UNIT 3. PLANE CURVES

Explicit formulas for plane curves, rotation number of a closed curve, osculating circle, evolute, involute, parallel curves, "Umlaufsatz". Convex curves and their characterization, the Four Vertex Theorem.

This unit and the following one are devoted to the study of curves in low dimensional spaces. We start with plane curves.

A plane curve $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ is given by two coordinate functions.

$$\gamma(t) = (x(t) , y(t)) \qquad t \in [a,b].$$

The curve γ is of general position if the vector γ' is a linearly independent "system of vectors". Since a single vector is linearly independent if and only if it is non-zero, the condition of being a curve of general type is equivalent to regularity for plane curves. From this point on we suppose that γ is regular.

The Frenet vector fields $\mathbf{t}_1, \mathbf{t}_2$ are denoted by $\mathbf{t}$ and $\mathbf{n}$ in classical differential geometry and they are called the (unit) tangent and the (unit) normal vector fields of the curve. There is only one curvature function of a plane curve $\kappa = \kappa_1$. Frenet formulas have the form

$$\mathbf{t}' = \upsilon \kappa \mathbf{n} ,$$

$$\mathbf{n}' = - \upsilon \kappa \mathbf{t} ,$$

where $\upsilon = |\gamma '|$.

Now let us find explicit formulas for $\mathbf{t}$, $\mathbf{n}$, κ. Obviously,

$$\mathbf{t} = \frac{1}{\upsilon} (x',y') = \frac{1}{\sqrt{x'^2+y'^2}} (x',y').$$

The normal vector $\mathbf{n}$ is the last vector of the Frenet basis so it is determined by the condition that $(\mathbf{t},\mathbf{n})$ is a positively oriented orthonormal basis, that is, in our case, $\mathbf{n}$ is obtained from $\mathbf{t}$ by a right angled rotation of positive direction. The right angled rotation in the positive direction takes the vector (a,b) to the vector (-b,a) (check this!) thus

$$\mathbf{n} = \frac{1}{\upsilon} (-y',x') = \frac{1}{\sqrt{x'^2+y'^2}} (-y',x').$$

To express κ, let us start from the equation

$$\gamma' = \upsilon \mathbf{t}.$$

Differentiating and using the first Frenet formula,

$$\gamma'' = v'\,\mathbf{t} + v\,\mathbf{t}' = v'\mathbf{t} + v^2\kappa\,\mathbf{n}.$$

Taking dot product with $\mathbf{n}$ and using $\langle\mathbf{t},\mathbf{n}\rangle = 0$ we get

$$\langle\gamma'',\mathbf{n}\rangle = v^2\kappa\ ,$$

which gives

$$\kappa = \frac{\langle\gamma'',\mathbf{n}\rangle}{v^2} = \frac{-x''y' + y''x'}{v^3} = \frac{\det\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}} .$$

Now we establish some facts concerning the curvature of a plane curve which probably will contribute to a better understanding of its geometrical meaning.

First we shall consider curves in an arbitrary dimensional space and investigate the following question. Given a point on a curve, find the circle which approximates the curve around the point with the highest possible accuracy. We have already solved a similar problem for k-planes and the scheme of our approach works for circles as well.

Definition. Let $C_1,C_2,\ldots;C$ be circles in $\mathbb{R}^n$, S_i and S the planes of C_i and C resp., O_i and O the centers of C_i and C resp., r_i and r the radii of C_i and C resp. We say that the sequence $C_1,C_2,\ldots$ tends to the circle C if the limits $\lim S_i$, $\lim O_i$, $\lim r_i$ exist and equal to S, O, r respectively.

Definition. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a smooth curve, $\bar{t} \in [a,b]$, and for any three different arguments $t_1,t_2,t_3 \in [a,b]$, denote by $C(t_1,t_2,t_3)$ the circle passing through $\gamma(t_1),\gamma(t_2),\gamma(t_3)$, provided that these points do not lie in a straight line. If the circles $C(t_1,t_2,t_3)$ are defined if t_1,t_2,t_3 lie in a sufficiently small neighborhood of $\bar{t}$, and their limit $\lim C(t_1,t_2,t_3) = C$ exists as t_1,t_2,t_3 tend to $\bar{t}$, then C is called the osculating circle of the curve γ at the point $\bar{t}$.

Theorem. Suppose that $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is a smooth curve of general type, $\bar{t} \in [a,b]$ is such that $\kappa_1(\bar{t}) \neq 0$. Then the osculating circle of γ at $\bar{t}$ exists, its plane is the osculating 2-plane of γ at $\gamma(\bar{t})$, its center is the point $\gamma(\bar{t}) + \frac{1}{\kappa_1(\bar{t})}\,\mathbf{t}_2(\bar{t})$, its radius is $\frac{1}{|\kappa_1(\bar{t})|}$.

