Unit 5. Hypersurfaces

Vector fields along hypersurfaces, tangential vector fields, derivations of vector fields with respect to a tangent direction, the Weingarten map, bilinear forms, the first and second fundamental forms of a hypersurface, principal directions and principal curvatures, mean curvature and the Gaussian curvature, Euler's formula.

Definition. Let $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$ be a parameterized hypersurface. A vector field along the hypersurface is a mapping $X:\Omega\longrightarrow T_*\mathbb{R}^n$ from the domain of parameters into the tangent bundle of $\mathbb{R}^n$ such that $X(u) \in T_{\mathbf{r}(u)}\mathbb{R}^n$ for any $u \in \Omega$. X is a tangential vector field, if X(u) is tangent to the hypersurface at $\mathbf{r}(u)$.

Since X(u) has the form $\big(\mathbf{r}(u),\tilde{X}(u)\big)$, where $\tilde{X}:\Omega\longrightarrow \mathbb{R}^n$, there is a one to one correspondence between vector fields along a parameterized hypersurface and smooth mappings of the domain of parameters into $\mathbb{R}^n$. Roughly speaking, if we are given a smooth mapping of the parameter domain into $\mathbb{R}^n$, we may think of it as a vector field along the hypersurface though formally it is not a vector field. In this way, the mappings $\mathbf{r}_1,\dots,\mathbf{r}_{n-1}$ and N should be thought of as vector fields along the hypersurface, the first n-1 of which are tangential.

Given a vector field along a hypersurface, we would like to express the speed of change of the vector field vectors as we move along the surface, in terms of the speed of our motion. This is achieved by the following.

Definition. Let $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$ be a parameterized hypersurface, $X:\Omega\longrightarrow \mathbb{R}^n$ be a vector field along it, $u_0 \in \Omega$, $\mathbf{v}$ a tangent vector of the hypersurface at $\mathbf{r}(u_0)$. We define the derivative $\partial_{\mathbf{v}}X$ of the vector field X in the direction $\mathbf{v}$ as $\partial_{\mathbf{v}}X = (X\circ u)'(0)$, where $u: [-1,1]\longrightarrow \Omega$ is a curve in the parameter domain such that $u(0) = u_0$ and $(\mathbf{r}\circ u)'(0) = \mathbf{v}$.

Since by the chain rule

$$(X\circ u)'(0) = \sum_{i=1}^{n-1} u_i'(0)\, X_i(u(0)) ,$$

where $(u_1,\dots,u_{n-1})$ are the components of u, $X_1,\dots,X_{n-1}$ are the partial derivatives of X, and by

$$\mathbf{v} = (\mathbf{r}\circ u)'(0) = \sum_{i=1}^{n-1} u_i'(0)\, \mathbf{r}_i(u(0))$$

the numbers $u'_1(0),\dots,u'_{n-1}(0)$ are the components of $\mathbf{v}$ in the basis $\mathbf{r}_1(u_0),\dots,\mathbf{r}_{n-1}(u_0)$ of the tangent space at $\mathbf{r}(u_0)$, we have the following formula

$$\partial_{\mathbf{v}}X = \sum_{i=1}^{n-1} v_i\, X_i(u_0),$$

where $v_1,\dots,v_{n-1}$ are the components of the vector $\mathbf{v}$ in the basis $\mathbf{r}_1(u_0),\dots,\mathbf{r}_{n-1}(u_0)$. This formula shows that the definition of $\partial_{\mathbf{v}}X$ is correct, i.e. independent of the choice of the curve $u(t)$.

We shall consider the local behavior of curvature on a hypersurface. The way in which a hypersurface curves around in $\mathbb{R}^n$ is closely related to the way the normal direction changes as we move from point to point.

Lemma. The derivative $\partial_{\mathbf{v}}N$ of the normal direction on a hypersurface with respect to a tangent vector $\mathbf{v}$ at $p = \mathbf{r}(u)$ is tangent to the hypersurface at $\mathbf{r}(u)$.

