Unit 11. Differentiation of Vector Fields

Affine connection at a point, global affine connection, Christoffel symbols, covariant derivation of vector fields along a curve, parallel vector fields and parallel translation, symmetric connections, Riemannian manifolds, compatibility with a Riemannian metric, the fundamental theorem of Riemannian geometry, Levi-Civita connection.

Although there is a natural way to differentiate a smooth function defined on a manifold with respect to a tangent vector, there is no natural way to differentiate vector fields. In fact, there are lot of possible rules for differentiating vector fields with respect to a tangent vector, and to choose one of them, (the most appropriate one), the differentiable manifold structure alone is not enough. A fixed rule for the differentiation of vector fields is itself an additional structure on the manifold, called an affine connection. Later we shall see that on Riemannian manifolds i.e. on manifolds the tangent spaces of which are equipped with a dot product we can introduce differentiation of vector fields in a natural way. A precise formulation of this statement is the "fundamental theorem of Riemannian geometry".

As far as only vector fields on an open domain of $\mathbb{R}^n$ are considered, the following definition seems to be quite natural.

Definition. The derivative of a smooth vector field X on an open subset U in $\mathbb{R}^n$ with respect to a tangent vector $Y \in T_p\mathbb{R}^n$ is defined by

$$\nabla_Y X = (X\circ\gamma)'(0) ,$$

where $\gamma : [-\varepsilon.\varepsilon] \longrightarrow U$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = Y$. We see that

$$\nabla_Y X = \sum_{i=1}^{n} Y(X_i)\, \partial_i(p) , \qquad (*)$$

where ∂_i denotes the i-th coordinate vector field on $\mathbb{R}^n$, X_i are the components of the vector field X. In particular, the value of $\nabla_Y X$ does not depend on the choice of γ.

It is easy to check that differentiation of vector fields has the following properties.

$$\nabla_{(Y_1+Y_2)}X = \nabla_{Y_1}X + \nabla_{Y_2}X \tag{1}$$

$$\nabla_{cY}X = c\,\nabla_Y X \tag{2}$$

$$\nabla_Y(X_1+X_2) = \nabla_Y X_1 + \nabla_Y X_2 \tag{3}$$

$$\nabla_Y(f\,X) = Y(f)\,X + f\,\nabla_Y X \tag{4}$$

$$\nabla_{X_1}X_2 - \nabla_{X_2}X_1 = [X_1, X_2] \qquad \text{(torsion free property)} \tag{5}$$

$$Y(\langle X_1, X_2\rangle) = \langle \nabla_Y X_1, X_2\rangle + \langle X_1, \nabla_Y X_2\rangle \qquad \text{(agreement with the metric)} \tag{6}$$

where $X_1, X_2 \in \mathfrak{X}(\mathbb{R}^n)$, $Y \in T_p\mathbb{R}^n$, $f \in \mathcal{F}(\mathbb{R}^n)$, $c \in \mathbb{R}$.

Now we shall study the general case. Let M be a smooth manifold. As we mentioned, there is no natural rule for derivation of vector fields on M, so we introduce such rules axiomatically, as operations satisfying some of the properties (1-6).

Definition. An affine connection at a point $p \in M$ is a mapping which assigns to each tangent vector $Y \in T_pM$ and each vector field $X \in \mathfrak{X}(M)$ a new tangent vector $\nabla_Y X \in T_pM$ called the covariant derivative of X with respect to Y and satisfies the following identities

$$\nabla_{(Y_1+Y_2)}X = \nabla_{Y_1}X + \nabla_{Y_2}X \tag{1}$$

$$\nabla_{cY}X = c\,\nabla_Y X \tag{2}$$

$$\nabla_Y(X_1+X_2) = \nabla_Y X_1 + \nabla_Y X_2 \tag{3}$$

$$\nabla_Y(f\,X) = Y(f)\,X + f(p)\,\nabla_Y X \tag{4}$$

where $X_1, X_2 \in \mathfrak{X}(M)$, $Y, Y_1, Y_2 \in T_pM$, $f \in \mathcal{F}(M)$, $c \in \mathbb{R}$.

