or it has at least $d + 2$ vertices and satisfies the following conditions:*

- *The graph $\bar{T} = (V, B \cup C \cup S)$ obtained by replacing every member of T by bars is globally rigid in \mathbb{R}^d ,*
- *T contains at least $(d + 1)/2$ non-cable members,*
- *The graph $(V, B \cup C)$ is connected.*

Proof. First suppose that T has at most $d + 1$ vertices. By adding bars as well as replacing some of the members by bars we may assume that the underlying graph of T is complete and T has exactly one non-bar member, denoted by uv . We shall show that the generic realizations of such a tensegrity graph are flexible. To this end, let (T, p) be a generic realization in \mathbb{R}^d . There exists an affine subspace $A \subseteq \mathbb{R}^d$ of codimension 2 that contains $p(w)$, $w \neq u, v$ and does not contain u, v . Then rotating u around A by a sufficiently small amount and in the suitable direction gives a non-trivial motion of (T, p) respecting the member constraints.

Now assume that T has at least $d + 2$ vertices. The fact that \bar{T} must be globally rigid in \mathbb{R}^d follows immediately from the definitions. The second condition follows from Lemma 3.1, since in a generic realization the convex hull contains at least $d + 1$ points from $\{p(u), u \in V\}$, all of which are extreme points. Finally, if $(V, B \cup C)$ were not connected, then we could translate one of its connected components by a sufficiently large amount to obtain a realization $(T, q) \prec (T, p)$ that is not congruent to (T, p) , contradicting global rigidity. \square

It turns out that the necessary conditions of Lemma 3.2 are sufficient for weak global rigidity in the $d = 1$ case. The proof of this is based on the following lemma. We note that this statement can be extended to (and follows from) a result regarding ear decompositions of matroids, based on e.g. [4, Theorem 5.2.9]. For the sake of completeness, we give a self-contained proof.

Lemma 3.3. *Let $G = (V, E)$ be a 2-connected graph. Then for any partition $E = E_1 \cup E_2$ of the edge set of G , where (V, E_1) is connected, there is an open ear decomposition of G , such that each ear, as well as the starting cycle, contains at most one edge from E_2 .*

Proof. We may assume that $T = (V, E_1)$ is a spanning tree by moving edges from E_1 to E_2 . Let the starting cycle of the ear decomposition be the unique cycle in the graph $G_1 + e$ for some $e \in E_2$. Suppose that we have a suitable ear decomposition of some subgraph H of G ; we shall show that if $V(H) \neq V$, then we can find an open ear extending H that contains one edge from E_2 .

Suppose, then, that $V(H) \neq V$. It follows from the connectedness of T that there is an edge $uv \in E_1$ with $u \in V(H), v \notin V(H)$. Observe that the existence of a suitable ear decomposition of H implies that $V(H)$ induces a connected subgraph of T . This implies that u cannot be a leaf vertex of T , so it is a cut-vertex of T . Let $A \subseteq V$ denote the set of vertices that are unreachable from $V(H) - u$ in $T - u$. Clearly, $v \in A$, so, in particular, A is nonempty. Now, since G is 2-connected, there must be an edge $e = u'v'$, necessarily from E_2 , such that $u' \in A$ and $u \neq v' \notin A$. Let P be the unique path in T from u to u' and P' the path from v' to u . Concatenating P , e and P' gives a cycle in G with precisely one edge from E_2 . This cycle must contain at least one other vertex from $V(H)$ besides u , since otherwise it would all lie in A , contradicting the choice of e . Thus the cycle contains a suitable open ear as a subgraph. \square

The proof of Lemma 3.3 actually implies that we can choose the starting cycle C in the ear decomposition such that it contains exactly one edge from E_2 .

We shall use the following simple construction in the next proof. Let C be a tensegrity graph such that its underlying graph is a cycle and it contains precisely one strut. Let v_1, \dots, v_n denote the vertices of C in cyclic order such that v_1v_n is the strut. By a *stretched cycle* we mean a realization (C, p) of C in \mathbb{R}^1 such that $p(v_1) < p(v_2) < \dots < p(v_n)$. It is known that a stretched cycle is universally rigid.

