Unit 12. Curvature

Curvature operator, curvature tensor, Bianchi identities, Riemann-Christoffel tensor, symmetry properties of the Riemann-Christoffel tensor, sectional curvature, Schur's theorem, space forms, Ricci tensor, Ricci curvature, scalar curvature, curvature tensor of a hypersurface.

If ∇ is an affine connection on a manifold M, then we may consider the operator

$$R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}:\mathfrak{X}(M)\longrightarrow \mathfrak{X}(M),$$

where $[\nabla_X,\nabla_Y]=\nabla_X\circ\nabla_Y-\nabla_Y\circ\nabla_X$ is the usual commutator of operators. The mapping that assigns to the vector fields X,Y the operator R(X,Y) is called the <u>curvature operator</u> of the connection. The assignment

$$\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\longrightarrow \mathfrak{X}(M)$$
$$(X,Y,Z)\longmapsto R(X,Y)(Z)$$

is called the <u>curvature tensor</u> of the connection. To reduce the number of brackets, we shall denote R(X,Y)(Z) simply by R(X,Y;Z). Thus, the letter R is used in two different meanings, later it will denote also a third mapping, but the number of arguments of R makes always clear which meaning is considered.

<u>Proposition</u>. The curvature tensor is linear over the ring of smooth functions in each of its arguments, and it is skew symmetric in the first two arguments.

<u>Proof</u>. Skew symmetry in the first two arguments is clear, since

$$R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} = -[\nabla_Y,\nabla_X] + \nabla_{[Y,X]} = -R(Y,X).$$

According to this, it suffices to check linearity of the curvature tensor in the first and third arguments.

Linearity in the first argument is proved by the following identities.

$$R(X_1+X_2,Y) = [\nabla_{X_1+X_2},\nabla_Y] - \nabla_{[X_1+X_2,Y]} = [\nabla_{X_1}+\nabla_{X_2},\nabla_Y] - \nabla_{[X_1,Y]+[X_2,Y]} =$$
$$= [\nabla_{X_1},\nabla_Y] + [\nabla_{X_2},\nabla_Y] - \nabla_{[X_1,Y]} - \nabla_{[X_2,Y]} = R(X_1,Y) + R(X_2,Y).$$

and

$$R(fX,Y;Z) = ([\nabla_{fX},\nabla_Y] - \nabla_{[fX,Y]})(Z) = f\nabla_X\nabla_Y Z-\nabla_Y(f\nabla_X Z)-\nabla_{f[X,Y]-Y(f)X}(Z) =$$
$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X(Z) =$$
$$= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = f\,R(X,Y;Z).$$

Additivity in the third argument is clear, since R(X,Y) is built up of the additive operators ∇_X, ∇_Y and their compositions. To have linearity, we need

$$
\begin{aligned}
R(X,Y;fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) = \\
&= \nabla_X(Y(f)Z+f\nabla_Y Z)-\nabla_Y(X(f)Z+f\nabla_X Z)-[X,Y](f)Z-f\nabla_{[X,Y]}Z = \\
&= XY(f)Z+Y(f)\nabla_X Z+X(f)\nabla_Y Z+f\nabla_X\nabla_Y Z -YX(f)Z-X(f)\nabla_Y Z-Y(f)\nabla_X Z-f\nabla_Y\nabla_X Z- \\
&\qquad -XY(f)Z+YX(f)Z - f\nabla_{[X,Y]}Z \\
&= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = f\, R(X,Y;Z). \blacksquare
\end{aligned}
$$

The proposition is a bit surprising, because the curvature tensor is built up from covariant derivations, which are not linear operators over the ring of smooth functions.

We have already introduced tensor fields over a hypersurface. We can introduce tensor fields over a manifold in the same manner. A tensor field $\mathcal{T}$ of type (k,ℓ) is an assignment to every point p of a manifold M a tensor $\mathcal{T}(p)$ of type (k,ℓ) over the tangent space T_pM. If $\partial_1,\ldots,\partial_n$ are the basis vector fields defined by a chart over the domain of the chart, and we denote by $dx^1(p)$, ..., $dx^n(p)$ the dual basis of $\partial_1(p),\ldots,\partial_n(p)$, then a tensor field is uniquely determined over the domain of the chart by the components

$$\mathcal{T}^{i_1\ldots i_k}_{j_1\ldots j_l}(p)=\mathcal{T}(p)(dx^{i_1}(p),\ldots,dx^{i_k}(p);\partial_{j_1}(p),\ldots,\partial_{j_l}(p)).$$

We say that the tensor field is smooth, if for any chart the functions $\mathcal{T}^{i_1\ldots i_k}_{j_1\ldots j_l}$ are smooth. We shall consider only smooth tensor fields.

