Unit 1. Basic Structures on $\mathbb{R}^n$, Length of Curves.

$\mathbb{R}^n$, addition of vectors and multiplication by scalars, vector spaces over $\mathbb{R}$, linear combinations, linear independence, basis, dimension, linear and affine linear subspaces, tangent space at a point, tangent bundle; dot product, length of vectors, the standard metric on $\mathbb{R}^n$; balls, open subsets, the standard topology on $\mathbb{R}^n$, continuous maps and homeomorphisms; simple arcs and parameterized continuous curves, reparameterization, length of curves, integral formula for differentiable curves, parameterization by arc length.

THE LINEAR STRUCTURE OF $\mathbb{R}^n$

Recall that $\mathbb{R}^n$ is the set of n-tuples of real numbers

$$\mathbb{R}^n = \{ (x_1,...,x_n) : x_i \in \mathbb{R} \}.$$

We shall denote the elements of $\mathbb{R}^n$ by underlined letters.

We know from 3-dimensional geometry that the introduction of a Cartesian coordinate system in space yields an identification between points, vectors and 3-tuples of real numbers (each point can be identified by its position vector and its coordinates). By analogy, we can think of the elements of $\mathbb{R}^n$ (which are just n-tuples of reals) as points of an n-dimensional space or vectors.

If $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{y} = (y_1,...,y_n)$ are two elements of $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is a real number then we define the sum and difference of $\mathbf{x}$ and $\mathbf{y}$ and the scalar multiple of $\mathbf{x}$ by

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1,...,x_n \pm y_n)$$

$$\lambda\, \mathbf{x} = (\lambda\, x_1,...,\lambda\, x_n).$$

$\mathbb{R}^n$ is a vector space over the field of real numbers with respect to these operations.

(Let us recall the definition of an abstract vector space. A set X is a vector space over the field of real numbers if X is equipped with a binary operation +, and for each $\lambda \in \mathbb{R}$ the multiplication of elements of X with λ is defined in such a way that the following identities are satisfied

$((\mathbf{x} + \mathbf{y}) + \mathbf{z}) = (\mathbf{x} + (\mathbf{y} + \mathbf{z}))$ (associativity);

$\exists\ \mathbf{0} \in X$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in X$;

$\forall\ \mathbf{x} \in X\ \ \exists\ -\mathbf{x} \in X$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$;

$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutativity);

$\lambda(\mathbf{x} + \mathbf{y}) = \lambda\,\mathbf{x} + \lambda\,\mathbf{y}$;

$(\lambda + \mu)\,\mathbf{x} = \lambda\,\mathbf{x} + \mu\,\mathbf{x}$;

$(\lambda\,\mu)\,\mathbf{x} = \lambda\,(\,\mu\,\mathbf{x}\,)$;

$1\,\mathbf{x} = \mathbf{x}$.

Exercise: Check that $\mathbb{R}^n$ is indeed a vector space.

A <u>linear combination</u> of some vectors $\mathbf{x}_1,\ldots,\mathbf{x}_k \in X$ is a vector of the form

$$\lambda_1\mathbf{x}_1 + \ldots + \lambda_k\mathbf{x}_k,$$

where the λ_i's are real numbers. The vectors $\mathbf{x}_1,\ldots,\mathbf{x}_k \in X$ are <u>linearly independent</u> if their linear combination can be $\mathbf{0}$ only if all coefficients λ_i are 0. A <u>basis</u> of X is a maximal set of linearly independent vectors. The <u>dimension</u> of the vector space X is the cardinality of a basis.)

Exercise: Show that $\mathbb{R}^n$ is n-dimensional.

A non-empty subset V of $\mathbb{R}^n$ is a <u>linear subspace</u> if for all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in \mathbb{R}$ we have $\mathbf{x} + \mathbf{y} \in V$ and $\lambda\,\mathbf{x} \in V$. Linear subspaces of $\mathbb{R}^n$ are vector spaces themselves, so their dimension is defined. A 0-dimensional subspace consists of the single point $\mathbf{0}$. 1-dimensional linear subspaces are straight lines passing through the origin, 2-dimensional linear subspaces are ordinary planes that go through the origin etc. Linear subspaces always go through the origin.

