Unit 10. The Lie Algebra of Vector Fields

Vector fields and ordinary differential equations; basic results of the theory of ordinary differential equations (without proof); the Lie algebra of vector fields and the geometric meaning of Lie bracket, commuting vector fields, Lie algebra of a Lie group.

Definition. A smooth vector field X over a differentiable manifold M is a smooth mapping of M into its tangent bundle, such that $X(p) \in T_pM$ for each $p \in M$.

Obviously, smooth vector fields over M form a real vector space with respect to the operations

$$(X + Y)(p) := X(p) + Y(p), \quad (\lambda X)(p) := \lambda X(p),$$

where X, Y are vector fields, $\lambda \in \mathbb{R}$, $p \in M$. We can multiply vector fields by smooth functions as well, by the rule

$$(f X)(p) := f(p) X(p).$$

We denote by $\mathfrak{X}(M)$ the vector space of smooth vector fields.

Associated to a local coordinate system $\mathbf{x} = (x_1,\ldots,x_n)$ on M, there is a basis of T_pM at each $p \in \operatorname{dom} \mathbf{x}$, formed by the tangent vectors $\left\{\partial_i(p)\colon 1\leqslant i\leqslant n\right\}$. The mapping $\partial_i : p \longmapsto \partial_i(p)$ gives a local smooth vector field in the domain of the chart for each i. Thus, every smooth vector field X can be written in the form

$$X = \sum_{i=1}^{n} X_i \, \partial_i,$$

where the X_i-s are smooth functions on the domain of $\mathbf{x}$. The functions X_i are called the components of the vector field X.

Given a smooth vector field X on a manifold M, we may pose the following problem. Find those smooth curves $\gamma : (a,b) \longrightarrow M$ on M for which the speed of γ at $t \in (a,b)$ is $X(\gamma(t))$. Such curves are called the integral curves of the vector field. Obviously, a restriction of an integral curve onto a subinterval is also an integral curve, therefore, it is enough to look for the maximal integral curves which can not be extended to an integral curve defined on a larger interval.

Trying to solve the problem, we find that it reduces to an ordinary differential equation of first order. Indeed, γ is an integral curve if and only if for each chart $\mathbf{x} = (x_1,\ldots,x_n)$, the "vector-valued" function

$$\mathbf{f} = \mathbf{x}\circ\gamma : (a,b) \longrightarrow \mathbb{R}^n$$

satisfies the differential equation

$$\mathbf{f}'(t) = (X_1\circ\mathbf{x}^{-1}\circ\mathbf{f}(t),\ldots,X_n\circ\mathbf{x}^{-1}\circ\mathbf{f}(t)).$$

Actually, finding integral curves of a vector field is the same problem as solving an ordinary differential equation, only the language of formulation is different. Translating the basic results of the theory of ordinary differential equations into the language of geometry we get the following theorems, we mention without proof.

Theorem.

i) (Existence and uniqueness of solutions). Let X be a smooth vector field on a differentiable manifold M. Then for each point $p \in M$ there exists a unique maximal integral curve γ_p: $(a,b) \longrightarrow M$ of the vector field X such that $0 \in (a,b)$ and $\gamma_p(0) = p$ (a and b might be $-\infty$ and ∞ respectively).

ii) (straightening vector fields). Let X be an arbitrary vector field on a manifold M and $p \in M$ such that $X(p) \neq 0$. Then there exists a chart $\mathbf{x} = (x_1,\ldots,x_n)$ around p for which $X = \partial_1$. This means that the derivative of the mapping $\mathbf{x}$ turns the vector field X into a constant vector field on $\mathbb{R}^n$.

iii) (Unboundedness of solutions). If a (or b) finite then no compact subset of M contains the image $\gamma_p((a,0))$ (or $\gamma_p((0,b))$).

iv) (differentiable dependence on the initial point). Let us define the set $U_t \subset M$ for $t \in \mathbb{R}$ as follows

$$U_t = \{p \in M : t \in \operatorname{dom} \gamma_p\}.$$

Then U_t is an open subset of M and the mapping $H_t : U_t \longrightarrow M$ defined by $H_t(p) = \gamma_p(t)$ is a diffeomorphism between U_t and U_{-t}. If, furthermore, the expression $H_{t_1}(H_{t_2}(p))$ is defined, then so is $H_{t_1+t_2}(p)$ and $H_{t_1}(H_{t_2}(p)) = H_{t_1+t_2}(p)$. The family $\{H_t : t \in \mathbb{R}\}$ is called the one-parameter family of diffeomorphisms or the flow generated by the vector field.

