UNIT 6. SURFACES IN THE 3-DIMENSIONAL SPACE

Umbilical, spherical and planar points, surfaces consisting of umbilics; surfaces of revolution, Beltrami's pseudosphere; lines of curvature, parameterizations for which coordinate lines are lines of curvature, Dupin's theorem, confocal second order surfaces; ruled and developable surfaces: equivalent definitions, basic examples, relations to surfaces with K=0, structure theorem.

A regular parameterized surface $\mathbf{r}:\Omega\longrightarrow\mathbb{R}^3$ (Ω is an open subset of the plane) has two principal curvatures $\kappa_1(u,v)$ and $\kappa_2(u,v)$ at each point $p = \mathbf{r}(u,v)$ of the surface. If $\kappa_1(u,v) \leqslant \kappa_2(u,v)$ then $\kappa_1(u,v)$ is the minimum of normal curvatures in different directions at p, while $\kappa_2(u,v)$ is the maximum of them. If $\kappa_1(u,v) < \kappa_2(u,v)$ then the principal directions corresponding to $\kappa_1(u,v)$ and $\kappa_2(u,v)$ are uniquely defined, however if $\kappa_1(u,v) = \kappa_2(u,v)$ then the normal curvature is constant in all directions and every direction is principal.

Definition. A point $p = \mathbf{r}(u,v)$ of a surface is called a umbilical point or umbilic if the principal curvatures at p are equal. A umbilical point p is said to be spherical if $\kappa_1(u,v) = \kappa_2(u,v) \neq 0$, and planar if $\kappa_1(u,v) = \kappa_2(u,v) = 0$.

The following theorem gives a characterization of those surfaces which have only umbilical points.

Theorem. A connected regular surface all points of which are umbilical is contained in a plane or sphere.

Proof. First we show that the principal curvature function $\kappa_1 = \kappa_2 = \kappa$ is constant along the surface. Fixing a parameterization $\mathbf{r}$, we have

$$N_u = -\kappa\,\mathbf{r}_u \quad\text{and}\quad N_v = -\kappa\,\mathbf{r}_v,$$

since $\mathbf{r}_u$ and $\mathbf{r}_v$ are principal directions as any tangent vector is. Differentiating the first equation with respect to v, the second with respect to u, we get

$$N_{uv} = -\kappa_v\,\mathbf{r}_u - \kappa\,\mathbf{r}_{uv} \quad\text{and}\quad N_{uv} = -\kappa_u\,\mathbf{r}_v - \kappa\,\mathbf{r}_{uv},$$

from which $\kappa_v\,\mathbf{r}_u = \kappa_u\,\mathbf{r}_v$. Since $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent, the last equation can hold only if $\kappa_u = \kappa_v = 0$, i.e. if κ is constant.

1st case: $\kappa \equiv 0$. In this case $N_u = N_v = 0$, therefore the normal vector is constant along the surface. The derivative of the function $\langle N,\mathbf{r}\rangle$ with respect to u and v is $\langle N, \mathbf{r}_u\rangle = \langle N, \mathbf{r}_v\rangle = 0$ because N is perpendicular to the tangent vectors $\mathbf{r}_u, \mathbf{r}_v$, hence $\langle N, \mathbf{r}\rangle$ is constant and the surface is contained in a plane with equation $\langle N, \mathbf{x}\rangle = \text{const}$.

2nd case: $\kappa \neq 0$. We claim that in this case the surface is contained in a sphere. The facts we have so far suggests that if the claim is true then the center of the sphere should be $\mathbf{r} + (1/\kappa)\, N$. Setting $\mathbf{p} = \mathbf{r} + (1/\kappa)\, N$, we have to make sure first that $\mathbf{p}$ does not depend on u and v. Indeed, differentiating with respect to u results

$$\mathbf{p}_u = \mathbf{r}_u + (1/\kappa)\, N_u = \mathbf{r}_u - (1/\kappa)\, \kappa\, \mathbf{r}_u = 0 \qquad (\kappa \text{ is constant!})$$

and similarly,

$$\mathbf{p}_v = \mathbf{r}_v + (1/\kappa)\, N_v = \mathbf{r}_v - (1/\kappa)\, \kappa\, \mathbf{r}_v = 0.$$

Now to show that the surface lies on a sphere centered at $\mathbf{p}$ we have to prove that the function $\|\mathbf{r}-\mathbf{p}\|$ is constant. This follows from

$$\frac{\partial}{\partial u} \|\mathbf{r}-\mathbf{p}\|^2 = 2 \langle \mathbf{r}_u, \mathbf{r}-\mathbf{p} \rangle = 2 \langle \mathbf{r}_u, (1/\kappa) N \rangle = 0,$$

and

$$\frac{\partial}{\partial v} \|\mathbf{r}-\mathbf{p}\|^2 = 2 \langle \mathbf{r}_v, \mathbf{r}-\mathbf{p} \rangle = 2 \langle \mathbf{r}_v, (1/\kappa) N \rangle = 0.$$

