Unit 13. Geodesics

Definition of geodesics, normal coordinates, variation of a curve, a variation formula for the length, description of spheres about a point with the help of normal coordinates, minimal property of geodesics.

We define the length of a smooth curve $\gamma:[a,b]\longrightarrow M$ lying on a Riemannian manifold $(M,\langle,\rangle)$ to be the integral

$$\ell(\gamma) = \int_a^b \sqrt{\langle\gamma'(t),\gamma'(t)\rangle}\, dt.$$

It is worth mentioning that the classical definition of length as the limit of the lengths of inscribed broken lines does not make sense, since the distance of points is not directly defined. The situation is just the opposite. We can define first the length of curves as a primary concept and derive from it a so called intrinsic metric $d(p,q)$, at least for connected Riemannian manifolds. $d(p,q)$ is the infimum of the lengths of all curves joining p to q. The metric enables us to define the length of "broken lines" given just by a sequence of vertices $P_1,\dots,P_N$ to be the sum of the distances between consecutive vertices. There is a theorem saying that the length of a smooth curve $\gamma:[a,b]\longrightarrow M$ is equal to the limit of the lengths of inscribed broken lines $\gamma(t_0),\gamma(t_1),\dots,\gamma(t_N)$, $a=t_0<t_1<\dots<t_N=b$ as the maximum of the distances $|t_i-t_{i-1}|$ tends to zero.

To find the analog of straight lines in the intrinsic geometry of a Riemannian manifold we have to characterize straight lines in a way that makes sense for Riemannian manifolds as well. Since the length of curves is one of the most fundamental concepts of Riemannian geometry, we can take the following characterization: a curve is a straight line if and only if for any two points on the curve, the segment of the curve bounded by the points is the shortest among curves joining the two points. A slight modification of this property could be used to distinguish a class of curves, but it is not clear whether such curves exist at all on a general Riemannian manifold.

For a physicist a straight line is the trajectory of a particle with zero acceleration or that of a light beam. This observation can also lye in the base of a definition. We only have to find a proper generalization of "acceleration" for curves lying in a Riemannian manifold. It seems quite natural to proceed as follows. The speed vectors of a curve yield a vector field along the curve. On the other hand, by the fundamental theorem of Riemannian geometry, the Riemannian metric determines a unique affine connection on the manifold which is symmetric and compatible with the metric. This connection allows us to differentiate vector fields along a curve with respect to the curve parameter. In particular, one can differentiate the speed vector field with respect to the curve parameter and may call the result the acceleration vector (or acceleration vector field along the curve).

Definition. Let M be a Riemannian manifold, γ be a curve on it and denote by $\frac{D}{dt}$ the covariant differentiation of vector fields along γ induced by the Levi-Civita connection. We say that γ is a geodesic if

$$\frac{D}{dt}\gamma' = 0.$$

Remark. More generally, if (M,∇) is a manifold with an affine connection, then curves satisfying $\frac{D}{dt}\gamma' = 0$ are said to be autoparallel. Geodesics are autoparallel curves for the Levi-Civita connection.

Proposition. The length of the speed vector of a geodesic is constant.

Proof. By the compatibility of the connection with the metric, parallel translation preserves length and angles between vectors. The definition of geodesics implies that the speed vector field is parallel along the curve, consequently consists of vectors of the same length. ∎

The proposition follows also from the equality

$$\frac{d}{dt}\langle\gamma', \gamma'\rangle = \langle\frac{D}{dt}\gamma', \gamma'\rangle + \langle\gamma', \frac{D}{dt}\gamma'\rangle = 0 .$$

As a consequence, we get that the property of "being geodesic" is not invariant under reparametrization. The parameter t of a regular geodesic is always related to the natural parameter s through an affine linear transformation i.e. $t = as + b$ for some $a,b \in \mathbb{R}$. This motivates the following definition.

Definition. A regular curve on a Riemannian manifold is a pre-geodesic if its natural reparameterization is geodesic.

In terms of a local coordinate system with coordinates $x_1,\dots,x_n$ a curve γ determines (and is determined by) n smooth functions $\gamma_i = x_i \circ \gamma$ $1 \leqslant i \leqslant n$. The equation $\frac{D}{dt}\gamma' = 0$ then takes the form

$$\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij} \circ \gamma \, \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} = 0 \qquad \text{for } 1 \leqslant k \leqslant n .$$

The existence of geodesics depends, therefore, on the solutions of a certain system of second order differential equations.