Tensors of valency (1,0) are the vector fields, tensors of valency (0,1) are the differential 1-forms. Thus, a differential 1-form assigns to every point of the manifold a linear function on the tangent space at that point. Differential 1-forms form a module over the ring of smooth functions, which we denote by $\Omega^1(M)$.

Every tensor field defines a multi-$\mathcal{F}(M)$-linear mapping

$$\Omega^1(M)\times\ldots\times\Omega^1(M)\times\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$$

and conversely, every such multi-$\mathcal{F}(M)$-linear mapping comes from a tensor field. (Check this!) Therefore, tensor fields can be identified with multi-$\mathcal{F}(M)$-linear mappings $\Omega^1(M)\times\ldots\times\Omega^1(M)\times\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$.

Tensors of type (1,k), that is multi-$\mathcal{F}(M)$-linear mappings

$$\Omega^1(M)\times\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$$

can be identified in a natural way with multi-$\mathcal{F}(M)$-linear mappings

$$\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M).$$

By this identification, $R:\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ corresponds to $\tilde{R}:\Omega^1(M)\times\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$, defined by $\tilde{R}(\omega;X_1,\ldots,X_k)=\omega(R(X_1,\ldots,X_k))$.

Using these identifications, the curvature tensor is a tensor field of valency (1,3) by the proposition. It is a remarkable consequence, that although the vectors $\nabla_X Z(p)$ and $\nabla_Y Z(p)$ are not determined by the vectors $X(p),Y(p),Z(p)$, to compute the value of $R(X,Y;Z)$ at p it suffices to know $X(p),Y(p),Z(p)$.

Beside skew-symmetry in the first two arguments, the curvature tensor has many other symmetry properties.

Theorem. (First Bianchi Identity). If R is the curvature tensor of a torsion free connection, then

$$R(X,Y;Z) + R(Y,Z;X) + R(Z,X;Y) = 0$$

for any three vector fields X,Y,Z.

Proof. Let us introduce the following notation. If $F(X,Y,Z)$ is a function of the vector fields X,Y,Z, then denote by $\circlearrowleft F(X,Y,Z)$ or $\underset{XYZ}{\circlearrowleft} F(X,Y,Z)$ the sum of the values of F at all cyclic permutations of the variables (X,Y,Z)

$$\circlearrowleft F(X,Y,Z) = F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y).$$

We shall use several times that behind the cyclic summation $\circlearrowleft$ we may cyclically rotate X,Y,Z in any expression

$$\circlearrowleft F(X,Y,Z) = \circlearrowleft F(Y,Z,X) = \circlearrowleft F(Z,X,Y).$$

The theorem claims vanishing of

$$\circlearrowleft R(X,Y;Z) = \circlearrowleft (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = \circlearrowleft (\nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[X,Y]}Z)$$
$$= \circlearrowleft (\nabla_X [Y,Z] - \nabla_{[X,Y]}Z) = \circlearrowleft (\nabla_Z [X,Y] - \nabla_{[X,Y]}Z) = \circlearrowleft [Z,[X,Y]],$$

but the latter expression is 0 according to the Jacobi identity on the Lie bracket of vector fields. (At the third and fifth equality we used the torsion free property of ∇.) ∎

The presence of an affine connection on a manifold allows us to differentiate not only vector fields, but also tensor fields of any type.

Definition. Let (M,∇) be a manifold with an affine connection. If $\omega \in \Omega^1(M)$ is a 1-form, X is a vector field, then we define the covariant derivative $\nabla_X\omega$ of ω with respect to X to be the 1-form

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y), \quad Y\in\mathfrak{X}(M).$$

In general, the covariant derivative $\nabla_X T$ of a tensor field

$$T:\Omega^1(M)\times\ldots\times\Omega^1(M)\times\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$$

of valency (k,ℓ) with respect to a vector field X is a tensor field of the same valency, defined by

$$(\nabla_X T)(\omega_1,\dots,\omega_k;X_1,\dots,X_\ell) = X(T(\omega_1,\dots,\omega_k;X_1,\dots,X_\ell)) - \\ - \sum_{i=1}^{k} T(\omega_1,\dots,\nabla_X\omega_i,\dots,\omega_k;X_1,\dots,X_\ell) - \\ - \sum_{j=1}^{\ell} T(\omega_1,\dots,\omega_k;X_1,\dots,\nabla_X X_j,\dots,X_\ell).$$

For the case of the curvature tensor, this definition gives

$$(\nabla_X R)(Y,Z;W) = \nabla_X(R(Y,Z;W)-R(\nabla_X Y,Z;W)-R(Y,\nabla_X Z;W)-R(Y,Z;\nabla_X W).$$