The theorem is proved. ∎

The next example shows how to compute the principal curvatures and directions for a surface of revolution. We consider a positive function $f: [a,b] \longrightarrow \mathbb{R}_+$ and the surface of revolution generated by rotation of its graph about x-axis. This surface can be parameterized by the mapping

$$\mathbf{r}(u,v) = \big(u,\ f(u) \cos v,\ f(u) \sin v \big).$$

The tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are obtained by partial differentiation

$$\mathbf{r}_u(u,v) = \big(1,\ f'(u) \cos v,\ f'(u) \sin v \big),$$
$$\mathbf{r}_v(u,v) = \big(0, - f(u) \sin v,\ f(u) \cos v \big).$$

The matrix of the first fundamental form with respect to the basis $\mathbf{r}_u, \mathbf{r}_v$ is

$$\mathcal{G} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u\rangle & \langle \mathbf{r}_u, \mathbf{r}_v\rangle \\ \langle \mathbf{r}_v, \mathbf{r}_u\rangle & \langle \mathbf{r}_v, \mathbf{r}_v\rangle \end{pmatrix} = \begin{pmatrix} 1 + f'^2(u) & 0 \\ 0 & f^2(u) \end{pmatrix}.$$

To obtain the matrix of the second fundamental form we need the normal vector field and the second order partial derivatives of $\mathbf{r}$.

$$\mathbf{r}_u \times \mathbf{r}_v = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & f'(u) \cos v & f'(u) \sin v \\ 0 & - f(u) \sin v & f(u) \cos v \end{pmatrix} =$$

$$= \big(f'(u)f(u),\ -f(u) \cos v,\ -f(u) \sin v \big)$$

$$N = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{1 + f'^2(u)}} \big(f'(u),\ - \cos v,\ - \sin v \big)$$

$$\mathbf{r}_{uu}(u,v) = \big(0,\ f''(u)\cos v,\ f''(u)\sin v \big),$$
$$\mathbf{r}_{uv}(u,v) = \big(0, -f'(u)\sin v,\ f'(u)\cos v \big),$$
$$\mathbf{r}_{vv}(u,v) = \big(0, -f(u)\cos v, -f(u)\sin v \big).$$

$$\mathcal{B} = \begin{pmatrix} \langle N,\mathbf{r}_{uu}\rangle & \langle N,\mathbf{r}_{uv}\rangle \\ \langle N,\mathbf{r}_{vu}\rangle & \langle N,\mathbf{r}_{vv}\rangle \end{pmatrix} = \frac{1}{\sqrt{1+f'^2(u)}} \begin{pmatrix} -f''(u) & 0 \\ 0 & f(u) \end{pmatrix}.$$

The matrix of the Weingarten map with respect to the basis $\mathbf{r}_u, \mathbf{r}_v$ is

$$\mathcal{L} = \mathcal{B}\,\mathcal{G}^{-1} = \begin{pmatrix} -f''(u)\big(1+f'^2(u)\big)^{-3/2} & 0 \\ 0 & f^{-1}(u)\big(1+f'^2(u)\big)^{-1/2} \end{pmatrix}.$$

As we see, the matrix of the Weingarten map is diagonal, consequently $\mathbf{r}_u, \mathbf{r}_v$ are eigenvectors the diagonal elements of $\mathcal{L}$ are eigenvalues of the Weingarten map. Thus the principal curvatures of the surface are

$$\kappa_1(u,v) = -\frac{f''(u)}{\big(1+f'^2(u)\big)^{3/2}} \qquad \kappa_2(u,v) = \frac{1}{f(u)\big(1+f'^2(u)\big)^{1/2}}\ .$$

We could have obtained this result in a more geometrical way. For any point p on the surface, the plane through p and the x-axis is a symmetry plane of the surface. Thus, reflection of a principal direction of the surface at p is also a principal direction (with the same principal curvature). The principal curvatures at p are either equal and then every direction is principal, or different and then the principal directions are unique. Since a direction is invariant under a reflection in a plane if and only if it is parallel or orthogonal to the plane, we may conclude that $\mathbf{r}_u$ and $\mathbf{r}_v$ are principal directions of the surface. Principal curvatures are the curvatures of the normal sections of the surface in the direction $\mathbf{r}_u, \mathbf{r}_v$.

The normal section of the surface in the direction $\mathbf{r}_u$ is the graph of f rotated about the x-axis (a meridian of the surface). Its curvature can be calculated according to the formula known for plane curves and gives κ_1 up to sign. The difference in sign is due to the fact that the unit normal of the surface and the principal normal of the meridian are opposite to one another.