Lemma 3.4. *Let T be a tensegrity graph and (T, p) a generic universally rigid tensegrity in \mathbb{R}^1 . Suppose that the tensegrity graph T' can be obtained from T by adding an open ear containing at most one strut. Then there is a generic universally rigid tensegrity (T', p') extending (T, p) .*

Proof. Let P denote the open ear added to T with endpoints u, v . If P is a single edge then the statement is trivial, so we assume that this is not the case. Since (T, p) is universally rigid, we may add a cable or strut uv to T without changing the universal rigidity of any extension (T', p') of (T, p) (if uv was already a member of T we may replace it instead by a bar and use essentially the same argument as in the following). If T contains a strut, then let uv be a cable, and otherwise let it be a strut. Now T' can be obtained by gluing T and the cycle $C = P + uv$ along $\{u, v\}$. Since C contains exactly one strut, it has a universally rigid realization as a stretched cycle. Let (C, p') be such a realization in which $p'(u) = p(u)$ and $p'(v) = p(v)$. Since gluing two universally rigid tensegrities in \mathbb{R}^1 along a pair of points yields a universally rigid tensegrity, gluing (T, p) and (C, p') along $\{u, v\}$ gives a universally rigid realization of T' . We can clearly choose (C, p') in such a way that the resulting tensegrity will be generic. \square

Theorem 3.5. *A tensegrity graph T has a generic globally rigid realization in \mathbb{R}^1 if and only if T is 2-connected, not all members are cables and there is no cut in T*

containing only struts. In fact, if these conditions hold, then T has a generic universally rigid realization in \mathbb{R}^1 .

Proof. Necessity is given by Lemma 3.2; we prove sufficiency by constructing a generic universally rigid realization of T . By Lemma 3.3, there is an open ear decomposition C, P_1, \dots, P_k of T such that each ear P_i contains at most one strut and the cycle C contains exactly one strut. We prove by induction on l that the subgraph of T spanned by C, P_1, \dots, P_l has a generic universally rigid realization. For $l = 0$, we can realize C as a stretched cycle. The induction step is given by Lemma 3.4. \square

Theorem 3.5 gives a polynomial-time checkable characterization of weak global rigidity in \mathbb{R}^1 . We also note that it implies that a tensegrity graph in \mathbb{R}^1 has a generic globally rigid realization if and only if it has a generic universally rigid realization in \mathbb{R}^1 . In the case of bar graphs this is known to hold in \mathbb{R}^d for all $d \geq 1$, see [3]. It would be interesting to see whether this extends to tensegrity graphs in the $d \geq 2$ case as well.

4 Strongly globally rigid tensegrity graphs

In this section we investigate strong global rigidity. As in the case of weak global rigidity, we first describe a necessary condition. Then we shall show that the same condition is sufficient in some cases (although not in general), and use this to prove that recognizing strongly globally rigid graphs in \mathbb{R}^d is co-NP-hard.

Lemma 4.1. *Let $T = (V, B, C, S)$ be strongly globally rigid in \mathbb{R}^d . Then for any d -separation (X, Y) of the bar subgraph (V, B) , $E(X - X \cap Y, Y - X \cap Y)$ contains a cycle with an odd number of cables.*

Proof. For clarity, we first focus on the case when $d = 1$ and the bar subgraph is connected. Suppose that the condition does not hold for some 1-separation (X, Y) of (V, B) with $X \cap Y = \{v\}$. We shall construct a realization (T, p) such that $(T, p') \preceq (T, p)$, where (T, p') is obtained by reflecting Y through v . Let us choose an arbitrary root vertex u_1, \dots, u_k for each connected component of the bipartite graph $H = (V, E(X - v, Y - v))$. We define (T, p) in the following way: first, set $p(v) = 0$. For a vertex $u \neq v$, if u can be reached in H from one of the root vertices using an even number of cables, then let $p(u) < 0$ be an arbitrary negative value. Otherwise let $p(u) > 0$ be an arbitrary positive value. This is well-defined, for if some vertex u can be reached from some root vertex using both an even and an odd number of cables, then the concatenation of these walks is a closed walk containing an odd number of cables, and this would necessarily contain a cycle with an odd number of cables. Moreover, we may choose the position of the vertices so that (T, p) can be translated into a generic framework.