Proof. We need to show that $\partial_{Y}N$ is orthogonal to $N(p)$. Indeed, differentiating the relation $1 \equiv \langle N , N \rangle$, we get

$$0 = \langle \partial_{\mathbf{v}}N , N \rangle + \langle N , \partial_{\mathbf{v}}N \rangle = 2 \langle \partial_{\mathbf{v}}N , N \rangle. \blacksquare$$

Definition. Let us denote by M the parameterized hypersurface $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^n$ and by T_pM the linear space of its tangent vectors at $p = \mathbf{r}(u_0)$. The linear map

$$L_p : T_pM \longrightarrow T_pM$$

defined for a fixed $p \in M$ by

$$L_p(\mathbf{v}) = - \partial_{\mathbf{v}} N$$

is called the Weingarten map or shape operator of M at p.

Before going on with the study of hypersurfaces, let us recall some definitions from linear algebra.

Definition. Let V be a vector space. A bilinear function or form on V is a mapping

$$B : V\times V\longrightarrow \mathbb{R}$$

satisfying the identities

$$B(\alpha\mathbf{x}_1+\beta\mathbf{x}_2,\mathbf{y}) = \alpha B(\mathbf{x}_1,\mathbf{y}) + \beta B(\mathbf{x}_2,\mathbf{y}),$$
$$B(\mathbf{x},\alpha\mathbf{y}_1+\beta\mathbf{y}_2) = \alpha B(\mathbf{x},\mathbf{y}_1) + \beta B(\mathbf{x},\mathbf{y}_2)$$

for any $\mathbf{x},\mathbf{x}_1,\mathbf{x}_2,\mathbf{y},\mathbf{y}_1,\mathbf{y}_2\in V$, $\alpha,\beta\in\mathbb{R}$.

B is said to be symmetric if

$$B(\mathbf{x},\mathbf{y}) = B(\mathbf{y},\mathbf{x}) \qquad \text{for all } \mathbf{x}\ \mathbf{y} \in V.$$

A symmetric bilinear function is positive definite if $B(\mathbf{x},\mathbf{x}) > 0$ for $\mathbf{x}\neq\mathbf{0}$.

A vector space equipped with a positive definite symmetric bilinear form is a Euclidean vector space.

For example, $\mathbb{R}^n$ with the usual dot product on it is a Euclidean vector space.

If $\mathbf{x}_1,\dots,\mathbf{x}_n$ is a basis of the vector space V and B is a bilinear function on V then the nxn matrix (b_{ij}) with entries $b_{ij} = B(\mathbf{x}_i,\mathbf{x}_j)$ is called the matrix representation of B with respect to the basis $\mathbf{x}_1,\dots,\mathbf{x}_n$. Fixing the basis we get a one to one correspondence between bilinear functions and nxn matrices. A bilinear form is symmetric if and only if its matrix representation with respect to a basis is symmetric.

Definition. The quadratic form of a bilinear function B is the function defined by the equality $Q_B(\mathbf{x}) = B(\mathbf{x},\mathbf{x})$.

Symmetric bilinear functions can be recovered from their quadratic forms with the help of the identity

$$B(\mathbf{x},\mathbf{y}) = \frac{1}{2}\Big(Q_B(\mathbf{x}+\mathbf{y}) - Q_B(\mathbf{x}) - Q_B(\mathbf{y})\Big).$$

Now we return to hypersurfaces. We define two bilinear forms on each tangent space of the hypersurface

Definition. Let M be a parameterized hypersurface $\mathbf{r}:\Omega\longrightarrow\mathbb{R}^n$, $u_0\in\Omega$, T_pM the linear space of tangent vectors of M at $p = \mathbf{r}(u)$, $L_p:T_pM \longrightarrow T_pM$ the Weingarten map. The first fundamental form of the hypersurface is the bilinear function I_p on T_pM obtained by restriction of the dot product onto T_pM

$$I_p(\mathbf{v},\mathbf{w}) = \langle\, \mathbf{v}\ ,\ \mathbf{w}\,\rangle \qquad \text{for } \mathbf{v},\mathbf{w}\in T_pM.$$