<u>Theorem</u>. (Second Bianchi Identity) The curvature tensor of a torsion free connection satisfies the following identity

$$\underset{XYZ}{\circlearrowleft} (\nabla_X R)(Y,Z;W) = (\nabla_X R)(Y,Z;W) + (\nabla_Y R)(Z,X;W) + (\nabla_Z R)(X,Y;W) = 0.$$

<u>Proof</u>. $(\nabla_X R)(Y,Z;W)$ is the value of the operator

$$\nabla_X\circ R(Y,Z)-R(\nabla_X Y,Z)-R(Y,\nabla_X Z)-R(Y,Z)\circ\nabla_X : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

on the vector field W, hence we have to prove vanishing of the operator

$$\underset{XYZ}{\circlearrowleft} \nabla_X\circ R(Y,Z)-R(\nabla_X Y,Z)-R(Y,\nabla_X Z)-R(Y,Z)\circ\nabla_X .$$

First, we have

$$\underset{XYZ}{\circlearrowleft}\nabla_X\circ R(Y,Z)-R(Y,Z)\circ\nabla_X=\underset{XYZ}{\circlearrowleft}(\nabla_X\nabla_Y\nabla_Z-\nabla_X\nabla_Z\nabla_Y-\nabla_X\nabla_{[Y,Z]})-(\nabla_Y\nabla_Z\nabla_X-\nabla_Z\nabla_Y\nabla_X-\nabla_{[Y,Z]}\nabla_X)=$$

$$\underset{XYZ}{\circlearrowleft}(\nabla_X\nabla_Y\nabla_Z-\nabla_X\nabla_Z\nabla_Y-\nabla_X\nabla_{[Y,Z]})-(\nabla_X\nabla_Y\nabla_Z-\nabla_X\nabla_Z\nabla_Y-\nabla_{[Y,Z]}\nabla_X)=\underset{XYZ}{\circlearrowleft} \nabla_{[Y,Z]}\nabla_X - \nabla_X\nabla_{[Y,Z]}.$$

On the other hand,

$$\underset{XYZ}{\circlearrowleft} -R(\nabla_X Y,Z)-R(Y,\nabla_X Z) = \underset{XYZ}{\circlearrowleft} R(\nabla_X Z,Y) - R(\nabla_X Y,Z) = \underset{XYZ}{\circlearrowleft} R(\nabla_Y X,Z) - R(\nabla_X Y,Z) =$$

$$\underset{XYZ}{\circlearrowleft} R(\nabla_Y X-\nabla_X Y,Z) = \underset{XYZ}{\circlearrowleft} R([Y,X],Z).$$

Combining these results,

$$\underset{XYZ}{\circlearrowleft} \nabla_X\circ R(Y,Z)-R(\nabla_X Y,Z)-R(Y,\nabla_X Z)-R(Y,Z)\circ\nabla_X$$

$$= \underset{XYZ}{\circlearrowleft} \nabla_{[Y,Z]}\nabla_X-\nabla_X\nabla_{[Y,Z]}+R([Y,X],Z)$$

$$= \underset{XYZ}{\circlearrowleft} \nabla_{[Y,Z]}\nabla_X-\nabla_X\nabla_{[Y,Z]}+R([Z,Y],X)$$

$$= \underset{XYZ}{\circlearrowleft} \nabla_{[Y,Z]}\nabla_X-\nabla_X\nabla_{[Y,Z]}+\nabla_{[Z,Y]}\nabla_X - \nabla_X\nabla_{[Z,Y]} - \nabla_{[[Z,Y],X]}$$

$$= \underset{XYZ}{\circlearrowleft} \nabla_{[[Y,Z],X]} = \nabla_{\circlearrowleft[[X,Y],Z]} = 0. \blacksquare$$

In the remaining part of this unit, we shall deal with Riemannian manifolds. If $(M,\langle,\rangle)$ is a Riemannian, manifold with Levi-Civita connection ∇, and R is the curvature tensor of ∇, then we can introduce a tensor $\tilde{R}$ of valency $(0,4)$, related to R by the equation

$$\tilde{R}(X,Y;Z,W) = \langle R(X,Y;Z),W \rangle.$$

$\tilde{R}$ is the *Riemann-Christoffel curvature tensor* of the Riemannian manifold. To simplify notation, we shall denote $\tilde{R}$ also by R. This will not lead to confusion, since the Riemann-Christoffel tensor and the ordinary curvature tensor have different number of arguments.

Levi-Civita connections are connections of special type, so it is not surprising, that the curvature tensor of a Riemannian manifold has stronger symmetries than that of an arbitrary connection. Of course, the general results can be applied to Riemannian manifolds as well, and yield

$$R(X,Y;Z,W) = -R(Y,X;Z,W);$$

$$\underset{XYZ}{\circlearrowleft} R(X,Y;Z,W) = 0.$$

In addition to these symmetries, we have the following ones.