Now, in (H, p) cables have endpoints with different signs, while struts have endpoints with the same sign. It follows that reflecting Y through v in (T, p) results in a tensegrity (T, p') that satisfies the member constraints of (T, p) . Thus, T is not strongly globally rigid, as needed.

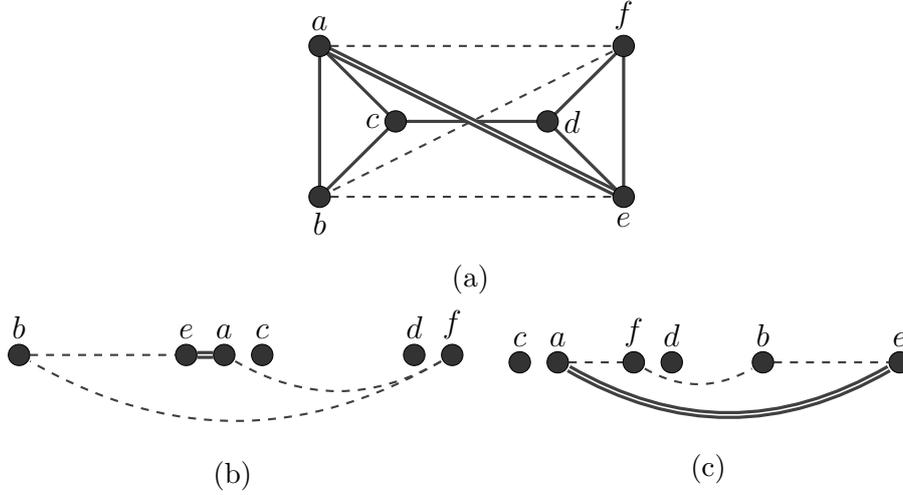


Figure 1: An example of a tensegrity graph which is strongly rigid in \mathbb{R}^1 and has the odd cycle property, but is not strongly globally rigid in \mathbb{R}^1 . In (b) and (c) the bars are not drawn. The tensegrity in (c) is obtained from the tensegrity in (b) by reflecting a and b through c , as well as e and f through d . It satisfies the member constraints of the tensegrity in (b), but is not congruent to it.

The general case can be shown analogously. Suppose that the condition does not hold for some d -separation (X, Y) of (V, B) . To define (T, p) , we first choose the points $p(v), v \in X \cap Y$ such that they lie in the $x_1 = 0$ hyperplane in a 'quasi-generic' position, i.e. they can be made generic by applying a suitable Euclidean isometry. Then we choose the first coordinates of $p(v), v \notin X \cap Y$ as in the previous special case, and the rest of the coordinates arbitrarily, in such a way that the whole framework is in a quasi-generic position. Now the framework (T, p') obtained by reflecting Y through the $x_1 = 0$ hyperplane satisfies the member constraints of (T, p) but is not congruent to it, which shows that T is not strongly globally rigid in \mathbb{R}^d . \square

If a tensegrity graph satisfies the necessary condition given in Lemma 4.1, we shall say that it has the (d -dimensional) *odd cycle property*. It follows from Lemma 4.1 that having the d -dimensional odd cycle property is a necessary condition for strong global rigidity in \mathbb{R}^d . Strong rigidity in \mathbb{R}^d is also clearly necessary. As Figure 1 shows, these conditions taken together are not sufficient in general, even in the $d = 1$ case; however, in the following lemmas we consider special cases where they do guarantee strong global rigidity.