Introducing the new functions $v_i = \frac{d\gamma_i}{dt}$ this system of n second order differential equations becomes a system of 2n first order equations

$$\begin{cases} \dfrac{d\gamma_k}{dt} = v_k \\ \\ \dfrac{dv_k}{dt} = - \displaystyle\sum_{i,j=1}^{n} \Gamma^k_{ij} \circ \gamma \, v_i v_j \end{cases} \qquad \text{for } 1 \leqslant k \leqslant n.$$

Applying the theorem the existence and uniqueness theorem for ordinary differential equations one obtains the following.

Proposition. For any point p on a Riemannian manifold M and for any tangent vector $X \in T_pM$, there exists a unique maximal geodesic γ defined on an interval containing 0 such that $\gamma(0) = p$ and $\gamma'(0) = X$.

If the maximal geodesic through a point p with initial velocity X is defined on an interval containing $[-\varepsilon,\varepsilon]$ then there is a neighborhood U of X in the tangent bundle such that every maximal geodesic started from a point q with initial velocity $Y \in T_qM$ is defined on $[-\varepsilon,\varepsilon]$.

Since a geodesic with zero initial speed can be defined on the whole real straight line, for each point p on the manifold one can find a positive δ such that for every tangent vector $X \in T_pM$ with $\| X \| < \delta$, the geodesic defined by the conditions $\gamma(0) = p$, $\gamma'(0) = X$ can be extended to the interval $[0,1]$.

This following notation will be convenient. Let $X \in T_pM$ be a tangent vector and suppose that there exists a geodesic $\gamma : [0,1] \longrightarrow M$ satisfying the conditions $\gamma(0) = p$, $\gamma'(0) = X$. Then the point $\gamma(1) \in M$ will be denoted by $\exp_p(X)$ and called the exponential of the tangent vector X.

Using the fact that for any positive c the curve $t \longmapsto \gamma(ct)$ is also a geodesic we see that the geodesic γ is described by the formula

$$\gamma(t) = \exp_p(tX).$$

As we have observed, $\exp_p(X)$ is defined provided that $\| X \|$ is small enough. In general however, $\exp_p(X)$ is not defined for large vectors X. This motivates the following.

Definition. A Riemannian manifold is geodesically complete if $\exp_p(X)$ is defined for all $p \in M$ and all vectors $X \in T_pM$. This is clearly equivalent to the following requirement that every geodesic segment should be possible to extend to an infinite geodesic.

Proposition. For a fixed point $p \in M$, the exponential map $\exp_p$ is a smooth map from an open neighborhood of $0 \in T_pM$ into the manifold. Furthermore, the restriction of it onto a (possibly even smaller) open neighborhood of $0 \in T_pM$ is a diffeomorphism.

Proof. Differentiability of the exponential mapping follows from the theorem on the differentiable dependence on the initial point for solutions of a system of ordinary differential equations. To show that $\exp_p$ is a local diffeomorphism, we only have to show that its derivative at the point $0 \in T_pM$ is a non-singular linear mapping (see Inverse Function Theorem). Since T_pM is a linear space, its tangent space $T_0(T_pM)$ at 0 can be identified with the vector space T_pM itself. Through this identification, the derivative of the exponential map at 0 maps $T_pM \approx T_0(T_pM)$ into T_pM. We show that this derivative is just the identity map of T_pM, hence non-singular. Let X be an element of the tangent space $T_pM \approx T_0(T_pM)$. To determine where X is taken by the derivative of the exponential mapping, we represent X as the speed vector of the curve $t \longmapsto \varphi(t) = tX$ at $t = 0$. The exponential mapping takes this curve to the geodesic curve $\gamma = \exp_p \circ \varphi$ $\gamma(t) = \exp_p(tX)$, the speed vector of which at $t = 0$ is X, so the derivative of the exponential map sends X to itself and this is what we claimed. ∎

By the proposition, we can introduce a local coordinate system, based on geodesics, about each point of the manifold as follows. We fix an *orthonormal* basis in the tangent space T_pM, which gives us an isomorphism $\iota : T_pM \longrightarrow \mathbb{R}^n$ that assigns to each tangent vector its components with respect to the basis, and then take $\iota \circ \exp_p^{-1}$. The map $\iota \circ \exp_p^{-1}$ is a diffeomorphism between an open neighborhood of p and that of the origin in $\mathbb{R}^n$, therefore, it is a smooth chart on M. Coordinate systems obtained this way are called normal coordinate systems, while the inverse of them we shall call normal parametrizations .