The second fundamental form of the hypersurface is the bilinear function II_p on T_pM defined by the equality

$$II_p(\mathbf{v},\mathbf{w}) = \langle\, L_p\mathbf{v}\ ,\ \mathbf{w}\,\rangle \qquad \text{for } \mathbf{v},\mathbf{w}\in T_pM.$$

The first fundamental form is obviously a positive definite symmetric bilinear function on the tangent space. Its matrix representation with respect to the basis $\mathbf{r}_1(u_0),\dots,\mathbf{r}_{n-1}(u_0)$ has entries $\langle\, \mathbf{r}_i(u_0)\ ,\ \mathbf{r}_j(u_0)\,\rangle$.

An important property of the Weingarten map and the second fundamental form is stated in the following theorem.

Theorem. The second fundamental form is symmetric, i.e.

$$\langle\, L_p\mathbf{v}\ ,\ \mathbf{w}\,\rangle = \langle\, \mathbf{v}\ ,\ L_p\mathbf{w}\,\rangle \qquad \text{for } \mathbf{v},\mathbf{w}\in T_pM,$$

or in other words, the Weingarten map is self-adjoint (with respect to the first fundamental form).

Proof. It is enough to prove that the matrix of the second fundamental form with respect to the basis $\mathbf{r}_1(u_0),\dots,\mathbf{r}_{n-1}(u_0)$ is symmetric.

Lemma. $II_p(\mathbf{r}_i(u_0),\mathbf{r}_j(u_0)) = \langle\, \mathbf{r}_{ij}(u_0)\ ,\ N(u_0)\,\rangle$.

Proof of Lemma. We know that the normal vector field N is perpendicular to any tangential vector field, thus

$$\langle N , \mathbf{r}_j \rangle \equiv 0.$$

Differentiating this identity with respect to the i-th parameter we get

$$\langle \partial_{\mathbf{r}_i} N , \mathbf{r}_j \rangle + \langle N , \mathbf{r}_{ji} \rangle \equiv 0,$$

from which

$$\langle N , \mathbf{r}_{ji} \rangle \equiv \langle - \partial_{\mathbf{r}_i} N , \mathbf{r}_j \rangle = \langle L_{\mathbf{r}} \mathbf{r}_i , \mathbf{r}_j \rangle. \blacksquare$$

Since by Young's theorem $\mathbf{r}_{ij} = \mathbf{r}_{ji}$, the lemma shows that the matrix of the second fundamental form is symmetric. $\blacksquare$

Comparing the identity proved in the lemma with the formula expressing the normal curvature of the hypersurface in a tangent direction $\mathbf{v}$ we see that the normal curvature is the quotient of the quadratic forms of the second and first fundamental forms

$$k(\mathbf{v}) = \frac{\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \langle N(u_0), \mathbf{r}_{ij}(u_0) \rangle v_i v_j}{\vartheta^2} = \frac{II_p(\mathbf{v},\mathbf{v})}{I_p(\mathbf{v},\mathbf{v})},$$

where $\mathbf{v} = \sum_{i=1}^{n-1} v_i \mathbf{r}_i(u_0)$ is a tangent vector of the hypersurface at $p = \mathbf{r}(u_0)$.

The expression

$$\boxed{k(\mathbf{v}) = \frac{II_p(\mathbf{v},\mathbf{v})}{I_p(\mathbf{v},\mathbf{v})}}$$

gives rise to a linear algebraic investigation of the normal curvature.

It is natural to ask at which directions the normal curvature attains its extrema. Since $k(\lambda\mathbf{v}) = k(\mathbf{v})$ for any $\lambda \neq 0$, it is enough to consider this question for the restriction of k onto the unit sphere S in the tangent space. The unit sphere of a Euclidean space is a compact (=closed and bounded) subset, thus by Weierstrass theorem, any continuous function defined on it attains its maximum and minimum.