Proof. Let us show first that $\gamma(t_1),\gamma(t_2),\gamma(t_3)$ do not lie in a straight line if t_1,t_2,t_3 are in a small neighborhood of $\bar{t}$. We know that the first and second order differences $\gamma_1(t_1,t_2)$ and $\gamma_2(t_1,t_2,t_3)$ tend to $\gamma'(\bar{t})$ and $\gamma''(\bar{t})/2$ respectively as t_1,t_2,t_3 tend to $\bar{t}$. Since $\gamma'(\bar{t})$ and $\gamma''(\bar{t})/2$ are not parallel, (otherwise γ would not be of general type when $n\geq 3$ or the curvature

$\kappa_1(\bar{t})$ would be zero when n = 2), neither are $\gamma_1(t_1,t_2)$ and $\gamma_2(t_1,t_2,t_3)$ if t_1,t_2,t_3 are close enough to $\bar{t}$. In this case the affine subspace spanned by $\gamma(t_1),\gamma(t_2)$ and $\gamma(t_3)$ has two non-parallel direction vectors, $\gamma_1(t_1,t_2)$ and $\gamma_2(t_1,t_2,t_3)$, thus it can not be a straight line.

Now suppose that $t_1<t_2<t_3$ are sufficiently close to $\bar{t}$ to guarantee that the circle $C(t_1,t_2,t_3)$ exists and denote by $\mathbf{p}$ and r its center and radius.

The plane of $C(t_1,t_2,t_3)$ is the plane spanned by $\gamma(t_1),\gamma(t_2),\gamma(t_3)$ and this plane converges to the osculating plane at $\bar{t}$.

Set

$$F(t) = |\gamma(t) - \mathbf{p}|^2 - r^2.$$

Since F has three different roots, applying the mean value theorem we find two numbers ξ_1 and ξ_2 such that $t_1<\xi_1<t_2<\xi_2<t_3$ and $F'(\xi_1) = F'(\xi_2) = 0$. By another application of the mean value theorem for F' we find $\xi_1< \eta < \xi_2$ such that $F''(\eta) = 0$. The equations $F'(\xi_1) = F''(\eta) = 0$ give

$$2 \langle\gamma'(\xi_1),\gamma(\xi_1)-\mathbf{p}\rangle = 0, \qquad (*)$$

and

$$\langle\gamma''(\eta),\gamma(\eta)-\mathbf{p}\rangle + \langle\gamma'(\eta),\gamma'(\eta)\rangle = 0. \qquad (**)$$

The first equation means that $\mathbf{p}$ can be written in the form $\mathbf{p} = \gamma(\xi_1) + \rho\, \mathbf{m}$, where $\rho \in \mathbb{R}$, $\mathbf{m}$ is a unit vector perpendicular to $\gamma'(\xi_1)$. Choosing a right orientation for $\mathbf{m}$, we may also assume that $\langle\mathbf{m},\mathbf{t}_2(\bar{t})\rangle$ is not negative.

<u>Lemma</u>. $\mathbf{m}$ tends to $\mathbf{t}_2(\bar{t})$ as t_1,t_2,t_3 tend to $\bar{t}$.

To prove the lemma, let us write $\mathbf{m}$ as a linear combination of the Frenet basis at $\bar{t}$

$$\mathbf{m} = \sum \alpha_i \mathbf{t}_i(\bar{t}), \text{ where } \alpha_i = \langle\mathbf{m},\mathbf{t}_i(\bar{t})\rangle.$$

a) $\alpha_1 = \langle\mathbf{m},\mathbf{t}_1(\bar{t})\rangle = \langle\mathbf{m},\mathbf{t}_1(\bar{t})-\mathbf{t}_1(\xi_1)\rangle$ tends to 0 since $\mathbf{t}_1(\bar{t})-\mathbf{t}_1(\xi_1)$ tends to $\mathbf{0}$.

b) Let $i \geqslant 3$. Since $\tilde{\mathbf{m}} = \mathbf{m} + (\gamma(\xi_1)-\gamma(t_1))$ is a direction vector of the plane spanned by $\gamma(t_1),\gamma(t_2),\gamma(t_3)$, it can be expressed as a linear combination of the form

$$\tilde{\mathbf{m}} = \lambda\, \gamma_1(t_1,t_2) + \mu\, \gamma_2(t_1,t_2,t_3).$$