The plane passing through p perpendicular to the x-axis intersects the surface in a circle the tangent of which at p is $\mathbf{r}_v$. The curvature of this circle is $\frac{1}{f(u)}$. The normal curvature $\kappa_2 = k(\mathbf{r}_v)$ of the surface in the direction $\mathbf{r}_v$ and the curvature of the circle intersection are related to one another by Meusnier's theorem as follows

$$\frac{1}{f(u)} = \frac{1}{\cos\alpha}\,\kappa_2\ ,$$

where α is the angle between the normal of the surface and the principal normal of the circle. As it is easy to see, α is the direction angle of the tangent to the meridian at p, that is, by elementary calculus

$\operatorname{tg} \alpha = f'(u)$, from which $\cos \alpha = \dfrac{1}{\sqrt{1 + f'^2(u)}}$.

Therefore, we get

$$\kappa_2 = \frac{1}{f(u)\left(1 + f'^2(u)\right)^{1/2}}$$

as before. The equation $\dfrac{1}{f(u)} = \dfrac{1}{\cos \alpha}\kappa_2$ has the following consequence.

Corollary. The second principal radius of curvature $\dfrac{1}{\kappa_2}$ of a surface of revolution at a given point p is the length of the segment of the normal of the surface between p and the x-axis intercept.

As an application, let us show that the surface of revolution generated by the tractrix has constant -1 Gaussian curvature. For this reason the surface is called pseudosphere. Its intrinsic geometry is locally the same as that of Bolyai's and Lobatchevsky's hyperbolic plane. This local model of hyperbolic geometry was discovered by Beltrami.

The tractrix is defined as the involute of the chain curve $\gamma(t)=(t, \operatorname{ch} t)$ touching the chain curve at (0,1). The length of the chain curve arc between $\gamma(0)$ and $\gamma(t)$ is

$$\int_0^t \sqrt{\|\gamma'(t)\|}\, dt = \int_0^t \sqrt{1 + \operatorname{sh}^2 t}\, dt = \int_0^t \operatorname{ch} t\, dt = \operatorname{sh} t.$$

This way, the tractrix has the parameterization

$$\hat{\gamma}(t) = \gamma(t) - \operatorname{sh} t \frac{\gamma'(t)}{\|\gamma'(t)\|} = (t, \operatorname{ch} t) - \operatorname{sh} t \frac{(1, \operatorname{sh} t)}{\operatorname{ch} t} = \left(t - \operatorname{th} t\ ,\ \frac{1}{\operatorname{ch} t}\right).$$

As we know from the theory of evolutes and involutes, the chain curve is the evolute of the tractrix, the segment $\gamma(t)\hat{\gamma}(t)$ is normal to the tractrix, and its length is the radius of curvature of the tractrix at $\hat{\gamma}(t)$. This implies from one hand that the first principal curvature of the pseudosphere is $\kappa_1 = -\dfrac{1}{\operatorname{sh} t}$. On the other hand, we obtain that the equation of the normal line of the tractrix at $\hat{\gamma}(t)$ is

$$\frac{y - \operatorname{ch} t}{x - t} = \operatorname{sh} t.$$

The x-intercept is

$$\left(t - \frac{\operatorname{ch} t}{\operatorname{sh} t}\ ,\ 0\right).$$

According to the general results on surfaces of revolution, the second principal radius of curvature of the pseudosphere is the distance between $\hat{\gamma}(t)$ and $(t - \operatorname{cth} t, 0)$, i.e.

$$\kappa_2^{-1} = \left|\left(\operatorname{cth} t - \operatorname{th} t,\ (\operatorname{ch} t)^{-1}\right|\right. = \left(\left(\frac{\operatorname{ch}^2 t - \operatorname{sh}^2 t}{\operatorname{ch} t \operatorname{sh} t}\right)^2 + \frac{1}{\operatorname{ch}^2 t}\right)^{1/2} =$$

$$= \left(\frac{1}{sh^2 t\ ch^2 t} + \frac{1}{ch^2 t} \right)^{1/2} = \frac{1}{sh\ t}.$$

This shows that $K = \kappa_1 \kappa_2 \equiv -1$.

Definition. A regular curve on a surface is said to be a line of curvature if the tangent vectors of the curve are principal directions.

There are many parameterizations of a hypersurface. In applications we should always try to find a parameterization that makes solving the problem easier. For theoretical purposes, it is good to know the existence of certain parameterizations that have nice properties. In what follows, we study parameterizations, for which coordinate lines are lines of curvature.

Theorem. The coordinate lines of a regular parameterization $\mathbf{r}: \Omega \longrightarrow \mathbb{R}^n$ of a hypersurface are lines of curvature if the matrices of the first and second fundamental forms are diagonal. The converse is also true if the principal curvatures of the hypersurface are different at each point.

