UNIT 4. 3D CURVES - CURVES ON HYPERSURFACES

Explicit formulas, projections of a space curve onto the coordinate planes of the Frenet basis, the shape of a curve around one of its points, hypersurfaces, regular hypersurface, tangent space and unit normal of a hypersurface, curves on hypersurfaces, normal sections, normal curvatures, Meusnier's theorem.

A 3-dimensional curve is a curve of general type if its first two derivatives are not parallel. From now on we shall suppose that the curves in question are all of general type.

The distinguished Frenet frame vector fields $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ of a 3-dimensional curve are denoted in classical differential geometry by $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ and they are called the (unit) tangent, the principal normal and the binormal vector fields of the curve respectively. These vector fields define a coordinate system at each point of the curve. The coordinate planes of this coordinate system are given the following names. We are already familiar with the plane that goes through a given curve point and spanned by the directions of the tangent and principal normal. It is the osculating plane of the curve. The plane that is spanned by the principal normal and the binormal is the plane that contains all straight lines orthogonally intersecting the curve at the given point. For this obvious reason, this plane is called the normal plane of the curve. The third coordinate plane, that is the plane spanned by the tangent and the binormal directions is the rectifying plane of the curve. The reason for this naming will become clear later. As we know from the general theory, a 3D curve of general type has two curvature functions κ_1, which is always positive and κ_2, which may have any sign. The first curvature κ_1 is denoted by the classics simply by κ and it is referred to as the curvature of the curve while the second curvature κ_2 is called the torsion of the curve and is denoted by τ.

Using the classical notation, Frenet formulas for a space curve can be written as follows.

$$\mathbf{t}' = \nu\,\kappa\,\mathbf{n}$$
$$\mathbf{n}' = -\nu\,\kappa\,\mathbf{t} + \nu\,\tau\,\mathbf{b}$$
$$\mathbf{b}' = -\nu\,\tau\,\mathbf{n}.$$

Now let us find explicit formulas for the computation of these vectors and curvatures in an economic way. The formulas we shall derive involve the "cross product" of vectors. Let us recall the definition and basic properties of this operation. The cross product of two vectors can be defined in a geometric and in an algebraic way. According to the geometric definition, the cross product **a**x**b** of the vectors **a** and **b** is **0** if **a** and **b** are parallel; if **a** and **b** are not parallel, then it is the vector defined by the following three conditions

i) **a**x**b** is perpendicular to both **a** and **b**;

ii) ‖**a**x**b**‖ is equal to the area of the parallelogram spanned by **a** and **b**;

iii) (**a**,**b**,**a**x**b**) is a positively oriented (right handed) basis of $\mathbb{R}^3$.

Algebraically we can introduce the cross product in the following way. Let $\mathbf{a} = (a_1,a_2,a_3)$, $\mathbf{b} = (b_1,b_2,b_3)$; $\mathbf{e}_1 = (1,0,0)$ $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$. The cross product of **a** and **b** is the determinant

$$\mathbf{a}\times\mathbf{b} = \det \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\ a_2b_3-a_3b_2\ ,\ a_3b_1-a_1b_3\ ,\ a_1b_2-a_2b_1\ ).$$

Here are some basic properties of the cross product.

i) $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if and only if **a** and **b** are parallel;

ii) $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$;

iii) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$;

iv) $\mathbf{c}\times(\mathbf{a}+\mathbf{b}) = \mathbf{c}\times\mathbf{a} + \mathbf{c}\times\mathbf{b}$;

v) $(\lambda\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\lambda\mathbf{b}) = \lambda(\mathbf{a}\times\mathbf{b})$ $\quad \lambda\in\mathbb{R}$.

<u>Exercise</u>. Show the equivalence of the geometric and algebraic definitions and prove basic properties of cross product.