Definition. A global affine connection (or briefly a connection) on M is a mapping which assigns to two smooth vector fields Y and X a new one $\nabla_Y X$ called the covariant derivative of the vector field X with respect to the vector field Y, having the following properties

$$\nabla_{(Y_1+Y_2)}X = \nabla_{Y_1}X + \nabla_{Y_2}X \tag{1'}$$

$$\nabla_{cY}X = c\,\nabla_Y X \tag{2'}$$

$$\nabla_Y(X_1+X_2) = \nabla_Y X_1 + \nabla_Y X_2 \tag{3'}$$

$$\nabla_Y(f\,X) = Y(f)\,X + f\,\nabla_Y X \tag{4'}$$

where $X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$, $f, c \in \mathcal{F}(M)$.

Lemma. For a global affine connection ∇, $X, Y \in \mathfrak{X}(M)$, $p \in M$, the tangent vector $(\nabla_Y X)(p)$ depends only on the behavior of X and Y in an open neighborhood of p.

Proof. Let us suppose that the vector fields X_1 and X_2 coincide on an open neighborhood U of p. Choose a smooth function $h \in \mathcal{F}(M)$ which is zero outside U and constant 1 on a neighborhood of p. Then we have $h(X_1-X_2) = 0$, consequently

$$0 = \nabla_Y(h(X_1-X_2)) = h\nabla_Y(X_1-X_2)+Y(h)(X_1-X_2).$$

Computing the right hand side at p we get

$$0 = (\nabla_Y X_1)(p) - (\nabla_Y X_2)(p).$$

Similarly, if the vector fields Y_1 and Y_2 coincide on an open neighborhood U of p and h is chosen as above, then we have

$$0 = h(Y_1-Y_2)X = \nabla_{h(Y_1-Y_2)}X = h\nabla_{Y_1} X - h\nabla_{Y_2} X ,$$

which yields $(\nabla_{Y_1} X)(p) - (\nabla_{Y_2} X)(p)$. ∎

The lemma shows that an affine connection can be restricted onto any open subset and can be recovered from its restrictions onto the elements of an open cover.

Let $x_1,\dots,x_n$ be local coordinates defined on an open subset U of M and $\partial_1 = \frac{\partial}{\partial x_1}, \quad ,\partial_n = \frac{\partial}{\partial x_n}$ be the corresponding basis vector fields on U. Given an affine connection ∇ on M, we can express the vector field $\nabla_{\partial_i}\partial_j$ as a linear combination of the basis vector fields

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ji}^{k}\, \partial_k .$$

The components Γ_{ji}^{k} are smooth functions called Christoffel symbols.

Proposition. The restriction of an affine connection onto an open coordinate neighborhood U is uniquely determined by the Christoffel symbols. Any n^3 smooth functions Γ_{ji}^{k} on U may be Christoffel symbols for an appropriate affine connection on U.

Proof. Let $X = \sum_{i=1}^{n} X_i\, \partial_i$, $Y = \sum_{j=1}^{n} Y_j\, \partial_j$ be two smooth vector fields on U. Then by the properties of affine connections $\nabla_Y X$ can be computed as follows

$$\nabla_Y X = \nabla_{\left(\sum_{j=1}^{n} Y_j\,\partial_j\right)} \left(\sum_{i=1}^{n} X_i\,\partial_i\right) = \sum_{j=1}^{n} Y_j\,\nabla_{\partial_j} \left(\sum_{i=1}^{n} X_i\,\partial_i\right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} Y_j \left(\partial_j(X_i)\partial_i + X_i \nabla_{\partial_j} \partial_i\right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(Y_j\,\partial_j(X_i)\partial_i + X_i Y_j \sum_{k=1}^{n} \Gamma^k_{ij}\,\partial_k\right) =$$

$$= \sum_{k=1}^{n} \left(Y(X_k) + \sum_{i=1}^{n} \sum_{j=1}^{n} X_i Y_j\,\Gamma^k_{ij}\right) \partial_k.$$

This formula shows that the knowledge of the Christoffel symbols enables us to compute the covariant derivative of any vector field with respect to any other one. On the other hand, if Γ^k_{ij} is an arbitrary smooth function on U for $1 \leqslant i,j,k \leqslant n$, then defining the covariant derivative of a vector field by the above formula, we obtain an affine connection on U. ∎

Observe, that in fact, the tangent vector $(\nabla_Y X)(p)$ depends only on the vector Y(p), so a global affine connection on a manifold defines an affine connection at each of its points. Furthermore, we do not need to know the vector field X everywhere on U to compute $(\nabla_Y X)(p)$. It is enough to know X at the points of a curve $\gamma\ [-\varepsilon,\varepsilon] \longrightarrow M$ such that $\gamma(0) = p$, $\gamma\,'(0)=Y(p)$.