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Since tangent vectors to a manifold at a point are identified with derivations at the point, vector fields can be considered to be differential operators assigning to a smooth function another smooth function by the formula

$$[X(f)](p) = [X(p)](f), \text{ where } X \in \mathfrak{X}(M),\ f \in \mathcal{F}(M),\ p \in M.$$

In this sense, a vector field X is *a linear mapping* $X:\mathcal{F}(M)\longrightarrow \mathcal{F}(M)$, *satisfying the "Leibniz' rule"*

$$X(fg) = X(f)g + fX(g).$$

Definition. Let A and B be two linear endomorphisms of a vector space V. Then the linear mapping $[A,B] = A \circ B - B \circ A$ is called the commutator of them.

Proposition. The commutator of linear mappings satisfies the following identities ($A,B,C \in \mathcal{E}nd((V)$, $\lambda \in \mathbb{R}$)

i) $[A+B,C] = [A,C]+[B,C]$ $\quad [C,A+B] = [C,A]+[C,B]$

$[\lambda A,B] = [A,\lambda B] = \lambda[A,B]$ (bilinearity)

ii) $[A,B] = -[B,A]$ (anti-commutation)

iii) $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]= 0$ (Jacobi identity)

Proof. We prove only iii), the rest is left to the reader.

$$\begin{aligned}
&[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = \\
&= [A,(B\circ C-C\circ B)]+[B,(C\circ A-A\circ C)]+[C,(A\circ B-B\circ A)] = \\
&= A\circ(B\circ C-C\circ B)-(B\circ C-C\circ B)\circ A + B\circ(C\circ A-A\circ C) - \\
&\quad -(C\circ A-A\circ C)\circ B + C\circ(A\circ B-B\circ A)-(A\circ B-B\circ A)\circ C = \\
&= A\circ B\circ C - A\circ C\circ B - B\circ C\circ A + C\circ B\circ A + \\
&\quad +B\circ C\circ A - B\circ A\circ C - C\circ A\circ B + A\circ C\circ B + \\
&\quad +C\circ A\circ B - C\circ B\circ A - A\circ B\circ C + B\circ A\circ C = \\
&= 0 \ \blacksquare
\end{aligned}$$

Definition. Let us suppose that a linear space L is endowed with a bilinear mapping $[\ ,\ ]:L\times L\longrightarrow L$ satisfying conditions i), ii), and iii) of the above proposition. Then the pair $(L, [,])$ is called a Lie algebra.

Proposition. Let X and Y be two smooth vector fields on a manifold M. Considering them to be linear endomorphisms of the vector space of smooth functions $\mathcal{F}(M)$, the commutator $[X,Y]$ of them is also a vector field.

Proof. The commutator $[X,Y]$ is a linear endomorphism of $\mathcal{F}(M)$ so we only have to check that it satisfies the Leibniz' rule. For $f,g \in \mathcal{F}(M)$ we have

$$[X,Y](fg) = (X\circ Y-Y\circ X)(fg) = X(Y(fg))-Y(X(fg)) =$$

$$= X\left(Y(f)g + fY(g) \right) + Y\left(X(f)g + fX(g) \right)$$

$= X\circ Y(f)g + Y(f)X(g) + X(f)Y(g) + fX\circ Y(g) - Y\circ X(f)g - X(f)Y(g) -$

$- Y(f)X(g) - fY\circ X(g) =$

$= [X,Y](f)g + f[X,Y](g)$. ∎

Corollary. The commutator of vector fields, which is generally called the Lie bracket of them, is a binary operation on $\mathfrak{X}(M)$, giving the space of vector fields a Lie algebra structure.