A subset of $\mathbb{R}^n$ is an <u>affine linear subspace</u> if it is a translate of a linear subspace. The <u>dimension</u> of an affine linear subspace is the dimension of the linear subspace from which it is obtained by a translation. k-dimensional affine linear subspaces of $\mathbb{R}^n$ will be called shortly <u>k-planes</u>. 0-planes are points, 1-planes are straight lines, 2-planes are the ordinary 2-dimensional planes of $\mathbb{R}^n$. (n-1)-planes of $\mathbb{R}^n$ are also called hyperplanes.

In 3-dimensional geometry, a vector is an equivalence class of directed segments, where two directed segments represent the same vector if they have the same direction and length. Given a point $\mathbf{p}$ and a vector $\mathbf{x}$, there is a unique directed segment that starts from $\mathbf{p}$ and represents $\mathbf{x}$. Returning to $\mathbb{R}^n$, we shall call a pair $(\mathbf{p},\mathbf{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ a <u>tangent vector</u> of $\mathbb{R}^n$ at $\mathbf{p}$ or a <u>vector based at</u> $\mathbf{p}$ (and we may think of this pair as a directed segment that initiates from $\mathbf{p}$ and represents the vector $\mathbf{x}$). The set of all tangent vectors of $\mathbb{R}^n$ at $\mathbf{p}$ form the <u>tangent space of</u> $\mathbb{R}^n$ <u>at</u> $\mathbf{p}$ and is denoted by $T_{\mathbf{p}}\mathbb{R}^n$. There is a one to one correspondence between $\mathbb{R}^n$ and $T_{\mathbf{p}}\mathbb{R}^n$ established by the "forgetting" mapping

$$T_{\mathbf{p}}\mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad (\mathbf{p},\mathbf{x}) \mapsto \mathbf{x},$$

with the help of which we can furnish $T_{\mathbf{p}}\mathbb{R}^n$ with a vector space structure in a natural way.

The tangent bundle $T_*\mathbb{R}^n$ of $\mathbb{R}^n$ is the disjoint union of all tangent spaces of $\mathbb{R}^n$: $T_*\mathbb{R}^n = \bigcup_{\mathbf{p}\in\mathbb{R}^n} T_{\mathbf{p}}\mathbb{R}^n$. The mapping π: $T_*\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(\mathbf{p},\mathbf{x}) \longmapsto \mathbf{p}$ that assigns to each tangent vector the point at which it is tangent is called the canonical projection of the tangent bundle.

THE METRIC OF $\mathbb{R}^n$

The linear structure of $\mathbb{R}^n$ is not enough to measure distance, area, etc. To measure e.g. the length of curves we need a metric. The standard metric on $\mathbb{R}^n$ is connected with a bilinear function called the dot product. The dot product of the vectors $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$ is defined by the formula

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i\, y_i .$$

With the help of the dot product one can define the norm or length of a vector and the distance of two points as follows

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}, \quad d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| .$$

Proposition. We have

i) $d(\mathbf{x}, \mathbf{y}) \geqslant 0$, and equality holds if and only if $\mathbf{x} = \mathbf{y}$;

ii) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$;

iii) $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geqslant d(\mathbf{x}, \mathbf{z})$ (triangle inequality)

for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

Proof. It is enough to prove iii), the rest is obvious. Introducing the vectors $\mathbf{a} = \mathbf{y} - \mathbf{x}$ and $\mathbf{b} = \mathbf{z} - \mathbf{y}$, iii) is equivalent to the inequalities

$$\|\mathbf{a}\| + \|\mathbf{b}\| \geqslant \|\mathbf{a} + \mathbf{b}\| ,$$

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| \geqslant \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\langle \mathbf{a}, \mathbf{b} \rangle ,$$

$$\|\mathbf{a}\|\,\|\mathbf{b}\| \geqslant \langle \mathbf{a}, \mathbf{b} \rangle .$$