<u>Definition</u>. Let $\gamma\colon [a,b] \longrightarrow M$ be a smooth curve in M. A smooth <u>vector field</u> X <u>along the curve</u> γ is a smooth mapping $X\colon [a,b] \longrightarrow TM$ which assigns to each $t \in [a,b]$ a tangent vector $X(t) \in T_{\gamma(t)}M$.

Now suppose that M is provided with an affine connection. Then any vector field X along γ determines a new vector field $\frac{DX}{dt}$ along γ called the <u>covariant derivative</u> of X. In terms of local coordinates, if $\partial_1,\dots,\partial_n$ are the basis vector fields determined by a chart and the functions $X_i\colon [a,b] \longrightarrow \mathbb{R}$ and $Y_i\colon [a,b] \longrightarrow \mathbb{R}$ assign to $t \in [a,b]$ the components of the vectors $X(\gamma(t))$ and $D_{\gamma'(t)}$ in the basis $\partial_1(\gamma(t)),\dots,\ \partial_n(\gamma(p))$, then $\frac{DX}{dt}$ is defined as follows

$$\frac{DX}{dt}(\tau) := \sum_{k=1}^{n} \left(X_k'(\tau) + \sum_{i=1}^{n} \sum_{j=1}^{n} X_i(\tau)Y_j(\tau)\,\Gamma^k_{ij}(\gamma(\tau))\right) \partial_k(\gamma(\tau)).$$

<u>Definition</u>. A vector field X along a curve γ is said to be a <u>parallel vector field</u> if the covariant derivative $\frac{DX}{dt}$ is identically zero.

Proposition. Given a curve γ and a tangent vector X_0 at the point $\gamma(0)$, there is a unique parallel vector field X along γ which extends X_0.

Proof. The proposition follows from results on ordinary differential equations. Using local coordinates, condition $\frac{DX}{dt} = 0$ yields a system of ordinary differential equations for the components of X

$$X_k' + \sum_{i=1}^{n} \sum_{j=1}^{n} X_i Y_j \Gamma^k_{ij} \circ \gamma = 0.$$

Since these equations are linear, the existence and uniqueness theorem for linear differential equations guaranties that the solutions of this system of differential equations are uniquely determined by the initial values $X_k(0)$ and can be defined for all relevant values of t. ∎

The vector $X_t = X(\gamma(t))$ is said to be obtained from X_0 by parallel translation along γ.

Definition. A connection is called symmetric or torsion free if it satisfies the identity

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Applying this identity to the case $X = \partial_i$, $Y = \partial_j$, since $[\partial_i, \partial_j] = 0$ one obtains the relation

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Conversely, if $\Gamma^k_{ij} = \Gamma^k_{ji}$ then using the expression of covariant derivative with the help of Christoffel symbols we get

$$\nabla_X Y - \nabla_Y X =$$

$$= \sum_{k=1}^{n} \left(X(Y_k) + \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i X_j \Gamma^k_{ij} \right) \partial_k - \sum_{k=1}^{n} \left(Y(X_k) + \sum_{i=1}^{n} \sum_{j=1}^{n} X_i Y_j \Gamma^k_{ij} \right) \partial_k$$

$$= \sum_{k=1}^{n} \left(X(Y_k) - Y(X_k) \right) \partial_k$$

$$= [X, Y].$$

There is a useful characterization of symmetry. Consider a "parameterized surface" in M that is a smooth mapping $s: R \longrightarrow M$ from a rectangular domain R of the plane $\mathbb{R}^2$ into M.

By a vector field X along s is meant a mapping which assigns to each $(x, y) \in R$ a tangent vector $X(x, y) \in T_{s(x,y)}M$.

As examples, the two standard vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on the plane give rise to vector fields $Ts(\frac{\partial}{\partial x})$ and $Ts(\frac{\partial}{\partial y})$ along s. These will be denoted briefly by $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$. Here $Ts: T\mathbb{R}^2 \longrightarrow TM$ denotes the derivative of s.

For any smooth vector field X along s the <u>covariant partial derivatives</u> $\frac{DX}{\partial x}$, $\frac{DX}{\partial y}$ are new vector fields along s constructed as follows. For each fixed y_0, restricting X to the curve $x \longmapsto s(x,y_0)$ one obtains a vector field along this curve. Its covariant derivative with respect to x is defined to be $\frac{DX}{\partial x}(x,y_0)$. This defines $\frac{DX}{\partial x}$ along the entire parameterized surface s.

