UNIT 2. Curvatures of a Curve

Convergence of k-planes, the osculating k-plane, curves of general type in $\mathbb{R}^n$, the osculating flag, vector fields, moving frames and Frenet frames along a curve, orientation of a vector space, the standard orientation of $\mathbb{R}^n$, the distinguished Frenet frame, Gram-Schmidt orthogonalization process, Frenet formulas, curvatures, invariance theorems, curves with prescribed curvatures.

One of the most important tools of analysis is linearization, or more generally, the approximation of general objects with easily treatable ones. E.g. the derivative of a function is the best linear approximation, Taylor's polynomials are the best polynomial approximations of the function around a point. Adapting this idea to the theory of curves, the following questions arise naturally. Given a curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ and a point $\gamma(\bar{t})$ on it, find the straight line or circle (or conic or polynomial curve of degree $\leqslant n$) that approximates the curve around $\gamma(\bar{t})$ best or find the k-plane (k-sphere, quadric surface etc.) that is tangent to the curve at $\gamma(\bar{t})$ with the highest possible order.

We shall deal now with the problem of finding the k-plane that fits to a curve at a given point best. The classical approach to this problem is the following. A k-plane is determined uniquely by (k+1) of its points that do not lie in a (k-1)-plane. Let us take k+1 points $\gamma(t_0), \gamma(t_1), \dots, \gamma(t_k)$ on the curve. If γ is a curve of "general type" then these points span a unique k-plane, which will be denoted by $A(t_0, \dots, t_k)$. The k-plane we look for is the limit position of the k-planes $A(t_0, \dots, t_k)$ as $t_0, \dots, t_k$ tend to $\bar{t}$. To properly understand the last sentence, we need a definition of convergence of k-planes.

Definition. Let $X_1, X_2, \dots;$ X be k-planes. We say that the sequence of k-planes $X_1, X_2, \dots$ tends to X if one can find points $\mathbf{p}_j \in X_j$ $\mathbf{p} \in X$ and linearly independent direction vectors $\mathbf{v}_1^j, \dots, \mathbf{v}_k^j$ of X_j and $\mathbf{v}_1, \dots, \mathbf{v}_k$ of X such that $\lim \mathbf{p}_j = \mathbf{p}$ and $\lim \mathbf{v}_i^j = \mathbf{v}_i$ for $i = 1, \dots, k$ as j tends to infinity.

Exercise.* Show that a sequence of k-planes can have at most one limit.

Solution. Suppose that the sequence $X_1, X_2, \dots$ has two limits X and Y. Then by the definition, one can find points $\mathbf{p}_j$, $\mathbf{q}_j \in X_j$ $\mathbf{p} \in X$ $\mathbf{q} \in Y$ and linearly independent direction vectors $\{\mathbf{v}_1^j, \dots, \mathbf{v}_k^j\}$ and $\{\mathbf{w}_1^j, \dots, \mathbf{w}_k^j\}$ of X_j and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of X and $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ of Y, such that $\lim \mathbf{p}_j = \mathbf{p}$, $\lim \mathbf{q}_j = \mathbf{q}$,

$\lim \mathbf{v}_i^j = \mathbf{v}_i$ and $\lim \mathbf{w}_i^j = \mathbf{w}_i$ for $i = 1,\dots,k$ as j tends to infinity. Since $\{\mathbf{v}_1^j,\dots,\mathbf{v}_k^j\}$ and $\{\mathbf{w}_1^j,\dots,\mathbf{w}_k^j\}$ span the same linear space, which contains the direction vector $\mathbf{p}_j-\mathbf{q}_j$, there exists a unique $k\times k$ matrix (a_{rs}^j) $1\leqslant r,s\leqslant k$ and a vector $(b_1^j,\dots,b_k^j)$ such that

$$\mathbf{v}_i^j = \sum_{s=1}^{k} a_{is}^j \mathbf{w}_s^j \quad \text{for } i=1,\dots,k. \qquad (*)$$