Theorem. The Riemann-Christoffel curvature tensor is skew-symmetric in the last two arguments

$$R(X,Y;Z,W) = -R(X,Y;W,Z).$$

Proof. By the compatibility of the connection and the metric, we have

$$X(Y(\langle Z,W\rangle)) = X(\langle\nabla_Y Z,W\rangle+\langle Z,\nabla_Y W\rangle) = \langle\nabla_X\nabla_Y Z,W\rangle+\langle\nabla_Y Z,\nabla_X W\rangle+\langle\nabla_X Z,\nabla_Y W\rangle+\langle Z,\nabla_X\nabla_Y W\rangle,$$

and similarly,

$$Y(X(\langle Z,W\rangle)) = \langle\nabla_Y\nabla_X Z,W\rangle+\langle\nabla_X Z,\nabla_Y W\rangle+\langle\nabla_Y Z,\nabla_X W\rangle+\langle Z,\nabla_Y\nabla_X W\rangle.$$

We also have

$$[X,Y](\langle Z,W\rangle) = \langle\nabla_{[X,Y]}Z,W\rangle + \langle Z,\nabla_{[X,Y]}W\rangle.$$

Subtracting from the first equality the second and the third one and applying $[X,Y] = X\circ Y - Y\circ X$, we obtain

$$0 = \langle\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,W\rangle+\langle Z,\nabla_X\nabla_Y W-\nabla_Y\nabla_X W-\nabla_{[X,Y]}W\rangle = R(X,Y;Z,W)+R(X,Y;W,Z).$$ ∎

The symmetries we have by now imply a further symmetry.

Theorem. Assume that R is an arbitrary tensor of valency $(0,4)$ having the following symmetry properties

$$R(X,Y;Z,W) = -R(Y,X;Z,W) = -R(X,Y;W,Z)$$

and

$$\underset{XYZ}{\circlearrowleft} R(X,Y;Z,W) = 0.$$

Then

$$R(X,Y;Z,W) = R(Z,W;X,Y).$$

Proof. Let us apply the Bianchi identity for the following five arrangements

$$R(X,Y;Z,W) + R(Y,Z;X,W) + R(Z,X;Y,W) = 0$$

$$R(X,Y;W,Z) + R(Y,W;X,Z) + R(W,X;Y,Z) = 0$$

$$R(X,W;Z,Y) + R(W,Z;X,Y) + R(Z,X;W,Y) = 0$$

$$R(Y,Z;W,X) + R(Z,W;Y,X) + R(W,Y;Z,X) = 0$$

$$2R(Y,W;Z,X) + 2R(W,Z;Y,X) + 2R(Z,Y;W,X) = 0.$$

Changing the order of letters in the first two and last two places to the alphabetical order, we obtain the following equalities.

$$-R(X,Y;W,Z) - R(Y,Z;W,X) + R(X,Z;W,Y) = 0$$

$$R(X,Y;W,Z) - R(W,Y;X,Z) + R(W,X;Y,Z) = 0$$

$$R(W,X;Y,Z) + R(W,Z;X,Y) - R(X,Z;W,Y) = 0$$

$$R(Y,Z;W,X) + R(W,Z;X,Y) - R(W,Y;X,Z) = 0$$

$$2R(W,Y;X,Z) - 2R(W,Z;X,Y) - 2R(Y,Z;W,X) = 0.$$

Adding these five equalities, we get the following equation after obvious simplifications.

$$2\,R(W,X;Y,Z) - 2\,R(Y,Z;W,X) = 0,$$

and this is the identity we wanted to prove. ▮

We know from linear algebra that a symmetric bilinear form is uniquely determined by its quadratic form. More generally, when a tensor has some symmetries, it can be reconstructed from its restriction to a suitable linear subspace of its domain. For tensors having the symmetries of a curvature tensor we have the following

<u>Proposition</u>. Let S_1 and S_2 be tensors (or tensor fields) of valency (0,4), satisfying the following relations

$$S_i(X,Y;Z,W) = -S_i(Y,X;Z,W) = -S_i(X,Y;W,Z);$$

$$\underset{XYZ}{\circlearrowleft} S_i(X,Y;Z,W) = 0.$$

Then if $S_1(X,Y;Y,X) = S_2(X,Y;Y,X)$ for every X and Y, then $S_1=S_2$.

<u>Proof</u>. Consider the difference $S = S_1-S_2$. S has the same symmetries as S_1 and S_2, $S(X,Y;Y,X) = 0$ for all X,Y and we have to show S = 0.