For a Riemannian manifold M, we can define the sphere of radius r centered at $p \in M$ as the set of points $q \in M$ such that $d(p,q) = r$, where $d(p,q)$ denotes the intrinsic distance of p and q. When the radius of the sphere is increasing, the topological type of the sphere changes at certain critical values of the radius. For small radii however, the intrinsic spheres are diffeomorphic to the ordinary spheres in $\mathbb{R}^n$, and what is more, we have the following.

Theorem. The normal parameterization of a manifold about a point p maps the sphere about the origin with radius r, provided that it is contained in the domain of the parameterization, diffeomorphically onto the intrinsic sphere centered at p with radius r .

We prove the theorem later.

Definition. A variation of a smooth curve $\gamma : [a,b] \longrightarrow M$ is a smooth mapping γ_* from the rectangular domain $[-\delta,\delta]\mathrm{x}[a,b]$ into M such that $\gamma_*(0,t) = \gamma(t)$ for all $t \in [a,b]$.

Given a variation of a curve we may introduce a one parameter family of curves γ_ε $\varepsilon \in [-\delta,\delta]$ by setting $\gamma_\varepsilon(t) = \gamma_*(\varepsilon,t)$. By our assumption these curves yield a deformation of the curve $\gamma_0 = \gamma$.

Theorem. Let γ_* be a variation of a geodesic γ. Let $\ell(\varepsilon)$ denote the length of the curve γ_ε. Then the following formula holds

$$\frac{d\ell}{d\varepsilon}(0) = < \frac{d\gamma_*}{d\varepsilon}(0,b), \frac{\gamma'(b)}{\|\gamma'(b)\|} > - < \frac{d\gamma_*}{d\varepsilon}(0,a), \frac{\gamma'(a)}{\|\gamma'(a)\|} >.$$

Proof. By the definition of the length of a curve, one has

$$\frac{d\ell}{d\varepsilon}(0) = \frac{d}{d\varepsilon}\int_a^b \left\| \frac{d\gamma_*}{dt}(\varepsilon,\tau)\right\| d\tau = \frac{d}{d\varepsilon}\int_a^b \sqrt{< \frac{d\gamma_*}{dt}(\varepsilon,\tau), \frac{d\gamma_*}{dt}(\varepsilon,\tau) >}\, d\tau$$

$$= \int_a^b \frac{d}{d\varepsilon}\sqrt{< \frac{d\gamma_*}{dt}(\varepsilon,\tau), \frac{d\gamma_*}{dt}(\varepsilon,\tau) >}\,\Bigg|_{\varepsilon=0} d\tau$$

$$= \int_a^b \frac{\frac{d}{d\varepsilon} < \frac{d\gamma_*}{dt}(\varepsilon,\tau), \frac{d\gamma_*}{dt}(\varepsilon,\tau) > \Big|_{\varepsilon=0}}{2\sqrt{< \frac{d\gamma_*}{dt}(0,\tau), \frac{d\gamma_*}{dt}(0,\tau) >}}\, d\tau .$$

With the help of the covariant differentiation induced by the Levi-Civita connection this expression can be written as follows.

$$\int_a^b \frac{\left< \frac{D}{d\varepsilon}\frac{d\gamma_*}{dt}(0,\tau), \frac{d\gamma_*}{dt}(0,\tau) \right>}{\| \gamma'(\tau) \|} d\tau = \int_a^b \left< \frac{D}{d\varepsilon}\frac{d\gamma_*}{dt}(0,\tau) , \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|} \right> d\tau$$

By the symmetry of the connection, this is equal to

$$\int_a^b \left< \frac{D}{dt}\frac{d\gamma_*}{d\varepsilon}(0,\tau) , \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|} \right> d\tau.$$