$$\mathbf{p}_j-\mathbf{q}_j = \sum_{s=1}^{k} b_s^j \mathbf{w}_s^j$$

The components a_{rs}^j of this matrix and the numbers b_s^j can be determined by solving the system (*) of linear equations, thus by Cramer's rule, they are rational functions (quotients of polynomials) of the components of the vectors $\mathbf{v}_i^j$ and $\mathbf{w}_i^j$. The denominator of the quotient expressing a_{rs}^j and b_r^j is a non-vanishing $k\times k$ minor of the matrix with rows $\mathbf{w}_1^j,\mathbf{w}_2^j,\dots,\mathbf{w}_k^j$. Using the facts that rational functions are continuous and that the denominator of the fraction expressing a_{rs}^j or b_r^j tends to a non-vanishing minor of the matrix with rows $\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k$ if we choose the minors mentioned above properly, we get that the limits $\lim a_{rs}^j = a_{rs}$, $\lim b_r^j = b_r$ exist as $j\to\infty$. Taking $j\to\infty$ in (*) we obtain

$$\mathbf{v}_i = \sum_{s=1}^{k} a_{is} \mathbf{w}_s \quad \text{for } i=1,\dots,k,$$

$$\mathbf{p}-\mathbf{q} = \sum_{s=1}^{k} b_s \mathbf{w}_s,$$

from which follows that the vectors $\mathbf{v}_1,\dots,\mathbf{v}_k$ and $\mathbf{w}_1,\dots,\mathbf{w}_k$ span the same linear space, i.e. the k-planes X and Y must be parallel and that the point $\mathbf{q}$ is a common point of them consequently X = Y. ▮

Remark. The set of all k-dimensional linear/affine subspaces of an n-dimensional linear space has a natural topology, which can easily be described with the help of the factor space topology construction.

Assume that a topological space (X,τ) is divided into a disjoint union of its subsets. Such a subdivision can always be thought of as a splitting of X into the equivalence classes of an equivalence relation ~ on X. Denoting by $Y = X/_\sim$ the set of equivalence classes we have a natural mapping $\pi: X\longrightarrow Y$ assigning to an element $x \in X$ its equivalence class $[x] \in Y$.

Proposition. The set

$$\tau' = \{\, U \subset Y : \pi^{-1}(U) \in \tau \,\}$$

is a topology on Y.

Proof. The proof follows from the following set theoretical identities.

i) $\pi^{-1}(\emptyset) = \emptyset, \quad \pi^{-1}(Y) = X$;

ii) $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$;

iii) $\pi^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} \pi^{-1}(U_i)$. ∎

<u>Definition</u>. The family τ' is called the <u>factor space topology</u> on Y.

Now consider the set

$$V(n,k) = \{(\mathbf{x}_1,\dots,\mathbf{x}_k) \in (\mathbb{R}^n)^k : \mathbf{x}_1,\dots,\mathbf{x}_k \text{ are linearly independent}\}.$$

$V(n,k)$ is an open subset in $(\mathbb{R}^n)^k = \mathbb{R}^{nk}$ hence it inherits a subspace topology from the standard topology of $\mathbb{R}^{nk}$. If we define two elements of $V(n,k)$ to be equivalent if they span the same k-dimensional linear subspace of $\mathbb{R}^n$, then the set $Gr(n,k) = V(n,k)/_\sim$ of equivalence classes is essentially the same as the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. This set becomes a topological space with the factor space topology. Topological spaces of the form $Gr(n,k)$ are called <u>Grassmann manifolds</u>.

<u>Exercise</u>. Show that $Gr(n,k)$ is homeomorphic to $Gr(n,n-k)$.

We can define <u>affine Grassmann manifolds</u> similarly. We set

$$\tilde{V}(n,k) = \{(\mathbf{x}_0,\dots,\mathbf{x}_k) \in (\mathbb{R}^n)^{k+1} : \mathbf{x}_0,\dots,\mathbf{x}_k \text{ are not in a (k-1)-plane}\},$$

furnish $\tilde{V}(n,k)$ with the subspace topology inherited from $\mathbb{R}^{n(k+1)}$ and define an equivalence relation on $\tilde{V}(n,k)$ by

$$(\mathbf{x}_0,\dots,\mathbf{x}_k) \sim (\mathbf{y}_0,\dots,\mathbf{y}_k) \Leftrightarrow \mathbf{x}_0,\dots,\mathbf{x}_k \text{ and } \mathbf{y}_0,\dots,\mathbf{y}_k \text{ span the same k-plane.}$$

$\tilde{V}(n,k)/_\sim$ is the set of affine k-dimensional subspaces that has the factor space topology on it.