The length of $\tilde{\mathbf{m}}$ tends to 1, $\gamma_1(t_1,t_2)$ and $\gamma_2(t_1,t_2,t_3)$ converge to two linearly independent vectors, thus, for any sequence of triples t_1,t_2,t_3 tending to $\bar{t}$, the corresponding sequence of λ's and μ's remain bounded. On the other hand,

$$\lim \langle\gamma_1(t_1,t_2),\mathbf{t}_i(\bar{t})\rangle = \langle\gamma'(\bar{t}),\mathbf{t}_i(\bar{t})\rangle = 0,$$
$$\lim \langle\gamma_2(t_1,t_2,t_3),\mathbf{t}_i(\bar{t})\rangle = \frac{1}{2} \langle\gamma''(\bar{t}),\mathbf{t}_i(\bar{t})\rangle = 0,$$

which implies

$$\lim \alpha_i = \lim \langle\mathbf{m},\mathbf{t}_i(\bar{t})\rangle = \lim \langle\tilde{\mathbf{m}},\mathbf{t}_i(\bar{t})\rangle = 0 \quad \text{for } i \geqslant 3.$$

c) We have $\alpha_2 = \langle \mathbf{m}, \mathbf{t}_2(\bar{t})\rangle \geqslant 0$ and $\| \mathbf{m} \|^2 = \sum \alpha_i^2 = 1$ by our assumptions. By a) and b), α_2 must tend to 1. This finishes the proof of the Lemma.

Substituting $\mathbf{p} = \gamma(\xi_1) + \rho\, \mathbf{m}$ into (**) we get

$$\langle\gamma''(\eta),\gamma(\eta)-\gamma(\xi_1)-\rho\, \mathbf{m} + \langle\gamma'(\eta),\gamma'(\eta)\rangle = 0,$$

from which

$$\rho = \frac{\langle\gamma''(\eta),\gamma(\eta)-\gamma(\xi_1)\rangle + \langle\gamma'(\eta),\gamma'(\eta)\rangle}{\langle\gamma''(\eta),\mathbf{m}\rangle} .$$

Since ξ_1,ξ_2 and η are sandwiched between t_1 and t_3, they converge to $\bar{t}$ if t_1,t_2 and t_3 tend to $\bar{t}$. In this case $\mathbf{p}$ tends to the vector

$$\gamma(\bar{t}) + \frac{\langle\gamma''(\bar{t}),\gamma(\bar{t})-\gamma(\bar{t})\rangle + \langle\gamma'(\bar{t}),\gamma'(\bar{t})\rangle}{\langle\gamma''(\bar{t}),\mathbf{t}_2(\bar{t})\rangle}\,\mathbf{t}_2(\bar{t}) = \gamma(\bar{t}) + \frac{1}{\kappa_1(\bar{t})}\,\mathbf{t}_2(\bar{t}).$$

The radius of the circle $C(t_1,t_2,t_3)$ is the distance between $\gamma(t_1)$ and $\mathbf{p}$. This will obviously tend to the distance between $\gamma(\bar{t})$ and $\gamma(\bar{t}) + \frac{1}{\kappa_1(\bar{t})}\,\mathbf{t}_2(\bar{t})$, i.e. to $\frac{1}{|\kappa_1(\bar{t})|}$ as t_1,t_2,t_3 tend to $\bar{t}$. ▮

From now on we restrict our attention to plane curves.

Definition. The center of the osculating circle is called the center of curvature, the radius of the osculating circle is called the radius of curvature of the given curve at the given point.

Definition. The locus of the centers of curvature of a curve is called the evolute of the curve. The evolute is defined for arcs along which the curvature is not zero.

If $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ is a curve with a nowhere zero curvature function κ and unit normal vector field $\mathbf{n}$, then the evolute can be parameterized by the mapping $\tilde{\gamma} : [a,b] \longrightarrow \mathbb{R}^2$, $\tilde{\gamma} = \gamma + (1/\kappa)\, \mathbf{n}$.

Exercise. Show that the evolute of the ellipse $\gamma(t) = (a \cos t, b \sin t)$ is the "affine astroid" $\tilde{\gamma}(t) = \left(\frac{a^2-b^2}{a} \cos^3 t \;,\; \frac{b^2-a^2}{b} \sin^3 t \right)$.

The evolute of a curve was introduced by Huygens in connection with his investigations on the propagation of wave fronts. If we generate a curvilinear wave on the surface of calm water (e.g. we drop a wire into it), the wave starts moving. Mathematically, consecutive positions of the wave front are described by the parallel curves of the original curve.

Definition. Let γ be a regular plane curve with normal vector field $\mathbf{n}$. A parallel curve of γ is a curve of the form $\gamma_d = \gamma + d\, \mathbf{n}$, where $d \in \mathbb{R}$ is a fixed real.

Experiments on wave fronts show that even if the initial wave front is a smooth curve, singularities may appear on the wave front during its motion which move for a while and then disappear.

Definition. Let γ be a smooth parameterized curve, t a point of its domain. We say that $\gamma(t)$ (or t) is a singular point (or singular parameter) of the curve γ if $\gamma'(t) = \mathbf{0}$.

Proposition. Singular points of the parallel curves of a regular curve γ sweep out the evolute of γ.