Proof. The matrix $\mathcal{L}$ of the Weingarten map is the quotient $\mathcal{B}\,\mathcal{G}^{-1}$ of the matrices of the first and second fundamental forms. If these matrices are diagonal, then so is $\mathcal{L}$. Obviously, the matrix of a linear map with respect to a basis is diagonal if and only the basis consists of eigenvectors of the linear map. In our case, we get that $\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}$ form an eigenvector basis for the Weingarten map, i.e. these vectors are principal directions. Since $\mathbf{r}_i$ is the tangent vector of the i-th family of coordinate lines, the coordinate lines are lines of curvature.

Now suppose that the coordinate lines are lines of curvature and that the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ corresponding to the principal directions $\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}$ are different at every point. In this case,

$$\kappa_i \langle \mathbf{r}_i, \mathbf{r}_j \rangle = \langle L\mathbf{r}_i, \mathbf{r}_j \rangle = \langle \mathbf{r}_i, L\mathbf{r}_j \rangle = \kappa_j \langle \mathbf{r}_i, \mathbf{r}_j \rangle,$$

$$(\kappa_i - \kappa_j)\langle \mathbf{r}_i, \mathbf{r}_j \rangle = 0,$$

and since $(\kappa_i - \kappa_j) \neq 0$ for $i \neq j$,

$$g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle = 0 \quad \text{if } i \neq j.$$

Hence, the matrix $\mathcal{G}$ of the first fundamental form is diagonal. The matrix $\mathcal{L}$ of the Weingarten map is also diagonal by our assumption, consequently the matrix $\mathcal{B} = \mathcal{L}\,\mathcal{G}$ of the second fundamental form is diagonal as well. ■

Theorem. Suppose that a regular parameterized surface in $\mathbb{R}^3$ has no umbilical points. Then every point of the surface has a neighborhood that admits a reparameterization with respect to which coordinate lines are lines of curvature.

Proof. (sketch) The complete proof of the theorem rests upon some results on ordinary differential equations, so we only indicate the geometrical part of the construction of such a parameterization. Since the surface contains no umbilics, principle directions are uniquely defined at each point and one can find two smooth unit vector fields ξ,η along the surface that show in the principal directions at every point.

As it is known, if we are given a tangential vector field ζ and a point p on a surface, then there exists a curve γ_p on the surface such that $\gamma_p(0) = p$ the speed vector $\gamma_p'(t)$ of the curve is just $\zeta(\gamma_p(t))$ for every t from the domain of γ_p. Such curves are called integral curves of the vector field ζ through p. Every integral curve is contained in a maximal one which is unique.

Let us denote by γ^1_* and γ^2_* the integral curves of the vector fields ξ and η through $*$. They are lines of curvature of the surface. Fix a point p and consider the parameterization that assigns to a pair of real numbers (u,v) the intersection point of the curves $\gamma^1_{(\gamma^2(v))}$ and $\gamma^2_{(\gamma^1(u))}$. This parameterization is well defined in a small neighborhood of the origin in $\mathbb{R}^2$. It is smooth and maps onto a neighborhood of p, while the coordinate lines are certain reparameterizations of the integral curves of the vector fields ξ and η i.e. coordinate lines are lines of curvature. ▮

Remarks.

The parameterization constructed above has also the property that the coordinate lines through p are parameterized by arc length.

The theorem does not hold for higher dimensions. (Study where the above proof breaks down.)

Now we give a description of lines of curvature on ellipsoids. Our approach, which is based on Dupin's theorem, works for any surfaces of second order. Dupin's theorem claims that if we have three families of surfaces such that the surfaces of any of the families foliate an open domain in $\mathbb{R}^3$, and surfaces from different families intersect one another orthogonally, then the intersection curves of surfaces from different families are lines of curvature. We can obtain families of surfaces in a natural way considering a curvilinear coordinate system on $\mathbb{R}^3$.

Definition. A curvilinear coordinate system on $\mathbb{R}^3$ is a one to one smooth mapping $\mathbf{r}:\Omega\longrightarrow \mathbb{R}^3$ from an open domain of $\mathbb{R}^3$ onto an open domain of $\mathbb{R}^3$ with smooth inverse. Ω is foliated by planes parallel to one of the three coordinate planes in $\mathbb{R}^3$. The images of these planes are the coordinate surfaces of the coordinate system $\mathbf{r}$. There are three families of coordinate surfaces. Each family foliates the same domain, the image of $\mathbf{r}$. Coordinate surfaces from different families intersect one another in coordinate lines. We say that $\mathbf{r}$ defines a triply orthogonal system of surfaces, if coordinate surfaces from different families intersect one another orthogonally, or equivalently, if $\langle \mathbf{r}_i, \mathbf{r}_j \rangle = 0$ for $i \neq j$.

Theorem. (Dupin's theorem). If the curvilinear coordinate system $\mathbf{r}: \Omega \longrightarrow \mathbb{R}^3$ defines a triply orthogonal system, then the coordinate lines are lines of curvature on the coordinate surfaces.