<u>Exercise</u>. Show that convergence of k-planes as defined above is the same as convergence with respect to the topology we have just constructed.

<u>Definition</u>. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a curve, $\bar{t} \in (a,b)$, $1 \leqslant k \leqslant n$. If the k-planes $A(t_0,\dots,t_k)$ are defined for parameters close enough to $\bar{t}$ and their limit exists as $t_0,\dots,t_k \to \bar{t}$, then the limit is called the <u>osculating k-plane</u> of the curve γ at $\bar{t}$. The osculating 1-plane of a curve is just the <u>tangent</u> of the curve. (The word "osculate" comes from the Latin "osculari – to kiss").

The definition of the osculating k-plane is justified by intuition. It is the k-plane that passes through k+1 points "infinitely close" to a given point. However, it would not be easy to determine the osculating k-plane using directly its definition. Fortunately, we have the following theorem.

Theorem. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a smooth curve, $\bar{t} \in (a,b)$, $1 \leqslant k \leqslant n$. If the derivatives $\gamma'(\bar{t}), \gamma''(\bar{t}), \ldots, \gamma^{(k)}(\bar{t})$ are linearly independent, then the osculating k-plane of γ is defined at $\bar{t}$ and it is the k-plane that passes through $\gamma(\bar{t})$ with direction vectors $\gamma'(\bar{t}), \gamma''(\bar{t}), \ldots, \gamma^{(k)}(\bar{t})$.

To prove the theorem we need some preparation.

Definition. Let $f : [a,b] \longrightarrow \mathbb{R}^n$ be a vector valued function, $t_0, t_1, \ldots \in [a,b]$ are different numbers. The higher order divided differences or difference quotients are defined recursively

$$f_0(t_0) := f(t_0)$$

$$f_1(t_0,t_1) := \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

$$\ldots$$

$$f_k(t_0,t_1,\ldots,t_k) := \frac{f_{k-1}(t_1,\ldots,t_k) - f_{k-1}(t_0,\ldots,t_{k-1})}{t_k - t_0}$$

Exercise. Show that the k-th order divided difference is a symmetric function of the variables $t_0,\ldots,t_k$ and has the following explicit form

$$f_k(t_0,t_1,\ldots,t_k) = \sum_{i=0}^{k} f(t_i) \frac{1}{\omega'(t_i)},$$

where $\omega(t)$ is the polynomial $(t-t_0)(t-t_1)\ldots(t-t_k)$, hence

$$\omega'(t_i) = (t_i-t_0)(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_k).$$

Lemma. If $f : [a,b] \longrightarrow \mathbb{R}$ is a smooth function, then there exists a number $\xi \in [a,b]$ such that

$$f_k(t_0,t_1,\ldots,t_k) = \frac{f^{(k)}(\xi)}{k!}.$$

Proof. Let $P(t)$ be the polynomial of degree $\leqslant k$ for which $f(t_i) = P(t_i)$ for $i=0,1,\ldots,k$. Such a polynomial exists and is unique. P is unique, since if Q is also a polynomial of degree $\leqslant k$ such that $f(t_i) = P(t_i) = Q(t_i)$ for $i=0,\ldots,k$, then the polynomial $P-Q$ has degree $\leqslant k$ and $k+1$ roots, which is possible only in the case when $P-Q \equiv 0$, $P \equiv Q$. We show the existence of P by an explicit construction. Set

$$P_i(t) = \frac{(t-t_0)(t-t_1)\ldots(t-t_{i-1})(t-t_{i+1})\ldots(t-t_k)}{(t_i-t_0)(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_k)}.$$

Obviously, P_i is a polynomial of degree k such that $P_i(t_j) = \delta_{ij}$ (Kronecker δ symbol denotes $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ if $i \neq j$). Thus, the polynomial

$$P(t) = \sum_{i=0}^{k} f(t_i) P_i(t)$$

is good for our purposes.