Proof. Since $\gamma_d' = \gamma' + d\,\mathbf{n}' = v\,\mathbf{t} - v\,d\,\kappa\,\mathbf{t} = (1 - d\,\kappa)\,v\,\mathbf{t}$, singular parameters are characterized by $1 - d\,\kappa(t) = 0$. Then the corresponding singular points $\gamma_d(t) = \gamma(t) + (1/\kappa(t))\,\mathbf{n}(t)$ lie on the evolute of the curve. It is also easy to show that any evolute point is a singular point of a suitable parallel curve. ∎

Exercise. Study the singularities on the parallel curves of an ellipse.

Proposition. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ be a regular plane curve with curvature $\kappa \neq 0$ such that $\kappa' > 0$, and evolute $\tilde{\gamma} = \gamma + (1/\kappa)\,\mathbf{n}$. Then the normal of γ at $t \in [a,b]$ is tangent to the evolute at $\tilde{\gamma}(t)$ and the length of the arc of the evolute between $\tilde{\gamma}(t_1)$ and $\tilde{\gamma}(t_2)$, $t_1 < t_2$, is the difference of the radii of curvatures $1/\kappa(t_1) - 1/\kappa(t_2)$.

Proof. The speed vector of $\tilde{\gamma}$ is

$$\tilde{\gamma}' = \gamma' + (1/\kappa)'\,\mathbf{n} + (1/\kappa)\,\mathbf{n}' = v\,\mathbf{t} + (1/\kappa)'\,\mathbf{n} - (1/\kappa)\,v\,\kappa\,\mathbf{t} = (1/\kappa)'\,\mathbf{n}.$$

This equation shows that $\tilde{\gamma}$ is a regular curve and its tangent at $\tilde{\gamma}(t)$ is parallel to the normal $\mathbf{n}(t)$ of γ, which proves the first part of the proposition. As for the length of the evolute, it is equal to the integral

$$\int_{t_1}^{t_2} |\tilde{\gamma}'(\tau)|\,d\tau = \int_{t_1}^{t_2} |(1/\kappa)'(\tau)\,\mathbf{n}(\tau)|\,d\tau = \int_{t_1}^{t_2} |(1/\kappa)'(\tau)|\,d\tau = \int_{t_1}^{t_2} -(1/\kappa)'(\tau)\,d\tau =$$

$$= 1/\kappa(t_1) - 1/\kappa(t_2).$$ ∎

The above proposition gives a method to construct the curve γ from its evolute. Suppose for simplicity that $\kappa > 0$. Take a thread of length $1/\kappa(a)$ and fix one of its ends to $\tilde{\gamma}(a)$. Then pulling the other end of the thread wrap it on the curve $\tilde{\gamma}$ starting from $\gamma(a)$. By the proposition, the moving end of the thread will slip along γ. Mathematically, the thread construction gives an involute of a curve.

Definition. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ be a unit speed curve with unit tangent vector field $\mathbf{t}$. An involute of the curve γ is a curve $\hat{\gamma}$ of the form $\hat{\gamma}(s) = \gamma(s) + (\ell - s)\,\mathbf{t}(s)$, where ℓ is a given real number.

A curve has many involutes corresponding to the different choices of the length ℓ of the thread.

Corollary. A curve satisfying the conditions of the previous proposition is an involute of its evolute (more exactly a reparameterization of it).

Proposition. Let γ be a unit speed curve with $\kappa > 0$, $\hat{\gamma}(s) = \gamma(s)+(\ell-s)\mathbf{t}(s)$ an involute of it such that ℓ is greater than the length of γ. Then the evolute of $\hat{\gamma}$ is γ.

Proof. We have

$$\hat{\gamma}'(s) = \mathbf{t}(s) - \mathbf{t}(s) + (\ell-s)\,\kappa(s)\,\mathbf{n}(s) = (\ell-s)\,\kappa(s)\,\mathbf{n}(s),$$
$$\hat{\gamma}''(s) = [(\ell-s)\,\kappa(s)]'\,\mathbf{n}(s) - (\ell-s)\,\kappa^2(s)\,\mathbf{t}(s).$$

The first equation implies that the Frenet frame $\hat{\mathbf{t}},\hat{\mathbf{n}}$ of $\hat{\gamma}$ is related to that of γ by $\hat{\mathbf{t}} = \mathbf{n}$, $\hat{\mathbf{n}} = -\mathbf{t}$. Computing the curvature $\hat{\kappa}$ of $\hat{\gamma}$,

$$\hat{\kappa}(s) = \frac{\langle\hat{\gamma}''(s),\hat{\mathbf{n}}(s)\rangle}{|\hat{\gamma}'(s)|^3} = \frac{(\ell-s)^2\kappa^3(s)}{(\ell-s)^3\kappa^3(s)} = \frac{1}{(\ell-s)}\ .$$

Thus, the evolute of $\hat{\gamma}$ is

$$\hat{\gamma} + (1/\hat{\kappa})\,\hat{\mathbf{n}} = \gamma + (\ell-s)\,\mathbf{t} - (\ell-s)\,\mathbf{t} = \gamma\ .\blacksquare$$

We formulate some further results on the evolute and involute as an exercise.