Definition. Let f be a differentiable function defined on the unit sphere S of a Euclidean vector space. We say that the vector $\mathbf{v} \in S$ is a critical point of f if for any curve $\gamma : [-1,1] \longrightarrow S$ on the sphere such that $\gamma(0) = \mathbf{v}$ the derivative of the composite function $f \circ \gamma$ vanishes at 0.

Clearly, local minimum and maximum points of a function are its critical points, but the converse is not true. The following proposition gives a characterization of critical points for the restriction of the normal curvature onto the unit sphere of the tangent space.

Proposition. Let V be a finite dimensional vector space with a positive definite symmetric bilinear function $\langle , \rangle$ and let $L: V \longrightarrow V$ be a self-adjoint linear transformation on V. Let $S = \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ and define $f: S \longrightarrow \mathbb{R}$ by $f(\mathbf{x}) = \langle L\mathbf{x}, \mathbf{x} \rangle$. Then $\mathbf{v} \in S$ is a critical point of f if and only if $\mathbf{v}$ is an eigenvector of L.

Proof. For any curve $\gamma: [-1,1] \longrightarrow S$ such that $\gamma(0) = \mathbf{v}$, we have

$$\frac{d}{dt} \langle L(\gamma(t)), \gamma(t) \rangle \Big|_{t=0} = \langle L(\gamma'(0)), \gamma(0) \rangle + \langle L(\gamma(0)), \gamma'(0) \rangle$$
$$= \langle L\gamma'(0), \mathbf{v} \rangle + \langle L\mathbf{v}, \gamma'(0) \rangle = 2 \langle L\mathbf{v}, \gamma'(0) \rangle.$$

This means that $\mathbf{v}$ is a critical point of f if and only if $L\mathbf{v}$ is orthogonal to every vectors of the form $\gamma'(0)$. Since the speed vectors $\gamma'(0)$ of spherical curves through $\mathbf{v} = \gamma(0)$ range over the tangent space of the sphere S, $\mathbf{v}$ is a critical point of f if and only if $L\mathbf{v}$ is orthogonal to the tangent space of S at $\mathbf{v}$ however, since the normal vector of this tangent space is $\mathbf{v}$, the latter condition is satisfied if and only if $L\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, i.e. $\mathbf{v}$ is an eigenvector of L. ▮

As an application of the proposition, let us prove the following theorem of linear algebra.

Theorem. Let V be a finite dimensional Euclidean vector space and let $L : V \longrightarrow V$ be a self-adjoint linear transformation on V. Then there exists an orthonormal basis of V consisting of eigenvectors of L.

Proof. By induction on the dimension n of V. For $n = 1$ the theorem is trivial. Assume that it is true for $n = k$. Suppose $n = k + 1$. By the proposition, there exists a unit vector $\mathbf{v}_1$ in V which is an eigenvector of L. Let $W = \mathbf{v}_1^{\perp} = \{\mathbf{w} \in V : \mathbf{v}_1 \perp \mathbf{w}\}$. Then $L(W) \subseteq W$ since we have

$$\langle L\mathbf{w}, \mathbf{v}_1 \rangle = \langle \mathbf{w}, L\mathbf{v}_1 \rangle = \langle \mathbf{w}, \lambda_1 \mathbf{v}_1 \rangle = \lambda_1 \langle \mathbf{w}, \mathbf{v}_1 \rangle = 0$$

for any $\mathbf{w} \in W$, where λ_1 is the eigenvalue belonging to $\mathbf{v}_1$. Clearly $L|_W$ is self-adjoint. Since $\dim(W) = \dim(V) - 1 = k$, the induction assumption implies that there exists an orthonormal basis $(\mathbf{v}_2, \ldots, \mathbf{v}_n)$ for W consisting of eigenvectors of $L|_W$. But each eigenvector of $L|_W$ is an eigenvector of L, so $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an orthonormal basis for V consisting of eigenvectors of L. ▮

Definition. For a hypersurface M in $\mathbb{R}^n$ parameterized by $\mathbf{r}$, $\mathbf{r}(u_0) = p \in M$, the eigenvalues $\kappa_1(p), \ldots, \kappa_{n-1}(p)$ of the Weingarten map $L_p: T_pM \longrightarrow T_pM$ are called the principal curvatures of M at p, the unit eigenvectors of L_p are called principal curvature directions.