The difference f-P has k+1 roots. Since by the mean value theorem the interval between two zeros of a smooth function contains an interior point at which the derivative vanishes, the derivative $f'-P'$ has at least k roots. Similarly, $f''-P''$ has at least k-1 roots, etc. $f^{(k)}-P^{(k)}$ vanishes at a certain point $\xi \in [a,b]$. The k-th derivative of a polynomial of degree k is k! times the coefficient of the highest power, from which

$$f^{(k)}(\xi) = P^{(k)}(\xi) = k! \sum_{i=0}^{k} f(t_i) \frac{1}{\omega'(t_i)} = k!\, f_k(t_0,t_1,\dots,t_k). \blacksquare$$

<u>Corollary</u>. Since ξ can be chosen from the interval spanned by the points $t_0,\dots,t_k$, if these points tend to $\bar{t} \in [a,b]$, then ξ also tends to $\bar{t}$, consequently $f_k(t_0,t_1,\dots,t_k) = \frac{f^{(k)}(\xi)}{k!}$ tends to $\frac{f^{(k)}(\bar{t})}{k!}$.

<u>Corollary</u>. Applying the previous corollary to the components of a vector valued function $f:[a,b]\longrightarrow \mathbb{R}^n$, we get $f_k(t_0,t_1,\dots,t_k)$ tends to $\frac{f^{(k)}(\bar{t})}{k!}$ as $t_0,\dots,t_k$ tend to $\bar{t}$.

<u>Proof of the theorem</u>. Let us recall that if $\mathbf{p}_0,\dots,\mathbf{p}_k$ are position vectors of k+1 points in $\mathbb{R}^n$, then the affine plane spanned by them consists of linear combinations the coefficients in which have sum equal to 1

$$A(\mathbf{p}_0,\dots,\mathbf{p}_k) = \{\alpha_0\mathbf{p}_0+\dots+\alpha_k\mathbf{p}_k \ :\ \alpha_0+\dots+\alpha_k=1\ \}.$$

The direction vectors of this affine plane are linear combinations $\alpha_0\mathbf{p}_0+\dots+\alpha_k\mathbf{p}_k$ such that $\alpha_0+\dots+\alpha_k=0$.

<u>Exercise</u>. Prove this.

We claim that if $\gamma:[a,b]\longrightarrow \mathbb{R}^n$ is a curve in $\mathbb{R}^n$, then $\gamma_k(t_0,\dots,t_k)$ is a direction vector of the affine linear subspace spanned by the points $\gamma(t_0),\dots\gamma(t_k)$. To see this, we have to show that $\sum_{i=0}^{k} \frac{1}{\omega'(t_i)} = 0$. Consider the function $f\equiv 1$ and construct the polynomial P of degree $\leqslant k$ which coincides with f at $t_0,\dots,t_k$ using the general formulae. By the above proposition, there exists a number ξ such that

$$0 = f^{(k)}(\xi) = P^{(k)}(\xi) = \sum_{i=0}^{k} \frac{1}{\omega'(t_i)} ,$$

as we wanted to show. This way, $\gamma(t_0)$ is a point and $\gamma_1(t_1,t_0),\dots,$ $\gamma_k(t_k,\dots,t_0)$ are direction vectors of the affine linear subspace spanned by the points $\gamma(t_0),\dots,\gamma(t_k)$. If γ is smooth, then $\gamma(t_0)$ tends to $\gamma(\bar{t})$, $\gamma_1(t_1,t_0)$ tends to $\gamma'(\bar{t})$, and so on, $\gamma_k(t_k,\dots,t_0)$ tends to $\gamma^{(k)}(\bar{t})/k!$ as the points $t_0,\dots,t_k$ tend to $\bar{t} \in [a,b]$. Since by our assumption the first k derivatives of γ are linearly independent at $\bar{t}$, so are the vectors $\gamma_1(t_1,t_0),\dots,\gamma_k(t_k,\dots,t_0)$ if $t_0,\dots,t_k$ are in a small neighborhood of $\bar{t}$ and in this case the k-plane $A(t_0,\dots,t_k)$ tends to the k-plane that passes through $\gamma(\bar{t})$ with direction vectors $\gamma'(\bar{t}),\gamma''(\bar{t})/2,\dots,\gamma^{(k)}(\bar{t})/k!$. ∎

Definition. Let V be an n-dimensional vector space. A flag in V is a sequence of linear subspaces $\{\mathbf{0}\}=V_0\subset V_1\subset\dots\subset V_n=V$ such that $\dim V_i = i$. An affine flag is a sequence of affine subspaces $A_0\subset A_1\subset\dots\subset A_n=V$ such that $\dim A_i = i$.