We have for any X,Y,Z

$$\begin{aligned} 0 = S(X,Y+Z;Y+Z,X) &= S(X,Y;Y,X) + S(X,Y;Z,X) + S(X,Z;Y,X) + S(X,Z;Z,X) = \\ &= S(X,Y;Z,X) + S(X,Z;Y,X) + \\ &\quad + (S(X,Y;Z,X) + S(Y,Z;X,X) + S(Z,X;Y,X)) \\ &= 2S(X,Y;Z,X). \end{aligned}$$

Now taking four arbitrary vectors (vector fields) X,Y,Z,W and using $S(X,Y;Z,X) \equiv 0$, we obtain

$$0 = S(X+W,Y;Z,X+W) = S(X,Y;Z,X) + S(X,Y;Z,W) + S(W,Y;Z,X) + S(W,Y;Z,W) =$$

$$= S(X,Y;Z,W) + S(W,Y;Z,X),$$

i.e., S is skew symmetric in the first and fourth variables. Thus,

$$S(X,Y;Z,W) = S(Y,X;W,Z) = - S(Z,X;W,Y) = S(Z,X;Y,W),$$

in other words, S is invariant under cyclic permutations of the first three variables. But the sum of the three equal quantities $S(X,Y;Z,W)$, $S(Y,Z;X,W)$ and $S(Z,X;Y,W)$ is 0 because of the Bianchi symmetry, thus $S(X,Y;Z,W)$ is 0. ▮

Exercise. Let S be a tensor of valency (0,4) having all the curvature tensor symmetries, and let $Q_S(X,Y) := S(X,Y;Y,X)$. Prove that $Q_S(X,Y)=Q_S(Y,X)$ and

$$\begin{aligned} 6S(X,Y;Z,W) = {} & Q_S(X+W,Y+Z)-Q_S(Y+W,X+Z) + \\ & +Q_S(Y+W,X)-Q_S(X+W,Y)+Q_S(Y+W,Z)-Q_S(X+W,Z)+ \\ & +Q_S(X+Z,Y)-Q_S(Y+Z,X)+Q_S(X+Z,W)-Q_S(Y+Z,W)+ \\ & +Q_S(X,Z)-Q_S(Y,Z)+Q_S(Y,W)-Q_S(X,W). \end{aligned}$$

Definition. Let M be a Riemannian manifold, p a point on M, X and Y two non-parallel tangent vectors at p. The number

$$K(X,Y) = \frac{R(X,Y;Y,X)}{|X|^2|Y|^2-\langle X,Y\rangle^2}$$

is called the sectional curvature of M at p, in the direction of the plane spanned by the vectors X and Y in T_pM.

The name assumes that K(X,Y) depends only on the plane spanned by the vectors X and Y. This is indeed so, since if $\mathbf{x}_1,\mathbf{y}_1$ and $\mathbf{x}_2,\mathbf{y}_2$ are two bases of a 2-dimensional linear space, then we can transform one of them into the other by a finite number of elementary basis transformations of the form

$$\mathbf{x} \to \alpha\mathbf{x} \ , \ \mathbf{y} \to \beta\mathbf{y}, \text{ where } \alpha\beta\neq 0; \ \mathbf{x} \to \mathbf{x}+\mathbf{y} \ , \ \mathbf{y} \to \mathbf{y} \ ; \ \mathbf{x} \to \mathbf{y} \ , \ \mathbf{y} \to \mathbf{x}$$

and we have the following proposition.

Proposition. If M is a Riemannian manifold with sectional curvature K, X and Y are tangent vectors at $p \in M$, α and β are non-zero scalars, then

(i) $K(X,Y) = K(X+Y,Y)$;

(ii) $K(X,Y) = K(\alpha X,\beta Y)$;

(iii) $K(X,Y) = K(Y,X)$.

Proof. (i) follows from

$$R(X+Y,Y;Y,X+Y)=R(X,Y;Y,X)+R(X,Y;Y,Y)+R(Y,Y;Y,X)+R(Y,Y;Y,Y)=R(X,Y;Y,X)$$

and

$$\begin{aligned} |X+Y|^2|Y|^2-\langle X+Y,Y\rangle^2 &= (|X|^2+|Y|^2+2\langle X,Y\rangle)|Y|^2- (\langle X,Y\rangle^2+2\langle X,Y\rangle|Y|^2+|Y|^4) = \\ &= |X|^2|Y|^2-\langle X,Y\rangle^2. \end{aligned}$$

(ii) follows from

$R(\alpha X,\beta Y;\beta Y,\alpha X) = \alpha^2\beta^2\, R(X,Y;Y,X)$

and

$|\alpha X|^2|\beta Y|^2-\langle\alpha X,\beta Y\rangle^2 = \alpha^2\beta^2(|X|^2|Y|^2-\langle X,Y\rangle^2)$.