Now let $\gamma:[a,b]\longrightarrow \mathbb{R}^3$ be a curve of general type. The unit tangent vector field **t** can be obtained by normalizing the speed vector γ'

$$\mathbf{t} = \frac{\gamma'}{\|\gamma'\|} \quad .$$

To obtain the principal normal **n** we can use the general method based on Gram Schmidt orthogonalization process

$$\mathbf{n} = \frac{\gamma'' - \langle\gamma'',\mathbf{t}\rangle\ \mathbf{t}}{\|\gamma'' - \langle\gamma'',\mathbf{t}\rangle\ \mathbf{t}\|} = \frac{\|\gamma'\|^2\ \gamma'' - \langle\gamma'',\gamma'\rangle\ \gamma'}{\|\ \|\gamma'\|^2\ \gamma'' - \langle\gamma'',\gamma'\rangle\ \gamma'\ \|} \quad ,$$

and after this we can compute the binormal as a cross product of **t** and **n**

$$\mathbf{b} = \mathbf{t}\times\mathbf{n}\ .$$

In practice however, it is more convenient to calculate the binormal first. The binormal vector is the unit normal vector of the osculating plane for which (**t**,**n**,**b**) is positively oriented. The osculating plane is spanned by the first two derivatives γ', γ'' of γ, furthermore the pair $(\mathbf{t}, \mathbf{n})$ defines the same orientation of the osculating plane as the pair (γ', γ'') so the basis $(\gamma', \gamma'', \mathbf{b})$ is positively oriented. Hence,

$$\mathbf{b} = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|} .$$

Having computed $\mathbf{b}$, $\mathbf{n}$ can be obtained as

$$\mathbf{n} = \mathbf{b} \times \mathbf{t} = \frac{(\gamma' \times \gamma'') \times \gamma'}{\|\gamma' \times \gamma''\| \; \|\gamma'\|} .$$

Before the computation of the curvature and torsion let us express the first three derivatives of γ as linear combinations of the Frenet vectors.

$$\gamma' = v\, \mathbf{t}$$

$$\gamma'' = v'\, \mathbf{t} + v\, \mathbf{t}' = v'\, \mathbf{t} + v^2 \kappa\, \mathbf{n}$$

$$\begin{aligned} \gamma''' &= v''\, \mathbf{t} + v' \mathbf{t}' + (v^2 \kappa)' \mathbf{n} + v^2 \kappa\, \mathbf{n}' = \\ &= v''\, \mathbf{t} + v' v \kappa\, \mathbf{n} + (v^2 \kappa)' \mathbf{n} + v^2 \kappa v (-\kappa\, \mathbf{t} + \tau\, \mathbf{b}) = \\ &= \left(v'' - v^3 \kappa^2\right) \mathbf{t} + \left(v' v \kappa + (v^2 \kappa)'\right) \mathbf{n} + \left(v^3 \kappa \tau\right) \mathbf{b} . \end{aligned}$$

From the first two equations

$$\gamma' \times \gamma'' = v\, \mathbf{t} \times \left(v'\, \mathbf{t} + v^2 \kappa\, \mathbf{n}\right) = v^3 \kappa\, \mathbf{b} .$$

Taking the length of these vectors and using that v and κ are positive,

$$\| \gamma' \times \gamma'' \| = v^3 \kappa ,$$

from which

$$\boxed{\kappa = \frac{\| \gamma' \times \gamma'' \|}{\| \gamma' \|^3}}$$

The torsion of the curve is involved only in the coefficient of $\mathbf{b}$ in the expression for γ'''. We can draw out the essential information for the torsion and get rid of the "rubbish" by taking the dot product of this expression with $\mathbf{b}$ or a vector parallel with $\mathbf{b}$. Since $\gamma' \times \gamma'' \parallel \mathbf{b}$, we get

$$\langle \gamma' \times \gamma'', \gamma''' \rangle = v^6 \kappa^2 \tau .$$

Combining this equation with the expression we have for the length of $\gamma' \times \gamma''$,

$$\boxed{\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{\| \gamma' \times \gamma'' \|^2}}$$

Recall that the numerator of this fraction is the determinant of the matrix the rows of which are $\gamma', \gamma'', \gamma'''$ and geometrically, it is the signed volume of the parallelepiped spanned by $\gamma', \gamma'', \gamma'''$, where the sign is positive if and only if $(\gamma', \gamma'', \gamma''')$ is positively oriented.