If the principal curvatures are ordered so that $\kappa_1(p) \leqslant \kappa_2(p) \leqslant \ldots \leqslant \kappa_{n-1}(p)$, the discussion above shows that $\kappa_{n-1}(p)$ is the maximal, $\kappa_1(p)$ is the minimal value of the normal curvature $k(\mathbf{v})$.

Theorem. (Euler's formula) Let $\mathbf{v}_1,\dots,\mathbf{v}_{n-1}$ be an orthonormal basis of T_pM consisting of principal curvature directions, $\kappa_1(p),\dots,\kappa_{n-1}(p)$ be the corresponding principal curvatures. Then the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in T_pM$, $\| \mathbf{v} \| = 1$, is given by

$$k(Y) = \sum_{i=1}^{n-1} k_i(p) <Y,Y_i>^2 = \sum_{i=1}^{n-1} k_i(p) \cos^2(\theta_i),$$

where $\theta_i = \arccos(<\mathbf{v},\mathbf{v}_i>)$ is the angle between $\mathbf{v}$ and $\mathbf{v}_i$.

Proof. Since $(\mathbf{v}_1,\dots,\mathbf{v}_n)$ is an orthonormal basis, the vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \sum_{i=1}^{n-1} <\mathbf{v},\mathbf{v}_i> \mathbf{v}_i.$$

Making use of this formula, we obtain

$$k(\mathbf{v}) = <L_p(\mathbf{v}),\mathbf{v}> = <L_p\left(\sum_{i=1}^{n-1} <\mathbf{v},\mathbf{v}_i> \mathbf{v}_i\right), \sum_{i=1}^{n-1} <\mathbf{v},\mathbf{v}_i> \mathbf{v}_i> =$$

$$= <\sum_{i=1}^{n-1} <\mathbf{v},\mathbf{v}_i> \kappa_i(p)\, \mathbf{v}_i, \sum_{i=1}^{n-1} <\mathbf{v},\mathbf{v}_i> \mathbf{v}_i> = \sum_{i=1}^{n-1} \kappa_i(p) <\mathbf{v},\mathbf{v}_i>^2. \blacksquare$$

The determinant and trace of the Weingarten map, that is the product and sum of the principal curvatures are of particular importance in differential geometry.

Definition. For M a hypersurface, $p \in M$, the determinant $K(p)$ of the Weingarten map L_p is called the Gaussian or Gauss-Kronecker curvature of M at p, $H(p) = 1/(n-1)\ \mathrm{trace}\ (L_p)$ is called the mean curvature.

When we compute the principal curvatures and directions of a hypersurface at a point we generally work with a matrix representation of the Weingarten map. Recall that if V is a linear space with basis $\mathbf{x}_1,\dots,\mathbf{x}_n$ and $L: V \longrightarrow V$ is a linear mapping then the matrix representation of V with respect to the basis $\mathbf{x}_1,\dots,\mathbf{x}_n$ is the nxn matrix (ℓ_{ij}) for which

$$L(\mathbf{x}_i) = \sum_{j=1}^{n} \ell_{ij}\mathbf{x}_j \quad i=1,2,\dots,n.$$

When we have a deal with a parameterized hypersurface $\mathbf{r}: \Omega \longrightarrow \mathbb{R}^n$, it is natural to take the basis $\mathbf{r}_1(u),\dots,\mathbf{r}_{n-1}(u)$ of the tangent space at $\mathbf{r}(u)$. Let us denote by $\mathcal{G} = (g_{ij})$, $\mathcal{B} = (b_{ij})$ and $\mathcal{L} = (\ell_{ij})$ the matrix representations of the first and second fundamental forms and the Weingarten map respectively, with respect to this basis (g_{ij}, b_{ij} and ℓ_{ij} are functions on the parameter domain). Components of $\mathcal{G}$ and $\mathcal{B}$ can be calculated according to the equations