Proof. We may consider without loss of generality the surface $(u,v) \longmapsto \mathbf{r}(u,v,w_0)$. It is enough to show that the matrices of the first and second fundamental forms of the surface with respect to the given parameterization are diagonal. The matrix of the first fundamental form is diagonal by our assumption $\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = 0$. The nondiagonal element of the matrix of the second fundamental form is $\langle \mathbf{r}_{12}, N \rangle$, where N is the unit normal of the surface. Since $\mathbf{r}_3$ is parallel to N, $\langle \mathbf{r}_{12}, N \rangle = 0$ will follow from $\langle \mathbf{r}_{12}, \mathbf{r}_3 \rangle = 0$. Differentiating the equation $\langle \mathbf{r}_1, \mathbf{r}_3 \rangle = 0$ with respect to the second variable yields

$$\langle \mathbf{r}_{12}, \mathbf{r}_3 \rangle + \langle \mathbf{r}_1, \mathbf{r}_{23} \rangle = 0$$

and similarly,

$$\langle \mathbf{r}_{23}, \mathbf{r}_1 \rangle + \langle \mathbf{r}_2, \mathbf{r}_{31} \rangle = 0$$
$$\langle \mathbf{r}_{31}, \mathbf{r}_2 \rangle + \langle \mathbf{r}_3, \mathbf{r}_{12} \rangle = 0.$$

Solving this system of linear equations for the unknown quantities $\langle \mathbf{r}_{12}, \mathbf{r}_3 \rangle$, $\langle \mathbf{r}_{23}, \mathbf{r}_1 \rangle$, $\langle \mathbf{r}_{31}, \mathbf{r}_2 \rangle$, we see that they are all zero. ∎

The canonical equation of an ellipsoid has the form

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1.$$

Suppose $A > B > C$. We can embed this surface into a triply orthogonal system of second order surfaces as follows. Consider the surface

$$F_\lambda: \quad \frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1.$$

F_λ is — an ellipsoid for $\lambda > -C$;

— a one sheeted hyperboloid for $-C > \lambda > -B$;

— a two sheeted hyperboloid for $-B > \lambda > -A$.

In accordance with these cases, we get three families of surfaces. Surfaces obtained by such a perturbation of the equation of a second order surface are called confocal second order surfaces.

Proposition. Let $(x,y,z) \in \mathbb{R}^3$ be a point for which $xyz \neq 0$. Then there exist exactly three λ-s, one from each of the intervals $(-C,+\infty)$, $(-B,-C)$,

$(-A,-B)$ such that $(x,y,z) \in F_\lambda$.

Proof. Condition $(x,y,z) \in F_\lambda$ is equivalent to

$$P(\lambda) = (A+\lambda)(B+\lambda)(C+\lambda) - \big(x^2(B+\lambda)(C+\lambda) + y^2(A+\lambda)(C+\lambda) + z^2(A+\lambda)(B+\lambda)\big) = 0.$$

This is an equation of degree three for λ, so the number of solutions is not more than three. To see that all the three solutions are real and located as it is stated, compute P at the nodes.

$$P(-A) = -x^2(B-A)(C-A) < 0$$
$$P(-B) = -y^2(A-B)(C-B) > 0$$
$$P(-C) = -z^2(A-C)(B-C) < 0$$

Furthermore, $F(\lambda) = \lambda^3 + \ldots$ implies $\lim_{\lambda\to\infty} F(\lambda) = +\infty$. Thus by Bolzano's theorem F has at least one root on each of the intervals $(-C,+\infty)$, $(-B,-C)$, $(-A,-B)$. ∎

Proposition. If $(x,y,z) \in F_\lambda \cap F_{\lambda'}$, $xyz \neq 0$, $\lambda \neq \lambda'$, then F_λ intersects $F_{\lambda'}$ orthogonally.

Lemma. If a nonempty subset M of $\mathbb{R}^3$ is defined by an equation

$$M = \{(x,y,z) \in \mathbb{R}^3 : F(x,y,z) = 0\},$$

so that the gradient vector field $\operatorname{grad} F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is not zero at points of M, then every point of M has a neighborhood (in M) which is the image of a regular parameterized surface. In this case the tangent plane of M at $p \in M$ is orthogonal to grad F (p).

The first part of the lemma is a direct application of the implicit function theorem, we omit details. Suppose that M admits a regular parameterization $\mathbf{r}$ around $p \in M$. Then $F\circ\mathbf{r} \equiv 0$. Differentiating with respect to the i-th variable (i=1,2) using the chain rule we obtain

$$0 = \frac{\partial F\circ\mathbf{r}}{\partial u_i} = \langle (\operatorname{grad} F)\circ\mathbf{r}\ ,\ \mathbf{r}_i \rangle,$$

hence grad F (p) is orthogonal to the tangent vectors $\mathbf{r}_1, \mathbf{r}_2$ that span the tangent space.