Now we are going to study the shape of the orthogonal projections of a curve onto the planes spanned by the vectors of the distinguished Frenet frame. For simplicity, suppose that the curve γ is parameterized by arc length and examine the curve around $\gamma(0)$. Since $v \equiv 1$, the formulas that express the derivatives of γ in terms of Frenet vectors reduce to the form

$$\gamma' = \mathbf{t}$$

$$\gamma'' = \mathbf{t}' = \kappa\, \mathbf{n}$$

$$\gamma''' = \kappa'\mathbf{n} + \kappa\, \mathbf{n}' = \kappa'\mathbf{n} + \kappa\, (-\kappa\, \mathbf{t} + \tau\, \mathbf{b}) = -\kappa^2\, \mathbf{t} + \kappa'\, \mathbf{n} + \kappa\, \tau\, \mathbf{b}\ .$$

We can approximate the curve γ around $\gamma(0)$ by its Taylor expansion.

$$\gamma(t) = \gamma(0) + \gamma'(0)t + \frac{\gamma''(0)}{2}\, t^2 + \frac{\gamma'''(0)}{6}\, t^3 + o(t^3).$$

Recall that the "little oh" notation $o(t^3)$ is used in the following sense. If f, g, and h are functions defined around a given point a, then we write

$$f(t) = g(t) + o(h(t))$$

if $\frac{f(t) - g(t)}{h(t)}$ tends to zero as t tends to a. Though a is not involved in the equality it is generally clear from the context what it is. For example, in our case $a = 0$.

Expressing the derivatives of γ with the help of Frenet vectors we get

$$\gamma(t) - \gamma(0) =$$

$$= \left(t - \kappa^2(0)\frac{t^3}{6}\right) \mathbf{t}(0) + \left(\kappa(0)\frac{t^2}{2} + \kappa'(0)\frac{t^3}{6}\right) \mathbf{n}(0) + \left(\kappa(0)\tau(0)\frac{t^3}{6}\right) \mathbf{b}(0) + o(t^3)\ .$$

Looking at this expansion we conclude that the projection of the curve on the osculating plane is well approximated by the parabola $t\mathbf{t}(0) + \kappa(0)\frac{t^2}{2}\mathbf{n}(0)$, (observe that the curvature of this parabola at $t = 0$ is $\kappa(0)$), the projection onto the normal plane has locally the same shape as the semicubical parabola $\kappa(0)\frac{t^2}{2}\mathbf{n}(0) \ + \frac{t^3}{6}\left(\kappa'(0)\ \ \mathbf{n}(0) \ + \ \kappa(0)\tau(0)\ \ \mathbf{b}(0)\right)$, in particular, it has a so called cusp singularity at $t = 0$, finally, the projection onto the rectifying plane has the Taylor expansion $\mathbf{t}(0) + \frac{t^3}{6}\ \left(\kappa^2(0)\kappa(0)\mathbf{t}(0)\ \tau(0)\ \mathbf{b}(0)\right) + o(t^3)$, so its shape is like the graph of a cubic function. It is easy to see, that the curvature of this projection is 0 at $t = 0$, thus it is almost straight around the origin. That is the reason why the rectifying plane was given just this name: projection of the curve onto the rectifying plane straightens the curve and "rectifying" means straightening.