The last inequality is known as the Cauchy-Schwartz-Bunyakovsky inequality and can be proved as follows. Consider the second degree polynomial of the variable λ defined by the equality

$$p(\lambda) = (\mathbf{a} + \lambda\mathbf{b})^2 = \lambda^2\|\mathbf{b}\|^2 + \lambda\, 2\langle \mathbf{a}, \mathbf{b} \rangle + \|\mathbf{a}\|^2 .$$

Since we have $p(\lambda) \geqslant 0$ for all λ, the discriminant of p must be non-positive, i.e.

$$0 \geqslant 4\langle \mathbf{a}, \mathbf{b}\rangle^2 - 4\,\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2,$$

and this is just what we wanted to prove. ∎

Remark. If X is a set and d is a function defined on the direct product X x X, such that conditions i), ii) and iii) are satisfied, then d is called a metric on X and the pair (X , d) is called a metric space.

THE TOPOLOGY OF $\mathbb{R}^n$

Definition. The open ball in $\mathbb{R}^n$ with center $\mathbf{x} \in \mathbb{R}^n$ and radius $\varepsilon > 0$ (or an ε-ball centered at $\mathbf{x}$) is the set $B_\varepsilon(\mathbf{x}) = \{ \mathbf{y} \in \mathbb{R}^n : d(\mathbf{x},\mathbf{y}) < \varepsilon \}$. Similarly, the closed ball in $\mathbb{R}^n$ with center $\mathbf{x} \in \mathbb{R}^n$ and radius $\varepsilon > 0$ (or a closed ε-ball centered at $\mathbf{x}$) is defined as the set $\overline{B}_\varepsilon(\mathbf{x})=\{\mathbf{y} \in \mathbb{R}^n: d(\mathbf{x},\mathbf{y}) \leqslant \varepsilon\}$.

Definition. A subset U of $\mathbb{R}^n$ is called open if for each $\mathbf{x} \in U$ there is a positive ε such that the ball $B_\varepsilon(\mathbf{x})$ is contained in U.

Definition A pair (X,τ) is said to be a topological space if X is a set, and τ is a family of subsets of X, called the open subsets of X, such that

i) the empty set $\emptyset$ and X are open;

ii) the intersection of two open subsets is also open;

iii) the union of an arbitrary family of open subsets is open.

The family τ of open subsets is called the topology on X.

Examples.

a) Let X be an arbitrary set. The discrete topology on X is the "maximal topology" on X, in which every subset is open.

b) The antidiscrete topology on X is the "minimal topology" on X, in which only the empty set and X are open.

c) The standard topology of $\mathbb{R}^n$.

Proposition. The family of open subsets defines a topology on $\mathbb{R}^n$. This topology is referred to as the standard topology on $\mathbb{R}^n$.

Proof. Obviously, the empty set and $\mathbb{R}^n$ are open. If U and V are open subsets, and $\mathbf{x}$ is a common point of theirs, then there exist positive numbers ε_1, ε_2 such that $B_{\varepsilon_1}(\mathbf{x}) \subset U$ and $B_{\varepsilon_2}(\mathbf{x}) \subset V$. Let ε be the minimum of ε_1 and ε_2. Then $B_\varepsilon(\mathbf{x}) \subset U \cap V$ showing that the intersection $U \cap V$ is open. Finally, let $\{U_i : i \in I\}$ be an arbitrary family of open sets, and $\mathbf{x}$ be an element of their union. Then we can find an index $j \in I$ and a positive ε such that $B_\varepsilon(\mathbf{x}) \subset U_j \subset \bigcup_{i\in I} U_i$, thus the union $\bigcup_{i\in I} U_i$ is open. ∎

d) The topology of a topological space defines a topology on every of its subsets by the following construction.

Proposition. Let Y be a subset of a topological space (X, τ). Then the family $\tau|_Y = \{ U \cap Y : U \in \tau \}$ defines a topology on Y.