Let us observe, that the function $t \longmapsto \left< \frac{d\gamma_*}{d\varepsilon}(0,t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right>$ is a primitive function (=indefinite integral) for the function to be integrated. Indeed, the derivative of this function is

$$\frac{d}{dt} \left< \frac{d\gamma_*}{d\varepsilon}(0,t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right> =$$

$$= \left< \frac{D}{dt}\frac{d\gamma_*}{d\varepsilon}(0,t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right> + \left< \frac{d\gamma_*}{d\varepsilon}(0,t), \frac{D}{dt}\frac{\gamma'(t)}{\|\gamma'(t)\|} \right>,$$

but the second term on the right hand side is zero since γ is geodesic. Consequently,

$$\frac{d\ell}{d\varepsilon}(0) = \int_a^b \left< \frac{D}{dt}\frac{d\gamma_*}{d\varepsilon}(0,\tau) , \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|} \right> d\tau =$$

$$= \left< \frac{d\gamma_*}{d\varepsilon}(0,b), \frac{\gamma'(b)}{\|\gamma'(b)\|} \right> - \left< \frac{d\gamma_*}{d\varepsilon}(0,a), \frac{\gamma'(a)}{\|\gamma'(a)\|} \right>. \blacksquare$$

<u>Theorem</u>. Let M be a Riemannian manifold, $p \in M$, and denote by S_r the sphere of radius r in T_pM centered at the zero tangent vector. Presume r is chosen to be so small that the exponential mapping is a diffeomorphism on a ball containing S_r and denote the exponential image of S_r by $\tilde{S}_r$. Then for any $X \in S_r$ the radial geodesic $t \longmapsto \exp_p(tX)$ is perpendicular to $\tilde{S}_r$.

<u>Proof</u>. Every tangent vector of $\tilde{S}_p$ can be obtained as the speed vector of a curve $\exp_p \circ \beta$ where β is a curve in S_r passing through $\beta(0) = X$. Given such a curve, let us define a variation of the geodesic $\gamma : t \longmapsto \exp_p(tX)$ in the following way

$$\gamma_*(\varepsilon, t) := \exp_p(t\beta(\varepsilon)).$$

For a fixed ε, the curve γ_ε is a geodesic of length r so $\ell(\varepsilon)$ is constant. Thus, the previous theorem implies that

$$0 = \frac{d\ell}{d\varepsilon}(0) = \left\langle \frac{d\gamma_*}{d\ \varepsilon}(0,1),\ \frac{\gamma\ '(1)}{\|\gamma\ '(1)\|} \right\rangle - \left\langle \frac{d\gamma_*}{d\ \varepsilon}(0,0),\ \frac{\gamma\ '(0)}{\|\gamma\ '(0)\|} \right\rangle.$$

Since $\gamma_*(\varepsilon,0) := \exp_p(0\ \beta(\varepsilon)) = p$ and , $\gamma_*(\varepsilon,1) := \exp_p(\beta(\varepsilon))$ we have $\frac{d\gamma_*}{d\ \varepsilon}(0,0) = \mathbf{0}$ and $\frac{d\gamma_*}{d\ \varepsilon}(0,1) = (\exp_p\circ\beta)'(0)$, therefore, we get

$$0 = \left\langle (\exp_p\circ\beta)'(0)\ ,\ \frac{\gamma\ '(1)}{\|\gamma\ '(1)\|} \right\rangle\ ,$$

showing that γ intersects $\tilde{S}_r$ orthogonally. ▮

Now we are ready to prove the theorem saying that $\tilde{S}_r$ is a sphere in the intrinsic geometry of the manifold. It is clear that for any point q on $\tilde{S}_r$ $d(p,q) \leqslant r$, since the radial geodesic from p to r has length r, so all we need is the following.

<u>Theorem</u>. If $\tilde{\gamma} : [a,b] \longrightarrow M$ is an arbitrary curve connecting p to a point of $\tilde{S}_r$, then its length is $\geqslant r$.

<u>Proof</u>. We may suppose without loss of generality that $\tilde{\gamma}(b)$ is the only intersection point of the curve with $\tilde{S}_r$ and $\tilde{\gamma}\ (t) \neq p$ for $t > a$. Then there is a unique curve γ in the tangent space T_pM such that $\tilde{\gamma} = \exp_p\circ\gamma$. Let N denote the vector field on $T_pM - \{\mathbf{0}\}$ consisting of unit vectors perpendicular to the spheres centered at the origin. N is the gradient vector field of the function $f : X \longmapsto \|\ X\ \|$ on T_pM. The theorem above shows that the exponential map takes N into a unit vector field $\tilde{N}$ on M, perpendicular to the sets $\tilde{S}_t$.