Now we shall study a problem which connects curve theory to surface theory. If a curve lyes on a surface, curvedness of the surface forces the curve to bend. Thus, curvedness of a surface can be detected by the curvatures of the curves lying on the surface. Heuristically clear that the curvature of a curve on a given surface should be the same as the curvature of the intersection curve of the surface and the osculating plane of the curve provided that the osculating plane is not tangent to the surface. This is indeed true and thus we may pose the question how to compute the curvature of the curve using only information on the surface and the position of the osculating plane of the curve. The existence of a formula that answers this question will prove our heuristics.

Definition. A parameterized hypersurface in $\mathbb{R}^n$ is a differentiable mapping $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$ from an open domain Ω of $\mathbb{R}^{n-1}$ into the n-dimensional space. Smooth curves on a parameterized hypersurface are curves of the form $\gamma(t) = \mathbf{r}(u(t))$, where the mapping $t\mapsto u(t)$ is a smooth curve lying in the parameter domain Ω. Curves of the form $t\mapsto \mathbf{r}(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_{n-1})$, where $u_1,\dots,u_{i-1},u_{i+1},\dots,u_{n-1}$ are fixed numbers are called the parameter lines or coordinate lines on the hypersurface. The speed vectors of the parameter lines $t\mapsto \mathbf{r}(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_{n-1})$, which are just the partial derivatives of the mapping $\mathbf{r}$ with respect to the i-th variable, will be denoted by $\mathbf{r}_i(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_{n-1})$.

Since we shall often work with formulas containing partial derivatives of a function it is convenient to introduce the shorthand convention that we shall denote the partial derivative of a multivariable function F with respect to its i-th variable by F_i. In general, the higher order partial derivative $\frac{\partial^k F}{\partial u_{i_1}\dots\partial u_{i_k}}$ of F will be denoted by $F_{i_1\dots i_k}$. If there is a danger of confusion with lower indices, the lower indices of the function will be separated from the indices of variables with respect to which we have to take the partial derivative by a comma. Thus, $\frac{\partial^k F_{j_1\dots j_\ell}}{\partial u_{i_1}\dots\partial u_{i_k}}$ will be denoted by $F_{j_1\dots j_\ell,i_1\dots i_k}$.

Definition. A parameterized hypersurface is regular if the vectors $\mathbf{r}_1(u),\dots,\mathbf{r}_{n-1}(u)$ are linearly independent for any $u\in\Omega$. In this case we also say that $\mathbf{r}$ is an immersion of the domain Ω into $\mathbb{R}^n$.

Definition. The tangent plane of a regular parameterized hypersurface at the point $\mathbf{r}(u)$ is the plane through $\mathbf{r}(u)$ spanned by the direction vectors $\mathbf{r}_1(u),\dots,\mathbf{r}_{n-1}(u)$. The unit normal vector of the hypersurface at the point $\mathbf{r}(u)$ is defined to be the unit normal vector $\mathbf{N}(u)$ of the tangent plane, for which $\mathbf{r}_1(u),\dots,\mathbf{r}_{n-1}(u)$, $N(u)$ is a positively oriented basis of $\mathbb{R}^n$.

For parameterized surfaces in $\mathbb{R}^3$, the unit normal vector field can be calculated with the help of cross product

$$N(u_1,u_2) = \frac{\mathbf{r}_1(u_1,u_2) \times \mathbf{r}_2(u_1,u_2)}{\| \mathbf{r}_1(u_1,u_2) \times \mathbf{r}_2(u_1,u_2)\|} .$$

To get a similar formula in higher dimensions, we need a suitable generalization of the cross product.