Exercise.

a) Suppose that the regular curves γ_1 and γ_2 have regular evolutes. Show that γ_1 and γ_2 are parallel if and only if their evolutes are the same.

b) Show that if two involutes of a regular curve are regular, then they are parallel.

Let $\gamma\colon[a,b]\longrightarrow \mathbb{R}^2$ be a *unit speed* curve. The direction angle $\alpha(s)$ of the tangent $\mathbf{t}(s)$ is determined only up to an integer multiple of 2π, however one can see easily the existence of a *differentiable* function $\alpha : [a,b]\longrightarrow \mathbb{R}$ such that $\alpha(s)$ is a direction angle of $\mathbf{t}(s)$ for all $s \in [a,b]$. Then

$$\mathbf{t}(s) = \big(\cos\alpha(s),\sin\alpha(s)\big).$$

Differentiating with respect to s,

$$\dot{\mathbf{t}} = \dot{\alpha}(-\sin\alpha,\cos\alpha).$$

If we compare this equality with Frenet equations, we see immediately that

$$\kappa = \dot{\alpha}\ ,$$

i.e. the curvature is the derivative of the direction angle of the tangent vector with respect to the arc length.

Definition. The total curvature of a curve is the integral of its curvature function with respect to arc length

$$\int_a^b \kappa(s)\,ds\ .$$

By the relation $\kappa = \dot{\alpha}$, the total curvature of a curve is $\alpha(b) - \alpha(a)$, thus it measures the rotation made by the tangent vector during the motion along the curve from the initial point to the end of it.

Definition. A curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is a smooth closed curve, if there exists a smooth mapping $\tilde{\gamma} : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\tilde{\gamma}|_{[a,b]} = \gamma$, and $\tilde{\gamma}$ is periodic with period b-a $\tilde{\gamma}(t + b - a) \equiv \tilde{\gamma}(t)$.

In other words, a curve is closed if it returns to its initial point in such a way that arriving at the end of the curve one can smoothly go through from the end to the beginning and thus continue the motion periodically until infinity. Since the tangent vector of a smooth closed curve is the same at the endpoints $\gamma(a)$ and $\gamma(b)$, the direction angles $\alpha(b)$ and $\alpha(a)$ differ in an integer multiple of 2π.

Definition. The integer $\big(\alpha(b)-\alpha(a)\big)/2\pi$ is called the rotation number of the closed curve γ.

Exercise. Construct a closed curve with an arbitrarily given rotation number $k \in \mathbb{Z}$.

Solving the exercise we may see that all our efforts to construct a curve with rotation number $\neq \pm 1$ having no self-intersection are in vain. The reason for this is the famous "Umlaufsatz" (rotation number theorem).

Definition. A curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is called simple if it has no self-intersection, i.e. $\gamma(t) \neq \gamma(t')$ whenever $t \neq t'$; γ is a simple closed curve if it is closed and we may have $\gamma(t) = \gamma(t')$ for $t \neq t'$ only in the case $\{t,t'\} = \{a,b\}$.

Theorem. (Umlaufsatz) The rotation number of a simple closed curve in the plane is equal to ± 1, or equivalently the total curvature of a simple closed curve is $\pm 2\pi$ hence independent of the actual shape of the curve!

The proof of theorem will use some new concepts borrowed from topology.

Definition. Let S^1 denote the unit circle $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$. The mapping $\pi : \mathbb{R} \longrightarrow S^1$ $\pi(t) = (\cos t, \sin t)$ is called the universal covering map of S^1. Given a continuous mapping $\varphi : X \longrightarrow S^1$ from a topological space to S^1, we say that φ has a lifting to $\mathbb{R}$ if there exist a continuous mapping $\bar{\varphi} : X \longrightarrow \mathbb{R}$ such that $\varphi = \bar{\varphi} \circ \pi$.

Exercise. Construct a continuous mapping from a topological space into the circle which has no lifting.

Lemma. Suppose that the image of the mapping $\phi : X \longrightarrow S^1$ does not cover the point $(\cos \alpha, \sin \alpha) \in S^1$ and that the restriction φ of ϕ onto a subspace

$Y \subset X$ has got a lifting $\bar{\varphi}$ such that $\bar{\varphi}(Y)$ is contained in an interval of the form $(\alpha+2k\pi, \alpha+2(k+1)\pi)$, $k \in \mathbb{Z}$. Then $\bar{\varphi}$ can be extended to a lifting $\bar{\phi}$ of ϕ. If furthermore Y is non-empty, X is path connected (i.e. any two points of X can be connected by a continuous curve lying in X) then the lifting $\bar{\phi}$ is unique, and maps X into the interval $(\alpha+2k\pi, \alpha+2(k+1)\pi)$.