Proposition. Let us choose a local coordinate system $(x_1,\dots,x_n)$ on M and denote by $\partial_1,\dots,\partial_n$ the associated coordinate vector fields. Then we have

i) $[\partial_i,\partial_j] = 0$;

ii) $[fX,gY] = fg[X,Y] + fX(g)Y - gY(f)X$ for each $X,Y \in \mathfrak{X}(M)$, $f,g \in \mathcal{F}(M)$;

iii) if $X = \sum_{i=1}^{n} f_i\partial_i$, $Y = \sum_{i=1}^{n} g_i\partial_i$ are arbitrary vector fields,

then

$$[X,Y] = \sum_{i=1}^{n} (X(g_i)-Y(f_i))\partial_i = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j}\Big)\partial_i .$$

Proof. i) The first part of the proposition is equivalent to Young's theorem, (known from multivariable calculus), which says that for any smooth function, defined on an open subset of $\mathbb{R}^n$, the mixed partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are equal.

ii) Let h be an arbitrary smooth function on M, and apply the operator $[fX,gY]$ to it.

$$[fX,gY](h) = fX(g\,Y(h))-gY(fX(h)) = fX(g)Y(h)+fgX(Y(h))- gY(f)X(h)- gfY(X(h))$$
$$= \Big(fg[X,Y] + fX(g)Y - gY(f)X\Big)(h).$$

iii) Using i) and ii) we get

$$[X,Y] = \left[\sum_{i=1}^{n} f_i\partial_i \; , \; \sum_{j=1}^{n} g_j\partial_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} [f_i\partial_i, g_j\partial_j] =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f_i\partial_i(g_j)\partial_j - g_j\partial_j(f_i)\partial_i = \sum_{i=1}^{n} (X(g_i) - Y(f_i))\partial_i . \ ∎$$

Suppose that we are given two vector fields X and Y on an open subset of $\mathbb{R}^n$. The corresponding flows H_s and G_t do not commute in general: $H_s\circ G_t \neq G_t\circ H_s$.

To measure the lack of commutation of the flows H_s and G_t, we consider the difference

$$\Phi(s,t;p) = G_t\circ H_s(p) - H_s\circ G_t(p), \text{ for a fixed point } p.$$

Φ is a differentiable function of s and t and it is **0** if t or s is zero. This means, that in the Taylor expansion of Φ around $(0,0;p)$

$$\Phi(s,t;p) = \Phi(0,0;p)+\left(s\frac{\partial\Phi}{\partial s}(0,0;p) + t\frac{\partial\Phi}{\partial t}(0,0;p)\right) +$$
$$+\left(\frac{s^2}{2}\frac{\partial^2\Phi}{\partial s^2}(0,0;p) + st\frac{\partial^2\Phi}{\partial s\partial t}(0,0;p) + \frac{t^2}{2}\frac{\partial^2\Phi}{\partial t^2}(0,0;p)\right) + o(s^2+t^2)$$

the only non-zero partial derivative is $\frac{\partial^2\Phi}{\partial s\partial t}(0,0;p)$.

Claim. Through the natural identification of the tangent space of $\mathbb{R}^n$ at p with the vectors of $\mathbb{R}^n$, the vector $\frac{\partial^2\Phi}{\partial s\partial t}(0,0;p)$ corresponds to the tangent vector $[X,Y](p)$.

Proof. Put $X = \sum_{i=1}^{n} f_i\partial_i$, $Y = \sum_{i=1}^{n} g_i\partial_i$, where ∂_i denotes the vector field $\frac{\partial}{\partial x_i}$. Let us compute first the vector $\frac{\partial^2}{\partial s\partial t} H_s\circ G_t(p)$ at $s = t = 0$.

We have $\left(\frac{\partial}{\partial s} H_s\circ G_t(p)\right)(0,t) = \left(\frac{\partial}{\partial s}\gamma_{G_t(p)}(s)\right)(0,t) = X(G_t(p))$.

Differentiating by t,

$$\frac{\partial^2}{\partial s\partial t} H_s\circ G_t(p)\Big|_{s=t=0} = \frac{d}{dt}X(G_t(p))\Big|_{t=0} = \sum_{i=1}^{n}\left(Y(f_i)\partial_i\right)(p).$$

A similar computation shows that

$$\frac{\partial^2}{\partial s\partial t} G_t\circ H_s(p)\Big|_{s=t=0} = \sum_{i=1}^{n}\left(X(g_i)\partial_i\right)(p).$$