Let $\mathbf{r}_i = (r_i^1,\dots,r_i^n) \in \mathbb{R}^n$, $i = 1,2,\dots,n-1$, be n-dimensional vectors, $\mathbf{e}_1,\dots,\mathbf{e}_n$ the standard basis of $\mathbb{R}^n$. The <u>exterior product</u> of the vectors $\mathbf{r}_1,\dots,\mathbf{r}_{n-1}$ is defined by the equality

$$\mathbf{r}_1\wedge\dots\wedge\mathbf{r}_{n-1} = \det \begin{vmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \\ r_1^1 & \dots & r_1^n \\ \dots & \dots & \dots \\ r_{n-1}^1 & \dots & r_{n-1}^n \end{vmatrix} .$$

<u>Exercise</u>. (cf. Ex. on page 18) Show that $\mathbf{r}_1\wedge\dots\wedge\mathbf{r}_{n-1}$ is orthogonal to $\mathbf{r}_1,\dots,\mathbf{r}_{n-1}$, it is different from $\mathbf{0}$ if and only if $\mathbf{r}_1,\dots,\mathbf{r}_{n-1}$ are linearly independent, and finally, $\mathbf{r}_1,\dots,\mathbf{r}_{n-1},(-1)^{n-1}\mathbf{r}_1\wedge\dots\wedge\mathbf{r}_{n-1}$ is a positively oriented basis of $\mathbb{R}^n$.

As a consequence of the exercise, for regular hypersurfaces we have

$$N(u) = (-1)^{n-1} \frac{\mathbf{r}_1(u)\wedge\dots\wedge\mathbf{r}_{n-1}(u)}{\| \mathbf{r}_1(u)\wedge\dots\wedge\mathbf{r}_{n-1}(u) \|} .$$

Consider the curve $\gamma(t) = \mathbf{r}(u(t))$ lying on the regular parameterized hypersurface $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$, where $u = (u_1,\dots,u_{n-1})$ is a curve in Ω. Express the first two derivatives of γ using Frenet formulas on one hand and the special form of γ as a surface curve on the other. Using the chain rule, we get

$$\vartheta\, \mathbf{t}_1 = \gamma' = \sum_{i=1}^{n-1} u_i'\, \mathbf{r}_i(u)$$

and

$$\vartheta' \mathbf{t}_1 + \vartheta^2\kappa_1\, \mathbf{t}_2 = \gamma'' = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} u_i' u_j'\, \mathbf{r}_{ij}(u) + \sum_{i=1}^{n-1} u_i''\, \mathbf{r}_i(u).$$

Multiplying the last equation by the normal vector of the hypersurface and using the fact that it is orthogonal to the tangent vectors $\mathbf{t}_1,\mathbf{r}_1,\dots,\mathbf{r}_{n-1}$, we obtain

$$\vartheta^2\kappa_1 \langle \mathbf{N}(u),\mathbf{t}_2\rangle = \langle N(u),\gamma''\rangle = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \langle N(u),\mathbf{r}_{ij}(u)) u_i' u_j' ,$$

from which

$$\boxed{\kappa_1 = \frac{1}{\langle \mathbf{N}(u),\mathbf{t}_2\rangle} \quad \frac{\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \langle N(u),\mathbf{r}_{ij}(u)\rangle u_i' u_j'}{\vartheta^2}}$$

Let us study this expression. We claim that the right hand side is determined by the osculating plane of the curve and the surface provided that the osculating plane is not tangent to the surface.

Let us start with the expression $k(\gamma') = \dfrac{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \langle N(u), \mathbf{r}_{ij}(u)\rangle\, u'_i u'_j}{\upsilon^2}$.

Since the quantities $\langle N(u),\mathbf{r}_{ij}(u)\rangle$ are determined by the parameterization of the hypersurface, the functions $u'_1,\ldots,u'_{n-1}$ are the components of the speed vector γ' of the curve with respect to the basis $\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}$ of the tangent space, υ is the length of the speed vector γ', $k(\gamma')$ depends only on the speed vector γ' of the curve (that justifies the notation $k(\gamma')$).