Definition. A curve $\gamma:[a,b]\longrightarrow\mathbb{R}^n$ is called a curve of general type in $\mathbb{R}^n$ if the first n-1 derivatives $\gamma'(t),\gamma''(t),\dots,\gamma^{(n-1)}(t)$ are linearly independent for all $t\in[a,b]$.

Definition. The osculating flag of a curve of general type at a given point is the affine flag consisting of the osculating k-planes for $k = 0,1,\dots,n-1$ and the whole space.

Our plan is the following. A curve of general type in $\mathbb{R}^n$ is not contained in any affine subspace of dimension $k < n-1$, so we may pose the question, how far is it from being contained in a k-plane. In other words, we want to measure the deviation of the curve from its osculating k-plane. One way to do this is that we measure how quickly the osculating flag rotates as we travel along the curve. Since the faster we travel along the curve the faster change we observe, it is natural to consider the speed of rotation of the osculating flag with respect to the unit speed parameterization of the curve. This will lead us to quantities that describe the way a curve is curved in space, which are called the curvatures of the curve. There is one question of technical character left: how can we measure the change of an affine subspace. This problem can be solved by introducing an orthonormal basis at each point in such a way that the first k basis vectors span the osculating k-plane at the point in question, then measuring the change of this basis.

Definition. A (smooth) vector field along a curve $\gamma:[a,b]\longrightarrow\mathbb{R}^n$ is a smooth mapping $\mathbf{v}:[a,b]\longrightarrow T_*\mathbb{R}^n$ such that $\mathbf{v}(t)\in T_{\gamma(t)}\mathbb{R}^n$ for all $t\in[a,b]$. (We shall deal only with smooth vector fields.)

Thus, $\mathbf{v}(t)$ is a vector based at $\gamma(t)$. If we forget about the initial point of $\mathbf{v}(t)$, which is, after all, determined by the parameter t, then $\mathbf{v}$ can be considered as simply a mapping from the parameter interval [a,b] to $\mathbb{R}^n$.

Definition. A moving frame along a curve $\gamma:[a,b]\longrightarrow\mathbb{R}^n$ is a collection of n vector fields $\mathbf{t}_1,\dots,\mathbf{t}_n$ along γ such that $\langle\mathbf{t}_i(t),\mathbf{t}_j(t)\rangle = \delta_{ij}$ for all $t\in[a,b]$.

There are many moving frames along a curve and most of them have nothing to do with the geometry of the curve. This is not the case for Frenet frames.

Definition. A moving frame $\mathbf{t}_1,\dots,\mathbf{t}_n$ along a curve γ is called a Frenet frame if for all k, $1\leqslant k\leqslant n$, $\gamma^{(k)}(t)$ is contained in the linear span of $\mathbf{t}_1(t),\dots,\mathbf{t}_k(t)$.

Exercise. Construct a curve which has no Frenet frame and one with infinitely many Frenet frames. Show that a curve of general type in $\mathbb{R}^n$ has exactly 2^n Frenet frames.

According to the exercise, a Frenet frame along a curve of general type is almost unique. To select a distinguished Frenet frame from among all of them, we use orientation.

Definition. Let $\mathbf{v}_1,\dots,\mathbf{v}_k$ and $\mathbf{w}_1,\dots,\mathbf{w}_k$ be two ordered bases of a linear space V. We say that they have the same orientation or they define the same orientation of V, if the kxk matrix (a_{ij}) defined by the system of equalities

$$\mathbf{v}_i = \sum_{j=1}^{k} a_{ij}\,\mathbf{w}_j \qquad \text{for } i = 1,2,\dots,k$$

has positive determinant.