Finally, (iii) comes from the equalities $R(X,Y;Y,X) = R(Y,X;X,Y)$ and $|X|^2|Y|^2-\langle X,Y\rangle^2=|Y|^2|X|^2-\langle Y,X\rangle^2$. ∎

Definition. Riemannian manifolds, the sectional curvature function of which is constant, called spaces of constant curvature or simply space forms. The space form is elliptic or spherical if $K > 0$, K is parabolic or Euclidean if $K=0$ and is hyperbolic if $K < 0$.

Typical examples are the n-dimensional sphere, Euclidean space and hyperbolic space. Further examples can be obtained by factorization with fixed point free actions of discrete groups.

The following remarkable theorem sounds similarly to the theorem saying that a connected surface consisting of umbilics is contained in a sphere or plane (page 51).

Theorem (Schur). If M is a connected Riemannian manifold, $\dim M \geqslant 3$ and the sectional curvature $K(X_p,Y_p)$, $X_p,Y_p\in T_pM$ depends only on p (and does not depend on the plane spanned by X_p and Y_p, then K is constant, that is, as a matter of fact, it does not depend on p either.

Proof. By the assumption,

$$R(X,Y;Y,X) = f\,(|X|^2|Y|^2-\langle X,Y\rangle^2)$$

for some function f. Our goal is to show that f is constant.

Consider the tensor field of valency (0,4) defined by

$$S(X,Y;Z,W) = f(\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle).$$

It is clear from the definition that S is skew-symmetric in the first and last two arguments. S has also the Bianchi symmetry. Indeed,

$$\underset{XYZ}{\circlearrowleft} S(X,Y;Z,W)=\underset{XYZ}{\circlearrowleft} f(\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle)=\underset{XYZ}{\circlearrowleft} f(\langle Y,W\rangle\langle Z,X\rangle-\langle X,Z\rangle\langle Y,W\rangle)=0.$$

We also have $R(X,Y;Y,X) = S(X,Y;Y,X)$, therefore $R = S$.

Set $\tilde{S}(X,Y;Z) = f(\langle Y,Z\rangle X -\langle X,Z\rangle Y)$. Then

$$\langle R(X,Y;Z),W\rangle =R(X,Y;Z,W) = S(X,Y;Z,W) = \langle\tilde{S}(X,Y;Z),W\rangle,$$

that is,

$$R(X,Y;Z) = \tilde{S}(X,Y;Z) \qquad \text{for all } X,Y,Z.$$

Differentiating with respect to a vector field U we get

$$(\nabla_U R)(X,Y;Z) = (\nabla_U\tilde{S})(X,Y;Z) = \nabla_U(\tilde{S}(X,Y;Z))-\tilde{S}(\nabla_U X,Y;Z)-\tilde{S}(X,\nabla_U Y;Z)-\tilde{S}(X,Y;\nabla_U Z).$$

Since

$$\nabla_U(\tilde{S}(X,Y;Z)) = U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + f\nabla_U(\langle Y,Z\rangle X - \langle X,Z\rangle Y) =$$
$$= U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + f(U\langle Y,Z\rangle X + \langle Y,Z\rangle\nabla_U X - U\langle X,Z\rangle Y - \langle X,Z\rangle\nabla_U Y) =$$
$$= U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) +$$
$$+ f(\langle\nabla_U Y,Z\rangle X + \langle Y,\nabla_U Z\rangle X + \langle Y,Z\rangle\nabla_U X - \langle\nabla_U X,Z\rangle Y - \langle X,\nabla_U Z\rangle Y - \langle X,Z\rangle\nabla_U Y) =$$
$$= U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) +$$
$$+ \tilde{S}(\nabla_U X,Y;Z) + \tilde{S}(X,\nabla_U Y;Z) + \tilde{S}(X,Y;\nabla_U Z),$$

we obtain

$$(\nabla_U R)(X,Y;Z) = (\nabla_U\tilde{S})(X,Y;Z) = U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

Using the second Bianchi identity, this gives us

$$\underset{UXY}{\circlearrowleft} U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) = \underset{UXY}{\circlearrowleft} (\nabla_U R)(X,Y;Z) = 0.$$

If $X \in T_pM$ is an arbitrary tangent vector to the manifold, then we can find non-zero vectors Y, $Z=U \in T_pM$ such that X,Y and U are orthogonal ($\dim M \geq 3$!). Then

$$0 = \underset{UXY}{\circlearrowleft} U(f)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) = X(f)\langle U,U\rangle Y - Y(f)\langle U,U\rangle X.$$

Since X and Y are linearly independent, $X(f)\langle U,U\rangle = Y(f)\langle U,U\rangle = 0$. $\langle U,U\rangle$ is positive, therefore $X(f) = Y(f) = 0$, yielding that the derivative of f in an arbitrary direction X is 0. This means that f is locally constant, and since M is connected, f is constant. ∎

The curvature tensor is a complicated object containing a lot of information about the geometry of the manifold. There are some useful ways to derive some simpler tensor fields from the curvature tensor. Of course, the simplification is paid by losing information.