Proof. (of proposition) We need to show

$$\operatorname{grad} F_\lambda(x,y,z) \perp \operatorname{grad} F_{\lambda'}(x,y,z),$$

or equivalently,

$$\frac{x^2}{(A+\lambda)(A+\lambda')} + \frac{y^2}{(B+\lambda)(B+\lambda')} + \frac{z^2}{(C+\lambda)(C+\lambda')} = 0.$$

We know that

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1 \quad \text{and} \quad \frac{x^2}{A+\lambda'} + \frac{y^2}{B+\lambda'} + \frac{z^2}{C+\lambda'} = 1.$$

Subtracting these equalities we obtain $(\lambda-\lambda')$ times the equality to prove. Since $\lambda \neq \lambda'$, we are ready. ∎

Regular surfaces swept out by a moving straight line are ruled surfaces. A bit more generally, we shall call a regular surface ruled, if every point of the surface has a neighborhood with a regular parameterization of the form

$$\mathbf{r}(u,v) = \gamma(u) + v\,\delta(u),$$

where γ is a smooth curve, called the directrix, δ is a nowhere zero vector field along γ. The straight lines $v \longmapsto \gamma(u_o) + v\,\delta(u_o)$ are the generators of the surface.

Theorem. The following propositions are equivalent for ruled surfaces:

(i) the normal vector field N is constant along the generators;

(ii) $\mathbf{r}_{uv}$ is tangential for the parameterization $\mathbf{r}(u) = \gamma(u) + v\,\delta(u)$;

(iii) the Gaussian curvature K is constant 0.

Proof. (i) $\Longrightarrow$ (iii) If N is constant along the generators, then $L(\mathbf{r}_v) = -N_v = \mathbf{0} = 0\,\mathbf{r}_v$, thus generators are lines of curvature and the corresponding principal curvature is 0 everywhere. From this follows that the Gaussian curvature is 0.

(iii) $\Longrightarrow$ (i) The normal section of a ruled surface in the direction of a generator is the generator itself. Hence, the normal curvature of the surface in the direction $\mathbf{r}_v$ of the generators is 0. If $\mathbf{r}_v$ were not a principal direction, then 0 would be strictly between the principal curvatures, in which case we would have $K < 0$. The contradiction shows that $\mathbf{r}_v$ is a principal direction at a given point, and thus $-N_v = L(\underline{r}_v) = 0\underline{r}_v = \underline{0}$, i.e. N is constant along the generators.

Remark. We have proved here that $K \leqslant 0$ for any ruled surface.

(iii) $\Longleftrightarrow$ (ii) According to the formula

$$K = \frac{\det \mathcal{B}}{\det \mathcal{G}},$$

$K = 0$ if and only if $\det \mathcal{B} = 0$. Since $\mathbf{r}_{vv} = 0$,

$$\det \mathcal{B} = \det \begin{pmatrix} \langle \mathbf{r}_{uu}, N\rangle & \langle \mathbf{r}_{uv}, N\rangle \\ \langle \mathbf{r}_{uv}, N\rangle & \langle \mathbf{r}_{vv}, N\rangle \end{pmatrix} = \det \begin{pmatrix} \langle \mathbf{r}_{uu}, N\rangle & \langle \mathbf{r}_{uv}, N\rangle \\ \langle \mathbf{r}_{uv}, N\rangle & 0 \end{pmatrix} = \langle \mathbf{r}_{uv}, N\rangle^2$$

and thus $\det \mathcal{B} = 0$ if and only if $\mathbf{r}_{uv}$ is orthogonal to N i.e. if $\mathbf{r}_{uv}$ is tangential. ▮

Definition. A ruled surface that satisfies one of the equivalent conditions of the previous theorem is called a developable surface.

Examples of *ruled but not developable surfaces:*

a) $\mathbf{r}(u,v) = (u, v, uv)$ (hyperbolic paraboloid);

b) $\mathbf{r}(u,v) = (a \cos u,\ b \sin u,\ 0) + v\,(-a \sin u,\ b \cos u,\ c)$ (one sheet hyperboloid)

c) $\mathbf{r}(u,v) = (0,0,u) + v\,(\cos u, \sin u,\ 0)$ (helicoid).