Having the same orientation is an equivalence relation on ordered bases, and there are two equivalence classes. Fixing one of the equivalence classes the elements of which will be called then positively oriented bases is an orientation of V.

Definition. The standard orientation of $\mathbb{R}^n$ is the orientation defined by the ordered basis $\mathbf{e}_1,\dots,\mathbf{e}_n$, where $\mathbf{e}_i = (0,\dots,0,\underset{i\text{-th place}}{\hat{1}},0,\dots,0)$.

Definition. A Frenet frame $\mathbf{t}_1,\dots,\mathbf{t}_n$ of a curve γ of general type in $\mathbb{R}^n$ is called a distinguished Frenet frame if for all k, $1\leqslant k\leqslant n-1$, the vectors $\mathbf{t}_1(t),\dots,\mathbf{t}_k(t)$ have the same orientation in their linear span as the vectors $\gamma'(t),\dots,\gamma^{(k)}(t)$, and the basis $\mathbf{t}_1(t),\dots,\mathbf{t}_n(t)$ is positively oriented with respect to the standard orientation of $\mathbb{R}^n$.

Proposition. A curve of general type possesses a unique distinguished Frenet frame.

Proof. We can determine the first n-1 vector fields of the distinguished Frenet frame by application of the Gram-Schmidt orthogonalization process to the first n-1 derivatives of γ. According to this, we set

$$\mathbf{t}_1 = \frac{\gamma'}{\|\gamma'\|}.$$

Suppose that $\mathbf{t}_1,\dots,\mathbf{t}_k$ have already been defined. Then $\mathbf{t}_{k+1}$ must be of the form

$$\mathbf{t}_{k+1} = \beta\gamma^{(k+1)} + (\alpha_1\mathbf{t}_1+\dots+\alpha_k\mathbf{t}_k), \tag{$*$}$$

where the coefficients β, $\alpha_1,\dots,\alpha_k$ are to be determined. Taking the dot product of both sides of (*) with $\mathbf{t}_i$ $(1\leqslant i\leqslant k)$, we obtain

$$0 = \beta <\gamma^{(k+1)},\mathbf{t}_i> + \alpha_i,$$

consequently,

$$\mathbf{t}_{k+1} = \beta \left(\gamma^{(k+1)} -(<\gamma^{(k+1)},\mathbf{t}_1>\mathbf{t}_1+\dots+<\gamma^{(k+1)},\mathbf{t}_k>\mathbf{t}_k)\right).$$

The parameter β must be used to normalize the vector which stands on the right of it. Thus,

$$\beta = \pm \| \gamma^{(k+1)} -(<\gamma^{(k+1)},\mathbf{t}_1>\mathbf{t}_1+\dots+<\gamma^{(k+1)},\mathbf{t}_k>\mathbf{t}_k)\|^{-1}.$$

Exercise. Show that in order to get a distinguished Frenet frame, we have to choose a positive β, i.e.

$$\mathbf{t}_{k+1} = \frac{\left(\gamma^{(k+1)} -(<\gamma^{(k+1)},\mathbf{t}_1>\mathbf{t}_1+\dots+<\gamma^{(k+1)},\mathbf{t}_k>\mathbf{t}_k)\right)}{\|\gamma^{(k+1)} -(<\gamma^{(k+1)},\mathbf{t}_1>\mathbf{t}_1+\dots+<\gamma^{(k+1)},\mathbf{t}_k>\mathbf{t}_k)\|} .$$

To finish the proof, we have to show that given n-1 mutually orthogonal unit vectors $\mathbf{t}_1,\dots,\mathbf{t}_{n-1}$ in $\mathbb{R}^n$, there is a unique vector $\mathbf{t}_n$ for which the vectors $\mathbf{t}_1,\dots,\mathbf{t}_n$ form a positively oriented orthonormal basis of $\mathbb{R}^n$. The condition that a vector is perpendicular to $\mathbf{t}_1,\dots,\mathbf{t}_{n-1}$ is equivalent to a system of n-1 linearly independent linear equation, the solutions of which form a 1-dimensional linear subspace (a straight line). There are exactly two opposite unit vectors parallel to a given straight line, and exactly one of them will fulfill the orientation condition. (Replacing a vector of an ordered basis by its opposite changes the orientation.) ∎