The proof of the proposition is straightforward from the following identities.

i) $\emptyset \cap Y = \emptyset, \qquad X \cap Y = Y$;

ii) $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y$;

iii) $\bigcup_{i \in I} (U_i \cap Y) = (\bigcup_{i \in I} U_i) \cap Y.$ ∎

Definition. The topology $\tau|_Y$ is called the subspace topology or the topology induced on Y by τ.

Definition. A subset Y of a topological space (X, τ) is said to be closed if X \ Y is open.

Warning. Most of the subsets of $\mathbb{R}^n$ are neither open nor closed.

Definition. We say that a subset U of X is a neighborhood of a point $x \in X$, if there is an open subset V such that $x \in V \subset U$.

Definition. Let (X,τ) and (Y,τ') be topological spaces. A mapping $f: X \longrightarrow Y$ is said to be continuous at the point $x \in X$ (with respect to the given topologies) if for each neighborhood U of f(x) $f^{-1}(U)$ is a neighborhood of x. The mapping f is continuous (with respect to the given topologies), if it is continuous at each point or equivalently if for each $U \in \tau'$ we have $f^{-1}(U) \in \tau$.

Exercise. Show that for $\mathbb{R}^n$ the above definition is equivalent to the "ε-δ" definition of continuity at a point: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous at $\mathbf{x} \in \mathbb{R}^n$ with respect to the standard topologies if and only if for any $\varepsilon > 0$ one can find a positive δ such that $|\mathbf{x}-\mathbf{x}'| < \delta$ implies $|f(\mathbf{x})-f(\mathbf{x}')| < \varepsilon$.

The map f is a homeomorphism, if it is a bijection (= one-to-one and onto) such that both f and f^{-1} are continuous.

We say that two topological spaces are homeomorphic or have the same topological type, if there is a homeomorphism between them.

Homeomorphic topological spaces are considered to be the same from the viewpoint of topology. Intuitively, two homeomorphic spaces are homeomorphic if a rubber model of one of them can be transformed into that of the other. We are allowed to stretch and shrink the model but not allowed to cut the model or glue pieces together. More exactly, we may cut the model somewhere only if later on we glue together the parts we get in the same way as they were joined. Of course, this description of homeomorphism is applicable only for "nice spaces" such as surfaces, curves etc. and by no means substitutes the precise definition.

For example, the circle, the perimeter of a square, and the trefoil knot are homeomorphic, so are a solid disc and a solid square, however a circle is not homeomorphic to a solid disc. Generally it is easy to show that two homeomorphic spaces are indeed homeomorphic: we only have to present a homeomorphism. However, to show that two topological spaces are not homeomorphic, we have to find a topological property, which is had by one of the spaces but is not by the other. For example, the fact that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $n \neq m$, is a deep theorem of topology (the "dimension invariance theorem"), the proof of which requires sophisticated techniques.

CURVES IN $\mathbb{R}^n$

Definition. A simple arc in a topological space is a subset Γ homeomorphic to a closed interval $[a,b]$ of $\mathbb{R}$. A parameterization of a simple arc is a homeomorphism $\gamma : [a,b] \longrightarrow \Gamma$.

Definition. A path or a continuous (parameterized) curve in a topological space is a continuous mapping of an interval $[a,b]$ into the space. The images of a and b are the initial and terminal points of the curve respectively. The path is said to connect the initial point to the terminal point.

As we see, there are two different approaches to the concept of curve. According to the intuitive-geometrical approach, curves are *topological spaces* or *subsets* of a topological space. The second approach however, which will be more suitable for our purposes, introduces curves as trajectories of a moving point. This view is reflected in the definition of a continuous curve. We stress that according to this definition, a curve is a *mapping* and not a set of points.

Remark. The image of a continuous curve $\mathbf{x} : [a,b] \longrightarrow \mathbb{R}^n$ as a point set may not look like a curve. The Italian mathematician Peano (1858-1932) constructed a continuous curve that passes through each point of a square. Such a pathology can not occur if we restrict ourselves to smooth curves.

Definition. A smooth (parameterized) curve in $\mathbb{R}^n$ is a smooth map $\mathbf{x} : [a,b] \longrightarrow \mathbb{R}^n$ from a closed interval $[a,b]$ into $\mathbb{R}^n$.