Lemma 4.2. *Let $T = (V, B, C, S)$ be a tensegrity graph such that the bar subgraph $G = (V, B)$ is obtained by taking some graphs G_1, \dots, G_k that are globally rigid in \mathbb{R}^d along with a sequence of d distinct vertices from each graph and identifying the corresponding vertices. Then T is strongly globally rigid in \mathbb{R}^d if and only if it has the odd cycle property.*

Proof. Necessity follows from Lemma 4.1, so we only need to show sufficiency. Let (T, p) be a generic realization of T in \mathbb{R}^d . First, note that in any d -separation (X, Y)

of G , X must be the union of some of the globally rigid graphs G_i which we used to construct G , while $X \cap Y$ must be the set of identified vertices. Moreover, the frameworks equivalent to (G, p) are (up to congruence) precisely those that arise via taking a d -separation (X, Y) of G and reflecting Y through the affine hyperplane H spanned by $\{p(v), v \in X \cap Y\}$. This is because the position of each globally rigid subgraph G_i is determined, up to reflection through H , by the position of the d identified vertices.

We shall show that none of the tensegrities obtained in this way satisfy the member constraints of (T, p) . Indeed, let (X, Y) be a d -separation, with H denoting, again, the affine hyperplane spanned by $\{p(v), v \in X \cap Y\}$. Then $E(X - X \cap Y, Y - X \cap Y)$ contains a cycle with an odd number of cables. It follows that there is either a cable $u_1 u_2 \in C$ in this cycle such that $p(u_1)$ and $p(u_2)$ lie on the same side of H , or a strut $u'_1 u'_2 \in S$ such that $p(u'_1)$ and $p(u'_2)$ lie on different sides of H . The length of this cable (strut, respectively) increases (decreases, resp.) if we reflect Y through H , so that the resulting tensegrity does not satisfy the member constraints of (T, p) , which is what we wanted to show. \square

The following lemma describes another situation where the odd cycle property and strong rigidity together characterize strong global rigidity. As this lemma is not needed for our result on the hardness of recognizing strongly globally rigid tensegrity graphs, its proof, along with some discussion, is given in Appendix A.

Lemma 4.3. *Let T be a tensegrity graph such that the bar subgraph $G = (V, B)$ has two connected components, each being globally rigid in \mathbb{R}^1 . Then T is strongly globally rigid in \mathbb{R}^1 if and only if it is strongly rigid and it has the 1-dimensional odd cycle property.*

Finally, we consider the decision problem d -OCP in which, given a tensegrity graph $T = (V, B, C, S)$ we want to decide whether T has the d -dimensional odd cycle property.

Theorem 4.4. *d -OCP is co-NP-complete for every $d \geq 1$, even for tensegrities in which the bar subgraph (V, B) is obtained by taking some graphs G_1, \dots, G_k that are globally rigid in \mathbb{R}^d along with a sequence of d distinct vertices from each graph and identifying the corresponding vertices.*

Proof. The d -OCP problem is indeed in co-NP, since given a d -separation (X, Y) of (V, B) we can check whether the tensegrity graph $H = (V, E(X - X \cap Y, Y - X \cap Y))$ contains a cycle with an odd number of cables by subdividing each strut by a vertex and checking whether the resulting graph contains an odd cycle (i.e. is not bipartite). Let us call a d -separation (X, Y) of (V, B) *pure* if $E(X - X \cap Y, Y - X \cap Y)$ does not contain a cycle with an odd number of cables. As a shorthand, we shall refer to cycles with an odd number of cables simply as *odd cycles*; this will not cause confusion, since we shall only consider bipartite subgraphs, which have no odd cycles in the usual sense.

The hardness proof is along the same lines as [9, Theorem 3.1]. We shall show that there is a polynomial-time reduction from the complement of 3-SAT to d -OCP. Consider an instance of 3-SAT given by the formula φ , containing n variables. We

shall construct a tensegrity graph in the following way. The bar subgraph $G = (V, B)$ will consist of $2n + 2$ sufficiently large vertex-disjoint complete graphs, glued along d vertices $Z = \{v_1, \dots, v_d\}$, with the sizes of these complete graphs to be fixed later. This construction ensures that every d -separation of the bar subgraph has Z as the separating vertex set. We shall refer to the connected components of $G - Z$ as $\mathcal{T}, \mathcal{F}, P_1, \dots, P_n, \overline{P}_1, \dots, \overline{P}_n$ (we use \mathcal{T} instead of T to avoid confusion with the tensegrity T).