Exercise. Show that if $\mathbf{e}_1,\dots,\mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$ and

$$\mathbf{t}_i=\alpha_{i1}\mathbf{e}_1+\dots+\alpha_{in}\mathbf{e}_n \text{ for } i=1,\dots,n-1,$$

then $\mathbf{t}_n$ can be obtained as the formal determinant of the matrix

$$\begin{vmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ & \dots & \\ \alpha_{(n-1)1} & \dots & \alpha_{(n-1)n} \\ \mathbf{e}_1 & & \mathbf{e}_n \end{vmatrix} .$$

Let γ be a curve of general type in $\mathbb{R}^n$ parameterized by arc length. Set

$$\dot{\mathbf{t}}_i = \sum_{j=1}^{n} \alpha_{ij}\, \mathbf{t}_i.$$

(By convention, derivation with respect to arc length is denoted by $\dot{}$ instead of ' .)

Proposition. Using the above notation, $\alpha_{ij} = 0$ provided $j > i+1$.

Proof. Since by construction the vector $\mathbf{t}_i$, $1\leqslant i\leqslant n-1$, is a linear combination of the vectors $\gamma^{\cdot},\dots,\gamma^{(i)}$, $\dot{\mathbf{t}}_i$ is a linear combination of the vectors $\gamma^{\cdot},\dots,\gamma^{(i+1)}$. Since the last vectors are linearly expressible in terms of the vectors $\mathbf{t}^{\cdot},\dots,\mathbf{t}^{(i+1)}$, this proves the proposition. ▌

Proposition. The matrix (α_{ij}) is skew-symmetric i.e. $\alpha_{ij} = -\alpha_{ji}$.

Proof. Since $\langle \mathbf{t}_i, \mathbf{t}_j\rangle \equiv \delta_{ij}$ is a constant function, differentiating we get

$$\langle \dot{\mathbf{t}}_i, \mathbf{t}_j\rangle + \langle \mathbf{t}_i, \dot{\mathbf{t}}_j\rangle = \alpha_{ij} + \alpha_{ji} = 0. \ ▌$$

Thus the only non-zero coefficients are $\alpha_{i,i+1} = -\alpha_{i+1,i}$. Setting

$$\kappa_1 = \alpha_{12},\ \kappa_2 = \alpha_{23}\ ,\dots,\ \kappa_{n-1} = \alpha_{n-1,n}$$

we therefore see that the following formulas hold

$$\begin{aligned}
\dot{\mathbf{t}}_1 &= \kappa_1\ \mathbf{t}_2\\
\dot{\mathbf{t}}_2 &= -\kappa_1\ \mathbf{t}_1 + \kappa_2\ \mathbf{t}_3\\
\dot{\mathbf{t}}_{n-1} &= -\kappa_{n-2}\ \mathbf{t}_{n-2} + \kappa_{n-1}\ \mathbf{t}_n\\
\dot{\mathbf{t}}_n &= -\kappa_{n-1}\ \mathbf{t}_{n-1}.
\end{aligned}$$

These formulas are called Frenet formulas for a curve in $\mathbb{R}^n$. The functions $\kappa_1,\dots,\kappa_{n-1}$ are called the curvatures of a curve.

Exercise. Show that $\kappa_1,\dots,\kappa_{n-2}$ are positive, while κ_{n-1} may have any sign.

Frenet formulas can easily be modified for smooth curves parameterized in an arbitrary way. If γ is a curve of general type, then it is regular, hence we may consider its reparameterization by arc length $\tilde{\gamma} = \gamma \circ s^{-1}$. If $\tilde{\kappa}_1,\dots,\tilde{\kappa}_{n-1}$ are the curvature functions of the unit speed curve $\tilde{\gamma}$ defined as above, then we define the curvature functions of the curve γ to be the functions $\kappa_1 = \tilde{\kappa}_1 \circ s,\dots,\kappa_{n-1} = \tilde{\kappa}_{n-1} \circ s$.