Proof. The restriction of π onto $(\alpha+2k\pi, \alpha+2(k+1)\pi)$ is a homeomorphism between $(\alpha+2k\pi, \alpha+2(k+1)\pi)$ and $S^1 \setminus \{(\cos\alpha, \sin\alpha)\}$. Thus $\bar{\phi}$ can be defined as $\bar{\phi} = (\pi|_{(\alpha+2k\pi, \alpha+2(k+1)\pi)})^{-1} \circ \phi$. If X is path connected, then so is its image under a continuous lifting $\bar{\phi}$. Consequently $\bar{\phi}(X)$ must be contained in one of the intervals $(\alpha+2k\pi, \alpha+2(k+1)\pi)$. If Y is non-empty then k is uniquely determined, hence $\bar{\phi}$ must have the form $(\pi|_{(\alpha+2k\pi, \alpha+2(k+1)\pi)})^{-1} \circ \phi$. ▮

Proposition. Any continuous mapping $\varphi : [a,b] \longrightarrow S^1$ from an interval into the circle has got a lifting.

Proof. Choose a partition $a = t_0 < t_1 < \ldots < t_k = b$ of the interval $[a,b]$ fine enough to insure that the restriction of φ onto $[t_i, t_{i+1}]$ does not cover the whole circle. This is possible because of the continuity of φ. Then using the lemma we can define a lifting $\bar{\varphi}$ of φ step by step, extending $\bar{\varphi}$ recursively to $[a, t_i]$, $i=1,2,\ldots,k$. ▮

Remark. If φ is differentiable, then so is its lifting.

Proposition. Any continuous mapping $\varphi : T \longrightarrow S^1$ from a rectangle $T = [a,b] \times [c,d]$ into the circle has got a lifting to $\mathbb{R}$.

Proof. Divide the rectangle into nxn small rectangles so that the image of any of the small rectangles does not cover the circle. Then we may define a lifting $\bar{\varphi}$ of φ recursively, applying at each step the lemma above. $\bar{\varphi}$ can be defined first on the small rectangles of the first row going from left to right then on the rectangles of the second row, etc.

1	2	3	...	n
n+1	n+2	n+3	...	2n
	.	.	. .	

Proof of the "Umlaufsatz". Let us choose a point p on the regular simple closed curve γ in such a way that the curve is contained on one side of the tangent at p and parameterize the curve by arc length starting from p. Denoting this parameterization also by $\gamma : [0,\ell] \longrightarrow \mathbb{R}^2$, we define a mapping $\varphi : [0,\ell] \times [0,\ell] \longrightarrow S^1$ by

$$\varphi(t_1,t_2) = \begin{cases} -\gamma'(0)/\|\gamma'(0)\| & \text{if } t_1=\ell,\ t_2=0, \\ \gamma'(0)/\|\gamma'(0)\| & \text{if } t_1=0,\ t_2=\ell, \\ \dfrac{\gamma(t_1)-\gamma(t_2)}{\|\gamma(t_1)-\gamma(t_2)\|} & \text{if } t_1>t_2 \text{ and } \{t_1,t_2\}\neq\{0,\ell\}, \\ \gamma'(t_1)/\|\gamma'(t_1)\| & \text{if } t_1=t_2, \\ \dfrac{\gamma(t_2)-\gamma(t_1)}{\|\gamma(t_2)-\gamma(t_1)\|} & \text{if } t_2>t_1 \text{ and } \{t_1,t_2\}\neq\{0,\ell\}. \end{cases}$$

It is easy to see that φ is continuous, so it has a continuous lifting $\bar{\varphi}$. If the function $\alpha\colon[0,\ell]\longrightarrow\mathbb{R}$ is defined by $\alpha(t) = \bar{\varphi}(t,t)$, then $\alpha(t)$ is a direction angle of the speed vector $\mathbf{t}(t)$ of γ thus $\alpha(\ell)-\alpha(0)$ is 2π times the rotation number of γ. Consider the functions $\xi(t) = \bar{\varphi}(0,t)$ and $\vartheta(t) = \bar{\varphi}(t,\ell)$. $\xi(t)$ is a direction angle of the unit vector $\gamma(t)-p\ /\ \|\gamma(t)-p\|$, $\vartheta(t)$ is a direction angle of its opposite. Thus ξ and ϑ differ only in a constant of the form $(2k+1)\pi$. Since the vectors $\gamma(t)-p\ /\ \|\gamma(t)-p\|$ point in a half-plane bounded by the tangent at p, the image of ξ is contained in an open interval of length 2π (see lemma). Thus $\xi(\ell)-\xi(0)$, which has obviously the form $(2m+1)\pi$ for some $m\in\mathbb{Z}$, must be equal to $\pm\pi$. Hence, we conclude that