Recall that a map $\mathbf{x} : U \longrightarrow \mathbb{R}^n$ defined on an open subset of $\mathbb{R}^n$ is said to be smooth or infinitely many times differentiable if the coordinate functions $x_1, x_2, \dots, x_n$ of $\mathbf{x} = (x_1, x_2, \dots x_n)$ have continuous partial derivatives of any order.

We say that a map $\mathbf{x} : A \longrightarrow \mathbb{R}^n$ defined on an *arbitrary* set $A \subset \mathbb{R}^n$ is smooth or infinitely many times differentiable if there exists an open set $U \subset \mathbb{R}^n$ and a smooth mapping $\tilde{\mathbf{x}} : U \longrightarrow \mathbb{R}^n$ such that $A \subset U$ and $\mathbf{x} = \tilde{\mathbf{x}}|_A$.

Definition. We say that the continuous curve $\mathbf{x}_1 : [a,b] \longrightarrow \mathbb{R}^n$ is obtained from the curve $\mathbf{x}_2 : [c,d] \longrightarrow \mathbb{R}^n$ by a reparameterization if there is a homeomorphism $\varphi : [a,b] \longrightarrow [c,d]$ such that $\varphi(a) = c$, $\varphi(b) = d$ and $\mathbf{x}_1 = \mathbf{x}_2 \circ \varphi$. A reparameterization is called regular if φ is smooth and $\varphi' > 0$.

Intuitively, reparameterization means that without changing the trajectory of a point we change the velocity, with which it moves along the trajectory.

Definition. The length of a continuous curve $\mathbf{x} : [a,b] \longrightarrow \mathbb{R}^n$ is the limit of the length of inscribed broken lines with vertices $\mathbf{x}(t_0), \mathbf{x}(t_1), \dots, \mathbf{x}(t_N)$ where $a = t_0 < t_1 < \dots < t_N = b$ and the limit is taken as $\max\limits_{1 \leqslant i \leqslant N} |t_i - t_{i-1}|$ tends to zero. Provided that this limit exists (it does not have to), the curve is called rectifiable.

The following theorem yields a formula that can be used in practice to compute the length of curves.

Theorem. A smooth curve $\mathbf{x} : [a,b] \longrightarrow \mathbb{R}^n$ is always rectifiable and its length is determined by the integral

$$\ell(\mathbf{x}) = \int_a^b \| \mathbf{x}'(t) \| \, dt.$$

Proof. Denoting by $x_1, x_2, \dots, x_n$ the coordinate functions of $\mathbf{x}$ the length of the broken line considered in the definition of the length of a curve is equal to

$$\lambda = \sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} (x_j(t_i) - x_j(t_{i-1}))^2}$$

By Lagrange's mean value theorem we can find real numbers ξ_{ij} such that

$$x_j(t_i) - x_j(t_{i-1}) = x_j'(\xi_{ij})\,(t_i - t_{i-1}), \quad t_{i-1} < \xi_{ij} < t_i.$$

Using these equalities we get

$$\lambda = \sum_{i=1}^{N} \left(\sqrt{\sum_{j=1}^{n} x_j'(\xi_{ij})^2} \right) (t_i - t_{i-1}).$$

Lemma. A continuous function $f : [a,b] \longrightarrow \mathbb{R}$ on a closed interval is uniformly continuous (i.e. for each positive ε one can find a positive δ such that $x, y \in [a,b]$, $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \varepsilon.$$

Proof of the lemma. Suppose to the contrary, that there is an $\varepsilon > 0$ for which we can not find a suitable δ. Then there exists a sequence of pairs of real numbers x_n, y_n such that $x_n, y_n \in [a,b]$, $|x_n - y_n| < 1/n$ but $|f(x_n)-f(y_n)| > \varepsilon$. By compactness of $[a,b]$, we can select a convergent subsequence $x_{i_n} \longrightarrow x$ of the sequence (x_n). Condition $|x_n - y_n| < 1/n$ ensures that $y_{i_n} \longrightarrow x$ as well and so, by the continuity of f at x we have $|f(x_{i_n})-f(y_{i_n})| \longrightarrow 0$. But this contradicts the condition $|f(x_n)-f(y_n)|>\varepsilon$. ▮

Let us return to the proof of the theorem. Fix a positive ε. By the lemma, we can find a positive δ such that $t,t' \in [a,b]$ and $|t - t'| < \delta$ imply $|x_j'(t) - x_j'(t')| < \varepsilon$ for all j.