Definition. Let (M,∇) be a manifold with an affine connection, R be the curvature tensor of ∇. The Ricci tensor Ric of the connection is a tensor field of valency (0,2) assigning to the vector fields X and Y the function $\mathrm{Ric}(X,Y)$ the value of which at $p \in M$ is the trace of the linear mapping

$$T_pM \longrightarrow T_pM,$$
$$Z_p \longmapsto R(Z_p,X(p);Y(p)), \qquad \text{where } Z_p \in T_pM.$$

Proposition. The Ricci tensor of a Riemannian manifold is a symmetric tensor

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X).$$

Proof. Let $e_1,\dots,e_n$ be an orthonormal basis in T_pM, where p is an arbitrary point in the Riemannian manifold M. We can compute the trace of a linear mapping $A:T_pM\longrightarrow T_pM$ by the formula

$$\operatorname{trace} A = \sum_{i=1}^{n} \langle A(e_i),e_i\rangle.$$

In particular,

$$\operatorname{Ric}(X,Y)(p) = \sum_{i=1}^{n} \langle R(e_i,X(p);Y(p)),e_i\rangle = \sum_{i=1}^{n} R(e_i,X(p);Y(p),e_i\rangle =$$

$$= \sum_{i=1}^{n} R(Y(p),e_i;e_i,X(p)\rangle = \sum_{i=1}^{n} R(e_i,Y(p);X(p),e_i\rangle = \operatorname{Ric}(Y,X)(p).\blacksquare$$

Since the Ricci tensor of a Riemannian manifold is symmetric, it is uniquely determined by its quadratic form $X\longmapsto \operatorname{Ric}(X,X)$.

Definition. Let $X_p \in T_pM$ be a non-zero tangent vector of a Riemannian manifold M. The Ricci curvature of M at p in the direction X_p is the number

$$r(X_p) = \frac{\operatorname{Ric}(X_p,X_p)}{|X_p|^2}.$$

Fixing an orthonormal basis $\frac{X_p}{|X_p|^2} = e_1,e_2,...,e_n$ we can express the Ricci curvature as follows

$$r(X_p) = \frac{\operatorname{Ric}(X_p,X_p)}{|X_p|^2} = \sum_{i=1}^{n} \frac{R(e_i,X_p;X_p,e_i\rangle}{|X_p|^2} = \sum_{i=2}^{n} K(X_p,e_i).$$

The meaning of this formula is that the Ricci curvature in the direction X_p is the sum of the sectional curvatures in the directions of the planes spanned by the vectors X_p and e_i, where e_i runs over an orthonormal basis of the orthogonal complement of X_p in T_pM. It is a nice geometrical corollary that this some is independent of the choice of the orthogonal basis.

With the help of a scalar product, one can associate to every bilinear function a linear transformation. For the case of the Riemannian metric and the Ricci tensor, we can find a unique $\mathcal{F}(M)$-linear transformation $\overline{\operatorname{Ric}}:\mathfrak{X}(M)\longrightarrow\mathfrak{X}(M)$ such that

$$\operatorname{Ric}(X,Y) = \langle X,\overline{\operatorname{Ric}}(Y)\rangle \text{ for every } X,Y\in\mathfrak{X}(M).$$

Definition. The scalar curvature s(p) of a Riemannian manifold M at a point p is the trace of the linear mapping $\overline{\operatorname{Ric}}:T_pM\longrightarrow T_pM$.

Let us find an expression for the scalar curvature in terms of the Ricci curvature and the sectional curvature. Let $e_1,...,e_n$ be an orthonormal basis in T_pM. Then

$$s(p) = \operatorname{trace} \overline{Ric} = \sum_{i=1}^{n} \langle \overline{Ric}(e_i), e_i \rangle = \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} r(e_i),$$

i.e. s(p) is the sum of Ricci curvatures in the directions of an orthogonal basis. Furthermore,

$$s(p) = \sum_{i=1}^{n} r(e_i) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} K(e_i, e_j) = 2 \sum_{1\leqslant i<j\leqslant n} K(e_i, e_j),$$

that is, the scalar curvature is twice the sum of sectional curvatures taken in the directions of all coordinate planes of an orthonormal coordinate system in T_pM.