$$\alpha(\ell)-\alpha(0) = \bar{\varphi}(\ell,\ell)-\bar{\varphi}(0,0) = \big(\bar{\varphi}(\ell,\ell)-\bar{\varphi}(0,\ell)\big) + \big(\bar{\varphi}(0,\ell)-\bar{\varphi}(0,0)\big) =$$
$$= \vartheta(\ell)-\vartheta(0) + \xi(\ell)-\xi(0) = 2\big(\xi(\ell)-\xi(0)\big) = \pm 2\pi.$$ ▮

Remark. With more work but using essentially the same idea, one can generalize the "Umlaufsatz" for piecewise smooth closed simple curves. The generalization says that for a simple closed polygon with smooth curvilinear edges, the sum of oriented external angles plus the sum of the total curvatures of the edges equals $\pm 2\pi$.

Definition. A simple closed curve γ is convex, if for any point $P=\gamma(\bar{t})$, the curve lies on one side of the tangent to γ at P. In other words the function $\langle\gamma(t),\underline{n}(\bar{t})\rangle$ must be $\geqslant 0$ or $\leqslant 0$ for all t.

Exercise. Show that a simple closed curve is convex if and only if every arc of the curve lies on one side of the straight line through the endpoints of the arc.

Convex curves can be characterized with the help of the curvature function.

Theorem. A simple closed curve is convex if and only if $\kappa\geqslant 0$ or $\kappa\leqslant 0$ everywhere along the curve.

Proof. Assume first that γ is a naturally parameterized convex curve. Let $\alpha(t)$ be a continuous direction angle for the tangent $\underline{t}(t)$. As we know,

$\alpha^{\cdot} = \kappa$, thus, it suffices to show that α is a weakly monotonous function. This follows if we show that if α takes the same value at two different parameters t_1, t_2, then α is constant on the interval $[t_1, t_2]$.

The rotation number of a simple curve is ± 1, hence the image of $\underline{t}$ covers the whole unit circle. As a consequence, we can find a point at which

$$\underline{t}(t_3) = -\underline{t}(t_1) = -\underline{t}(t_2).$$

If the tangent lines at t_1, t_2, t_3 were different, then one of them would be between the others and this tangent would have points of the curve on both sides. This contradicts convexity, hence two of these tangents say the tangents at $P = \gamma(t_i)$ and $Q = \gamma(t_j)$ coincide.

We claim that the segment $\overline{PQ}$ is an arc of γ. It is enough to prove that this segment is in the image of γ. Assume to the contrary that a point $X \in \overline{PQ}$ is not covered by γ. Drawing a line $e \neq PQ$ through X, we can find at least two intersection points R and S of e and the curve, since e separates P and Q and γ has two essentially disjoint arcs connecting P to Q. Since PQ is a tangent of γ, the point R and S must lie on the same side of it. As a consequence, we get that one of the triangles PQR and PQS, say the first one is inside the other. However, this leads to a contradiction, since for such a configuration the tangent through S necessarily separates two vertices of the triangle PQR, which lie lie on the curve.

If γ is defined on the interval $[a,b]$, then $\gamma(a) = \gamma(b)$ is either on the segment $\overline{PQ}$ or not. The first case is not possible, because then α would be constant on the intervals $[a, t_1]$ and $[t_2, b]$, yielding

$$\alpha(a) = \alpha(t_1) = \alpha(t_2) = \alpha(b)$$

and

$$\text{rotation number} = (\alpha(b) - \alpha(b))/2\pi = 0.$$

In the second case α is constant on the interval $[t_1, t_2]$, as we wanted to show.

Now to prove the converse, assume that γ is a simple closed curve with $\kappa \geqslant 0$ everywhere and assume to the contrary that γ is not convex (the case $\kappa \leqslant 0$ can be treated analogously). Then we can find a point $P = \gamma(t_1)$, such that the tangent at P has curve points on both of its sides. Let us find on each side a curve point, say $Q = \gamma(t_2)$ and $R = \gamma(t_3)$ respectively, lying at maximal distance from the tangent at P. Then the tangents at P, Q and R are different and parallel. Since the unit tangent vectors $\underline{t}(t_i)$, $i=1,2,3$ have parallel directions, two of them, say $\underline{t}(t_i)$ and $\underline{t}(t_j)$ must be equal. $A = \gamma(t_i)$ and $B = \gamma(t_j)$ divide the curve into two arcs. Denoting by K_1 and K_2

the total curvatures of these arcs, we deduce that these total curvatures have the form $K_1 = 2k_1\pi$, $K_2 = 2k_2\pi$, where $k_1, k_2 \in \mathbb{Z}$, since the unit tangents at the ends of the arcs are equal. On the other hand, we have $k_1 + k_2 = 1$ by the Umlaufsatz and $k_1 \geqslant 0$, $k_2 \geqslant 0$ by the assumption $\kappa \geqslant 0$. This is possible only if one of the total curvatures K_1 or K_2 is equal to zero. Since $\kappa \geqslant 0$, this means that $\kappa = 0$ along one of the arcs between A and B. But then this arc would be a straight line segment, implying that the tangents at A and B coincide. The contradiction proves the theorem. ∎