Subtracting these equalities we get

$$\frac{\partial^2\Phi}{\partial s\partial t}(0,0;p) = \left(\sum_{i=1}^{n}(X(g_i) - Y(f_i))\partial_i\right)(p) = [X,Y](p). \blacksquare$$

Now returning to the Taylor expansion of Φ, we see that

$$\Phi(s,t;p) = st\frac{\partial^2\Phi}{\partial s\partial t}(0,0;p) + o(s^2+t^2) = st[X,Y](p) + o(s^2+t^2)$$

In particular, we obtain the following expression for $[X,Y](p)$.

$$\boxed{[X,Y](p) = \lim_{t\to 0}(G_t\circ H_t(p) - H_t\circ G_t(p))/t^2.}$$

Definition. We say that two vector fields are commuting if their Lie bracket is the zero vector field.

Theorem. Let $\{H_t : t \in \mathbb{R}\}$ and $\{G_t : t \in \mathbb{R}\}$ be the one-parameter families of diffeomorphisms generated by the vector fields X and Y respectively and suppose that the vector fields X and Y are commuting. Then the diffeomorphisms H_s and G_t are commuting as well in the following sense.

For each point p of the manifold there exists a positive ε (depending on p) such that for any pair of real numbers s,t satisfying the inequality $|s| + |t| \leqslant \varepsilon$ the expressions $H_s(G_t(p))$ and $G_t(H_s(p))$ are defined and coincide: $H_s(G_t(p)) = G_t(H_s(p))$.

Proof. If both X and Y vanishes at p then $H_s(p) = G_t(p) = p$ for any s and t and thus the assertion holds trivially. We may thus suppose that one of the vectors X(p), Y(p), say X(p) is not zero. By the theorem on straightening vector fields we may suppose that the manifold is an open subset of $\mathbb{R}^n$, with coordinates $(x_1,\dots,x_n)$, and the vector field X coincides with the basis vector field ∂_1 .Let $Y = \sum_{i=1}^{n} g_i\partial_i$ be the splitting of Y into a linear combination of the basis vector fields ∂_i. By the formula for the Lie bracket of vector fields we have

$$0 = [X,Y] = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_1}\partial_i \implies \frac{\partial g_i}{\partial x_1} = 0 \quad \text{for each } i.$$

Consequently, the functions g_i do not depend on x_1, thus the vector field Y is invariant under translations parallel to the vector $\mathbf{e}_1=(1,0,\dots,0)$. This implies that if γ is an integral curve of the vector field Y then so is $\gamma + t\mathbf{e}_1$ for any t (the domain of $\gamma + t\mathbf{e}_1$ is an open subset of the domain of γ). On the other hand, the diffeomorphism H_t is just a translation by the vector $t\mathbf{e}_1$. So, for small s and t, we have

$$G_t(H_s(p)) = G_t(p + s\mathbf{e}_1) = \gamma_{p + s\mathbf{e}_1}(t) = (\gamma_p + s\mathbf{e}_1)(t) = \gamma_p(t) + s\mathbf{e}_1 =$$

$$= G_t(p) + s\mathbf{e}_1 = H_s(G_t(p)). \blacksquare$$

THE LIE ALGEBRA OF A LIE GROUP

Let $F:M \longrightarrow N$ be a diffeomorphism between two manifolds. F' defines a bijection between $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$, and this bijection is a Lie algebra isomorphism.

Let G be a Lie group, $g \in G$. Denote by L_g the left translation by g, i.e., $L_g:G \longrightarrow G$, $L_g(h) = gh$. L_g is a diffeomorphism, its inverse is $L_{(g^{-1})}$. A vector field $X\in\mathfrak{X}(G)$ is called left invariant if $L_g'(X) = X$ for all $g \in G$. Since $L_g'(X) = X$ and $L_g'(Y) = Y$ implies $L_g'[X,Y] = [L_g'(X),L_g'(Y)] = [X,Y]$, left invariant vector fields form a Lie subalgebra of $\mathfrak{X}(G)$.

Definition. The Lie algebra of left invariant vector fields of a Lie group is called the Lie algebra of the Lie group.