If $\mathbf{x}$ is an arbitrary vector field along the curve γ, s denotes the arc length parameter, t denotes an arbitrary parameter, then $\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \frac{d\mathbf{x}}{ds}\,\|\gamma'\|$. Thus, denoting by υ the length of the speed vector γ', we obtain the following more general formulas

$$\begin{aligned}
\mathbf{t}'_1 &= \upsilon\ \kappa_1\ \mathbf{t}_2\\
\mathbf{t}'_2 &= \upsilon\ (-\kappa_1\ \mathbf{t}_1 + \kappa_2\ \mathbf{t}_3)\\
&\quad \cdot\ \cdot\ \cdot\\
\mathbf{t}'_{n-1} &= \upsilon\ (-\kappa_{n-2}\ \mathbf{t}_{n-2} + \kappa_{n-1}\ \mathbf{t}_n)\\
\mathbf{t}'_n &= -\upsilon\ \kappa_{n-1}\ \mathbf{t}_{n-1}.
\end{aligned}$$

We formulate some theorems concerning the curvatures of a curve. The first two theorems are intuitively clear and can be proved mechanically. The third theorem is of great theoretical importance, the proof of which uses the existence and uniqueness theorem for the solution of linear differential equations.

Proposition. (Invariance under isometries) Let γ be a curve of general type in $\mathbb{R}^n$, $I:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ an isometry (distance preserving bijection). Then the curvature functions $\kappa_1,\dots,\kappa_{n-2}$ of the curves γ and $I\circ\gamma$ are the same. The last curvatures κ_{n-1} of these curves coincide if I is orientation preserving and they differ (only) in sign if I is orientation reversing.

Proposition. (Invariance under reparameterization) If $\tilde{\gamma}$ is a regular reparameterization of the curve γ i.e. $\tilde{\gamma} = \gamma\circ h$ for some strictly monotone function h, then the curvature functions of $\tilde{\gamma}$ and γ are related to one another by $\tilde{\kappa}_i = \kappa_i\circ h$.

Exercise.

a) Assume $h:[a,b]\longrightarrow [c,d]$ is a smooth bijection between the intervals $[a,b]$ and $[c,d]$ with $h'<0$, $\gamma:[c,d]\longrightarrow \mathbb{R}^n$ is a curve of general type. How are the curvatures of γ and $\tilde{\gamma} = \gamma\circ h$ related to one another?

b) Assume a curve γ of general type in $\mathbb{R}^n$ lies in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. Then we can compute the curvatures $\kappa_1,\dots,\kappa_{n-1}$ of this curve considering γ to be a curve in $\mathbb{R}^n$ and also we may compute the curvatures $\tilde{\kappa}_1,\dots,\tilde{\kappa}_{n-2}$ of this curve considering γ to be a curve in $\mathbb{R}^{n-1}$. What is the relationship between these two sets of numbers?

Theorem. Given n-2 positive smooth functions $\kappa_1,\dots,\kappa_{n-2}$ and a smooth function κ_{n-1} on an interval $[a,b]$, there exists a unit speed curve of general type in $\mathbb{R}^n$ the curvatures of which are the prescribed functions $\kappa_1,\dots,\kappa_{n-1}$. This curve is unique up to isometries of the space.

Further Exercises

2-1. Describe the curve, called astroid, given by the parameterization $\gamma:[0,2\pi]\longrightarrow \mathbb{R}^2$, $\gamma(t) = (\cos^3(t), \sin^3(t))$. Is the astroid a smooth curve? Is it regular? Is it a curve of general type? If the answer is no for a property, characterize those arcs of the astroid which have the property. Compute the length of the astroid. Show that the segment of a tangent lying between the axis intercepts has the same length for all tangents.

2-2. Show that curvatures $\kappa_1,\dots,\kappa_{n-2}$ of a curve of general type in $\mathbb{R}^n$ are positive.

2-3. Find distinguished Frenet's basis and the equation of the osculating 2-plane of the elliptical helix $t \longmapsto$ (a cos t, b sin t, c t) at the point (a,0,0) (a, b and c are given positive numbers).

2-4. Suppose that a curve of general type in $\mathbb{R}^n$ is contained in an n-1 dimensional affine subspace. Show that $\kappa_{n-1} \equiv 0$.