Definition. A point $\gamma(t)$ of a regular plane curve γ is called a vertex if $\kappa'(t) = 0$.

Vertices of a curve correspond to the singular points of the evolute.

By compactness, the curvature function of a closed curve attains somewhere its maximum and minimum, hence every closed curve has at least two vertices.

Exercise. Find a parameterization of Bernulli's lemniscate

$$\{P \in \mathbb{R}^2 : \overline{PA}\ \overline{PB} = 1/4\ (\overline{AB})^2\},$$

where $A \neq B$ are given points in the plane, plot the curve and show that it is a closed curve with exactly two vertices. Determine the rotation number of the lemniscate.

Theorem (Four Vertex Theorem). A convex closed curve has at least 4 vertices.

This result is sharp, since an ellipse has exactly four vertices.

Proof. Local maxima and minima of the curvature function yield vertices. One can always find a local minimum on an arc bounded by two local maxima, hence if we have two local maxima or minima of the curvature then we must have at least four vertices. Thus we have to exclude the case when the curvature function has one absolute maximum at A and one absolute minimum at B, and strictly monotonous on the arcs bounded by A and B. In this case, choose a coordinate system with origin at A and x-axis AB.

The arcs of the curve bounded by A and B do not cut the x-axis at points other then A and B. Indeed, if there were a further intersection point C, then the curve would be split into three arcs by A,B and C in such a way that on each arc we could find a point at which the tangent to the curve is parallel to the straight line ABC. If the three tangent at these points were different then the one in the middle would cut the curve apart contradicting to convexity, if two of the tangents coincided then we could find a straight line segment contained in the curve yielding an infinite number of vertices.

If the two arcs bounded by A and B lied on the same side of AB then the line AB would be a common tangent of the curve at A and B. In this case the segment $\overline{AB}$ would be contained in the curve, yielding an infinite number of vertices as before.

We conclude that for a suitable orientation of the y-axis, $y(t)\kappa'(t)\geqslant 0$ for every $t \in [a,b]$, where $\gamma(t) = (x(t),y(t))$ $t \in [a,b]$ is a unit speed parameterization of the curve. Hence we get

$$\int_a^b y(t)\kappa'(t)dt > 0.$$

Integrating by parts,

$$\int_a^b y(t)\kappa'(t)dt = \Big[y(t)\kappa(t)\Big]_a^b - \int_a^b y'(t)\kappa(t)dt = -\int_a^b y'(t)\kappa(t)dt.$$

The unit tangent vector field of the curve is $\underline{t} = (x',y')$, the unit normal vector field is $\underline{n} = (-y',x')$, hence by the first Frenet formula,

$$x'' = -\kappa y'.$$

Integrating,

$$-\int_a^b y'(t)\kappa(t)dt = \int_a^b x''(t)dt = \Big[x''(t)\Big]_a^b = 0.$$

This is a contradiction since a positive number can not be equal to 0. ∎

Further Exercises

3-1. Find the points on the ellipse $\gamma(t) = (a\cos t, b\sin t)$ at which the curvature is minimal or maximal $(a > b > 0)$.

3-2. The curve "cardioid" is the trajectory of a peripheral point of a circle rolling about a fixed circle of the same radius.

- Find a smooth parameterization of the cardioid.
- Compute its length.
- Show that its evolute is also a cardioid.

3-3. The "chain curve" is the graph of the hyperbolic cosine function

$$\mathrm{ch}(x) = \frac{e^x + e^{-x}}{2}.$$

- Determine the involute of the chain curve touching the chain curve at (0,1). (This curve is called "tractrix".)

- Let the tangent of the tractrix at P intersect the x-axis at Q. Show that the segment PQ has unit length.

3-4. Let γ be a simple regular closed curve of length ℓ with curvature function κ. Choose a real number d such that $1 \geqslant \kappa\, d$. How long is the parallel curve $\gamma_d = \gamma + d\,\mathbf{n}$?

3-5. Let γ be a regular plane curve for which the curvature function and its derivative are positive. Show that for any $t_1<t_2$ from the parameter domain of γ the osculating circle of γ at $\gamma(t_1)$ contains the osculating circle at $\gamma(t_2)$.