If $X\in\mathfrak{X}(G)$ is left invariant, then $X(g)=L_g'(X(e))$, thus, a left invariant vector field is uniquely determined by the vector $X(e) \in T_eG$. (e is the unit element of the group G.) Since every vector in T_eG extends to a left invariant vector field this way, the assignment $X\longmapsto X(e)$ yields a linear isomorphism between the vector space of left invariant vector fields on G and T_eG. As a consequence, we obtain that *the Lie algebra of a Lie group is finite dimensional, its dimension is the same as that of the Lie group.*

As an example, let us determine the Lie algebra of $Gl(n,\mathbb{R})$. $Gl(n,\mathbb{R})$ is an open subset in $Mat(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, so its manifold structure is given by one chart, the embedding. Tangent spaces at different points can be identified with the linear space $Mat(n,\mathbb{R})$. For $A \in Gl(n,\mathbb{R})$, the left translation $M\longmapsto AM$ extends to a linear transformation of the whole linear space $Mat(n,\mathbb{R})$. The derivative of a linear transformation of a linear space is the linear transformation itself, if we identify the tangent spaces at different points with the linear space, so a left invariant vector field $X:Gl(n,\mathbb{R})\longrightarrow Mat(n,\mathbb{R})$ has the form $X(A) = AM$, where $M \in Mat(n,\mathbb{R})$ is a fixed matrix.

The integral curves of a left invariant vector field on $Gl(n,\mathbb{R})$ can be described with the help of the exponential function for matrices. If M is an arbitrary square matrix, then we define e^M as the sum

$$\sum_{n=0}^{\infty} \frac{M^n}{n!} .$$

If we define the curve $\gamma_A:\mathbb{R}\longrightarrow Gl(n,\mathbb{R})$ by

$$\gamma_A(t) = A\, e^{Mt},$$

then we obtain an integral curve of the vector field $X(A) = AM$. Indeed,

$$\gamma_A'(t) = Ae^{Mt}M = X(Ae^{Mt}) = X(\gamma_A(t)).$$

The flow generated by the left invariant vector field X consists of the diffeomorphisms

$$H_t(A) = Ae^{Mt},$$

that is, H_t is a right translation by e^{Mt}.

Now let us take two left invariant vector fields $X(A)=AM$ and $Y(A)=AN$ and consider the flows H_t and G_t generated by them.

Computing $G_t \circ H_t(A) - H_t \circ G_t(A)$ up to $o(t^2)$, we get

$$G_t \circ H_t(A) - H_t \circ G_t(A) = A(e^{Mt}e^{Nt} - e^{Nt}e^{Mt}) = A(I+Mt+\frac{1}{2}(Mt)^2)(I+Nt+\frac{1}{2}(Nt)^2) - A(I+Nt+\frac{1}{2}(Nt)^2)(I+Mt+\frac{1}{2}(Mt)^2) + o(t^2) = A(MN-NM)t^2 + o(t^2).$$

We obtain, that *the Lie algebra of* $Gl(n,\mathbb{R})$ *is isomorphic to the Lie algebra of all matrices with Lie bracket* $[M,N] = MN-NM$.

Further Exercises

Exercise 10-1. Let ∂_1 and ∂_2 be the two coordinate vector fields on $\mathbb{R}^2$ determined by the identity mapping. Describe the vector fields

$$X(x_1,x_2) = x_1\partial_1 + x_2\partial_2$$
$$Y(x_1,x_2) = x_2\partial_1 - x_1\partial_2,$$

compute their Lie bracket, and determine the flows generated by them.

Exercise 10-2. Show that the Lie algebra of $SO(n)$ is isomorphic to the Lie algebra of skew-symmetric nxn matrices with Lie bracket $[X,Y]=XY-YX$.

Exercise 10-3. Show that $\mathbb{R}^3$ endowed with the cross-product x is a 3-dimensional Lie algebra isomorphic to the Lie algebra of $SO(3)$.

Exercise 10-4. For $\mathbf{v} \in \mathbb{R}^3$, let $X_{\mathbf{v}}$ denote the vector field on $\mathbb{R}^3$, defined by

$$X_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \times \mathbf{x}.$$

Describe the flow generated by $X_{\mathbf{v}}$, and prove that

$$[X_{\mathbf{v}}, X_{\mathbf{w}}] = - X_{\mathbf{v}\times\mathbf{w}}.$$

Exercise 10-5. Show that the Lie algebra of left invariant vector fields on a Lie group is isomorphic to the Lie algebra of right invariant vector fields.