To finish this unit with, let us study the curvature tensor of a hypersurface M in $\mathbb{R}^n$. As we observed at the end of the previous unit, the Levi-Civita connection $\tilde{\nabla}$ of a hypersurface can be expressed as $\tilde{\nabla} = \mathbf{P}\circ\partial$, where ∂ is the derivation rule of vector fields along the hypersurface as defined in Unit 5 (page 43), $\mathbf{P}$ is the orthogonal projection of a tangent vector of $\mathbb{R}^n$ at a hypersurface point onto the tangent space of the hypersurface at that point. ∂ is essentially the Levi Civita connection of $\mathbb{R}^n$, thus, as the curvature of $\mathbb{R}^n$ is 0,

$$\partial_X\circ\partial_Y-\partial_Y\circ\partial_X = \partial_{[X,Y]}$$

for any tangential vector fields $X,Y \in \mathfrak{X}(M)$.

We have

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z = \mathbf{P}(\partial_X\tilde{\nabla}_Y Z) &= \mathbf{P}(\partial_X(\partial_Y Z - \langle\partial_Y Z, \mathbf{N}\rangle\mathbf{N})) = \\ &= \mathbf{P}(\partial_X\partial_Y Z) - \mathbf{P}(X(\langle\partial_Y Z, \mathbf{N}\rangle)\,\mathbf{N}) - \mathbf{P}(\langle\partial_Y Z, \mathbf{N}\rangle\partial_X\mathbf{N}) \\ &= \mathbf{P}(\partial_X\partial_Y Z) - \langle\partial_Y Z, \mathbf{N}\rangle\partial_X\mathbf{N},\end{aligned}$$

where $X,Y,Z \in \mathfrak{X}(M)$.

Similarly,

$$\tilde{\nabla}_Y\tilde{\nabla}_X Z = \mathbf{P}(\partial_Y\partial_X Z) - \langle\partial_X Z, \mathbf{N}\rangle\partial_Y\mathbf{N}.$$

Combining these equalities with

$$\tilde{\nabla}_{[X,Y]}Z = \mathbf{P}(\partial_{[X,Y]}Z)$$

we get the following expression for the curvature tensor R of M

$$\begin{aligned}R(X,Y;Z) &= (\tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z) - \tilde{\nabla}_{[X,Y]}Z = \\ &= \mathbf{P}((\partial_X\partial_Y Z - \partial_Y\partial_X Z) - \partial_{[X,Y]}Z) - \langle\partial_Y Z, \mathbf{N}\rangle\partial_X\mathbf{N} + \langle\partial_X Z, \mathbf{N}\rangle\partial_Y\mathbf{N} \\ &= \langle\partial_X Z, \mathbf{N}\rangle\partial_Y\mathbf{N} - \langle\partial_Y Z, \mathbf{N}\rangle\partial_X\mathbf{N}.\end{aligned}$$

Since $\langle Z,\mathbf{N}\rangle$ is constant zero,

$$0 = X(\langle Z,\mathbf{N}\rangle) = \langle \partial_X Z,\mathbf{N}\rangle + \langle Z,\partial_X\mathbf{N}\rangle$$

and

$$0 = Y(\langle Z,\mathbf{N}\rangle) = \langle \partial_Y Z,\mathbf{N}\rangle + \langle Z,\partial_Y\mathbf{N}\rangle.$$

Putting these equalities together we deduce that

$$R(X,Y;Z) = \langle Z,\partial_Y\mathbf{N}\rangle\partial_X\mathbf{N} - \langle Z,\partial_X\mathbf{N}\rangle\partial_Y\mathbf{N} = \langle Z,L(Y)\rangle L(X) - \langle Z,L(X)\rangle L(Y).$$

Comparing the formula

$$R(X,Y;Z) = \langle Z,L(Y)\rangle L(X) - \langle Z,L(X)\rangle L(Y)$$

relating the curvature tensor to the Weingarten map on a hypersurface with Gauss' equations proved in unit 7 we see that the curvature tensor R coincides with the curvature tensor defined there. This way, the last equation can also be considered as a coordinate free display of Gauss' equations.

Further Exercises

Exercise 12-1. Consider tensors of valency (0,4) over an n-dimensional vector space V that satisfy

$$S(X,Y;Z,W) = - S(Y,X;Z,W) = - S(X,Y;W,Z);$$
$$\underset{XYZ}{\circlearrowleft} S(X,Y;Z,W) = 0.$$

Prove that these tensors form a linear space and determine the dimension of this space.

Exercise 12-2. Prove that if X_1 and X_2 are two nonparallel principal directions at a given point p of a hypersurface M, κ_1,κ_2 are the corresponding principal curvatures, then

$$K(X_1,X_2) = \kappa_1\kappa_2.$$

What is the minimum and maximum of K(X,Y), when X and Y run over T_pM?

Exercise 12-3. Express the Ricci curvature of a hypersurface in $\mathbb{R}^{n+1}$ in a principal direction in terms of the principal curvatures.

Exercise 12-4. Express the scalar curvature of a hypersurface in terms of the principal curvatures.