Next, we add cables and struts between these components in such a way that in a pure d -separation (X, Y) of G , X must contain exactly one of \mathcal{T} and \mathcal{F} , exactly one of P_i and \overline{P}_i for each $i = 1, \dots, n$, and each clause of φ must have a variable in it such that the corresponding component P_i or \overline{P}_i is contained in the same side of the d -separation as \mathcal{T} . Thus we shall have a correspondence between pure d -separations of T and valid solutions of φ . This can be done as follows. First, fix some vertices $t \in \mathcal{T}, f \in \mathcal{F}$, and for each of the remaining components, take two vertices u_1, u_2 from the component and add the odd cycle $u_1t, tu_2, u_2f \in C, fu_1 \in S$. This ensures that \mathcal{T} and \mathcal{F} must belong to different sides of a pure d -separation. Next, for each $i = 1, \dots, n$, take unused vertices (i.e. ones which have no incident cables and struts) $u \in P_i, \overline{u} \in \overline{P}_i, t_1, t_2 \in \mathcal{T}, f_1, f_2 \in \mathcal{F}$, and put an odd cycle on $\{u, \overline{u}\}, \{t_1, t_2\}$ and on $\{u, \overline{u}\}, \{f_1, f_2\}$ as in the previous step. These odd cycles make sure that P_i and \overline{P}_i can not belong to the same side of a pure d -separation. Finally, for each clause $\{x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, x_k^{\varepsilon_k}\}$ of φ , where $\varepsilon_i, \varepsilon_j, \varepsilon_k \in \{-1, 1\}$, take unused vertices $t_1, t_2, t_3 \in \mathcal{T}, x \in P_i^{\varepsilon_i}, y \in P_j^{\varepsilon_j}, z \in P_k^{\varepsilon_k}$, (here $P_i^1 = P_i$ and $P_i^{-1} = \overline{P}_i$) and add the odd cycle $t_1x, xt_2, t_2y, yt_3, t_3z \in C, zt_3 \in S$ to T . Then in a pure d -separation at least one of the components corresponding to the variables in this clause must belong to the same side as \mathcal{T} . Note that since we always put the odd cycles onto previously unused vertices, the only odd cycles in the resulting tensegrity are the ones that we specified.

It follows that φ can be satisfied if and only if there exists a pure d -separation, which is the same as T not having odd cycle property. We only used linearly many vertices in the number of clauses and variables, so the size of T is polynomial in the size of the 3-SAT instance. \square

Combining Lemma 4.2 and Theorem 4.4 yields the following hardness result regarding strong global rigidity.

Corollary 4.5. *For any $d \geq 1$, recognizing strongly globally rigid tensegrity graphs in \mathbb{R}^d is co-NP-hard.* \square

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Appendix

A Proof of Lemma 4.3

We recall the characterization of strong rigidity in \mathbb{R}^1 from [9]. We say that a tensegrity graph $T = (V, B, C, S)$ has the *alternating cycle property* if every bipartition $(U, V - U)$ of V is such that $E(U, V - U)$ contains either a bar or an *alternating cycle*, that is, a cycle in which cables and struts alternate.

Theorem A.1. [9] *A tensegrity graph is strongly rigid in \mathbb{R}^1 if and only if it has the alternating cycle property.*

The next lemma asserts that strong rigidity in \mathbb{R}^1 actually implies “orientation-preserving” global rigidity for all realizations.