$$g_{ij} = < \mathbf{r}_i , \mathbf{r}_j >,$$

$$b_{ij} = < N , \mathbf{r}_{ij} > \quad \text{(cf. Lemma above)}.$$

The relationship between the matrices $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{L}$ follows from the following equalities

$$b_{ij} = \langle L_{\mathbf{r}}\mathbf{r}_i , \mathbf{r}_j \rangle = \langle \sum_{k=1}^{n-1} \ell_{ik}\mathbf{r}_k , \mathbf{r}_j \rangle = \sum_{k=1}^{n-1} \ell_{ik} \langle \mathbf{r}_k , \mathbf{r}_j \rangle = \sum_{k=1}^{n-1} \ell_{ik}\, g_{kj}$$

expressing that $\mathcal{B} = \mathcal{L}\,\mathcal{G}$. $\mathcal{G}$ is the matrix of a positive definite bilinear function, hence it is invertible (its determinant is positive). Multiplying the equation $\mathcal{B} = \mathcal{L}\,\mathcal{G}$ with the inverse of $\mathcal{G}$ we get the expression

$$\boxed{\mathcal{L} = \mathcal{B}\,\mathcal{G}^{-1}}$$

for the matrix of the Weingarten operator.

Corollary. The Gaussian curvature of a hypersurface is equal to

$$\boxed{K = \frac{\det \mathcal{B}}{\det \mathcal{G}}}\ .$$

Recall from linear algebra that in order to determine the eigenvalues of a linear mapping with matrix representation $\mathcal{L}$ one has to find the roots of the characteristic polynomial $p_{\mathcal{L}}(\lambda) = \det(\mathcal{L} - \lambda I)$, where I denotes the identity matrix.

Having determined the eigenvalues of the linear mapping, components of eigenvectors with respect to the fixed basis are obtained as non-zero solutions of the linear system of equations $\mathcal{L}\,\mathbf{v} = \lambda\,\mathbf{v}$ where λ is a nonzero eigenvalue of $\mathcal{L}$.

Further Exercices

5-1. Determine the Weingarten map for a sphere of radius r at one of its points.

5-2. Find the normal curvature $k(\mathbf{v})$ for each tangent direction $\mathbf{v}$, the principal curvatures and the principal curvature directions, and compute the Gaussian and mean curvatures of the following surfaces at the given point p.

$(x_1^2/a^2)+(x_2^2/b^2)+(x_3^2/c^2) = 1$, $p = (a,0,0)$ (ellipsoid);

$(x_1^2/a^2)+(x_2^2/b^2)-(x_3^2/c^2) = 1$, $p = (a,0,0)$ (one-sheeted hyperboloid);

$x_1^2+\left(\sqrt{x_2^2+x_3^2} - 2\right)^2 = 1$ $p = (0,3,0)$ or $p = (0,1,0)$ (torus).

5-3. Suppose that the principal curvatures of a parameterized surface in $\mathbb{R}^3$ vanish. Show that the surface is a part of a plane.

5-4. Find the Gaussian curvature $K: M \longrightarrow \mathbb{R}$ for the following surfaces

$x_1^2+x_2^2-x_3^2 = 0$, $x_3 > 0$ (cone)

$(x_1^2/a^2)+(x_2^2/b^2)-(x_3^2/c^2) = 1$ (hyperboloid);

$(x_1^2/a^2)+(x_2^2/b^2)-x_3 = 0$ (elliptic paraboloid);

$(x_1^2/a^2)-(x_2^2/b^2)-x_3 = 0$ (hyperbolic paraboloid).

5-5. Let M be a (hyper)surface in $\mathbb{R}^3$, $p \in M$. Show that for each $\mathbf{v}, \mathbf{w} \in T_pM$,

$$L_p(\mathbf{v}) \times L_p(\mathbf{w}) = K(p)\ \mathbf{v} \times \mathbf{w}.$$