We can estimate the length of a curve as follows

$$\ell(\gamma) = \int_a^b \|\tilde{\gamma}\ '(\tau)\|\ d\tau \geqslant \int_a^b \langle\ \tilde{\gamma}\ '(\tau),\ \tilde{N}(\tilde{\gamma}(\tau))\ \rangle\ d\tau.$$

Since $\langle\ \tilde{\gamma}\ '(\tau),\ \tilde{N}(\tilde{\gamma}(\tau))\ \rangle$ is the component parallel to $\tilde{N}(q)$ of the speed vector $\tilde{\gamma}\ '(\tau)$ with respect to the splitting $T_qM = \mathbb{R}\ \tilde{N}(q) \oplus T_q\tilde{S}_*$ at $q = \tilde{\gamma}\ (\tau)$, it is equal to the component parallel to N(X) of the speed vector $\gamma\ '(\tau)$ with respect to the splitting

$$T_X(T_pM) = \mathbb{R}\ N(X) \oplus T_XS_*$$

at $X = \gamma\ (\tau)$. Therefore,

$$\langle\ \tilde{\gamma}\ '(\tau),\ \tilde{N}(\tilde{\gamma}(\tau))\ \rangle = \langle\ \gamma\ '(\tau),\ N(\gamma(\tau))\ \rangle = \langle\ \gamma\ '(\tau),\ \mathrm{grad}\ f(\gamma(\tau))\ \rangle$$
$$= \frac{d}{d\ t}\ f\circ\gamma\ (\tau),$$

and

$$\int_a^b \langle \tilde{\gamma}'(\tau), \tilde{N}(\tilde{\gamma}(\tau)) \rangle \, d\tau = \int_a^b \frac{d}{dt} f\circ\gamma\,(\tau)\, d\tau = \|\gamma(b)\| - \|\gamma(a)\| = r. \blacksquare$$

The proof also shows that the equality $\ell(\gamma) = r$ holds only for curves perpendicular to the spheres $\tilde{S}_*$.

Exercise. Show that such curves are pre-geodesics.

Theorem. A smooth curve $\gamma : [a,b] \longrightarrow M$ parameterized by the natural parameter in a Riemannian manifold is geodesic if and only if there is a positive ε such that for any two values $t_1, t_2 \in [a,b]$ such that $|t_1 - t_2| < \varepsilon$ the restriction of γ onto $[t_1, t_2]$ is a curve of minimal length among curves joining $\gamma(t_1)$ to $\gamma(t_2)$.

Remark. It is not true in general, that a geodesic curve is the a curve of minimal length among curves joining the same endpoints. To see this, it is enough to consider a long arc on a great circle on the sphere.

Further Exercises

Exercise 13-1. Show that a regular curve in a hypersurface $M \subset \mathbb{R}^{n+1}$ is a geodesic if and only if its ordinary accaleration $\gamma''(t)$ is perpendicular to $T_{\gamma(t)}M$ for every t. γ is a pre-geodesic if and only if $\gamma''(t)$ is contained in the plane spanned by $\gamma'(t)$ and the normal vector of M at $\gamma(t)$.

Exercise 13-2. Show that great circles on the sphere and helices on a cylinder are pre-geodesics.

Exercise 13-3. Find a regular pre-geodesic on the cone $x^2+y^2=z^2$, different from straight lines.

Exercise 13-4. Show that straight lines on a hypersurface are pre-geodesics.

Exercise 13-5. Show that symmetry planes of a surface in $\mathbb{R}^3$ intersect the surface in pre-geodesic lines.

Exercise 13-6. Using the results of Exercise 7-2 write the differential equation of geodesics on a surface of revolution with respect to the usual parameterization. Derive from the equations *Claireaut's theorem*: For a pre-geodesic curve on a surface of revolution the quantity $d \cos\alpha$ is constant, where d denotes the distance of the curve point from the axis of symmetry, α is the angle between the speed vector of the curve and the circle of rotation passing through the curve point.