Lemma A.2. *Suppose that T is a strongly rigid tensegrity graph in \mathbb{R}^1 and let (T, p) be a realization in \mathbb{R}^1 . Suppose that $(T, p') \preceq (T, p)$ is another framework such that (T, p') can be obtained from (T, p) by translating each of the connected components of the bar subgraph $G = (V, B)$ by some amount. Then (T, p') is congruent to (T, p) .*

Proof. By applying a translation to all of (T, p') we may suppose that each component of G is translated in the positive direction and that some component remains stationary. Let $\emptyset \neq U \subseteq V$ be the set of vertices that remain stationary. If $U = V$, then we are done, so suppose for contradiction that $U \subsetneq V$. Since T is strongly rigid, $E(U, V - U)$ contains an alternating cycle C . For convenience, let us orient the cables in $H = (V, E(U, V - U))$ from U to $V - U$, and the struts in the reverse direction, so that C becomes a directed cycle.

Let $l_s(p)$ and $l_s(p')$ denote the total length of the struts in C in (T, p) and (T, p') , respectively. Similarly, let $l_c(p)$ and $l_c(p')$ denote the total length of the cable in C in the respective realizations. Now each cable in (H, p) must point in the negative direction, since otherwise the length of the cable would increase as we apply the translations to (T, p) , contradicting $(T, p') \preceq (T, p)$. It follows that $l_s(p) \geq l_c(p)$. Similarly, each strut in (H, p') must point in the negative direction, since otherwise the translations would have shortened the strut, so that $l_c(p') \geq l_s(p')$. We also trivially have $l_c(p) \geq l_c(p')$ and $l_s(p) \leq l_s(p')$. Thus we have the following chain of inequalities:

$$l_c(p') \leq l_c(p) \leq l_s(p) \leq l_s(p') \leq l_c(p').$$

Now, either there is a strut in C which is longer in (H, p') than in (H, p) , or a cable in C that is shorter in (H, p') than in (H, p) , implying that one of the first and third inequalities must be strict, a contradiction. \square

Proof of Lemma 4.3. Necessity follows from Lemma 4.1 and, in the case of strong rigidity, the definitions. Let us denote the connected components of G by X and Y and let (T, p) be a generic realization on the line. It is not difficult to see that the equivalent realizations of (G, p) arise, up to congruence, by either translating Y by some amount, or by reflecting Y through a point $x \in \mathbb{R}^1$. In the first case, proper translations are ruled out by Lemma A.2. In the second case, if there is a cable in $E(X, Y)$ with both endpoints on the same side of x (and different from x), then reflecting Y through x would lengthen this cable, thus the resulting tensegrity does not satisfy the member constraints of (T, p) . Similarly, if there is a strut in $E(X, Y)$ with endpoints on different sides of x (and different from x), then the reflected image of this strut would be shorter. Now suppose that neither of these cases hold, so that each cable crosses x and each strut has both endpoints on the same side of x , where in both cases x is allowed to be an endpoint of the given edge. By moving to the nearest vertex, we may suppose $x = p(v)$ for some $v \in V$ without destroying this property. But the odd cycle property applied to the 1-separation $(X + v, Y + v)$ implies that either there is cable in $E(X - v, Y - v)$ with both endpoints on one side of $p(v)$ and different from $p(v)$, or a strut with endpoints on different sides of $p(v)$ and different from $p(v)$, a contradiction. \square

Lemmas 4.2 and 4.3 show that the 1-dimensional odd cycle property ensures, in effect, that if we start from a generic tensegrity in \mathbb{R}^1 and apply a reflection through some point (not necessarily in the configuration) to some of the 2-connected components, the resulting tensegrity does not satisfy the member constraints of the original one, except in the case when we reflected all or none of the 2-connected components. Figure 1 shows that if two or more reflections are involved then this is not true in general. This leaves open the case when the bar subgraph consists of three or more disjoint graphs that are globally rigid in \mathbb{R}^2 . It is not difficult to see that in this case, given a generic realization, the equivalent frameworks arise (up to congruence) by applying a translation to some of the connected components, and then reflecting some of the connected components through some point. Note that, by Lemma A.2, if the tensegrity is strongly rigid, then a proper translation does not preserve the member constraints; still, it is unclear whether the composition of a translation and a reflection can yield a non-congruent framework that is dominated by the original one.