Suppose that the approximating broken line is fine enough in the sense that $|t_i - t_{i-1}| < \delta$ for all i. Then we have by the triangle inequality

$$\left| \sqrt{\sum_{j=1}^{n} x_j'(\xi_{ij})^2} - \sqrt{\sum_{j=1}^{n} x_j'(t_i)^2} \right| \leqslant$$

$$\leqslant \sqrt{\sum_{j=1}^{n} (x_j'(\xi_{ij})-x_j'(t_i))^2} \leqslant \varepsilon\sqrt{n}.$$

Making use of this estimation we see that

$$\left| \lambda - \sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} x_j'(t_i)^2}\,(t_i-t_{i-1}) \right| \leqslant \varepsilon\sqrt{n}\,(b - a) \qquad (*)$$

In this formula

$$\sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} x_j'(t_i)^2}\,(t_i-t_{i-1}) = \sum_{i=1}^{N} \|\mathbf{x}'(t_i)\|\,(t_i-t_{i-1})$$

is just an integral sum which converges to the integral $\int_a^b \|\mathbf{x}'(t)\|\,dt$ when $\max_i |t_i - t_{i-1}|$ tends to zero. In this case the inequality $(*)$ guarantees that the length λ of the inscribed broken lines also tends to this integral. ▮

Let us consider the function $s\colon [a,b]\longrightarrow [0,\ell(\mathbf{x})]$

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\,d\tau ,$$

where $\mathbf{x}\colon [a,b]\longrightarrow \mathbb{R}^n$ is a smooth curve. $s(t)$ is the length of the arc of the curve between $\mathbf{x}(a)$ to $\mathbf{x}(t)$ and is not a strictly monotonous function of t in general. This fact motivates the following definition.

Definition. A smooth curve is said to be regular if $\mathbf{x}'(t) \neq 0$ for all t.

If $\mathbf{x}$ is a regular curve then s defines a regular reparameterization of $\mathbf{x}$. The map $\mathbf{x}\circ s^{-1}\colon [0,\ell(\mathbf{x})]\longrightarrow \mathbb{R}^n$ is referred to as the natural or unit speed parameterization of the curve **x** or as the parameterization of **x** by arc length. The second name is justified by the fact that the speed vector

$$(\mathbf{x}\circ s^{-1})'(t) = \mathbf{x}'(s^{-1}(t))\,(s^{-1})'(t) = \mathbf{x}'(s^{-1}(t))\frac{1}{s'(s^{-1}(t))} = \frac{\mathbf{x}'(s^{-1}(t))}{\|\mathbf{x}'(s^{-1}(t))\|}$$

of this parameterization has unit module at each point.

Further Exercises

1-1. Show that $\mathbb{R}^n$ and the open balls in $\mathbb{R}^n$ (with the subspace topology) are homeomorphic.

1-2. Show that the "punctured sphere" $S^2-\{p\}$, where

$$S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\} \quad \text{and } p \in S^2$$

furnished with the subspace topology is homeomorphic to the plane $\mathbb{R}^2$.

1-3. Using the intuitive description of homeomorphism classify (without proof) the capital letters of the alphabet up to homeomorphism. (The answer depends on the font type you choose, so choose your favorite font.)

1-4. The curve cycloid is the trajectory of a peripheral point of a circle that rolls along a straight line. Find a parameterization of the cycloid and compute the length of one of its arcs.

1-5. Find the natural parameterization of the helix

$$\gamma(t) = (a \cos t,\ a \sin t,\ b\ t).$$