<u>Proposition</u>. A connection is symmetric if and only if

$$\frac{D}{\partial x}\frac{\partial s}{\partial y} = \frac{D}{\partial y}\frac{\partial s}{\partial x}$$

for any parameterized surface s in M.

<u>Proof</u>. Let us choose a local coordinate system $(x_1,\dots,x_n)$ on M. The mapping s is given by n functions $s_i = x_i \circ s$. The vector field $\frac{\partial s}{\partial y}$ has the form

$$\frac{\partial s}{\partial y} = \sum_{i=1}^{n} \frac{\partial s_i}{\partial y}(\partial_i \circ s).$$

The partial covariant derivative of this vector field with respect to x is equal to

$$\frac{D}{\partial x}\frac{\partial s}{\partial y} = \frac{D}{\partial x}\left(\sum_{i=1}^{n}\frac{\partial s_i}{\partial y}(\partial_i \circ s)\right) = \sum_{i=1}^{n}\left(\frac{\partial^2 s_i}{\partial x \partial y}\partial_i \circ s + \frac{\partial s_i}{\partial y}\frac{D}{\partial x}(\partial_i \circ s)\right) =$$

$$= \sum_{i=1}^{n}\frac{\partial^2 s_i}{\partial x\partial y}\partial_i \circ s + \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial s_i}{\partial y}\frac{\partial s_j}{\partial x}\Gamma_{ij}^k \circ s\right)\partial_k \circ s.$$

This formula shows that interchanging the role of x and y we obtain the same vector field for any s if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$. ▮

Roughly speaking, the torsion free condition halves the degree of freedom in the choice of Christoffel symbols, a symmetric connection is uniquely determined by $n\frac{n(n+1)}{2}$ arbitrarily chosen functions, nevertheless, the space of symmetric affine connections on a manifold is still infinite dimensional.

We can reduce further the degree of freedom putting condition (6) on the connection. This condition however does not make sense on an arbitrary manifold, because dot product of tangent vectors at a given point is not defined in general.

<u>Definition</u>. Let M be a differentiable manifold. Suppose that each tangent space T_pM of M is equipped with a positive definite symmetric bilinear form $\langle,\rangle_p$ so that for any two smooth vector fields $X,Y: M \longrightarrow TM$ the function $M \ni p \longmapsto \langle X(p),Y(p)\rangle_p$ is smooth. Then M together with this structure is a

Riemannian manifold, the system of bilinear forms on the tangent spaces is the Riemannian metric on M.

Example. A hypersurface of $\mathbb{R}^n$ together with the first fundamental form is a Riemannian manifold.

Definition. A connection ∇ on M is compatible with the Riemannian metric if parallel translation preserves inner products. In other words, for any curve γ and any pair X, Y of parallel vector fields along γ, the inner product $\langle X, Y\rangle$ should be constant.

Lemma. Suppose that the connection is compatible with the metric. Let V, W be any two vector fields along γ. Then

$$\frac{d}{dt}\langle V,W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle .$$

Proof. Choose parallel vector fields $X_1,\dots,X_n$ along γ which are orthonormal at one point of γ and hence at every point of γ. Then the given fields V and W can be expressed as $\sum_{i=1}^{n} v_i X_i$ and $\sum_{i=1}^{n} w_i X_i$ respectively (where $v_i = \langle V, X_i\rangle$ is a real valued function). It follows that $\langle V, W\rangle = \sum_{i=1}^{n} v_i w_i$ and that

$$\frac{DV}{dt} = \sum_{i=1}^{n} \frac{dv_i}{dt} X_i, \qquad \frac{DW}{dt} = \sum_{i=1}^{n} \frac{dw_i}{dt} X_i.$$

Therefore

$$\left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle = \sum_{i=1}^{n}\left(\frac{dv_i}{dt} w_i + v_i \frac{dw_i}{dt}\right) = \frac{d}{dt}\langle V,W\rangle. \blacksquare$$

Corollary. An affine connection on a Riemannian manifold is compatible with the metric if and only if for any vector fields X_1, X_2 on M and any tangent vector $Y \in T_pM$ we have

$$Y(\langle X_1, X_2\rangle) = \langle \nabla_Y X_1, X_2\rangle + \langle X_1, \nabla_Y X_2\rangle.$$

Theorem. (Fundamental theorem of Riemannian geometry.) A Riemannian manifold possesses one and only one symmetric connection which is compatible with its metric.