Definition. Let $\mathbf{v}$ be an arbitrary tangent vector of the regular parameterized hypersurface $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$ at $\mathbf{r}(u)$, The intersection curve of the hypersurface and the plane through $\mathbf{r}(u)$ spanned by direction vectors $N(u)$ and $\mathbf{v}$ is called the normal section of the hypersurface in the direction $\mathbf{v}$. Giving the cutting normal plane an orientation by the ordered basis $(\mathbf{v}, N(u))$, we may consider the signed curvature of the normal section, which will be called the normal curvature of the hypersurface in the direction $\mathbf{v}$ and will be denoted by $k(\mathbf{v})$.

Applying the above general formulas for normal sections one may see easily that the normal curvature of a parameterized hypersurface in the direction $\mathbf{v} = v_1\, \mathbf{r}_1(u) +\ldots+ v_{n-1}\, \mathbf{r}_{n-1}(u)$ is just

$$k(\mathbf{v}) = \frac{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \langle N(u), \mathbf{r}_{ij}(u)\rangle\, v_i v_j}{\upsilon^2},$$

where $\upsilon = \|\mathbf{v}\|$. Since $k(\lambda\mathbf{v}) = k(\mathbf{v})$ for any $\lambda \neq 0$, the normal curvature depends only on the straight line of $\mathbf{v}$.

Returning to the curve γ we see that $k(\gamma')$ is determined by the tangent line of γ at the given point which is the intersection of the osculating plane of γ and the tangent space of the hypersurface.

Since the osculating plane and the tangent line determines the second Frenet vector $\mathbf{t}_2$ uniquely up to sign, we conclude that the the curvature $\kappa_1 = \left(1/\langle N(u),\mathbf{t}_2\rangle\right) k(\gamma')$ of the curve is determined by the osculating plane up to sign and since the curvature κ_1 is positive, both $\mathbf{t}_2$ and κ_1 are determined uniquely (and not only up to sign) by the osculating plane.

To finish this unit with, we formulate an obvious consequence of the formula expressing the curvature of a curve lying on a hypersurface.

Corollary. (Meusnier's theorem) If the osculating plane of a curve γ lying on a hypersurface is not contained in the tangent space of the hypersurface at a given point $\gamma(t) = \mathbf{r}(u(t))$, then the curvature of the curve and the normal curvature of the surface in the direction $\gamma'(t)$ are related to one another by the equation $\kappa_1(t) = \frac{1}{\cos\alpha} k(\gamma'(t))$, where α is the angle between the normal vector $N(u(t))$ of the hypersurface and the second Frenet vector $\mathbf{t}_2(t)$ of the curve.

Further Exercises

4-1. Given a unit speed curve of general type in $\mathbb{R}^3$ with distinguished Frenet frame $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, find a vector field ω along the curve such that $\dot{\mathbf{t}}_i = \omega \times \mathbf{t}_i$ holds for i=1,2,3. (ω is called the Darboux vector field of the curve.)

4-2. Suppose that the osculating planes of a curve of general type in $\mathbb{R}^3$ have a point in common. Show that the curve is a plane curve.

4-3. Suppose that the normal planes of a regular curve in $\mathbb{R}^3$ go through a fixed point O. Show that the curve lies on a sphere centered at O.

4-4. Let γ be a curve of general type in $\mathbb{R}^n$, $\mathbf{t}_1, \ldots \mathbf{t}_n$ its distinguished Frenet frame, $0 \leqslant k \leqslant n$. By the definition of the distinguished Frenet frame, the k-th derivative of γ can be expressed as a linear combination of $\mathbf{t}_1, \ldots, \mathbf{t}_k$ as

$$\gamma^{(k)} = c_1 \mathbf{t}_1 + \ldots + c_k \mathbf{t}_k ,$$

where $c_1, \ldots, c_k$ are suitable functions. Show that

$$c_k = |\gamma'|^k \kappa_1 \kappa_2 \ldots \kappa_{k-1} .$$

4-5. Compute the curvatures of the "moment curve" $\gamma(t) = (t, t^2, \ldots, t^n)$ at $t = 0$. (Hint: Use previous exercise.)