Examples of *developable surfaces:*

a) Cylinders over a curve. Let γ be a regular space curve, $\mathbf{v} \neq \mathbf{0}$ a vector nowhere tangent to γ. We define a cylinder over γ by the parameterization

$$\mathbf{r}(u,v) = \gamma(u) + v\,\mathbf{v}.$$

Since $\mathbf{r}_{uv} = \mathbf{0}$ is tangential, cylinders over a curve are developable.

b) Cones over a curve. Let γ be a regular curve, $\mathbf{p}$ be the position vector of a point not lying on any tangent to the curve. The cone over γ with vertex $\mathbf{p}$ is defined by the parameterization

$$\mathbf{r}(u,v) = v\,\gamma(u) + (1-v)\,\mathbf{p}\ .$$

The cone is regular only in the domain $v \neq 0$. The tangent plane of the cone is spanned by the vectors

$$\mathbf{r}_u(u,v) = v\,\gamma'(u) \quad \text{and} \quad \mathbf{r}_v(u,v) = \gamma(u) - \mathbf{p}.$$

Since

$$\mathbf{r}_{uv}(u,v) = \gamma'(u) = (1/v)\,\mathbf{r}_u(u,v),$$

cones are developable.

c) Tangential developables. Let γ be a curve of general type in $\mathbb{R}^3$. We show that the regular part of the surface swept out by the tangent lines of γ is developable. Indeed, the surface can be parameterized by

$$\mathbf{r}(u,v) = \gamma(u) + v\,\gamma'(u).$$

Partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_u(u,v) = \gamma'(u) + v\,\gamma''(u) \quad \text{and} \quad \mathbf{r}_v(u,v) = \gamma'(u).$$

Since γ is of general type, $\gamma'(u)$ and $\gamma''(u)$ are linearly independent, hence singularities of the surface of tangent lines are located along the generating curve γ.

Since $\mathbf{r}_{uv}(u,v) = \gamma''(u) = (1/v)\big(\mathbf{r}_u(u,v) - \mathbf{r}_v(u,v)\big)$ for $v \neq 0$, the regular part of the surface of tangent lines is developable. Surfaces of this type are called tangential developables.

As we proved in the theorem above, a ruled surface with Gaussian curvature equal to zero is developable. The following theorem states that in most cases the condition of being ruled follows from $K \equiv 0$.

Theorem. If the Gaussian curvature of a surface is 0 everywhere and the surface contains no planar point, then it is developable.

Proof. Gaussian curvature is positive at spherical points so the surface contains no umbilics. Therefore we may consider a parameterization $\mathbf{r}$ around any point $\mathbf{p}$ such that $\mathbf{p} = \mathbf{r}(0,0)$, coordinate lines are lines of curvature and the coordinate lines through p are unit speed curves (see unit 6). Suppose that $\mathbf{r}_u$ corresponds to the nonzero principal curvature $\kappa_1 = \kappa \neq 0$. Then

$$N_u = -\kappa\, \mathbf{r}_u \qquad \text{and} \qquad N_v = 0\, \mathbf{r}_v = \mathbf{0}\ .$$

The second equation shows that N is constant along v-coordinate lines. What we have to show is that v-coordinate lines are straight lines. For this purpose, it is enough to show that $\mathbf{r}$ is linear in v, i.e. $\mathbf{r}_{vv} = \mathbf{0}$. We prove this in a tricky way using the fact that the only vector which is perpendicular to each vectors of a basis is the zero vector. According to this proposition, it suffices to show that $\mathbf{r}_{vv}$ is orthogonal to the vectors N, $\mathbf{r}_u$, $\mathbf{r}_v$.

(i) $\mathbf{r}_{vv} \perp N$. This equation follows from

$$0 = \frac{\partial}{\partial v} \langle \mathbf{r}_v, N \rangle = \langle \mathbf{r}_{vv}, N \rangle + \langle \mathbf{r}_v, N_v \rangle = \langle \mathbf{r}_{vv}, N \rangle.$$

(ii) $\mathbf{r}_{vv} \perp \mathbf{r}_u$. Since lines of curvature are orthogonal,

$$0 = \frac{\partial}{\partial v} \langle \mathbf{r}_u, \mathbf{r}_v \rangle = \langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle + \langle \mathbf{r}_u, \mathbf{r}_{vv} \rangle.$$

On the other hand,

$$\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle = \langle -\frac{\partial}{\partial v} \big((1/\kappa)\, N_u\big), \mathbf{r}_v \rangle = \frac{\partial}{\partial v} \big((1/\kappa)\big)\, \kappa \langle \mathbf{r}_u, \mathbf{r}_v \rangle + (1/\kappa)\langle N_{vu}, \mathbf{r}_v \rangle = 0.$$

Combining these two equalities we get $\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle = 0$.

(iii) $\mathbf{r}_{vv} \perp \mathbf{r}_v$. This will follow from the observation that v-coordinate lines are all parameterized by arc length, i.e. $\| \mathbf{r}_v \| \equiv 1$. We know that $\| \mathbf{r}_v(0,v) \| = 1$ by the construction of $\mathbf{r}$. We also have

$$\frac{\partial}{\partial u} \langle \mathbf{r}_v, \mathbf{r}_v \rangle = 2\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle = 0,$$

showing that $\langle \mathbf{r}_v, \mathbf{r}_v \rangle$ does not depend on u. Thus,

$$\| \mathbf{r}_v(u,v) \| = \| \mathbf{r}_v(0,v) \| = 1 \text{ for every } u,v.$$

Now differentiating with respect to v,

$$0 = \frac{\partial}{\partial v} \langle \mathbf{r}_v, \mathbf{r}_v \rangle = 2\langle \mathbf{r}_{vv}, \mathbf{r}_v \rangle.$$

This completes the proof. ▮

We finish the investigation of developable surfaces with a structure theorem stating that every developable surface is made up of pieces of cylinders, cones and tangential developables.