Proof. Applying the compatibility condition to the basis vector fields $\partial_i, \partial_j, \partial_k$, corresponding to a fixed chart on the manifold and setting $\langle \partial_j, \partial_k\rangle = g_{jk}$ one obtains the identity

$$\partial_i\, g_{jk} = \langle \nabla_{\partial_i}\partial_j\,,\ \partial_k \rangle + \langle \partial_j\,,\ \nabla_{\partial_i}\partial_k \rangle .$$

permuting i,j and k this gives three linear equations relating the three quantities

$$\langle \nabla_{\partial_i}\partial_j\,,\ \partial_k \rangle\,, \quad \langle \nabla_{\partial_j}\partial_k\,,\ \partial_i \rangle\,, \quad \langle \nabla_{\partial_k}\partial_i\,,\ \partial_j \rangle .$$

(There are only three such quantities since $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$.) These equations can be solved uniquely; yielding the <u>first Christoffel identity</u>

$$\langle \nabla_{\partial_i}\partial_j\,,\ \partial_k \rangle = \frac{1}{2}\,(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) .$$

The left hand side of this identity is equal to $\sum_{l=1}^{n} \Gamma^l_{ij}\, g_{lk}$. Multiplying by the inverse (g^{kl}) of the matrix (g_{lk}) this yields the <u>second Christoffel identity</u>

$$\Gamma^l_{ij} = \sum_{k=1}^{n} \frac{1}{2}\,(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})\, g^{kl}.$$

Thus the connection is uniquely determined by the metric.

Conversely, defining Γ^l_{ij} by this formula, one can verify that the resulting connection is symmetric and compatible with the metric. This completes the proof. ▮

The unique symmetric affine connection which is compatible with the metric on a Riemannian manifold is called the <u>Levi-Civita connection</u>.

The connection ∇ we introduced on open subsets of $\mathbb{R}^n$ is just the Levi-Civita connection of $\mathbb{R}^n$.

Consider now a parameterized hypersurface $\mathbf{r}:\Omega \longrightarrow \mathbb{R}^n$. It is a Riemannian manifold with respect to the first fundamental form. The basis vector fields $\mathbf{r}_i$ through suitable identifications are the same as the basis vector fields ∂_i corresponding to the chart $\mathbf{r}^{-1}$. Comparing the formulae

$$\partial_{\mathbf{r}_i}\mathbf{r}_j = \mathbf{r}_{ij} = \sum_{k=1}^{n-1} \Gamma^k_{ij}\, \mathbf{r}_k + h_{ij}\, \mathbf{N}$$

and

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n-1} g^{kl}(g_{li,j} + g_{lj.i} - g_{ij,l})$$

proved in unit 7 with the formulae derived for the Levi-Civita connection we may conclude that the Christoffel symbols of a hypersurface introduced previously in unit 7 are the Christoffel symbols of the Levi-Civita connection of the hypersurface. Furthermore, denoting by $\tilde{\nabla}$ the Levi-Civita connection of the hypersurface, we see that for tangential vector fields X, Y $\tilde{\nabla}_Y X$ is the tangential component of $\partial_Y X$.

$$\tilde{\nabla}_Y X = \partial_Y X - \langle \partial_Y X , \mathbf{N} \rangle \mathbf{N}$$

Further Exercises

Exercise 11-1. Show that if ∇ and $\tilde{\nabla}$ are two affine connections on a manifold M, then their difference $S(X,Y) = \nabla_X Y - \tilde{\nabla}_X Y$ is an $\mathcal{F}(M)$-bilinear mapping. (In other words, S is a tensor field of valency $(1,2)$). Conversely, the sum of a connection and an $\mathcal{F}(M)$-bilinear mapping $S: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is a connection. (According to these statements, affine global connections form an affine space over the linear space of $(1,2)$-tensor fields.)

Exercise 11-2. Show that the <u>torsion</u> $T: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ of a connection ∇ defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is an $\mathcal{F}(M)$-bilinear mapping (i.e., T is a tensor field).

Exercise 11-3. Show that if T is the torsion of an affine connection ∇ the $\nabla - T/2$ is a symmetric connection.

Exercise 11-4. Check that the connection defined by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n-1} g^{kl}(g_{li,j} + g_{lj.i} - g_{ij,l})$$

is symmetric and compatible with the metric.