Theorem. Let $\mathbf{r}: [a,b] \times [c,d] \longrightarrow \mathbb{R}^3$ be a developable surface without planar points and suppose that the parameterization $\mathbf{r}$ of the surface is the one we used in the proof of the previous theorem. Then there exists a nowhere dense closed subset A of [a,b] the complement of which is a union of open intervals $[a,b]\setminus A = I_1 \cup I_2 \cup \ldots$ such that the restriction of $\mathbf{r}$ onto $I_n \times [c,d]$ is a part of a cylinder or cone or a tangential developable.

Proof. As it was proved above, $\mathbf{r}$ has the form $\mathbf{r}(u,v) = \mathbf{a}(u) + v\mathbf{b}(u)$, where $\mathbf{b}(u)$ is a unit vector field along the curve $\mathbf{a}(u)$. We have

$$\mathbf{r}_u(u,v) = \mathbf{a}'(u) + v\, \mathbf{b}'(u) \qquad \mathbf{r}_v(u,v) = \mathbf{b}(u) \qquad \mathbf{r}_{uv}(u,v) = \mathbf{b}'(u).$$

By the definition of developable surfaces, $\mathbf{r}_{uv}(u,v) = \mathbf{b}'(u)$ must be tangential to the surface, i.e. it lies in the plane spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$. $\mathbf{r}_u$ is orthogonal to $\mathbf{r}_v$ as they are lines of curvature, furthermore, $\mathbf{r}_{uv}$ is also orthogonal to $\mathbf{r}_v$ since $0 = \langle \mathbf{b}(u),\mathbf{b}(u)\rangle' = 2\langle \mathbf{b}(u),\mathbf{b}'(u)\rangle$. For there is only one direction in a plane which is orthogonal to a given nonzero vector, $\mathbf{r}_u$ and $\mathbf{r}_{uv}$ must be parallel: $\mathbf{b}'(u)\|\mathbf{a}'(u)+v\mathbf{b}'(u)$, or equivalently, $\mathbf{b}'(u)\|\mathbf{a}'(u)$. Hence, $\mathbf{b}'(u) = c(u)\ \mathbf{a}'(u)$ for some function $c\colon [a,b]\longrightarrow \mathbb{R}$. Now let A be the set of those roots of $c\ c'$ in $[a,b]$, which do not have a neighborhood consisting of only roots of $c\ c'$. A is closed and nowhere dense in $[a,b]$. If $[a,b]\setminus A$ is the union of the disjoint open intervals $I_1, I_2, \dots$, then for the restriction of c onto I_n, we have one of the following possibilities:

(i) the restriction is identically 0;

(ii) the restriction is a nonzero constant;

(iii) the restriction is strictly monotone and nowhere zero.

In the first case, $\mathbf{b}'(u) = 0$ and thus $\mathbf{b}$ is constant on I_n, thus the restriction of $\mathbf{r}$ onto $I_n\times[c,d]$ is a part of a cylinder.

In the second case, the point $\mathbf{p}(u) = \mathbf{a}(u)-(1/c)\mathbf{b}(u)$ does not depend on u. Indeed, $\mathbf{p}'(u) = \mathbf{a}'(u)-(1/c)\mathbf{b}'(u) = \mathbf{0}$. Furthermore, the point $\mathbf{p}$ lies on every generator of the surface, so this case serves a part of a cone.

Finally, consider the curve $\gamma(u) = \mathbf{a}(u)-(1/c(u))\mathbf{b}(u)$ for the last case. As

$$\gamma'(u) = \mathbf{a}'(u) - (1/c(u))'\mathbf{b}(u) - (1/c(u))\mathbf{b}'(u) = -(1/c(u))'\mathbf{b}(u)\|\mathbf{b}(u),$$

the tangent of γ at $\gamma(u)$ coincides with the generator $v\longmapsto \mathbf{r}(u,v)$ of the surface. γ is of general type, since $\gamma'(u)\times\gamma''(u)=\left((1/c(u))'\right)^2\mathbf{b}(u)\times\mathbf{b}'(u) = \left((1/c(u))'\right)^2 c(u)\ \mathbf{b}(u)\times\mathbf{a}'(u) \neq \mathbf{0}$. We conclude that in the third case the restriction of $\mathbf{r}$ onto $I_n\times[c,d]$ is a part of the tangential developable generated by the curve of general type γ. ∎